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PENSION FUNDING WITH MOVING AVERAGE RATES OF RETURN

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Abstract

In the context of the model of pension funding introduced by Dufresne in 1986, explicit expressions are found for the first two moments of fund level and total contributions, when (1) actuarial gains and losses are amortized over N years, and (2) arithmetic rates of return on assets form a moving average process. The results are obtained via a Markovian representation for the bilinear process obtained for the actuarial losses. One conclusion is that the dependence between successive rates of return may have very significant effects on the financial results obtained.

KEYWORDS: BILINEAR PROCESSES; MARKOVIAN REPRESENTATION; MOVING AVERAGE PROCESS; PENSION FUNDING

1. Introduction

We consider the model for the evolution of the assets and liabilities of a defined benefit pension plan studied by Dufresne (1986a). Its main features are that rates of return are random, but the population and plan are stationary. Two methods of determining total contributions were described: (i) proportional control, meaning that the normal cost has an adjustment equal to a fixed fraction of the unfunded liability, and (ii) amortization of gains and losses, which involves calculating each year's unexpected deviation from actuarial expected values, and liquidating each such amount separately over a period of N years. Method (i) is a simplified view of some of the practices of actuaries in the UK; method (ii), however, is part of the actual rules imposed in Canada and the United States for the financing of defined benefit plans. Both methods may be seen as controls applied to the pension funding process (see Dufresne (1993) and (1994) for more on this subject).

Dufresne (1989) was able to calculate explicit expressions for the first moment of fund level and contributions gains and losses are amortized, in the case where the rates of return on assets are i.i.d.; the second moments were obtained only when, moreover, the mean rate of return is equal to the valuation rate of interest. In this paper, we generalize these results to the case where arithmetic rates of return form a moving average ("MA" in the sequel) process of any order, with no restriction on their expected value. This is done by showing that the process representing the actuarial losses is a bilinear time series, and thus has Markovian representation in higher dimension. (Observe that Markovian representations had been used by Dufresne (1990), in a model where geometric rates of return are MA.) We also investigate whether the processes obtained have stationary versions.

Other references on the same general pension funding model include:

— i.i.d. rates of returns: Dufresne (1986a, 1986b, 1988), Haberman (1993b), Haberman and Zimbidis (1993);

— autoregressive rates of return: Dufresne (1993), Haberman (1990a[1], 1990b[1], 1991[1], 1992[1], 1993a, 1994), Gerrard and Haberman (1996), Cairns and Parker (1997).

Haberman and Wong (1997) obtain first and second moments when geometric rates of return form a MA process of order 1 or 2, and proportional control is applied. Bédard (1997) solves this problem in general, for MA(q) geometric rates of return, q an arbitrary positive integer. (The case of MA(q) arithmetic rates of return with proportional control is mathematically simpler).

Section 2 gives the required background on bilinear processes, and then Section 3 shows how they are applied to pension funding. Section 4 presents and analyses some numerical examples, Section 5 concludes the paper.

2. Bilinear processes

An early reference on bilinear processes is Granger and Andersen (1978). We give other references as needed below.

Definition. *A one-dimensional process $X = \{X_t\}$ is a* ***bilinear process of orders*** *p, q, P, Q, denoted $X \sim \mathrm{BL}(p, q, P, Q)$, if it satisfies*

$$X_t = \sum_{k=1}^{p} a_k X_{t-k} + \sum_{k=1}^{q} b_k e_{t-k} + \sum_{j=0}^{Q} \sum_{k=1}^{P} \beta_{jk} e_{t-j} X_{t-k} + \alpha, \qquad (1)$$

where $\{e_t\}$ is i.i.d., and $\{a_k\}$, $\{b_k\}$, $\{\beta_{jk}\}$, α are constants.

We do not assume the errors $\{e_t\}$ to be Gaussian, nor to have mean zero. Note that many authors set $\beta_{0k} = 0$ for all k.

Definition. *The process X in (1) is said to have a* ***bilinear Markovian representation*** *if it satisfies*

$$Z_t = A(e_t) Z_{t-1} + H(e_t) \qquad (2a)$$

$$X_t = B(e_t) Z_{t-1} + K(e_t), \qquad (2b)$$

where

— *Z_t is a column vector of dimension n;*

— *$A(e_t)$, $H(e_t)$, $B(e_t)$, $K(e_t)$ are matrices or vectors of polynomials of finite degree in e_t only, with respective dimensions $n \times n$, $n \times 1$, $1 \times n$, and 1×1;*

— *e_t and Z_{t-k} are independent for every $k \geq 1$.*

Note that bilinear Markovian representations are not unique. Pham (1986) shows that every bilinear process has a Markovian representation (2). We state the

[1] In those papers, some of the formulas for the moments of contributions and fund levels in the case of autoregressive rates of return are incorrect.

result in Theorem 1. His proof gives an explicit construction of the representation, which we apply in Section 3.

Theorem 1. *Every bilinear process has a bilinear Markovian representation.*

The first and second moments of a bilinear process $\{X_t; t \geq 0\}$ can be found recursively from any Markovian representation (2):

$$\begin{aligned} \mathsf{E}X_t &= \mathsf{E}B(e_t)\,\mathsf{E}Z_{t-1} + \mathsf{E}K(e_t) \\ \mathsf{E}Z_t &= \mathsf{E}A(e_t)\,\mathsf{E}Z_{t-1} + \mathsf{E}H(e_t). \end{aligned}$$

Here the independence of e_t and Z_{t-1} is seen to be essential, and it is of course required that the first moments of $A(e_t)$, $B(e_t)$, $K(e_t)$, $H(e_t)$, and the initial conditions $\mathsf{E}X_0, \mathsf{E}X_{-1}, \ldots$ are all finite. In the sequel, we make frequent use of the following matrix operations:

- $\mathrm{vec}\, M$ is the vector obtained from a matrix M by stacking its columns of a matrix one on top of the other (the first column of M is at the top of $\mathrm{vec}\, M$, and so on);
- $M \otimes N$ is the Kronecker product of matrices $M_{m\times n}$ and $N_{p\times q}$, defined as the $mp \times nq$ matrix whose (i,j)th block is $M_{ij}N$, $1 \leq i \leq m$, $1 \leq j \leq n$.

We also use the property (Nicholls & Quinn, p.11) $\mathrm{vec}(MNP) = (P' \otimes M)\mathrm{vec}\, N$.

To simplify the equations, let $\bar{A} = \mathsf{E}A(e_t)$, $\overline{A \otimes A} = \mathsf{E}[A(e_t) \otimes A(e_t)]$, and so on. Iterating the last equation above yields

$$\mathsf{E}Z_t = (I + \bar{A} + \bar{A}^2 + \cdots \bar{A}^{t-1})\bar{H} + \bar{A}^t\mathsf{E}Z_0 = (I - \bar{A})^{-1}(I - \bar{A}^t)\bar{H} + \bar{A}^t\mathsf{E}Z_0,$$

provided $\bar{I} - A$ is invertible. If

$$\rho(\bar{A}) = (\text{maximum modulus of eigenvalues of } \bar{A}) < 1,$$

then

$$\lim_{t\to\infty} \mathsf{E}X_t = \bar{B}(I - \bar{A})^{-1}\bar{H} + \bar{K}.$$

The same can be done for second moments; provided $\overline{A \otimes A}$, $\overline{A \otimes H}$, $\overline{H \otimes A}$, $\overline{H \otimes H}$, $\overline{B \otimes B}$, $\overline{B \otimes K}$, $\overline{K \otimes K}$ are all finite,

$$\begin{aligned} \mathsf{E}X_t^2 &= \overline{B \otimes B}\,\mathrm{vec}\,\mathsf{E}Z_tZ_t' + 2\overline{B \otimes K}\,\mathsf{E}Z_{t-1} + \overline{K \otimes K} \\ \mathrm{vec}\,\mathsf{E}Z_tZ_t' &= \overline{A \otimes A}\,\mathrm{vec}\,\mathsf{E}Z_{t-1}Z_{t-1}' + (\overline{H \otimes A} + \overline{A \otimes H})\mathsf{E}Z_{t-1} + \overline{H \otimes H}. \end{aligned}$$

If, moreover, $\rho(\bar{A}) < 1$ and $\rho(\overline{A \otimes A}) < 1$, then

$$\lim_{t\to\infty} \mathrm{vec}\,\mathsf{E}Z_tZ_t' = (I - \overline{A \otimes A})^{-1}(\overline{H \otimes A} + \overline{A \otimes H}) \lim_{t\to\infty} \mathsf{E}Z_t + \overline{H \otimes H}$$

whence

$$\lim_{t\to\infty} \mathsf{E}X_t^2 = \overline{B \otimes B} \lim_{t\to\infty} \mathrm{vec}\,\mathsf{E}Z_tZ_t' + 2\overline{B \otimes K} + \overline{K \otimes K}.$$

A less obvious question is whether stationary solutions of (2a) $\{Z_t;\ t \in \mathbb{Z}\}$ exist and are unique, and under what conditions. This problem was considered by Conlisk (1974), Nicholls and Quinn (1982), Pham (1985) and Guégan (1987), among others. We quote the following results from those papers.

Theorem 2. *Suppose* $\{e_t;\ t \in \mathbb{Z}\}$ *is i.i.d.,* $\rho(\bar{A}) < 1$ *and* $\rho(\overline{A \otimes A}) < 1$*, and* $\mathsf{E}\,\overline{H \otimes H}$ *is finite. Then (2a) has a unique solution* $\{Z_t;\ t \in \mathbb{Z}\}$*, given by*

$$Z_t \;=\; \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1} A(e_{t-k}) \right) H(e_{t-n}), \qquad t \in \mathbb{Z}.$$

The series on the right converges in L^2 *and a.s., and the process is strictly stationary. The process* $\{X_t;\ t \in \mathbb{Z}\}$ *in (2b) is then also strictly stationary, and has finite second moments if* $\overline{B \otimes B}$ *and* $\mathsf{E}\,K_t^2$ *are finite.*

Remark. The assumptions (i) $\{e_t\}$ Gaussian and (ii) $\mathsf{E}\,H(e_t) = 0$ made in Guégan (1987) are unnecessary. □

3. Amortization of gains and losses

The model is the same as in Dufresne (1986, 1989); we summarize the essential results required. An individual actuarial method (for example, Projected Unit Credit, Entry Age Normal) is applied to a stationary population. The assumptions are:

— there is no inflation on benefits nor on salaries, and the benefit formula is unchanging over time;

— except for rates of return on assets, all actuarial assumptions are realized;

— the population is stationary;

— the valuation rate of interest is fixed throught time;

— the initial unfunded liability is nil.

The following notation is used.

$a_{\overline{m}\rceil}$	Value of m-year annuity-immediate at rate i, equal to $(1-(1+i)^{-N})/i$
$\ddot{a}_{\overline{m}\rceil}$	Value of m-year annuity-due at rate i, equal to $(1-(1+i)^{-N})(1+i)/i$
ADJ	Adjustment made to normal cost (control)
AL	Actuarial liability, or reserve (constant)
BP	Annual benefit payments (constant)
C	Total annual contribution, equal to $NC + ADJ$
F	Value of fund's assets
i	Valuation rate of interest (constant)
L	Actuarial loss
N	Amortization period (a constant positive integer)

NC	Normal cost, or pure premium (constant)
R	Rate of return on assets
r	Average rate of return, equal to $\mathsf{E}\,R_t$
u_k	$= a_{\overline{N-k}\rceil}/\ddot{a}_{\overline{N}\rceil}$
UL	Unfunded actuarial liability, equal to $AL - F$

The evolution through time of assets and liabilities is described by:

$$\begin{aligned} F_t &= (1+R_t)(F_{t-1}+C_{t-1}-BP) &(3)\\ AL &= (1+i)(AL+NC-BP). &(4)\end{aligned}$$

The actuarial loss L_t is defined as the unexpected increase in the unfunded liability, *relative to actuarial assumptions*:

$$L_t = UL_t - \mathsf{E}^A[UL_t\,|\,\mathcal{F}_{t-1}], \tag{5}$$

where "E^A" stands for "expectation according to actuarial assumptions," and $\mathcal{F}_{t-1}$ is the σ-field representing information up to (and including) time $t-1$. Observe that losses may be negative. In our model the actuarial rate of interest is i, meaning that

$$\mathsf{E}^A[R_t\,|\,\mathcal{F}_{t-1}] = i.$$

The adjustment to the pure premium NC is defined as

$$ADJ_t = \sum_{k=0}^{N-1}\frac{L_{t-k}}{\ddot{a}_{\overline{N}\rceil}}, \tag{6}$$

since $L_s/\ddot{a}_{\overline{N}\rceil}$ is the level amount required to amortize L_s over N years, at rate of interest i. It is intuitively clear (and can be verified mathematically, see Dufresne (1989, 1994)) that at any date t the unfunded liability is equal to the unamortized portion of past actuarial losses:

$$UL_t = \sum_{k=0}^{N-1}\frac{\ddot{a}_{\overline{N-k}\rceil}}{\ddot{a}_{\overline{N}\rceil}}L_{t-k}. \tag{7}$$

Thus the total contribution and the fund level may be expressed as

$$C_t = NC + \sum_{k=0}^{N-1}\frac{L_{t-k}}{\ddot{a}_{\overline{N}\rceil}}, \qquad F_t = AL - \sum_{k=0}^{N-1}\frac{\ddot{a}_{\overline{N-k}\rceil}}{\ddot{a}_{\overline{N}\rceil}}L_{t-k}. \tag{8}$$

From (3) and (4), we get

$$\begin{aligned} UL_t &= (1+R_t)(AL-F_{t-1}-ADJ_{t-1}) - (R_t-i)(AL+NC-BP)\\ &= (1+i)(UL_{t-1}-AJ_{t-1}) + (R_t-i)[UL_{t-1}-ADJ_{t-1}-AL/(1+i)],\end{aligned}$$

which, together with (5), (6) and (7), yields

$$
\begin{aligned}
L_t &= (R_t - i)[UL_{t-1} - ADJ_{t-1} - AL/(1+i)] \\
&= (R_t - i)\left(\sum_{k=1}^{N-1} \frac{a_{\overline{N-k}|}}{\ddot{a}_{\overline{N}|}} L_{t-k} - \frac{AL}{1+i}\right). \qquad (9)
\end{aligned}
$$

At this point, we are ready to state our assumption regarding rates of return: if $\{R_t\} \sim$MA(q), then

$$
R_t = r + \sum_{j=0}^{q} d_j e_{t-j},
$$

where $\{d_j\}$ are constants ($d_0 = 1$, $d_q \neq 0$), and $\{e_t\}$ are zero-mean i.i.d. random variables (with an otherwise arbitrary distribution). Inserting this expression into (9), we obtain

$$
\begin{aligned}
L_t = &\sum_{k=1}^{N-1} (r-i)\frac{a_{\overline{N-k}|}}{\ddot{a}_{\overline{N}|}} L_{t-k} - \sum_{j=0}^{q} d_j \frac{AL}{1+i} e_{t-j} \\
&+ \sum_{k=1}^{N-1}\sum_{j=0}^{q} d_j \frac{a_{\overline{N-k}|}}{\ddot{a}_{\overline{N}|}} e_{t-j} L_{t-k} - (r-i)\frac{AL}{1+i}, \qquad (10)
\end{aligned}
$$

and thus $\{L_t\} \sim \mathrm{BL}(N-1, q, N-1, q)$, with (see Eq. (1))

$$
\begin{aligned}
a_k &= (r-i)\frac{a_{\overline{N-k}|}}{\ddot{a}_{\overline{N}|}}, & b_j &= -d_j \tfrac{AL}{1+i}, \\
\beta_{jk} &= d_j \frac{a_{\overline{N-k}|}}{\ddot{a}_{\overline{N}|}}, & \alpha &= -(r-i)\frac{AL}{1+i}.
\end{aligned} \qquad (11)
$$

(To simplify the notation, in the sequel we let $u_k = a_{\overline{N-k}|}/\ddot{a}_{\overline{N}|}$.) From Theorem 1, there is a Markovian representation

$$
Z_t = A(e_t)Z_{t-1} + H(e_t) \qquad (12a)
$$

$$
L_t = B(e_t)Z_{t-1} + K(e_t). \qquad (12b)
$$

Once a Markovian representation for $\{L_t\}$ has been found, the moments of $\{F_t\}$ and $\{C_t\}$ can be obtained from those of $\{L_t\}$. For finite t this is done recursively, starting at $t = 0$. We only specify the limits of those moments as $t \to \infty$. We assume that the required moments of e_t are finite, and to simplify the notation, we drop the "$\lim_{t\to\infty}$," and write the moments of a stationary version of the processes. From Section 2,

$$
\begin{aligned}
z &= \mathsf{E}\, Z_t = (I - \bar{A})^{-1}\bar{H} \\
\Lambda &= \mathsf{E}\, Z_t Z_t' \\
\operatorname{vec}\Lambda &= (I - \overline{A \otimes A})^{-1}\left[(\overline{A \otimes H} + \overline{H \otimes A})z + \overline{H \otimes H}\right] \\
\ell &= \mathsf{E}\, L_t = \bar{B}(I - \bar{A})^{-1}\bar{H} + \bar{K}
\end{aligned}
$$

First,

$$\begin{aligned}
\mathsf{E}\,L_t^2 &= \mathsf{E}\,(B_t\,Z_{t-1}+K_t)\,(Z_{t-1}'\,B_t'+K_t') \\
&= \mathsf{E}\,B_t\,Z_{t-1}\,Z_{t-1}'\,B_t' + \mathsf{E}\,B_t\,Z_{t-1}\,K_t' + \mathsf{E}\,K_t\,Z_{t-1}'\,B_t' + \mathsf{E}\,K_t\,K_t' \\
&= \overline{B\otimes B}\,\mathrm{vec}\,\Lambda + 2\overline{B\otimes K}\,z + \overline{K\otimes K}.
\end{aligned}$$

Second, for $n \geq 1$,

$$\begin{aligned}
\mathsf{E}\,L_t\,L_{t+n} &= \mathsf{E}\,L_t\,(B_{t+n}\,Z_{t+n-1}+K_{t+n}) \\
&= \bar{B}\,\mathsf{E}\,L_t\,Z_{t+n-1} + \ell\,\bar{K}.
\end{aligned}$$

We find

$$\begin{aligned}
\mathsf{E}\,L_t\,Z_{t+n} &= \bar{A}\,\mathsf{E}\,L_t\,Z_{t+n-1} + \ell\,\bar{H} \\
&= (\cdots) \;=\; \bar{A}^n\,\mathsf{E}\,L_t\,Z_t + (I+\cdots+\bar{A}^{n-1})\ell\,\bar{H}, \\
\mathsf{E}\,L_t\,Z_t &= \mathsf{E}\,(A_t\,Z_{t-1}+H_t)(Z_{t-1}'B_t'+K_t') \\
&= \mathsf{E}\,A_t\,Z_{t-1}\,Z_{t-1}'B_t' + \mathsf{E}\,A_t\,Z_{t-1}\,K_t' + \mathsf{E}\,H_t\,Z_{t-1}'B_t' + \mathsf{E}\,H_t\,K_t' \\
&= \overline{B\otimes A}\,\mathrm{vec}\,\Lambda + (\overline{K\otimes A}+\overline{B\otimes H})z + \overline{H\otimes K},
\end{aligned}$$

and thus

$$\mathsf{E}\,L_t\,L_{t+n} \;=\; \bar{B}\left[\bar{A}^{n-1}(\overline{B\otimes A}\,\mathrm{vec}\,\Lambda + (\overline{K\otimes A}+\overline{B\otimes H})z + \overline{H\otimes K}) + \sum_{k=0}^{n-2}\bar{A}^k\ell\,\bar{H}\right] + \ell\,\bar{K}.$$

Therefore:

$$\Gamma(n) \;=\; \mathsf{Cov}(L_t, L_{t+n}) \;=\; \begin{cases} \overline{B\otimes B}\,\mathrm{vec}\,\Lambda + 2\overline{B\otimes K}\,z + \overline{K\otimes K} - \ell^2, & n=0 \\ \bar{B}\left[\bar{A}^{|n|-1}M + \displaystyle\sum_{k=0}^{|n|-2}\bar{A}^k\ell\,\bar{H}\right] + \ell\,\bar{K} - \ell^2, & n\neq 0, \end{cases} \tag{13}$$

with $M = \overline{B\otimes A}\,\mathrm{vec}\,\Lambda + (\overline{K\otimes A}+\overline{B\otimes H})z + \overline{H\otimes K}$. As a partial check, we calculate

$$\begin{aligned}
\mathsf{E}\,L_t\,L_{t+1} &= \mathsf{E}\,(B_{t+1}\,Z_t+K_{t+1})(Z_{t-1}'\,B_t'+K_t') \\
&= \bar{B}\,\mathsf{E}\,Z_t\,Z_{t-1}'\,B_t' + \bar{B}\,\mathsf{E}\,Z_t\,K_t' + \bar{K}\,z'\,\bar{B}' + \bar{K}\,\bar{K}'.
\end{aligned}$$

From $Z_t \;=\; A_t\,Z_{t-1}+H_t$, we get

$$\begin{aligned}
\mathsf{E}\,Z_t\,Z_{t-1}'\,B_t' &= \mathsf{E}\,A_t\,Z_{t-1}\,Z_{t-1}'\,B_t' + \mathsf{E}\,H_t Z_{t-1}'\,B_t' \\
&= \overline{B\otimes A}\,\mathrm{vec}\,\Lambda + \overline{B\otimes H}\,z
\end{aligned}$$

$$\mathsf{E}\,Z_t\,K_t' \;=\; \mathsf{E}\,A_t\,Z_{t-1}\,K_t' + \mathsf{E}\,H_t\,K_t' \;=\; \overline{K\otimes A}\,z + \mathsf{E}\,H_t\,K_t'$$

and

$$\begin{aligned}
\mathsf{E}\,L_t\,L_{t+1} &= \bar{B}(\overline{B\otimes A}\,\mathrm{vec}\,\Lambda + \overline{B\otimes H}\,z + \overline{K\otimes A}\,z + \mathsf{E}\,H_t\,K_t') + \bar{K}\,z'\,\bar{B}' + \bar{K}\,\bar{K}' \\
&= \bar{B}(\overline{B\otimes A}\,\mathrm{vec}\,\Lambda + \overline{B\otimes H}\,z + \overline{K\otimes A}\,z + \overline{H\otimes K}) + \bar{K}\,\ell.
\end{aligned}$$

From (8) and (13), we immediately obtain

$$\mathsf{E}(C_t) = NC + \frac{N\ell}{\ddot{a}_{\overline{N|}}}, \qquad \mathsf{E}(F_t) = AL - \ell \sum_{j=0}^{N-1} \frac{\ddot{a}_{\overline{N-j|}}}{\ddot{a}_{\overline{N|}}}$$

$$\mathsf{Cov}(C_t, C_{t+n}) = \frac{1}{(\ddot{a}_{\overline{N|}})^2} \sum_{j,k=0}^{N-1} \Gamma(j-k+n)$$

$$\mathsf{Cov}(F_t, F_{t+n}) = \frac{1}{(\ddot{a}_{\overline{N|}})^2} \sum_{j,k=0}^{N-1} \ddot{a}_{\overline{N-j|}}\, \ddot{a}_{\overline{N-k|}}\, \Gamma(j-k+n).$$

We have thus shown the following:

Theorem 3. *Suppose a pension fund operates according to Eqs. (3) and (8), with the assumptions made above regarding the population and rates of return. Then*

(a) the process $\{L_t\}$ has a Markovian representation (12);

(b) if $\rho(\bar{A}) < 1$ and $\rho(\overline{A \otimes A}) < 1$, then $\{L_t\}$, $\{C_t\}$ and $\{F_t\}$ have strictly stationary versions for $t \in \mathbb{N}$ or $t \in \mathbb{Z}$ (all three processes realized over the same probability space;

(c) if, moreover, $\overline{B \otimes B}$ and $\mathsf{E}\, K_t^2$ are finite, then $\{L_t\}$, $\{C_t\}$ and $\{F_t\}$ have finite second moments.

We now give three specific Markovian representations for $\{L_t\}$, when rates rates of return assumed to be moving average of order 0, 1 and 2.

<u>***Case*** $\{R_t - r\} \sim \mathrm{MA}(0)$</u>

Here the Markovian representation is obvious:

$$Z_t = (L_{t-(N-2)}, \ldots, L_t)', \qquad H(e_t) = (0, \ldots, 0, ge_t + \alpha)',$$

$$A(e_t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 \\ (r-i+e_t)u_{N-1} & \cdots & & \cdots & (r-i+e_t)u_2 & (r-i+e_t)u_1 \end{pmatrix}_{(N-1)\times(N-1)},$$

$$B_t = (r-i+e_t)(u_{N-1}, \cdots, u_1), \qquad K(e_t) = -(r-i+e_t)AL/(1+i).$$

<u>***Case*** $\{R_t - r\} \sim \mathrm{MA}(1)$</u>

From Eqs.(10) and (11), we get:

$$Z_t = \begin{pmatrix} L_{t-(N-2)} \\ L_{t-(N-3)} \\ \vdots \\ L_{t-1} \\ L_t \\ \sum_{k=1}^{N-1} (a_k L_{t-k+1} + e_t \beta_{1k} L_{t-k+1}) + b_1 e_t \end{pmatrix},$$

$$A(e_t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & 0 & 1 & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \\ 0 & a_{N-1} & & \cdots & a_2 & a_1 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 0 \\ u_{N-1}\beta_{11} & \cdots & \cdots & u_2\beta_{11} & u_1\beta_{11} & 0 \end{pmatrix} e_t^2$$

$$+ \begin{pmatrix} 0 & \cdots & & & \cdots & 0 \\ 0 & \ddots & & & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & 0 \\ u_{N-1} & \cdots & \cdots & u_2 & u_1 & 0 \\ a_1 u_{N-1} & a_1 u_{N-2} + \beta_{1,N-1} & \cdots & a_1 u_2 + \beta_{13} & a_1 u_1 + \beta_{12} & \beta_{11} \end{pmatrix} e_t,$$

$$H(e_t) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ b_0 e_t + \alpha \\ (a_1 b_0 + b_1 + \beta_{11}\alpha) e_t + \beta_{11} b_0 \, e_t^2 + a_1 \alpha \end{pmatrix},$$

$$B(e_t) = (u_{N-1} e_t, u_{N-2} e_t, \cdots, u_2 e_t, u_1 e_t, 1), \qquad K(e_t) = g\, e_t + \alpha.$$

Case $\{R_t - r\} \sim \mathrm{MA}(2)$

In this case, Eq.(10) becomes

$$L_t = \sum_{k=1}^{N-1} a_k L_{t-k} + e_t \sum_{k=1}^{N-1} u_k L_{t-k} + e_{t-1} \sum_{k=1}^{N-1} \beta_{1k} L_{t-k} + e_{t-2} \sum_{k=1}^{N-1} \beta_{2k} L_{t-k} + \sum_{k=0}^{2} b_k e_{t-k} + \alpha, \qquad (14)$$

where $a_k = (r-i)u_k$, $b_j = -d_j AL/(1+i)$, $\beta_{1k} = d_1 u_k$, $\beta_{2k} = d_2 u_k$ and $\alpha = -(r-i)AL/(1+i)$.

Following the procedure given by Pham (1986), we set

$$\begin{aligned}
\beta'_{1k} &= \beta_{1,k+1}, \qquad k = 0,1,\ldots,N-2,\\
\beta'_{2k} &= \beta_{2,k+2}, \qquad k = 0,1,\ldots,N-3,\\
\beta''_{11} &= \beta_{21}.
\end{aligned}$$

and rewrite (14) as

$$\begin{aligned}
L_t &= \sum_{k=1}^{N-1} a_k L_{t-k} + e_t \sum_{k=1}^{N-1} u_k L_{t-k} + \sum_{j=0}^{N-2}\sum_{k=1}^{2} \beta'_{kj} e_{t-k} L_{t-j-k} + \beta''_{11} e_{t-2} L_{t-1}\\
&\qquad + \sum_{k=0}^{2} b_k e_{t-k} + \alpha.
\end{aligned}$$

The Markovian representation obtained is as follows:

$$\begin{aligned}
Z_t &= \begin{pmatrix} Z_t^{(0)} \\ Z_t^{(0)} e_t \\ e_t \end{pmatrix} = \begin{pmatrix} A^{(0)}(e_t) & K^{(0)}(e_t) & D^{(0)}(e_t) \\ A^{(0)}(e_t)e_t & K^{(0)}(e_t)e_t & D^{(0)}(e_t)e_t \\ & 0_{1\times(2N+3)} & \end{pmatrix} Z_{t-1} + \begin{pmatrix} B^{(0)}(e_t) \\ B^{(0)}(e_t)e_t \\ e_t \end{pmatrix}\\
L_t &= (u_{N-1}e_t, \cdots, u_2 e_t, u_1 e_t, 1, 0, \cdots, 0)_{1\times(2N+3)} Z_{t-1} + b_0 e_t + \alpha,
\end{aligned}$$

where

$$Z_t^{(0)} = A^{(0)}(e_t) Z_{t-1}^{(0)} + B^{(0)}(e_t) + (K^{(0)}(e_t) Z_{t-1}^{(0)} + D^{(0)}(e_t)) e_{t-1},$$

$$Z_t^{(0)} = \begin{pmatrix} L_{t-(N-2)} \\ L_{t-(N-3)} \\ \vdots \\ L_{t-1} \\ L_t \\ \sum_{k=1}^{N-1} a_k L_{t-k+1} + \sum_{k=1}^{2} (b_k + \sum_{j=0}^{N-2} \beta'_{kj} L_{t-j-k+1}) e_{t+1-k} + \beta''_{11} e_{t-1} L_t \\ \sum_{k=2}^{N-1} a_k L_{t-k+2} + b_2 e_t + \sum_{j=0}^{N-2} \beta'_{2j} L_{t-j} e_t \end{pmatrix},$$

$$
\begin{aligned}
&A^{(0)}(e_t)\\
&= \begin{pmatrix}
0 & 1 & 0 & \cdots & & \cdots & 0\\
0 & 0 & 1 & 0 & \cdots & \cdots & 0\\
\vdots & \ddots & & & & \ddots & \vdots\\
0 & \cdots & \cdots & 0 & 1 & 0 & 0\\
0 & \cdots & & \cdots & 0 & 1 & 0\\
0 & \cdots & \cdots & \cdots & 0 & a_1 & 1\\
0 & 0 & a_{N-1} & \cdots & a_3 & a_2 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & \cdots & & & \cdots & 0\\
\vdots & \ddots & & & & \vdots\\
\vdots & & & & \ddots & \vdots\\
0 & \cdots & & & \cdots & 0\\
u_{N-1}\beta'_{10} & \cdots & & u_1\beta'_{10} & 0 & 0\\
u_{N-1}\beta'_{20} & \cdots & \cdots & u_1\beta'_{20} & 0 & 0
\end{pmatrix} e_t^2\\
&+ \begin{pmatrix}
0 & \cdots & & & & \cdots & 0\\
\vdots & \ddots & & & & & \vdots\\
\vdots & & & & & \ddots & \vdots\\
0 & \cdots & & & & \cdots & 0\\
u_{N-1} & u_{N-2} & \cdots & \cdots & u_1 & 0 & 0\\
a_1u_{N-1} & a_1u_{N-2}+\beta'_{1,N-2} & \cdots & a_1u_2+\beta'_{12} & a_1u_1+\beta'_{11} & \beta'_{10} & 0\\
a_2u_{N-1} & a_2u_{N-2}+\beta'_{2,N-2} & \cdots & a_2u_2+\beta'_{22} & a_2u_1+\beta'_{21} & \beta'_{20} & 0
\end{pmatrix} e_t,
\end{aligned}
$$

$$
B^{(0)}(e_t) = \begin{pmatrix}
0\\
\vdots\\
\vdots\\
0\\
\alpha + b_0 e_t\\
a_1\alpha + (a_1b_0 + b_1 + \alpha\beta'_{10})e_t + \beta'_{10}b_0e_t^2\\
a_2\alpha + (a_2b_0 + b_2 + \alpha\beta'_{20})e_t + \beta'_{20}b_0e_t^2
\end{pmatrix},
$$

$$
K^{(0)}(e_t) = \begin{pmatrix}
0 & \cdots & & & & \cdots & 0\\
\vdots & \ddots & & & & & \vdots\\
\vdots & & & & & \ddots & \vdots\\
0 & \cdots & & & & \cdots & 0\\
u_{N-1}\beta''_{11}e_t & \cdots & \cdots & u_2\beta''_{11}e_t & \beta''_{11} + u_1\beta''_{11}e_t & 0 & 0\\
0 & \cdots & \cdots & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
D^{(0)}(e_t) = \begin{pmatrix}
0\\
\vdots\\
\vdots\\
0\\
\beta''_{11}\alpha + \beta''_{11}b_0e_t\\
0
\end{pmatrix}.
$$

4. Numerical examples

In the numerical examples given below, $R_t = r + e_t + d_1 e_{t-1}$, where e_t has a Beta(2,2) distribution over $(-b, b)$, that is, a density equal to

$$\frac{3}{4b^3}(b^2 - x^2)\mathbf{1}_{(-b,b)}(x).$$

Here b is adjusted according to the given values of $\mathsf{Var}\, R_t$ and d_1. In all cases $\mathsf{P}(R_t > -1) = 1$, $\rho(\bar{A}) < 1$, $\rho(\overline{A \otimes A}) < 1$, and $\overline{B \otimes B}$, $\overline{H \otimes H}$ and $\mathsf{E}\, K_t^2$ are finite. Of course this choice of distribution for the errors is only a matter of computational convenience, and other choices are possible. Computations show that the moments of contributions and fund levels approach their limiting values very rapidly. This is because the fund already starts at a level equal to AL; hence $\mathsf{Var}\, L_t$ converges very fast as t increases; typically, for instance, $\mathsf{Var}\, L_t$ is very close to its limit value after just a few iterations. This in turn means that after N years or so $\mathsf{Var}\, F_t$ and $\mathsf{Var}\, C_t$ are also very close to their limits.

The examples chosen show the dependence of the moments of contributions and fund levels on N, and on the assumptions regarding the rates of return R_t. The assumptions are comparable to the ones in Dufresne (1989); they are made for illustrative purposes only.

Population	English Life Table No. 13 (Males), stationary, constant salaries
Entry Age	30 (only)
Retirement age	65
Benefits	Straight life annuity (2/3 of salary)
Valuation method	Entry Age Normal
Valuation rate of interest	$i = .01$
Actuarial liability	$AL = 451\%$ of payroll
Normal cost	$NC = 14.5\%$ of payroll

(*N.B.* We imagine here that monetary amounts have initially been deflated by the index for the increase of salaries, so that the limits as time goes to infinity of contributions and reserves are finite. Thus, the valuation rate of interest is net of the rate of increase of salaries, which is why i it is set at such a low level.)

The limits of the standard deviations of F_t and C_t as $t \to \infty$ are shown in Tables 1 to 4. *As in Section 3, we drop the "$\lim_{t\to\infty}$" in front of* $\mathsf{E}\, F_t$*, etc., in effect dealing with stationary versions of the processes.*

Example 1

First, suppose rates of return are i.i.d., and $\mathsf{E}\, R_t = i$. Then it is possible to find the first and second moments of F_t and C_t without a state-space representation. Taking expectations on both sides of (9), we find $\mathsf{E}\, L_t = 0$, and so $\mathsf{E}\, C_t = NC$ and $\mathsf{E}\, F_t = AL$ (this is legitimate if it is known that $\rho(\bar{A}) < 1$ and $\rho(\overline{A \otimes A}) < 1$, and thus that a second-order stationary solution exists). Next, multiplying by L_{t-n},

$n \geq 1$, we get $\mathsf{E}\, L_t\, L_{t-n} = 0$. Thus

$$\begin{aligned}
\mathsf{Var}\, L_t &= \frac{\sigma^2 AL^2/(1+i)^2}{1-\sigma^2 \sum_{j=1}^{N-1} u_j^2} \\
\mathsf{Var}\, F_t &= \sum_{j=0}^{N-1} \left(\frac{\ddot{a}_{\overline{N-j|}}}{\ddot{a}_{\overline{N|}}} \right)^2 \mathsf{Var}\, L_t \\
\mathsf{Var}\, C_t &= \frac{N}{(\ddot{a}_{\overline{N|}})^2} \mathsf{Var}\, L_t.
\end{aligned}$$

Table 1 shows the results when $i = .01$ and σ, the standard deviation of R_t, is either 5 % or 10 %. The variability of contributions decreases with N increasing, while the variablility of the fund level increases; this is the usual trade-off effect, noted in Dufresne (1986a).

Table 1

Example 1. $\{R_t - r\} \sim \mathrm{MA}(0)$, $r = i = .01$

m	$\sigma = .05$				$\sigma = .10$			
	$\frac{\mathsf{E}F}{AL}$	$\frac{\sqrt{\mathsf{Var}F}}{AL}$	$\frac{\mathsf{E}C}{NC}$	$\frac{\sqrt{\mathsf{Var}C}}{NC}$	$\frac{\mathsf{E}F}{AL}$	$\frac{\sqrt{\mathsf{Var}F}}{AL}$	$\frac{\mathsf{E}C}{NC}$	$\frac{\sqrt{\mathsf{Var}C}}{NC}$
5	100.0%	7.4%	100.0%	70.3%	100.0%	14.8%	100.0%	141.3%
10	100.0	9.9	100.0	51.1	100.0	19.9	100.0	103.3
15	100.0	11.9	100.0	42.8	100.0	24.2	100.0	87.2
20	100.0	13.7	100.0	38.1	100.0	28.0	100.0	78.1

Example 2

The second example illustrates what happens when the rates of return are i.i.d. but $\mathsf{E}\, R_t \neq i$. All moments of contributions and fund levels are affected. The first moments may be obtained without a state-space representation; from Eq. (9),

$$\begin{aligned}
\mathsf{E}\, L_t &= (r-i) \sum_{k=1}^{N-1} u_k \mathsf{E}\, L_{t-k} - (r-i)AL/(1+i) \\
&= -\frac{(r-i)AL/(1+i)}{1-(r-i)\sum_{k=1}^{N-1} u_k},
\end{aligned}$$

which yields the first moments of C and F. Table 2 shows that the first moments of F and C are very significantly affected by the change from $\mathsf{E}\, R_t = .01$ to $\mathsf{E}\, R_t = .03$ (the fund is on average larger, and the contributions lower). Variances are also greatly affected. The variances in Table 2 are all larger than the corresponding ones in Table 1. Note that the matrices in the Markovian representation involve polynomials of degree 1 in e_t, and so the first two moments of L_t depend on $\mathsf{E}\, e_t$ and $\mathsf{E}\, e_t^2$ only.

Table 2

Example 2. $\{R_t - r\} \sim \mathrm{MA}(0)$, $r = .03, i = .01$

m	$\sigma = .05$				$\sigma = .10$			
	$\frac{\mathsf{E} F}{AL}$	$\frac{\sqrt{\mathsf{Var} F}}{AL}$	$\frac{\mathsf{E} C}{NC}$	$\frac{\sqrt{\mathsf{Var} C}}{NC}$	$\frac{\mathsf{E} F}{AL}$	$\frac{\sqrt{\mathsf{Var} F}}{AL}$	$\frac{\mathsf{E} C}{NC}$	$\frac{\sqrt{\mathsf{Var} C}}{NC}$
5	106.2%	7.9	34.6%	75.1%	106.2%	15.8%	34.6%	150.9%
10	112.2	11.4	29.2	59.7	112.2	23.1	29.2	120.8
15	118.9	15.1	23.1	55.1	118.9	30.7	23.1	112.5
20	126.6	19.1	16.1	54.4	126.6	39.4	16.1	112.2

Example 3

In this example, $R_t = r + e_t + e_{t-1}$, so that

$$\mathsf{Corr}(R_t, R_{t-1}) = +\frac{1}{2}.$$

Table 3 shows that this not affect expected values much, but that variances are significantly different than in Example 1, where $\mathsf{Corr}(R_t, R_{t-1}) = 0$. The variances of both contributions and fund levels are higher, in all cases, in the presence of dependence between successive rates of return. The explanation is that the losses $\{L_t\}$ are positively correlated. Note that the matrices in the Markovian representation involve polynomials of degree 2 in e_t, and so the first two moments of L_t depend on $\mathsf{E}\, e_t^k$ for $1 \le k \le 4$. It was noted that $\mathsf{Corr}(L_t, L_{t+1})$ is very close to .5 (for all N), but that $\mathsf{Corr}(L_t, L_{t+k})$ was very close to 0 for all $k \ge 0$.

Table 3

Example 3. $\{R_t - r\} \sim \mathrm{MA}(1)$, $r = .01, i = .01$, $d_1 = 1$

m	$\sigma = .05$				$\sigma = .10$			
	$\frac{\mathsf{E}F}{AL}$	$\frac{\sqrt{\mathsf{Var}F}}{AL}$	$\frac{\mathsf{E}C}{NC}$	$\frac{\sqrt{\mathsf{Var}C}}{NC}$	$\frac{\mathsf{E}F}{AL}$	$\frac{\sqrt{\mathsf{Var}F}}{AL}$	$\frac{\mathsf{E}C}{NC}$	$\frac{\sqrt{\mathsf{Var}C}}{NC}$
5	100.3%	9.7%	96.9%	94.7%	101.2%	19.8%	87.4%	192.5%
10	100.6	13.6	96.4	71.2	102.5	28.3	85.3	148.8
15	101.0	16.7	96.1	60.7	103.9	35.9	84.2	130.8
20	101.3	19.5	96.0	54.6	105.3	43.2	83.3	121.8

Example 4

In this example, $R_t = r + e_t - e_{t-1}$, so that

$$\mathsf{Corr}(R_t, R_{t-1}) = -\frac{1}{2}.$$

As in Example 3, it is seen (Table 4) that expected values are not much different from the case of i.i.d. rates of return, but that variances are significantly affected. The variances of both contributions and fund levels are lower, in all cases, in the presence of dependence between successive rates of return. It was noted that $\mathsf{Corr}(L_t, L_{t+1})$ is very close to -.5, but that, as in Example 3, correlations at larger lags were very close to 0.

Table 4

Example 4. $\{R_t - r\} \sim \mathrm{MA}(1)$, $r = .01, i = .01$, $d_1 = -1$

m	$\sigma = .05$				$\sigma = .10$			
	$\frac{\mathsf{E}F}{AL}$	$\frac{\sqrt{\mathsf{Var}F}}{AL}$	$\frac{\mathsf{E}C}{NC}$	$\frac{\sqrt{\mathsf{Var}C}}{NC}$	$\frac{\mathsf{E}F}{AL}$	$\frac{\sqrt{\mathsf{Var}F}}{AL}$	$\frac{\mathsf{E}C}{NC}$	$\frac{\sqrt{\mathsf{Var}C}}{NC}$
5	99.7%	3.8%	103.1%	31.4%	98.8%	7.7%	112.4%	63.0%
10	99.4	3.7	103.6	16.1	97.6	7.3	114.1	32.3
15	99.1	3.6	103.8	11.0	96.4	7.2	114.8	22.0
20	98.7	3.6	104.0	8.5	95.2	7.1	115.3	16.9

5. Conclusion

We have shown how to calculate the moments of contributions and fund levels in a simple pension model, when rates of return are a moving average process of order up to 2. The same principles apply for higher order MA processes, with Markovian representations in higher dimensions. Even though each actuarial loss is separately amortized in full, it is seen the method of amortization of losses is unable to keep average contributions and fund levels equal to the normal cost and the actuarial liablilty, respectively, when rates of return are on average different from the valuation rate of interest (Example 2). We saw that the variances of C and F are significantly affected by the dependence between successive rates of return (Examples 3 and 4).

These considerations may be important when choosing actuarial assumptions, or when actuarial funding legislation is put into place. In particular, requiring the use of a valuation rate of interest lower than average rates of return does not imply that that fund levels would on average equal the actuarial liability computed at the valuation rate.

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