# The distributions of some quantities for Erlang(2) risk models

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**Abstract**

We study the distributions of [1] the first time that the surplus reaches a given level and [2] the duration of negative surplus in a Sparre Andersen risk process with the inter-claim times being Erlang(2) distributed. These distributions can be obtained through the inversion of Laplace transforms using the inversion relationship for the Erlang(2) risk model given by Dickson and Li (2010).

**Keywords:** Sparre Andersen risk model; Erlang(2) inter-claim times; Generalised Lundberg equation; Duration of negative surplus; First hitting time; Laplace transform.

## 1 Introduction

Consider a Sparre Andersen insurance surplus process

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i\ , \quad t \geq 0\ , \tag{1.1}$$

where $u \geq 0$ is the initial surplus, $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random variables with common distribution function (d.f.) $P = 1 - \bar{P}$ and density function $p$, representing claim amounts. We denote by $\mu_k = E[X_1^k]$ the $k$-th moment of $X_1$. The counting process $\{N(t);\, t \geq 0\}$ denotes the number of claims up to time $t$ and is defined as $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$, where the inter-claim times $\{W_i\}_{i=1}^{\infty}$ are assumed to be i.i.d. random variables with common density function $k$. Further, we assume that $\{W_i\}_{i=1}^{\infty}$ and $\{X_i\}_{i=1}^{\infty}$ are independent.

In this paper, we consider a risk model in which the inter-claim times are Erlang(2) distributed, i.e. $k(t) = \beta^2 t e^{-\beta\, t}$, $t > 0$, $\beta > 0$. We also assume that $c\,E\,[W_1] > E\,[X_1]$, so $2c > \beta\,\mu_1$, providing a positive safety loading factor.

Define $T$ to be the time of ruin from initial surplus $u$ so that $T = \inf\{t : U(t) < 0 \,|\, U(0) = u\}$, with $T = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. We define $\psi(u) = P(T < \infty)$ to be the ultimate ruin probability.

Let

$$G(u, y) = P(T < \infty,\ |U(T)| \leq y \,|\, U(0) = u)$$

so that $G(u, y)$ represents the probability that ruin occurs from initial surplus $u$ with a deficit at ruin of no more than $y$. We denote its (defective) density as $g(u, y)$. Let $Y$ denote the deficit at ruin given that ruin has occurred. Then $\bar{G}(u, y) = G(u, y)/\psi(u)$ and $\bar{g}(u, y) = g(u, y)/\psi(u)$ are the distribution function and the density function of $Y$, respectively.

In this paper we consider the distribution of the first hitting time of the surplus process through level $x > 0$, starting from $u = 0$, and the related problem of the distribution of the duration of periods of negative surplus. This leads to the problem of calculating the distribution of the total duration of negative surplus. By a symmetry argument, the distribution of the first hitting time for our risk model is the same as the distribution of the ruin time in a dual risk model given by

$$U_D(t) = x - c\,t + \sum_{i=1}^{N(t)} X_i\ , \quad t \geq 0\ ,$$

where the condition $2c > \beta\,\mu_1$ for the insurance surplus process would be replaced by $2c < \beta\,\mu_1$. See Cheung (2010), for example, for a discussion of dual risk models.

Let $\hat{a}(s) = \int_0^\infty e^{-sx}\, a(x)\, dx$ denote the Laplace transform of a function $a$.

# 2 The first hitting time

Define

$$T_x = \inf\{t > 0 : U(0) = 0,\, U(t) \geq x\}, \qquad x > 0,$$

to be the first time that the surplus process upcrosses through level $x$, and define for $\delta > 0$

$$R_\delta(x) = \mathbb{E}\left[e^{-\delta T_x} \middle| U(0) = 0\right]$$

to be the Laplace transform of $T_x$. Li (2008) shows that

$$R_\delta(x) = \frac{r_2 - \delta/c}{r_2 - r_1} e^{-r_1 x} + \frac{r_1 - \delta/c}{r_1 - r_2} e^{-r_2 x}, \qquad x > 0, \tag{2.1}$$

where $r_1$ and $r_2$ are the two positive solutions of Lundberg's fundamental equation:

$$\left(s - \frac{\beta+\delta}{c}\right)^2 = \frac{\beta^2}{c^2}\,\hat{p}(s). \tag{2.2}$$

We know from Dickson and Hipp (2001) that $0 < r_1 < (\beta+\delta)/c < r_2$ and

$$\begin{aligned} r_1 &= \frac{\beta+\delta}{c} - \frac{\beta}{c}\hat{q}(r_1), \\ r_2 &= \frac{\beta+\delta}{c} + \frac{\beta}{c}\hat{q}(r_2), \end{aligned}$$

where $\hat{q}(s) = \sqrt{\hat{p}(s)}$.

Let us rewrite equation (2.1) in a different form:

$$\begin{aligned} R_\delta(x) &= e^{-r_1 x} - \frac{r_1 - \delta/c}{r_1 - r_2}\left(e^{-r_1 x} - e^{-r_2 x}\right) \\ &= e^{-r_1 x} - \frac{\beta}{c}(1 - \hat{q}(r_1))\frac{e^{-r_1 x} - e^{-r_2 x}}{r_1 - r_2}. \end{aligned} \tag{2.3}$$

Similarly, we can write

$$R_\delta(x) = e^{-r_2 x} - \frac{\beta}{c}(1 + \hat{q}(r_2))\frac{e^{-r_1 x} - e^{-r_2 x}}{r_1 - r_2}. \tag{2.4}$$

These lead to a third formulation, namely

$$R_\delta(x) = \frac{1}{2}\left(e^{-r_1 x} + e^{-r_2 x}\right) + \frac{\beta}{2c}\left(2 - \hat{q}(r_1) + \hat{q}(r_2)\right)\frac{e^{-r_1 x} - e^{-r_2 x}}{r_2 - r_1}. \tag{2.5}$$

This is our preferred equation for $R_\delta(x)$ since it is symmetric in $r_1$ and $r_2$, unlike equations (2.3) and (2.4). As we shall see, the main task in inverting $R_\delta(x)$ is dealing with $(e^{-r_1 x} - e^{-r_2 x})/(r_2 - r_1)$. This term arises in related Laplace transforms for the Erlang(2) risk model, as shown by Sun (2005).

We can apply the method in Dickson and Li (2010) to invert the Laplace transform in (2.5) to obtain the distribution of $T_x$. Dickson and Li (2010) show that if

$$\hat{f}(r_1) = \int_0^\infty e^{-r_1 t} f(t)dt = \hat{g}(\delta) = \int_0^\infty e^{-\delta t} g(t)dt, \tag{2.6}$$

then

$$g(t) = ce^{-\beta t} f(ct) + \sum_{n=1}^\infty \frac{\beta^n t^{n-1} e^{-\beta t}}{n!}\int_0^{ct} y q^{n*}(ct - y) f(y)dy, \tag{2.7}$$

where $q^{n*}$ denotes the $n$-fold convolution of $q$. Further, if

$$\hat{f}(r_2) = \int_0^\infty e^{-r_2 t} f(t)dt = \hat{h}(\delta) = \int_0^\infty e^{-\delta t} h(t)dt, \tag{2.8}$$

then

$$h(t) = ce^{-\beta t} f(ct) + \sum_{n=1}^{\infty} \frac{(-\beta)^n t^{n-1} e^{-\beta t}}{n!} \int_0^{ct} y q^{n*}(ct-y) f(y) dy. \tag{2.9}$$

From these results, Dickson and Li (2012) observe that $\hat{q}(r_1) - \hat{q}(r_2)$ is the inverse of a function $n$ given by

$$n(t) = 2c \sum_{m=1}^{\infty} \frac{(\beta t)^{2m-1}\, e^{-\beta t}}{(2m)!} q^{2m*}(ct).$$

Similarly, $\hat{q}(r_1) + \hat{q}(r_2)$ is the inverse of a function $m$ given by

$$m(t) = 2c \sum_{n=0}^{\infty} \frac{(\beta t)^{2n}\, e^{-\beta t}}{(2n+1)!} q^{(2n+1)*}(ct).$$

# 3 The density of the first hitting time

In this section we first find general expressions for the density of the first hitting time $T_x$ by inverting its Laplace transform, then we simplify our results for Erlang(2) claims. Consider first inversion of $e^{-r_1 x}$. The equation satisfied by $r_1$ is

$$\beta + \delta - cr_1 = \beta\, \hat{q}(r_1),$$

which is of exactly the same form as Lundberg's fundamental equation for the classical risk model. It therefore follows from Dickson and Willmot (2005) that

$$\begin{aligned} e^{-r_1 x} &= e^{-(\delta+\beta)x/c} + \sum_{n=1}^{\infty} \frac{(\beta/c)^n}{n!} x \int_0^\infty (y+x)^{n-1} e^{-(\delta+\beta)(y+x)/c} q^{n*}(y) dy \\ &= e^{-(\delta+\beta)x/c} + \int_{x/c}^\infty e^{-\delta t}\, \frac{x}{t}\, e(ct-x,t)dt \end{aligned}$$

where

$$e(x,t) = \sum_{n=1}^{\infty} e^{-\beta t} \frac{(\beta t)^n}{n!} q^{n*}(x)$$

is a compound Poisson density function. Thus, the inverse of $e^{-r_1 x}$ is

$$h_{x+}(t) = \begin{cases} e^{-\beta x/c} & \text{for } t = x/c, \\ \frac{x}{t}\, e(ct - x, t) & \text{for } t > x/c. \end{cases}$$

This is just the well-known result from the classical risk model where the Laplace transform of $T_x$ is $e^{-\rho x}$ where $\rho$ is the unique positive solution of Lundberg's fundamental equation for that model. (See Gerber and Shiu (1998).)

Similarly, the inverse of $e^{-r_2 x}$ is $h_{x-}(t)$ where

$$h_{x-}(t) = \begin{cases} e^{-\beta x/c} & \text{for } t = x/c, \\ \frac{x}{t} \sum_{n=1}^{\infty} e^{-\beta t} \frac{(-1)^n (\beta t)^n}{n!} q^{n*}(ct - x) = \bar{h}_x(t) & \text{for } t > x/c. \end{cases} \tag{3.1}$$

Hence the inverse of $\left(e^{-r_1 x} + e^{-r_2 x}\right)/2$ is

$$\eta_x(t) = \begin{cases} e^{-\beta x/c} & \text{for } t = x/c, \\ \frac{x}{t} \sum_{n=1}^{\infty} e^{-\beta t} \frac{(\beta t)^{2n}}{(2n)!} q^{2n*}(ct - x) = \bar{\eta}_x(t) & \text{for } t > x/c, \end{cases} \tag{3.2}$$

and the inverse of $\left(e^{-r_1 x} - e^{-r_2 x}\right)/2$ is

$$\gamma_x(t) = \frac{x}{t} \sum_{n=1}^{\infty} e^{-\beta t} \frac{(\beta t)^{2n-1}}{(2n-1)!} q^{(2n-1)*}(ct - x) \qquad \text{for } t > x/c. \tag{3.3}$$

Now let $A_x(t)$ be such that

$$\int_0^\infty e^{-\delta t} A_x(t) dt = \frac{e^{-r_1 x} - e^{-r_2 x}}{r_2 - r_1}.$$

To find $A_x(t)$ we note that

$$\begin{aligned} \frac{e^{-r_1 x} - e^{-r_2 x}}{r_2 - r_1} &= -e^{-r_1 x} \frac{e^{-(r_2 - r_1)x} - 1}{r_2 - r_1} \\ &= -e^{-r_1 x} \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!} (r_2 - r_1)^{n-1} \\ &= -e^{-r_1 x} \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!} \left(\frac{\beta}{c}\right)^{n-1} (\hat{q}(r_1) + \hat{q}(r_2))^{n-1} \\ &= -e^{-r_1 x} \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!} \left(\frac{\beta\, \hat{m}(\delta)}{c}\right)^{n-1}. \end{aligned}$$

Similarly,

$$\frac{e^{-r_1 x} - e^{-r_2 x}}{r_2 - r_1} = e^{-r_2 x} \sum_{n=1}^{\infty} \frac{x^n}{n!} \left(\frac{\beta\, \hat{m}(\delta)}{c}\right)^{n-1}, \tag{3.4}$$

giving

$$
\begin{aligned}
\frac{e^{-r_1 x} - e^{-r_2 x}}{r_2 - r_1} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^n}{n!} \left( \frac{\beta\, \hat{m}(\delta)}{c} \right)^{n-1} \left( (-1)^{n-1} e^{-r_1 x} + e^{-r_2 x} \right) \\
&= \frac{x}{2} \left( e^{-r_1 x} + e^{-r_2 x} \right) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \left( \frac{\beta\, \hat{m}(\delta)}{c} \right)^{2n} \left( e^{-r_1 x} + e^{-r_2 x} \right) \\
&\quad - \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \left( \frac{\beta\, \hat{m}(\delta)}{c} \right)^{2n-1} \left( e^{-r_1 x} - e^{-r_2 x} \right).
\end{aligned}
$$

Hence by formulae (3.2) and (3.3), for $t > x/c$

$$
A_x(t) = x \left( \bar{\eta}_x(t) + \sum_{n=1}^{\infty} \frac{(\beta x/c)^{2n}}{(2n+1)!}\, \bar{\eta}_x * m^{2n*}(t) - \sum_{n=1}^{\infty} \frac{(\beta x/c)^{2n-1}}{(2n)!}\, \gamma_x * m^{(2n-1)*}(t) \right),
$$

with $A_x(x/c) = x\, e^{-\beta x/c}$. Finally, let $f_{T_x}$ be the density function of $T_x$. Then inversion of (2.5) gives

$$
f_{T_x}(t) = \bar{\eta}_x(t) + \frac{\beta}{c} A_x(t) - \frac{\beta}{2c}\, n * A_x(t) \tag{3.5}
$$

for $t > x/c$, with

$$
\Pr(T_x = x/c) = \eta_x(x/c) + \frac{\beta}{c} A_x(x/c) = e^{-\beta x/c} \left(1 + \beta x/c\right) = \Pr(W_1 > x/c).
$$

Although formula (3.5) is a general result, it is a formula that is not easily implemented, even for simple claim size distributions like the exponential distribution. As an alternative approach to finding $A_x(t)$, we can use the fact that $\hat{m}(\delta) = \hat{q}(r_1) + \hat{q}(r_2)$ to write formula (3.4) as

$$
\begin{aligned}
& e^{-r_2 x} \left( x + \sum_{m=1}^{\infty} \left( \frac{\beta}{c} \right)^m \frac{x^{m+1}}{(m+1)!} \left( \hat{q}(r_1) + \hat{q}(r_2) \right)^m \right) \\
=\ & x e^{-r_2 x} \left( 1 + \sum_{m=1}^{\infty} \left( \frac{\beta}{c} \right)^m \frac{x^m}{(m+1)!} \sum_{j=0}^{m} \binom{m}{j} \hat{q}(r_1)^{m-j}\, \hat{q}(r_2)^j \right).
\end{aligned}
$$

Now let $B_{k,m}(t)$ be such that

$$
\int_0^{\infty} e^{-\delta t}\, B_{k,m}(t)\, dt = \hat{q}(r_1)^k\, \hat{q}(r_2)^m
$$

for $k = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ so that the inverse of

$$\sum_{m=1}^{\infty} \left(\frac{\beta}{c}\right)^m \frac{x^m}{(m+1)!} \sum_{j=0}^{m} \binom{m}{j} \hat{q}(r_1)^{m-j}\, \hat{q}(r_2)^j$$

is a function $B_x$ given by

$$B_x(t) = \sum_{m=1}^{\infty} \left(\frac{\beta}{c}\right)^m \frac{x^m}{(m+1)!} \sum_{j=0}^{m} \binom{m}{j} B_{m-j,j}(t).$$

This gives

$$\begin{aligned} A_x(t) &= x\, h_{x-}(t) + x\, h_{x-} * B_x(t) \\ &= \begin{cases} x\, e^{-\beta x/c} & \text{for } t = x/c, \\ x\, \bar{h}_x(t) + x\, \bar{h}_x * B_x(t) & \text{for } t > x/c. \end{cases} \end{aligned} \quad (3.6)$$

From Dickson and Li (2010, 2012) we know how to find $B_{k,m}(t)$ for certain claim size distributions.

**Example 1** *Let $p(x) = \alpha^2 x e^{-\alpha x}$, so that $q(x) = \alpha e^{-\alpha x}$ and $\hat{q}(r) = \alpha/(\alpha+r)$. Dickson and Li (2012) define $C_{n,m}(t)$ be the inverse of $1/(r_1+\alpha)^{n+1}(r_2+\alpha)^{m+1}$, and show that*

$$C_{n,m}(t) = c^2 t (ct)^{n+m} e^{-(\beta+\alpha c)t} \sum_{l=0}^{\infty} \frac{(\alpha\beta c t^2)^l\, \sigma_{l,n,m}}{\Gamma(n+m+2l+2)}$$

*where*

$$\sigma_{l,n,m} = \sum_{j=0}^{l} (-1)^{l-j} \binom{n+2j+1}{j} \frac{n+1}{n+2j+1} \binom{m+2(l-j)+1}{l-j} \frac{m+1}{m+2(l-j)+1}.$$

*Using these results, the inverse of*

$$\sum_{m=1}^{\infty} \left(\frac{\beta}{c}\right)^m \frac{x^m}{(m+1)!} \sum_{j=0}^{m} \binom{m}{j} \hat{q}(r_1)^{m-j}\, \hat{q}(r_2)^j$$

*is*

$$B_x(t) = \sum_{m=1}^{\infty} \left(\frac{\alpha\,\beta}{c}\right)^m \frac{x^m}{(m+1)!} \sum_{j=0}^{m} \binom{m}{j} C_{m-j-1,j-1}(t).$$

*As $q^{n*}(x) = \alpha^n\, x^{n-1}\, e^{-\alpha x}/\Gamma(a)$, it is straightforward to write down expressions for $\bar{\eta}_x(t)$, $\bar{h}_x(t)$ and $n(t)$, and to express these in terms of hypergeometric functions. Although the convolutions in formulae (3.5) and (3.6) lead to rather messy formulae, it is not a difficult task to evaluate these convolutions by numerical integration. Of course, numerical implementation requires that infinite sums are truncated at an appropriate point.*

# 4 The distribution of the duration of periods of negative surplus

## 4.1 Conditional distributions

In this section, we study the density of the duration of a period of negative surplus, given that the surplus process falls below 0. For the surplus process $\{U(t)\}_{t\geq 0}$, if ruin occurs at time $T$, the process will cross level 0 at time $T+\bar{T}_1$ for the first time, where $\bar{T}_1$ is the duration of the first period of negative surplus. Let $\bar{T}_j$, for $j=2,3,\ldots$, be the duration of the $j$-th period of negative surplus. The distribution of $\bar{T}_j$ depends on the initial surplus, as this affects the phase of the Erlang(2) inter-arrival distribution when the surplus upcrosses level 0 before the surplus drops below 0 for the $j$-th time.

Let

$$D_\delta(u) = E\left[e^{-\delta \bar{T}_1} \,|\, T<\infty\right]$$

be the Laplace transform of $\bar{T}_1$ with respect to $\delta\,(\geq 0)$. Then

$$\begin{aligned} D_\delta(u) &= \int_0^\infty E\left[e^{-\delta \bar{T}_1}|Y=y\right]\,\bar{g}(u,y)\,dy \\ &= \int_0^\infty E[e^{-\delta T_y}]\,\bar{g}(u,y)\,dy \\ &= \int_0^\infty R_\delta(y)\,\bar{g}(u,y)dy. \end{aligned} \tag{4.1}$$

Substituting (2.5) into (4.1) yields

$$D_\delta(u) = \frac{1}{2}(\hat{\bar{g}}(u,r_1)+\hat{\bar{g}}(u,r_2)) + \frac{\beta}{2c}\,(2-\hat{q}(r_1)+\hat{q}(r_2))\,\frac{\hat{\bar{g}}(u,r_1)-\hat{\bar{g}}(u,r_2)}{r_2-r_1}, \tag{4.2}$$

where $\hat{\bar{g}}(u,r_i)=\int_0^\infty e^{-r_i y}\bar{g}(u,y)dy$, for $i=1,2$, are the Laplace transforms of $\bar{g}(u,y)$ with respect to $r_i$.

Let $\psi_i(u)$ and $g_i(u,y)$ be the probability of ruin and the defective density of the deficit at ruin for a modified Erlang(2) surplus process in which the distribution of the time to the first claim is Erlang($i$), for $i=1,2$. Clearly $\psi_2(u)=\psi(u)$ and $g_2(u,y)=g(u,y)$. Then $\bar{g}_i(u,y)=g_i(u,y)/\psi_i(u)$, $i=1,2$, are the densities of the deficit at ruin given that ruin has occurred for the surplus process with the distribution of the time to the first claim being Erlang($i$), for $i=1,2$, respectively.

Let $D_\delta^{(i)}$, $i=1,2$, be the Laplace transform of $\bar{T}_j$ for $j=2,3,\ldots$, given that the distribution of the time to the next claim from the last recovery time prior to the occurrence of the $j$-th period of negative surplus is Erlang($i$). Then

$$D_\delta^{(i)} = \int_0^\infty E\left[e^{-\delta \bar{T}_j} | Y = y\right] \bar{g}_i(0,y)\, dy, \quad i = 1, 2.$$

Clearly $D_\delta^{(2)} = D_\delta(0)$, i.e. the density of $\bar{T}_j$ for $j = 2, 3, \ldots$, is the same as that of $\bar{T}_1$ for $u = 0$ if the distribution of the time to the next claim from the last recovery time prior to the occurrence of the $j$-th period of negative surplus is Erlang(2).

In what follows, we show how to invert $D_\delta(u)$ and $D_\delta^{(1)}$ with respect to $\delta$ to obtain the conditional density function of $\bar{T}_1$ and $\bar{T}_j$ for $j = 2, 3, \ldots$ in two examples. We revisit these examples in Section 4.3.

**Example 2** *Let* $p(x) = \alpha e^{-\alpha x}$ *so that* $q(x) = \sqrt{\alpha/(x\pi)}e^{-\alpha x}$, $x > 0$, *and* $\hat{q}(s) = (\alpha/(s+\alpha))^{1/2}$. *It is well known that* $\bar{g}(u,y) = \alpha e^{-\alpha y}$, *leading to*

$$D_\delta(u) = \frac{1}{2}\left(\frac{\alpha}{r_1+\alpha} + \frac{\alpha}{r_2+\alpha}\right) + \frac{\beta}{2c}\left(2 - \hat{q}(r_1) + \hat{q}(r_2)\right) \frac{\alpha}{(r_1+\alpha)(r_2+\alpha)}. \tag{4.3}$$

*It follows from Dickson and Li (2010) that the inverse of the first term with respect to* $\delta$ *is*

$$c\alpha e^{-(\beta+\alpha c)t} \sum_{m=0}^{\infty} \frac{(c\alpha\beta^2 t^3)^m}{(2m)!(m+1)!} = c\alpha e^{-(\beta+\alpha c)t} {}_0F_2\left(\frac{1}{2}, 2; \frac{c\alpha\beta^2 t^3}{4}\right),$$

*where*

$${}_pF_q(B_1, B_2, \ldots, B_p, C_1, C_2, \ldots, C_q; Z) = \sum_{m=0}^{\infty} \frac{(B_1)_m (B_2)_m \ldots (B_p)_m}{(C_1)_m (C_2)_m \ldots (C_q)_m} \frac{Z^m}{m!}$$

*is the generalised hypergeometric function, with* $(a)_n = \Gamma(a+n)/\Gamma(a)$, *and the inverse of*

$$\frac{\beta\,\alpha}{c(r_1+\alpha)(r_2+\alpha)}$$

*is*

$$c\alpha\beta t\, e^{-(\beta+\alpha c)t} {}_0F_2\left(\frac{3}{2}, 2; \frac{c\alpha\beta^2 t^3}{4}\right). \tag{4.4}$$

*Last,*

$$\frac{\beta\alpha}{2c}\frac{\hat{q}(r_1) - \hat{q}(r_2)}{(r_1+\alpha)(r_2+\alpha)} = \frac{\beta\alpha^{3/2}}{2c(r_1+\alpha)(r_2+\alpha)}\left(\frac{1}{(r_1+\alpha)^{1/2}} - \frac{1}{(r_2+\alpha)^{1/2}}\right). \tag{4.5}$$

*From Dickson and Li (2010), the inverse of* $(r_1+\alpha)^{-1/2}$ *is a function* $V_1(t)$ *given by*

$$V_1(t) = \frac{e^{-(\beta+\alpha c)t}}{2}\sum_{m=0}^{\infty}\frac{\beta^m t^{m-1}}{m!}\,\frac{\alpha^{m/2}(ct)^{(m+1)/2}}{\Gamma\left(\frac{m+1}{2}+1\right)}$$

*and the inverse of* $(r_2+\alpha)^{-1/2}$ *is a function* $W_1(t)$ *given by*

$$W_1(t) = \frac{e^{-(\beta+\alpha c)t}}{2}\sum_{m=0}^{\infty}\frac{(-\beta)^m\, t^{m-1}}{m!}\,\frac{\alpha^{m/2}(ct)^{(m+1)/2}}{\Gamma\left(\frac{m+1}{2}+1\right)}$$

*meaning that the inverse of* $(r_1+\alpha)^{-1/2}-(r_2+\alpha)^{-1/2}$ *is a function, say* $g(t)$*, given by*

$$g(t) = \sum_{n=0}^{\infty}\gamma_n\, e^{-(\beta+\alpha c)t}\, t^{3n+1}$$

*where*

$$\gamma_n = \frac{\beta^{2n+1}\,\alpha^{n+1/2}\,c^{n+1}}{(2n+1)!\,\Gamma(n+2)}\,.$$

*Hence the inverse of (4.5) is the convolution of* $g$ *with a function* $h$*, where from (4.4)*

$$h(t) = \sum_{m=0}^{\infty} h_m\, e^{-(\beta+\alpha c)t}\, t^{3m+1}$$

*where*

$$h_m = \frac{\beta^{2m+1}\alpha^{m+3/2}c^{m+1}}{(2m+2)!\,m!}.$$

*The Laplace transform (with parameter* $s$*) of the product of the* $m$*-th term of* $h(t)$ *and the* $n$*-th term of* $g(t)$ *is*

$$h_m\,\gamma_n\,\frac{\Gamma(3m+2)\,\Gamma(3n+2)}{(\beta+\alpha c+s)^{3(m+n)+4}}$$

*giving*

$$h*g(t) = \sum_{l=0}^{\infty}\sigma_l\,\frac{e^{-(\beta+\alpha c)t}\,t^{3l+3}}{\Gamma(3l+4)}$$

*where*

$$\begin{aligned}\sigma_l &= \sum_{m=0}^{l} h_m\,\gamma_{l-m}\,\Gamma(3m+2)\,\Gamma(3\,(l-m)+2)\\ &= \alpha^{l+2}\,\beta^{2l+2}\,c^{l+2}\sum_{m=0}^{l}\frac{(3m+1)!}{(2m+2)!\,m!}\,\frac{(3(l-m)+1)!}{(2(l-m)+1)!(l-m+1)!}\end{aligned}$$

$$= \quad \frac{\alpha^{l+2}\,\beta^{2l+2}\,c^{l+2}}{2}\sum_{m=0}^{l}\binom{3m+2}{m}\frac{2}{3m+2}\binom{3(l-m)+2}{l-m+1}\frac{1}{3(l-m)+2}. \tag{4.6}$$

*In order to express our final answer in terms of hypergeometric functions (and hence make computation straightforward), it is necessary to express the sum in (4.6) in closed form. As shown in the Appendix,*

$$\sum_{m=0}^{l}\binom{3m+2}{m}\frac{2}{3m+2}\binom{3(l-m)+2}{l-m+1}\frac{1}{3(l-m)+2}=\frac{4\,(3l+3)!}{l!\,(2l+4)!}$$

*giving*

$$h*g(t)=2\,e^{-(\beta+\alpha c)t}\sum_{l=0}^{\infty}\alpha^{l+2}\,\beta^{2l+2}\,c^{l+2}\,\frac{t^{3l+3}}{l!\,(2l+4)!}$$

*Simplification gives*

$$h*g(t)=\frac{1}{12}\,\alpha^2\,\beta^2\,c^2\,t^3\,e^{-(\beta+\alpha c)t}\,{}_0F_2\left(\frac{5}{2},3;\frac{c\alpha\beta^2t^3}{4}\right)$$

*and hence the inverse of (4.3) gives the density of $\bar{T}_1$ as*

$$\begin{aligned} f_{\bar{T}_1}(u,t) \quad &= \quad c\alpha e^{-(\beta+\alpha c)t}\left({}_0F_2\left(\frac{1}{2},2;\frac{c\alpha\beta^2t^3}{4}\right)+\beta t\,{}_0F_2\left(\frac{3}{2},2;\frac{c\alpha\beta^2t^3}{4}\right)\right) \\ &\quad -\frac{1}{12}\,\alpha^2\,\beta^2\,c^2\,t^3\,e^{-(\beta+\alpha c)t}\,{}_0F_2\left(\frac{5}{2},3;\frac{c\alpha\beta^2t^3}{4}\right). \end{aligned}$$

**Remarks**

1. The distribution of the duration of the first period of negative surplus is independent of the initial surplus $u$ due to the memoryless property of the exponential distribution.

2. As $\bar{g}_i(u,y)=\bar{g}(u,y)=\alpha e^{-\alpha y}$, $i=1,2$, the distribution of the duration of other periods of negative surplus is the same as that of the duration of the first period of negative surplus.

**Example 3** *We now consider the case when $p(x)=\alpha^2xe^{-\alpha x}$. It follows from Li and Garrido (2004) or Sun (2005) that*

$$g(u,y)=\alpha e^{-\alpha y}\left(a_{11}e^{-R_1u}+a_{12}e^{-R_2u}\right)+\alpha^2ye^{-\alpha y}\left(a_{21}e^{-R_1u}+a_{22}e^{-R_2u}\right),$$

*where* $-R_1$ *and* $-R_2$ *are the negative solutions of*

$$\left(s-\frac{\beta}{c}\right)^2=\frac{\beta^2}{c^2}\left(\frac{\alpha}{s+\alpha}\right)^2, \tag{4.7}$$

*and*

$$\begin{aligned} a_{11} &= \frac{\beta^2}{c^2\alpha(r+\alpha)^2}\frac{\alpha\,(3\alpha+2r)-R_1(r+2\alpha)}{R_2-R_1}, \\ a_{12} &= \frac{\beta^2}{c^2\alpha(r+\alpha)^2}\frac{\alpha\,(3\alpha+2r)-R_2(r+2\alpha)}{R_1-R_2}, \\ a_{21} &= \frac{\beta^2}{c^2\alpha\,(r+\alpha)}\frac{\alpha-R_1}{R_2-R_1}, \\ a_{22} &= \frac{\beta^2}{c^2\alpha(r+\alpha)}\frac{\alpha-R_2}{R_1-R_2}, \end{aligned}$$

*with* $r>0$ *being the positive solution of equation (4.7). The ultimate ruin probability can be obtained as*

$$\psi(u)=\int_0^\infty g(u,y)dy=(a_{11}+a_{21})\;e^{-R_1u}+(a_{12}+a_{22})\;e^{-R_2u},\qquad u\geq 0.$$

*Then*

$$\bar{g}(u,y)=\frac{g(u,y)}{\psi(u)}=\theta(u)\,\alpha e^{-\alpha y}+(1-\theta(u))\,\alpha^2ye^{-\alpha y},$$

*where*

$$\theta(u)=\frac{a_{11}e^{-R_1u}+a_{12}e^{-R_2u}}{\psi(u)},\quad u\geq 0.$$

*Further*

$$\hat{\bar{g}}(u,r_i)=\theta(u)\frac{\alpha}{r_i+\alpha}+(1-\theta(u))\left(\frac{\alpha}{r_i+\alpha}\right)^2,\quad i=1,2,$$

*so that formula (4.2) simplifies to*

$$\begin{aligned} D_\delta(u) \;=\;& \frac{\alpha\theta(u)}{2}\left(\frac{1}{r_1+\alpha}+\frac{1}{r_2+\alpha}\right)+\frac{\alpha^2\,(1-\theta(u))}{2}\left(\frac{1}{(r_1+\alpha)^2}+\frac{1}{(r_2+\alpha)^2}\right) \\ &+\frac{\alpha\,\beta\,\theta(u)}{c\,(r_1+\alpha)(r_2+\alpha)}+\frac{\alpha^2\,\beta\,(1-\theta(u))}{c\,(r_1+\alpha)(r_2+\alpha)^2}+\frac{\alpha^2\,\beta\,(1-\theta(u))}{c\,(r_1+\alpha)^2(r_2+\alpha)} \\ &+\frac{\alpha^2\,\beta\,\theta(u)}{2c\,(r_1+\alpha)^2(r_2+\alpha)}-\frac{\alpha^2\,\beta\,\theta(u)}{2c\,(r_1+\alpha)(r_2+\alpha)^2} \\ &+\frac{\alpha^3\,\beta\,(1-\theta(u))}{2c\,(r_1+\alpha)^3(r_2+\alpha)}-\frac{\alpha^3\,\beta\,(1-\theta(u))}{2c\,(r_1+\alpha)(r_2+\alpha)^3}. \end{aligned} \tag{4.8}$$

*Then we can express the density of $\bar{T}_1$ as*

$$\begin{aligned} f_{\bar{T}_1}(u,t) &= \frac{\alpha\theta(u)}{2}\left(C_{0,-1}(t)+C_{-1,0}(t)\right)+\frac{\alpha^2\left(1-\theta(u)\right)}{2}\left(C_{1,-1}(t)+C_{-1,1}(t)\right) \\ &+\frac{\alpha\,\beta\,\theta(u)}{c}C_{0,0}(t)+\frac{\alpha^2\,\beta\left(1-\theta(u)/2\right)}{c}C_{1,0}(t)+\frac{\alpha^2\,\beta\left(1-3\theta(u)/2\right)}{c}C_{0,1}(t) \\ &+\frac{\alpha^3\,\beta\left(1-\theta(u)\right)}{2c}\left(C_{2,0}(t)-C_{0,2}(t)\right). \end{aligned}$$

*As shown in Dickson and Li (2010, 2012) each of these $C_{n,m}$ functions can be expressed in terms of hypergeometric functions.*
*To find the density of the duration of other periods of negative surplus, we need only find $g_1(0,y)$ as $g_2(0,y)=g(0,y)$. Dickson and Li (2012) give the bivariate Laplace transform of the time of ruin and the deficit at ruin for the modified surplus process in which the initial surplus is 0 and the distribution of the time to the first claim is exponential with parameter $\beta$. Setting $\delta=0$ and inverting this Laplace transform gives*

$$\begin{aligned} g_1(0,y) &= \frac{c\beta(r+\alpha)^2-\beta^2(r+\alpha)-\beta^2\alpha}{c^2\alpha(r+\alpha)^2}\alpha e^{-\alpha y} \\ &+\frac{c\beta(r+\alpha)+\beta^2}{c^2\alpha(r+\alpha)}\alpha^2 y e^{-\alpha y}, \quad y>0. \end{aligned}$$

*Then*

$$\psi_1(0)=\int_0^\infty g_1(0,y)dy=\frac{2\beta c(r+\alpha)^2-\beta^2\alpha}{\alpha c^2(r+\alpha)^2},$$

*and*

$$\bar{g}_1(0,y)=\frac{g_1(0,y)}{\psi_1(0)}=\theta_1\alpha e^{-\alpha y}+(1-\theta_1)\alpha^2 y e^{-\alpha y}, \quad y>0,$$

*where*

$$\theta_1=\frac{c(r+\alpha)^2-\beta(r+\alpha)-\beta\alpha}{2c(r+\alpha)^2-\beta\alpha}.$$

*Then $D_\delta^{(1)}=\int_0^\infty E[e^{-\delta\bar{T}_j}|Y=y]\bar{g}_1(0,y)dy=\int_0^\infty R_\delta(y)\bar{g}_1(0,y)dy$ and we can find an expression for $D_\delta^{(1)}$ by replacing $\theta(u)$ by $\theta_1$ in (4.8). Let $f_{\bar{T}_2}^{(1)}$ be the conditional density of the duration of other periods of negative surplus, conditioning on there being one phase of the Erlang distribution until the next claim following the previous upcrossing of the surplus process through 0.*

*Then*

$$\begin{aligned} f^{(1)}_{\bar{T}_2}(t) &= \frac{\alpha\theta_1}{2}\left(C_{0,-1}(t)+C_{-1,0}(t)\right)+\frac{\alpha^2(1-\theta_1)}{2}\left(C_{1,-1}(t)+C_{-1,1}(t)\right) \\ &\quad +\frac{\alpha\beta\,\theta_1}{c}C_{0,0}(t)+\frac{\alpha^2\beta\,(1-\theta_1/2)}{c}C_{1,0}(t)+\frac{\alpha^2\beta\,(1-3\theta_1/2)}{c}C_{0,1}(t) \\ &\quad +\frac{\alpha^3\beta\,(1-\theta_1)}{2c}\left(C_{2,0}(t)-C_{0,2}(t)\right), \quad t>0. \end{aligned}$$

## 4.2 The number of periods of negative surplus

We now define $N$ to be the number of periods of negative surplus, and we aim to find the distribution of $N$. For $0 \le u < x$, define $q_j(u,x)$ to be the probability that the surplus process hits level $x$ from initial surplus $u$ in phase $j$ for $j = 1, 2$. It follows from Li (2008, Equation (2.14)) that

$$q_j(u,x) = \mathbf{e}_1\mathbf{H}e^{-\boldsymbol{\Delta}(x-u)}\mathbf{H}^{-1}\mathbf{e}_j^{\top}, \quad j=1,2,$$

where $\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)$, $\boldsymbol{\Delta} = \mathrm{diag}(0,r)$, and

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ 1 & \frac{\beta-cr}{\beta} \end{pmatrix},$$

with $r > 0$ being the positive solution to equation (2.2) with $\delta = 0$. After some simplification we have

$$\begin{aligned} q_1(u,x) &= 1-\frac{\beta}{cr}+\frac{\beta}{cr}e^{-r(x-u)}, \\ q_2(u,x) &= \frac{\beta}{cr}-\frac{\beta}{cr}e^{-r(x-u)}. \end{aligned}$$

Let $\boldsymbol{\Gamma}(u) = [\gamma_{i,j}(u)]_{i,j=1}^2$ be a $2\times 2$ matrix with element $(i,j)$ being the probability that ruin occurs from initial surplus $u$ with $i$ phases until the first claim and that when the surplus upcrosses through 0 the number of phases until the next claim is $j$. Then

$$\begin{aligned} \gamma_{i,j}(u) &= \int_0^\infty g_i(u,y)q_{3-j}(0,y)dy \\ &= \begin{cases} \frac{\beta}{cr}\left(\psi_i(u)-\hat{g}_i(u,r)\right) & j=1, \\ \left(1-\frac{\beta}{cr}\right)\psi_i(u)+\frac{\beta}{cr}\hat{g}_i(u,r), & j=2. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} \Pr(N=0) &= 1-\psi(u), \\ \Pr(N=n) &= \mathbf{e}_2\,\boldsymbol{\Gamma}(u)\left[\boldsymbol{\Gamma}(0)\right]^{n-1}\boldsymbol{\phi}^{\top}(0), \quad n=1,2,\ldots, \end{aligned}$$

where $\phi(0) = (\phi_1(0), \phi_2(0))$ with $\phi_i(0) = 1 - \psi_i(0)$.

Now let $\boldsymbol{\alpha}_u = \mathbf{e}_2 \, \boldsymbol{\Gamma}(u) = \left(\frac{\beta}{cr}\left(\psi(u) - \hat{g}(u,r)\right), (1 - \frac{\beta}{cr})\psi(u) + \frac{\beta}{cr}\hat{g}(u,r)\right)$ and let $\mathbf{I}$ be the $2 \times 2$ identity matrix. Note that $\boldsymbol{\phi}^\top(0) = [\mathbf{I} - \boldsymbol{\Gamma}(0)]\mathbf{1}^\top$, where $\mathbf{1} = (1,1)$. Then the distribution of $N$ can be re-expressed as

$$\begin{aligned}\Pr(N = 0) &= 1 - \boldsymbol{\alpha}_u \mathbf{1}^\top, \\ \Pr(N = n) &= \boldsymbol{\alpha}_u \left[\boldsymbol{\Gamma}(0)\right]^{n-1} \left[\mathbf{I} - \boldsymbol{\Gamma}(0)\right] \mathbf{1}^\top, \quad n = 1, 2, \ldots.\end{aligned}$$

This shows that $N$ follows a discrete phase-type distribution with representation $(\boldsymbol{\alpha}_u, \boldsymbol{\Gamma}(0))$.

**Example 4** *Let $p(x) = \alpha e^{-\alpha x}$. Then it is well-known, e.g. Grandell (1991), that $\psi(u) = (1 - R/\alpha)e^{-Ru}$ where $-R$ is the unique negative solution of*

$$\left(s - \frac{\beta}{c}\right)^2 = \frac{\beta^2}{c^2} \frac{\alpha}{s + \alpha}$$

*and from*

$$\phi_1(u) = \phi_2(u) - \frac{c}{\beta} \frac{d}{du} \phi_2(u)$$

*(see, for example, Dickson and Li (2012)) we obtain $\psi_1(u) = (1 + cR/\beta)\psi_2(u)$. As $g_i(u,y) = \psi_i(u)\, \alpha e^{-\alpha y}$, we have*

$$\gamma_{ij}(u) = \begin{cases} \left(\frac{\beta}{cr} - \frac{\beta\,\alpha}{cr(\alpha + r)}\right)\psi_i(u), & j = 1, \\ \left(1 - \frac{\beta}{cr} + \frac{\beta\,\alpha}{cr(\alpha + r)}\right)\psi_i(u), & j = 2. \end{cases}$$

*When $\beta = 2$, $\alpha = 1$ and $c = 1.2$ we have $\Pr(N = 0) = 1 - \psi(u)$, and for $n = 1, 2, 3, \ldots$*

$$\Pr(N = n) = 0.1698(0.8302^{n-1})\psi(u).$$

## 4.3 Unconditional distributions

We can use the same ideas as in Section 4.2 to find the distribution of the $j$-th period of negative surplus for $j = 2, 3, 4, \ldots$. Let $\mathbf{a}_j$ be a vector given by

$$\mathbf{a}_j = \mathbf{e}_2 \, \boldsymbol{\Gamma}(u) \, \boldsymbol{\Gamma}(0)^{j-2}.$$

Then for $i = 1, 2$ the $i$-th element of $\mathbf{a}_j$ is the probability that the $(j-1)$-th upcrossing of the surplus process through 0 occurs with $i$ phases until the next claim. Thus, with probability $(\mathbf{a}_j)_1 \, \psi_1(0)$ the density of $\bar{T}_j$ is $f^{(1)}_{\bar{T}_2}(t)$ and with probability $(\mathbf{a}_j)_2\,\psi_2(0)$ it is $f_{\bar{T}_1}(0,t)$. Thus, given that there is a $j$-th period of negative surplus, its density is

$$g_j(t) = \xi_j\; f^{(1)}_{\bar{T}_2}(t) + (1-\xi_j)\, f_{\bar{T}_1}(0,t) \tag{4.9}$$

where

$$\xi_j = \frac{(\mathbf{a}_j)_1\,\psi_1(0)}{(\mathbf{a}_j)_1\,\psi_2(0) + (\mathbf{a}_j)_2\,\psi_2(0)}\,.$$

In principle, this allows us to specify the distribution of the total duration of negative surplus. The following examples illustrate contrasting situations.

**Example 5** *Let $p(x) = \alpha e^{-\alpha x}$. Then we have noted that $\{\bar{T}_j\}_{j=1}^{\infty}$ are i.i.d. random variables. Consequently, the distribution of the total duration of negative surplus is compound phase-type, meaning that we can use the recursive formula proposed by Wu and Li (2010) to calculate this distribution by discretising the distribution of $\bar{T}_1$.*

**Example 6** *Let $p(x) = \alpha^2 x e^{-\alpha x}$. In this case we can calculate the elements of $\mathbf{\Gamma}(u)$ using*

$$g_1(u,y) = g(u,y) - \frac{c}{\beta}\frac{d}{du}g(u,y).$$

*Table 4.1 shows the weight $\xi_j$ for different values of $j$ when $u = 0$ and 10, with $\alpha = 2$ and $c = 1.1$. As $j$ increases, the value of $\xi_j$ for each value of*

| $j$ | $\xi_j$ $(u=0)$ | $\xi_j$ $(u=10)$ |
|---|---|---|
| 2 | 0491300 | 0.489598 |
| 3 | 0.527392 | 0.527342 |
| 4 | 0.528454 | 0.528453 |
| 5 | 0.528486 | 0.528486 |
| 6 | 0.528487 | 0.528487 |
| 7 | 0.528487 | 0.528487 |

Table 4.1: Values of $\xi_j$

*$u$ is unchanged (to the number of decimal places shown), suggesting that the effect of the initial surplus on the distribution of $\bar{T}_j$ quickly diminishes as $j$ increases. The total duration of negative surplus is now a random sum where the quantities being summed are not i.i.d. random variables. In such a situation convolutions must be calculated numerically and there does not appear to be a neat approach to calculating the distribution of the total duration of negative surplus.*

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# APPENDIX

In order evaluate the sum in (4.6), we make use of the generalised binomial series $\mathcal{B}_t(z)$ described in Graham et al (1994), defined by

$$\mathcal{B}_t(z) = \sum_{k=0}^{\infty} \binom{tk+1}{k} \frac{z^k}{tk+1}$$

and satisfying

$$\mathcal{B}_t(z)^r = \sum_{k=0}^{\infty} \binom{tk+r}{k} \frac{r\, z^k}{tk+r}.$$

We can write the sum in (4.6) in an obvious notation as

$$\sum_{m=0}^{l} \binom{3m+2}{m} \frac{2}{3m+2} \binom{3(l-m)+2}{l-m+1} \frac{1}{3(l-m)+2} = \sum_{m=0}^{l} a_m\, b_{l-m}\,.$$

Define $A(z) = \sum_{m=0}^{\infty} a_m\, z^m$ and $B(z) = \sum_{m=0}^{\infty} b_m\, z^m$. Then $A(z) = \mathcal{B}_3(z)^2$ and

$$\begin{aligned} B(z) &= \sum_{n=0}^{\infty} \binom{3n+2}{n+1} \frac{z^n}{3n+2} = \sum_{m=1}^{\infty} \binom{3m-1}{m} \frac{z^{m-1}}{3m-1} \\ &= z^{-1} \sum_{m=0}^{\infty} \binom{3m-1}{m} \frac{z^m}{3m-1} + z^{-1} \\ &= z^{-1}(1 - \mathcal{B}_3(z)^{-1}). \end{aligned}$$

Thus, if $C(z) = A(z)B(z) = \sum_{m=0}^{\infty} c_l\, z^l$, then $c_l = \sum_{m=0}^{l} a_m\, b_{l-m}$, but we also have

$$C(z) = \mathcal{B}_3(z)^2\, z^{-1}(1 - \mathcal{B}_3(z)^{-1}) = z^{-1}\mathcal{B}_3(z)^2 - z^{-1}\mathcal{B}_3(z).$$

The coefficient of $z^l$ in $C(z)$ is thus the coefficient of $z^{l+1}$ in $\mathcal{B}_3(z)^2 - \mathcal{B}_3(z)$, giving

$$c_l = \binom{3l+5}{l+1} \frac{2}{3l+5} - \binom{3l+4}{l+1} \frac{1}{3l+4} = \frac{4\,(3l+3)!}{l!\,(2l+4)!}\,.$$

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