

**Notes on the Construction of Geometric
Representations of Confidence Intervals of Ratios
using Stata, Gauss and Eviews**

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Notes on the Construction of Geometric Representations of Confidence Intervals of Ratios using Stata, Gauss and Eviews

Supplementary Materials for:

A Geometric Comparison of the Delta and Fieller Confidence Intervals

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Abstract:

These notes demonstrate how one can define constrained extrema problems whose solutions can be interpreted as the Delta and the Fieller confidence intervals for the ratio of parameters that are estimated by normally distributed parameter estimates. Also included in these notes are the details of the derivation of the slopes of a constraint ellipse that is common to both constrained extrema problems. In addition, these notes provide an example of how one might generate a graphic representation of both constrained extrema problems using the Stata, Gauss and Eviews statistical computer programs.

Key words: Fieller method, Delta method, marginal ellipse

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1. Introduction

A statistic defined as the ratio of two parameters, which are estimated by normally distributed random variables, is often encountered in applied work. The Delta method has been nominated as the most common technique for drawing inferences for such nonlinear combinations. The primary alternative is the Fieller method (or theorem) (1932, 1944, and 1954) which is derived from the properties of a ratio of bivariate normally distributed random variables (see Marsaglia (1965) and Hinkley (1969) for a detailed discussion of these cases and Zerbe (1978) for an application to the general linear model). In these notes we demonstrate the derivation of the two constrained extrema problems whose solutions are these bounds as well as the derivation of an expression for the slopes of the constraint ellipse. We also present an example of how the constraint ellipse may be constructed geometrically with a number of widely available computer programs. This paper provides details for the analysis given in Hirschberg and Lye (2010).

2. The confidence interval for a linear combination of parameters as the solution to a constrained extrema problem.

It can be shown that the $100(1-\alpha)\%$ confidence bounds for a linear combination of a vector of parameters whose estimates are normally distributed random variables are the extrema of the Lagrangian function as proposed by Durand (1954) and Scheffé (1959 appendix III) defined as:

$$L = (\mathbf{a}'\boldsymbol{\beta} + \mathbf{c}) - \lambda \left((\mathbf{B} - \boldsymbol{\beta})' \boldsymbol{\Sigma}^{-1} (\mathbf{B} - \boldsymbol{\beta}) - z^2 \right). \quad (1)$$

In the general case of a linear combination of a k dimensional parameter vector $\boldsymbol{\beta}$ that is estimated by a normally distributed random vector \mathbf{B} :

$$\mathbf{B}_{k \times 1} \sim N(\boldsymbol{\beta}_{k \times 1}, \boldsymbol{\Sigma}_{k \times k}). \quad (2)$$

Where $\boldsymbol{\Sigma}$ is assumed non-singular and the linear combination is defined as $\theta = \mathbf{a}'\boldsymbol{\beta} + c$ where \mathbf{a} is a $k \times 1$ constant vector and c is a scalar constant. We propose that the $100(1-\alpha)\%$ confidence interval for θ , can be found from the solutions for the maximum and minimum of the Lagrangian function defined by (1). Where z is the appropriate z -statistic for the $100(1-\alpha)\%$ confidence bound (i.e. for $\alpha = .05$ $z = 1.96$), and $z^2 = \chi_1^2$ the square of which is equivalent to a chi-square distributed random variable with one degree of freedom.

Taking the first derivatives of L with respect to the parameters and the multiplier and setting them equal to zero, we find the following first order conditions which can be solved for the extrema of $\theta = \mathbf{a}'\boldsymbol{\beta} + c$ that satisfy the constraint:

$$\frac{\partial L}{\partial \boldsymbol{\beta}} = \mathbf{a} - \lambda 2\boldsymbol{\Sigma}^{-1}(\mathbf{B} - \boldsymbol{\beta}) = 0, \quad (3a)$$

$$\frac{\partial L}{\partial \lambda} = (\mathbf{B} - \boldsymbol{\beta})' \boldsymbol{\Sigma}^{-1}(\mathbf{B} - \boldsymbol{\beta}) - z^2 = 0. \quad (3b)$$

Rewriting (3a) we find that:

$$\mathbf{B} - \boldsymbol{\beta} = \frac{1}{2}\lambda^{-1}\boldsymbol{\Sigma}\mathbf{a}. \quad (4)$$

Which can then be substituted into (3b) to solve for $\frac{1}{4}\lambda^{-2}$:

$$\frac{1}{4}\lambda^{-2} = z^2 (\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a})^{-1}. \quad (5)$$

By assumption the covariance matrix is positive definite thus $\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a} \geq 0$ and we can take the square-root of both sides of (5) to obtain two values for $\frac{1}{2}\lambda^{-1}$: $\frac{1}{2}\lambda^{-1} = \pm z(\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a})^{-\frac{1}{2}}$ and solving (4)

for β : $\beta = \mathbf{B} - \frac{1}{2}\lambda^{-1}\Sigma\mathbf{a}$. By pre-multiplying both sides of this equation by \mathbf{a}' , adding c , and then substituting for the optimal value of $\frac{1}{2}\lambda^{-1}$ we define the expressions for the two extrema values of the constrained linear combination as:

$$\begin{aligned}\theta_{(1-\alpha/2)} &= (\mathbf{a}'\mathbf{B} + c) + z(\mathbf{a}'\hat{\Sigma}\mathbf{a})^{1/2} \\ \theta_{\alpha/2} &= (\mathbf{a}'\mathbf{B} + c) - z(\mathbf{a}'\hat{\Sigma}\mathbf{a})^{1/2}\end{aligned}\quad (6)$$

where $\hat{\Sigma}$ is an estimate of the covariance. Thus, (6) is the usual expression for the $100(1-\alpha)\%$ confidence interval of a linear combination of parameters that are estimated by normally distributed random variables.

3. The Fieller confidence interval for a ratio of parameters as the solution to a constrained extrema problem.

Following Von Luxburg and Franz (2004) the bounds of the ratio of two parameters estimated by normally distributed variables can be found as the maximum and minimum from the following Lagrangian function:

$$\mathbf{L} = \frac{\beta_1}{\beta_2} - \lambda \left[\begin{matrix} (b_1 - \beta_1) & (b_2 - \beta_2) \end{matrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix} - z^2 \right]. \quad (7)$$

If we substitute ω_{ij} as the elements of the inverse of the covariance matrix, $\psi = \beta_1/\beta_2$, and

$\beta_1 = \beta_2\psi$, (7) can be written in terms of ψ as:

$$\mathbf{L} = \psi - \lambda \left((b_2 - \beta_2) (\omega_{22}(b_2 - \beta_2) + \omega_{12}(b_1 - \psi\beta_2)) + (b_1 - \psi\beta_2) (\omega_{12}(b_2 - \beta_2) + \omega_{11}(b_1 - \psi\beta_2)) - z^2 \right). \quad (8)$$

The first order partial derivatives of \mathbf{L} with respect to β_2 , ψ , and λ :

$$\frac{\partial \mathbf{L}}{\partial \psi} = 2\lambda (\beta_2^2 \omega_{12} - b_2 \beta_2 \omega_{12} - b_1 \beta_2 \omega_{11} + \psi \beta_2^2 \omega_{11}) + 1, \quad (9a)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2b_1b_2\omega_{12} + b_1^2\omega_{11} + b_2^2\omega_{22} - z^2 - \beta_2(2b_1\omega_{12} + 2b_2\omega_{22} - \beta_2\omega_{22}) - 2\psi\beta_2(b_1\omega_{11} + b_2\omega_{12} - \beta_2\omega_{12}) + \psi^2\beta_2^2\omega_{11}, \quad (9b)$$

$$\frac{\partial \mathcal{L}}{\partial \beta_2} = 2\lambda(\beta_2(\omega_{22} + 2\psi\omega_{12} + \psi^2\omega_{11}) - b_2\omega_{22} - b_1\omega_{12} - \psi(b_1\omega_{11} + b_2\omega_{12})). \quad (9c)$$

The first order conditions for the extrema are obtained by setting these partial derivatives to zero.

First we can solve $\frac{\partial \mathcal{L}}{\partial \beta_2} = 0$ for β_2 using the expression in (9c). Then we substitute for β_2 in

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ using the expression in (9b) which results in a quadratic equation in ψ . The roots of this

quadratic are given by:

$$\Psi_{\alpha/2}, \Psi_{(1-\alpha/2)} = \frac{(b_1b_2\omega_{12}^2 - z^2\omega_{12} - b_1b_2\omega_{11}\omega_{22})^{-1} \pm z\sqrt{z^2(\omega_{12}^2 - \omega_{11}\omega_{22}) + 2b_1b_2\omega_{11}\omega_{12}\omega_{22} + b_2^2\omega_{11}\omega_{22}^2 - 2b_1b_2\omega_{12}^3 - b_1^2\omega_{11}\omega_{12}^2 + b_1^2\omega_{11}\omega_{22} - b_2^2\omega_{12}^2\omega_{22}}}{z^2\omega_{11} + b_2^2\omega_{12}^2 - b_2^2\omega_{11}\omega_{22}} \quad (10)$$

Alternatively, by using the correspondence between the covariance matrix and its inverse defined as:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = (\sigma_{12}^2 - \sigma_{11}\sigma_{22})^{-1} \begin{bmatrix} -\sigma_{22} & \sigma_{12} \\ \sigma_{12} & -\sigma_{11} \end{bmatrix} \quad (11)$$

We can show that the roots for the constrained extrema problem solution in (10) are equal to bounds determined by the Fieller method as:

$$\Psi_{\alpha/2}, \Psi_{(1-\alpha/2)} = \frac{z^2\sigma_{12} - b_1b_2 \pm z\sqrt{-z^2\sigma_{11}\sigma_{22} - 2b_1b_2\sigma_{12} + z^2\sigma_{12}^2 + b_2^2\sigma_{11} + b_1^2\sigma_{22}}}{z^2\sigma_{22} - b_2^2}. \quad (12)$$

4. The determination of the slopes of the constraint ellipse.

Following Marks (1982) we derive the slopes of tangents to the constraint ellipse defined by:

$$z^2 = [(b_1 - \beta_1) \quad (b_2 - \beta_2)] \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} \begin{bmatrix} (b_1 - \beta_1) \\ (b_2 - \beta_2) \end{bmatrix}. \quad (13)$$

Solving this quadratic for β_1 we obtain:

$$\beta_1 = \frac{1}{\omega_{11}} \left(b_1 \omega_{11} + b_2 \omega_{12} - \beta_2 \omega_{12} \pm \sqrt{2b_2\beta_2\omega_{11}\omega_{22} + z^2\omega_{11} - 2b_2\beta_2\omega_{12}^2 - b_2^2\omega_{11}\omega_{22} - \beta_2^2\omega_{11}\omega_{22} + b_2^2\omega_{12}^2 + \beta_2^2\omega_{12}^2} \right) \quad (14)$$

where b_1 and b_2 are the estimated parameters, ω_{ij} are elements of the inverse of the covariance of the regression parameters, and z^2 is the critical value of the Normal distribution for a two tailed test. When the inverse of the covariance matrix is defined in terms of the correlation coefficient ($\rho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$) we have:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{\sigma_{11}} & \frac{\rho}{\sigma_{12}} \\ \frac{\rho}{\sigma_{12}} & \frac{1}{\sigma_{22}} \end{bmatrix}. \quad (15)$$

And take the first derivative of β_1 with respect to the value of β_2 we define the slopes of the ellipse as:

$$\frac{\partial\beta_1}{\partial\beta_2} = \sqrt{\frac{\sigma_{11}}{\sigma_{22}}} \left(\rho \pm (b_2 - \beta_2) \sqrt{\frac{\rho^2 - 1}{b_2^2 - 2b_2\beta_2 + \beta_2^2 - t^2\sigma_{22}}} \right). \quad (16)$$

Which when evaluated at the estimates where $\beta_2 = b_2$ and $\sigma_{ij} = \hat{\sigma}_{ij}$ results in: $\frac{\partial\beta_1}{\partial\beta_2} = \hat{\rho} \sqrt{\frac{\hat{\sigma}_{11}}{\hat{\sigma}_{22}}}$.

5. An example of the construction of the constraint ellipse using Stata, Gauss, and Eviews.

The data for this example is the file *californian.dta* and from Stock and Watson's text (2007, p. 114). The data is for 420 school districts in the year 1998 on the average fifth grade test scores (y) ($y = testscr$ divided by 10) and the average annual per capita income in the school district measured in tens of thousands of 1998 dollars (z) ($z = avginc$ divided by 10). The regression of interest is:

$$y = \beta_0 + \beta_1 z + \beta_2 z^2 + \varepsilon. \quad (17)$$

The ratio of interest in this example is the turning point of the quadratic function which determines the level of income per capita at which the relationship between test scores and income changes sign. This level of income is defined as a ratio of parameters defined by:

$$\psi = \frac{\beta_1}{-2\beta_2}. \quad (18)$$

The 100(1- α)% joint confidence ellipse is frequently produced in most statistics packages that draw ellipses rather than the marginal ellipse (see Ruud 2000, page 230 for an example of the distinction between joint and marginal ellipses). The joint ellipse is specified as

$$(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})' [\text{cov}(\hat{\boldsymbol{\delta}})]^{-1} (\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}) = 2F_{\alpha}^{-1}(2, T - K). \quad (19)$$

where $\boldsymbol{\delta} = \begin{bmatrix} \beta_1 \\ -2\beta_2 \end{bmatrix}$ and $\hat{\boldsymbol{\delta}}$ is the corresponding OLS estimate. However, the marginal

100(1- α)% confidence ellipse that is used in (1) and (7), is given by:

$$(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})' [\text{cov}(\hat{\boldsymbol{\delta}})]^{-1} (\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}) = F_{\alpha}^{-1}(1, T - K). \quad (20)$$

The marginal 100(1- α)% confidence ellipse as defined in (20) can be found from the joint ellipse (19) by specifying an equivalent confidence level such that $F_{\alpha}^{-1}(1, T - K) = 2F_{\alpha}^{-1}(2, T - K)$.

The Table below lists the appropriate confidence levels to use in order to construct the appropriate marginal ellipse when the computer package is designed to plot only joint ellipses (such as the case of Eviews).

<i>df</i>	100(1- $\tilde{\alpha}$)%	
	100(1- α)% = 95%	100(1- α)% = 90%
10	86.7%	75.8%
20	86.0%	75.0%
30	85.8%	74.7%
40	85.6%	74.6%
50	85.6%	74.5%
100	85.5%	74.3%
200	85.4%	74.2%
5000	85.4%	74.2%

Table 1. The correspondence between 100(1- $\tilde{\alpha}$)% and 100(1- α)% for various sample sizes.

From Table 1 we can define an approximate rule that in a moderately sized sample (>150) to obtain a 95% marginal confidence ellipse (i.e. corresponding to $\alpha = 0.05$ in (20)), the value of $\tilde{\alpha}$ in (19) should be set to 0.15, which corresponds to an 85% joint confidence ellipse.

5.1 Stata Program

In Stata we rewrite the regression equation defined in (17) as

$$y = \gamma_0 + \gamma_1 z + \gamma_2 \left(-\frac{z^2}{2} \right) + \varepsilon. \quad (21)$$

So that the turning point defined in (18) is now the ratio of the parameters:

$$\psi = \frac{\gamma_1}{\gamma_2}, \quad (22)$$

which can be estimated using the OLS estimates $\hat{\gamma}_1$ and $\hat{\gamma}_2$ from (21). The program that generates a plot of the ellipse is given below. The ellipse is generated by calling upon the program *ellip* from Alexandersson (2004). For large samples, to obtain the appropriate dimensions of the marginal confidence ellipse the appropriate boundary constant is a chi square with 1 degree of freedom. The options *xlab* and *ylob* are used to plot the ellipse over appropriate values.

```

use "c:\cubic\californian.dta"
generate z2 = -0.5*z^2
regress y z z2
ellip z z2, coefs c(chi2 1) ylab(0 5 10 15 20 25 30 35 40 45) xlab(0 1 2 3 4 5 6 7 8 9 10 11)

```

5.2 Eviews program (versions 5, 6 and 7)

The regression specified in (21) is defined in the estimation menu as follows:

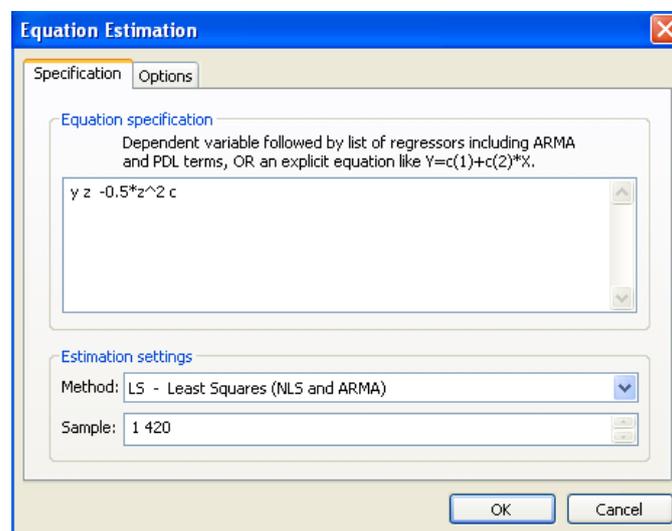


Figure 1 The regression specification dialogue window in Eviews

Then to obtain the 95% marginal confidence ellipse, we use *Confidence Ellipse* available under the *view/Coefficient Tests (Coefficient diagnostics in version 7)* option and specify the confidence level as 0.85. The estimated coefficients correspond to $c(2)$, the estimate of the parameter for the regressor defined as $(-0.5*z^2)$ and $c(1)$, the estimate of the parameter for z . Note that the scales on both the x and y axes can be altered in the default graph by clicking on them.

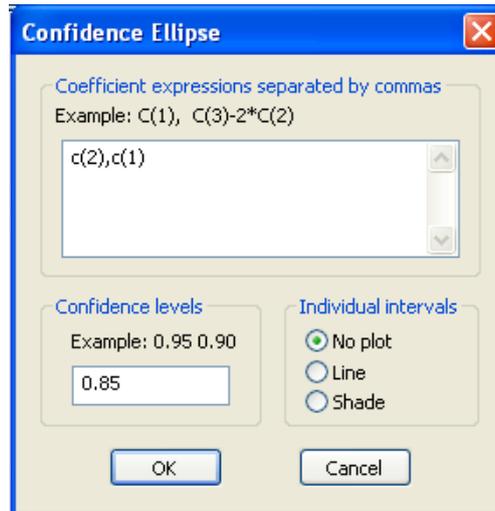


Figure 2 The ellipse dialogue window for the analysis of regression results in Eviews.

5.3 Gauss Program

Gauss is a general purpose computer program for the computation of linear algebra with a similar syntax to MATLAB, Proc IML in SAS and R. In the Gauss program listed below we estimate (21) where we refer to the OLS estimates of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ as $g1$ and $g2$ respectively. *fstat* uses the function *invf* to compute the critical value from the F distribution. Note to obtain the 95% critical value for the appropriate confidence ellipse this is required to be set to 0.15. The program to compute the confidence ellipse is modified from Hill and Adkins (2001, pages 47 & 62). Note that *scale* is used in the plotting routine to plot the ellipse over the appropriate values.

```

/* Read in data and generate data for regression*/
num = 420;k=3; df=num-k;
load data[num,2]= c:\cubic\ellipse.txt;

    y = data[.,1];
    z = data[.,2];
    z2 = -0.5*z.^2;
/* Run Regression*/
    __con=1;
{nam,m,b,stb,vc,std,sig,cx,rsq,resid,dbw} = ols(0,y,z~z2);

/* Obtain estimates and variances and covariances */

```

```

g1 = b[2,1];
g2 = b[3,1];

cov = zeros(2,2);
    cov[1,1] = vc[2,2];
    cov[1,2] = vc[2,3];
    cov[2,1] = cov[1,2];
    cov[2,2] = vc[3,3];

/* Choose critical value for ellipse*/
fstat = invf(2,df,.15);
fn invf(df1,df2,alpha) = 0.95 + 0.05*
minindc( abs( cdffc( seqa(1,.05,2000),df1,df2)- alpha ));

/* Generate ellipse */
a = inv(cov);
q = a[1,1]*a[2,2] - a[1,2]*a[1,2];
lb = g2 - sqrt(2*fstat*a[1,1]/q);
ub = g2 + sqrt(2*fstat*a[1,1]/q);
beta2 = seqa(lb,(ub-lb)/100,101);
csq = (g2 - beta2)^2*( - q/a[1,1]^2)+ fstat*2/a[1,1];
c = sqrt(abs(csq));
betala = g1 + (g2-beta2)*a[1,2]/a[1,1] + c;
betalb = g1 + (g2-beta2)*a[1,2]/a[1,1] - c;

d1 = beta2|rev(beta2);
d2 = betala|rev(betalb);

/* Plot the ellipse and estimated values of the numerator and denominator of
ratio*/
library pgraph;
- paxes = 1;
- pdate = 0;
- pmcolor=9;
let xx = 0 12;
let yy = 0 48;
scale(xx,yy);

xlabel("denominator of ratio");
ylabel("numerator of ratio");
xy(d1,d2);
- psym = zeros(1,7);
- psym[1] = g2;
- psym[2] = g1;
- psym[3] = 8;
- psym[4] = 1;
- psym[5] = 5;
- psym[6] = 1;
- psym[7] = 2;

```

5.4 Construction of Fieller intervals.

Once the ellipse has been drawn it can be pasted into a standard drawing package such as Microsoft Visio or one can construct the bounds using the printed version. For the figure shown below, the default graph from Gauss was imported to Microsoft Visio where the additional lines were added to obtain the lower and upper bounds of the Fieller interval. Note it is necessary to have the origin (0,0) included in the graph to construct the intervals. See Hirschberg and Lye (2010) for details as to the steps for construction of the interval.

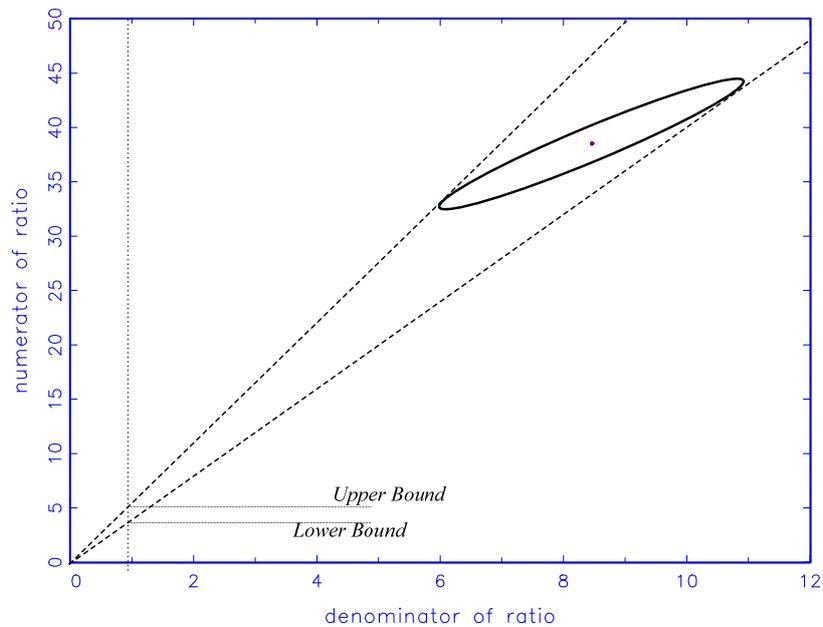


Figure 3 Example of ellipse generated by GAUSS routine with lines added with Microsoft Visio.

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