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Upper Bounds for Ultimate Ruin Probabilities in the Sparre Andersen Model with Interest

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Upper bounds for ultimate ruin probabilities in the Sparre Andersen model with interest

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Abstract

We consider the Sparre Andersen model modified by the inclusion of interest on the surplus. Exponential type upper bounds for the ultimate ruin probability are derived by martingale and recursive techniques. Applications of the results to the compound Poisson model are given. Numerical comparisons of upper bounds derived by each technique are presented.

Key words: Sparre Andersen model; compound Poisson model; force of interest; ruin probability; adjustment coefficient; Lundberg's inequality; martingale; optional stopping theorem.

1 Introduction

Consider the Sparre Andersen risk model. Let $X_1, X_2, \ldots,$ denote the inter-claim times, and let $T_n = \sum_{k=1}^{n} X_k$ denote the time of the nth claim, with $T_0 = 0$. Let Y_n be the amount of the nth claim.

We assume that $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are independent sequences of i.i.d. non-negative random variables. $\{X_n,\, n \geq 1\}$ have a common distribution $G(x) = \Pr\{X \leq x\}$ with $G(0) = 0$, and $\{Y_n,\, n \geq 1\}$ have a common distribution $F(x) = \Pr\{Y \leq x\}$ with $F(0) = 0$. Let $\bar{F}(x) = 1 - F(x)$.

The number of claims up to time t is denoted by $N(t) = \sup\{n : T_n \leq t\}$. Then the aggregate claim amount up to time t is

$$Z(t) = \sum_{n=1}^{N(t)} Y_n \,.$$

If the insurer's initial surplus is $u \geq 0$, then the Sparre Andersen model is given by

$$U(t) = u + ct - Z(t)\,, \quad t \geq 0, \tag{1.1}$$

where $c > 0$ is the rate of premium income and $U(0) = u$. See, for example, Grandell (1991). We assume that the positive net profit condition holds in this model, namely $c\,E(X) > E(Y)$.

In this paper, we consider the Sparre Andersen model modified by the inclusion of interest. We assume that the insurer receives interest on its surplus at a constant continuously compounded force of interest $\delta > 0$.

Let the time of ruin for this modified surplus process be $\tau_\delta = \inf\{t : U_\delta(t) < 0\}$, where $U_\delta(t)$ is the surplus at time t with $U_\delta(0) = u$. We denote by $\psi_\delta(u)$ the ultimate ruin probability when the force of interest is δ. Then

$$\psi_\delta(u) = \Pr\{\tau_\delta < \infty\} = \Pr\{\cup_{t \geq 0}\,(U_\delta(t) < 0)\}\,. \tag{1.2}$$

Since ruin can occur only at the time of a claim, we have

$$\psi_\delta(u) = \Pr\{\cup_{n=1}^{\infty}\,(U_\delta(T_n) < 0)\} = \Pr\{\cup_{n=1}^{\infty}\,(V_\delta(T_n) < 0)\} \tag{1.3}$$

where $V_\delta(T_n) = U_\delta(T_n)e^{-\delta T_n}$ is the present value at time 0 of $U_\delta(T_n)$.

We first consider expressions for $U_\delta(T_n)$ and $V_\delta(T_n)$. Recalling the notation for the present and accumulated values of an annuity payable continuously, we denote

$$\bar{a}_{\overline{t|}}^{(\delta)} = \begin{cases} (1 - e^{-\delta t})/\delta, & \text{if } \delta > 0\,, \\ t\,, & \text{if } \delta = 0\,, \end{cases}$$

and $\bar{s}^{(\delta)}_{\overline{t}|} = \bar{a}^{(\delta)}_{\overline{t}|} e^{\delta t}$, or, equivalently,

$$\bar{s}^{(\delta)}_{\overline{t}|} = \begin{cases} (e^{\delta t} - 1)/\delta, & \text{if } \delta > 0\,, \\ t\,, & \text{if } \delta = 0\,. \end{cases}$$

Since $\delta > 0$ is a constant, we have

$$\begin{aligned} U_\delta(T_1) &= ue^{\delta X_1} + c(e^{\delta X_1} - 1)/\delta - Y_1\,, \\ U_\delta(T_2) &= U_\delta(T_1)e^{\delta X_2} + c(e^{\delta X_2} - 1)/\delta - Y_2 \\ &= ue^{\delta(X_1+X_2)} + c(e^{\delta(X_1+X_2)} - 1)/\delta - Y_1 e^{\delta X_2} - Y_2\,, \\ &\cdots\cdots \end{aligned}$$

$$U_\delta(T_n) = U_\delta(T_{n-1})e^{\delta X_n} + c(e^{\delta X_n} - 1)/\delta - Y_n \tag{1.4}$$

$$= ue^{\delta T_n} + c(e^{\delta T_n} - 1)/\delta - \sum_{k=1}^{n} Y_k \exp\left\{\delta \sum_{i=k+1}^{n} X_i\right\}, \tag{1.5}$$

where we adopt the convention that $\sum_a^b = 0$ when $b < a$. Thus,

$$V_\delta(T_n) = U_\delta(T_n)e^{-\delta T_n} = u + c(1 - e^{-\delta T_n})/\delta - \sum_{k=1}^{n} Y_k \exp\left\{-\delta \sum_{i=1}^{k} X_i\right\} \tag{1.6}$$

$$= u + c\,\bar{a}^{(\delta)}_{\overline{T_n}|} - \sum_{k=1}^{n} Y_k e^{-\delta T_k} \tag{1.7}$$

with $V_\delta(T_0) = u$.

In fact, $\{U_\delta(T_n),\ n \geq 0\}$ is an embedded discrete surplus process of the Sparre Andersen model modified by the inclusion of interest. A similar expression to (1.4) for $U_\delta(T_n)$ for the compound Poisson model modified by the inclusion of interest has been given by Sundt and Teugels (1995) and by Vázquez-Abad (2000).

Further, we define

$$\psi_\delta(u;\, n) = \Pr\{\cup_{k=1}^{n}\,(U_\delta(T_k) < 0)\} = \Pr\{\cup_{k=1}^{n}\,(V_\delta(T_k) < 0)\}\,. \tag{1.8}$$

Then

$$\lim_{n\to\infty} \psi_\delta(u;\, n) = \psi_\delta(u)\,. \tag{1.9}$$

In fact, $\psi_\delta(u;\, n)$ is the probability that ruin occurs no later than the nth claim.

Exact solutions for the ruin probability $\psi_\delta(u)$ are difficult to find. In this paper, we instead derive upper bounds for $\psi_\delta(u)$ by two different methods, which are martingale techniques and recursive techniques, when suitable adjustment coefficients exist in the modified

Sparre Andersen model. As applications of the results, upper bounds for the ruin probability in the compound Poisson model are given, which are more amenable to calculation than those given by Sundt and Teugels (1995, 1997). Numerical examples are given to illustrate the applications of these upper bounds.

2 Upper bounds by martingales

Unlike the process $\{U(T_n),\, n \geq 0\}$ in the Sparre Andersen model of (1.1), the processes $\{U_\delta(T_n),\, n \geq 0\}$ given by (1.5) and $\{V_\delta(T_n),\, n \geq 0\}$ given by (1.7) do not have stationary and independent increments. Further, for any $R > 0$, the process $\{\exp\{-RV_\delta(T_n)\}\ ,\, n \geq 0\}$ is not a martingale. However, we can show that there exists a constant $R_1 > 0$ such that $\{\exp\{-R_1 V_\delta(T_n)\}\ ,\, n \geq 0\}$ is a super-martingale. Hence, using similar arguments to those in the martingale proof of Lundberg's inequality and the optional stopping theorem for super-martingales, we can derive an exponential upper bound for ψ_δ. First, we recall a modification of Proposition A.2.5 of Lamberton and Lapeyre (1996) concerning conditional expectations.

Lemma 2.1 *Let X and Y be two independent random vectors. For any non-negative (or bounded) Borel function f,*

$$E[f(X,Y)|\sigma(X)] = g(X), \ \ a.s.$$

where the function g, defined by

$$g(x) = E[f(x,Y)],$$

is a Borel function. In other words, under the assumptions, we can compute $E[f(X,Y)|\sigma(X)]$ as if X was a constant vector. □

Throughout this section, and the next two, we assume that $E(e^{tY})$ exists for $0 < t < \xi$, and that $lim_{t\to\xi}E(e^{tY}) = \infty$.

Lemma 2.2 *There exists a unique positive number, R_1, such that*

$$E\left[\exp\{-R_1(c\,\bar{a}^{(\delta)}_{\overline{X|}} - Ye^{-\delta X})\}\right] = 1\,. \tag{2.1}$$

Proof. This follows by considering the properties of the function

$$h(r) = E\left[\exp\{-r(c\,\bar{a}^{(\delta)}_{\overline{X|}} - Y e^{-\delta X})\}\right].$$

Theorem 2.1 *Let R_1 be defined as in Lemma 2.2. Then, for any $u \geq 0$,*

$$\psi_\delta(u) \;\leq\; e^{-R_1 u}. \tag{2.2}$$

Proof. First, by (1.7), we have

$$\begin{aligned} V_\delta(T_{n+1}) &= V_\delta(T_n) + c\left(e^{-\delta T_n} - e^{-\delta T_{n+1}}\right)/\delta - Y_{n+1}e^{-\delta T_{n+1}} \\ &= V_\delta(T_n) + e^{-\delta T_n}\left[c\,\bar{a}^{(\delta)}_{\overline{X_{n+1}|}} - Y_{n+1}e^{-\delta X_{n+1}}\right]. \end{aligned}$$

Let $\mathcal{F}_n = \sigma\{T_1, \ldots, T_n\}$. Then, for any $n \geq 0$,

$$\begin{aligned} E\left[e^{-R_1 V_\delta(T_{n+1})} \,|\, \mathcal{F}_n\right] &= e^{-R_1 V_\delta(T_n)} E\left[e^{-R_1 e^{-\delta T_n}\left[c\bar{a}^{(\delta)}_{\overline{X_{n+1}|}} - Y_{n+1}e^{-\delta X_{n+1}}\right]} \,|\, \mathcal{F}_n\right] \\ &= e^{-R_1 V_\delta(T_n)}\, E\left[\left(e^{-R_1\left[c\bar{a}^{(\delta)}_{\overline{X_{n+1}|}} - Y_{n+1}e^{-\delta X_{n+1}}\right]}\right)^{e^{-\delta T_n}} \,|\, \mathcal{F}_n\right] \\ &\leq e^{-R_1 V_\delta(T_n)} \left(E\left[e^{-R_1\left[c\bar{a}^{(\delta)}_{\overline{X_{n+1}|}} - Y_{n+1}e^{-\delta X_{n+1}}\right]} \,|\, \mathcal{F}_n\right]\right)^{e^{-\delta T_n}} &(2.3) \\ &= e^{-R_1 V_\delta(T_n)} \left(E\left[e^{-R_1\left[c\bar{a}^{(\delta)}_{\overline{X_{n+1}|}} - Y_{n+1}e^{-\delta X_{n+1}}\right]}\right]\right)^{e^{-\delta T_n}} &(2.4) \\ &= e^{-R_1 V_\delta(T_n)}, &(2.5) \end{aligned}$$

which implies that $\{e^{-R_1 V_\delta(T_n)},\, n \geq 0\}$ is a super-martingale, where the inequality (2.3) follows from $0 < e^{-\delta T_n} \leq 1$, Lemma 2.1 and Jensen's inequality for conditional expectations; equality (2.4) holds since X_{n+1} and Y_{n+1} are independent of $\mathcal{F}_n$; and the equality (2.5) follows from (2.1).

We know that $\tau_\delta \wedge n$ is a bounded stopping time since τ_δ is a stopping time. Thus, by the optional stopping theorem for super-martingales, we get

$$E\left[\exp\{-R_1 V_\delta(T_{\tau_\delta \wedge n})\}\right] \;\leq\; E\left[\exp\{-R_1 V_\delta(T_0)\}\right] \;=\; \exp\{-R_1 u\}. \tag{2.6}$$

However,

$$E\left[\exp\{-R_1 V_\delta(T_{\tau_\delta \wedge n})\}\right] \;\geq\; E\left[\exp\{-R_1 V_\delta(T_{\tau_\delta \wedge n})\}\, I(\tau_\delta \leq n)\right]$$

$$
\begin{aligned}
&= E\left[\exp\left\{-R_1 V_\delta(T_{\tau_\delta})\right\} I(\tau_\delta \le n)\right] \\
&\ge E\left[I(\tau_\delta \le n)\right] \\
&= \psi_\delta(u;\, n)\,. \qquad (2.7)
\end{aligned}
$$

Hence equations (2.6) and (2.7) yield

$$
\psi_\delta(u;\, n) \;\le\; e^{-R_1 u}\,, \qquad (2.8)
$$

which gives (2.2) by letting $n \to \infty$. □

It can be checked that if $\delta \downarrow 0$ in (2.1), then R_1 reduces to the adjustment coefficient for the Sparre Andersen model, which we denote by R_0. It is well known that R_0 satisfies

$$
E\left[\exp\left\{-R_0(cX - Y)\right\}\right] = 1\,. \qquad (2.9)
$$

Thus, Theorem 2.1 is a generalisation of Lundberg's inequality for the Sparre Andersen model.

Moreover, we note that the distribution of $c\,\bar{a}^{(\delta)}_{\overline{X}|} - Ye^{-\delta X}$ in (2.1) is that of the discounted value of the gain between two consecutive claims. However, the distribution of $c\,\bar{s}^{(\delta)}_{\overline{X}|} - Y$ is that of the accumulated value of the gain between two consecutive claims. Hence, we expect a different upper bound for ψ_δ if we replace $c\,\bar{a}^{(\delta)}_{\overline{X}|} - Ye^{-\delta X}$ in (2.1) by $c\,\bar{s}^{(\delta)}_{\overline{X}|} - Y$. Such an upper bound, derived by a different method, is given in Section 3.

3 Upper bounds by recursive techniques

In this section, we derive a different upper bound to that of the previous section by recursive techniques. Numerical comparisons between this upper bound and that in Theorem 2.1 are given in Section 4.

Lemma 3.1 *There exists a unique positive number, R_2, such that*

$$
E\left[\exp\left\{-R_2(c\,\bar{s}^{(\delta)}_{\overline{X}|} - Y)\right\}\right] = 1\,. \qquad (3.1)
$$

Proof. This follows by considering the properties of the function

$$
h(r) = E\left[\exp\left\{-r(c\,\bar{s}^{(\delta)}_{\overline{X}|} - Y)\right\}\right].
$$

Theorem 3.1 *Let R_2 be defined as in Lemma 3.1. Then, for any $u \geq 0$,*

$$\psi_\delta(u) \leq \beta E[\exp\{R_2 Y\}] E\left[\exp\left\{-R_2\left(ue^{\delta X} + c\, \bar{s}^{(\delta)}_{\overline{X}|}\right)\right\}\right] \tag{3.2}$$

where

$$\beta^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{R_2 y} dF(y)}{e^{R_2 t}\bar{F}(t)}. \tag{3.3}$$

In particular, if F is NWUC (new worse than used in convex ordering), then for any $u \geq 0$,

$$\psi_\delta(u) \leq E\left[\exp\left\{-R_2\left(ue^{\delta X} + c\, \bar{s}^{(\delta)}_{\overline{X}|}\right)\right\}\right]. \tag{3.4}$$

Proof. First, we condition on X_1 and Y_1 to obtain the following recursive equation for $\psi_\delta(u;\, n)$:

$$\begin{aligned}
\psi_\delta(u;\, n+1) &= E\left[\psi_\delta(ue^{\delta X_1} + c\, \bar{s}^{(\delta)}_{\overline{X_1}|} - Y_1;\, n)\right] \\
&= \int_0^\infty \int_0^\infty \psi_\delta(ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|} - y;\, n)\, dF(y) dG(x) \\
&= \int_0^\infty \left[\bar{F}(ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|}) + \int_0^{ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|}} \psi_\delta(ue^{\delta x} + c\, \bar{s}^{\delta}_{\overline{x}|} - y;\, n)\, dF(y)\right] dG(x).
\end{aligned} \tag{3.5}$$

Also, from the definition of β above we know that for any $x \geq 0$,

$$\begin{aligned}
\bar{F}(x) &\leq \beta e^{-R_2 x} \int_x^\infty e^{R_2 y} dF(y) && (3.6) \\
&\leq \beta e^{-R_2 x} E\left(e^{R_2 Y}\right). && (3.7)
\end{aligned}$$

Thus, by equation (3.7), we have

$$\begin{aligned}
\psi_\delta(u;\, 1) &= \Pr\{Y_1 > ue^{\delta X_1} + c\, \bar{s}^{(\delta)}_{\overline{X_1}|}\} \\
&= \int_0^\infty \bar{F}(ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|})\, dG(x) && (3.8) \\
&\leq \beta E\left(e^{R_2 Y}\right) \int_0^\infty \exp\left\{-R_2\left(ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|}\right)\right\} dG(x) \\
&= \beta E\left(e^{R_2 Y}\right) E\left[\exp\left\{-R_2\left(ue^{\delta X} + c\, \bar{s}^{(\delta)}_{\overline{X}|}\right)\right\}\right].
\end{aligned}$$

Under an inductive hypothesis, for some integer $n > 1$ we assume that

$$\psi_\delta(u;\, n) \leq \beta E\left(e^{R_2 Y}\right) E\left(\exp\left\{-R_2\left(ue^{\delta X} + c\, \bar{s}^{(\delta)}_{\overline{X}|}\right)\right\}\right). \tag{3.9}$$

Using equation (3.1) and the fact that $e^{\delta X} \geq 1$ we see that

$$\psi_\delta(u;\, n) \;\leq\; \beta E\left(e^{R_2 Y}\right) E\left(\exp\left\{-R_2\left(u + c\, \bar{s}^{(\delta)}_{\overline{X}|}\right)\right\}\right) = \beta e^{-R_2 u}\,. \tag{3.10}$$

Thus, by equations (3.5), (3.6) and (3.10), we have

$$\begin{aligned}
\psi_\delta(u;\, n+1) &= \int_0^\infty \beta \exp\left\{-R_2\left(ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|}\right)\right\} \int_{ue^{\delta x}+c\,\bar{s}^{(\delta)}_{\overline{x}|}}^\infty e^{R_2 y}\, dF(y)dG(x) \\
&\quad + \int_0^\infty \int_0^{ue^{\delta x}+c\,\bar{s}^{(\delta)}_{\overline{x}|}} \beta \exp\left\{-R_2\left(ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|} - y\right)\right\} dF(y)dG(x) \\
&= \beta \int_0^\infty \exp\left\{-R_2\left(ue^{\delta x} + c\, \bar{s}^{(\delta)}_{\overline{x}|}\right)\right\} \int_0^\infty e^{R_2 y}\, dF(y)dG(x) \\
&= \beta E\left(e^{R_2 Y}\right) E\left(\exp\left\{-R_2\left(ue^{\delta X} + c\, \bar{s}^{(\delta)}_{\overline{X}|}\right)\right\}\right)\,.
\end{aligned}$$

Hence equation (3.9) holds for any $n \geq 1$. Thus, inequality (3.2) follows by letting $n \to \infty$ in (3.9). In addition, inequality (3.4) follows from inequality (3.2) and the well known fact that if F is NWUC, then $\beta = \left[E\left(e^{R_2 Y}\right)\right]^{-1}$. See, for example, Willmot and Lin (2000). □

The upper bounds in Theorem 3.1 are different to that in Theorem 2.1. Also, since $\left[E\left(e^{R_2 Y}\right)\right]^{-1} \leq \beta \leq 1$ and $e^{\delta X} \geq 1$, we have the following simplified but weaker upper bound for ψ_δ.

Corollary 3.1 *Under the conditions of Theorem 3.1, for any* $u \geq 0$, $\psi_\delta(u) \leq e^{-R_2 u}$.

Proof. From (3.2), we have

$$\begin{aligned}
\psi_\delta(u) &\leq \beta E\left(e^{R_2 Y}\right) E\left(\exp\left\{-R_2\left(u + c\, \bar{s}^{(\delta)}_{\overline{X}|}\right)\right\}\right) \\
&= \beta\, e^{-R_2 u} E\left(e^{R_2 Y}\right) E\left(\exp\left\{-R_2 c\, \bar{s}^{(\delta)}_{\overline{X}|}\right\}\right) \\
&= \beta e^{-R_2 u} \;\leq\; e^{-R_2 u}\,.
\end{aligned}$$

□

Furthermore, it can be seen that if $\delta \downarrow 0$ in equation (3.1), then R_2 is reduced to R_0 in equation (2.9). Thus, Theorem 3.1 is also a generalisation of Lundberg's inequality for the Sparre Andersen model without interest. In the next section we give applications of Theorem 3.1 to the ruin probability in the compound Poisson model modified by the inclusion of interest, a model which has been studied by many authors: see, for example, Dickson and Waters (1999), Sundt and Teugels (1995, 1997), Vázquez-Abad (2000), and references therein. In addition, numerical examples in Section 4 show that the upper bounds in Theorem 3.1 appear to be better than that in Theorem 2.1.

4 Applications to the compound Poisson model

An important special case of the Sparre Andersen model is the compound Poisson model, in which $G(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. In this case, the positive net profit condition is $c > \lambda E(Y)$.

We denote by $\psi_\delta^*(u)$ the ruin probability in the compound Poisson model modified by the inclusion of interest, and we denote the moment generating function of Y as $M_Y(t) = E\left(e^{tY}\right)$. Then equation (2.1) is equivalent to

$$E\left[\exp\left\{-R_1 c\, a_{\overline{X}|}^{(\delta)}\right\} M_Y(R_1 e^{-\delta X})\right] = 1 . \tag{4.1}$$

First, we apply Lemma 2.1 and Theorem 2.1 to the compound Poisson risk model modified by interest.

Lemma 4.1 *There exists a unique positive number, κ_1, such that*

$$\int_0^{c/\delta} e^{-\kappa_1 y} (1 - \delta y/c)^{\lambda/\delta - 1} M_Y\left[\kappa_1(1 - \delta y/c)\right] dy = c/\lambda . \tag{4.2}$$

Theorem 4.1 *Let κ_1 be as in Lemma 4.1. Then for any $u \geq 0$,*

$$\psi_\delta^*(u) \ \leq \ e^{-\kappa_1 u} . \tag{4.3}$$

Proof. In the compound Poisson model, X is an exponential random variable with mean $1/\lambda$. Thus,

$$\begin{aligned} & E\left[\exp\left\{-\kappa_1 c\, a_{\overline{X}|}^{(\delta)}\right\} M_Y(\kappa_1 e^{-\delta X})\right] \\ = & \int_0^\infty \exp\left\{-\kappa_1 c(1 - e^{-\delta x})/\delta\right\} M_Y(\kappa_1 e^{-\delta x})\, \lambda e^{-\lambda x}\, dx \\ = & \frac{\lambda}{c} \int_0^{c/\delta} e^{-\kappa_1 y} (1 - \delta y/c)^{\lambda/\delta - 1} M_Y\left[\kappa_1(1 - \delta y/c)\right] dy , \end{aligned} \tag{4.4}$$

where (4.4) follows from the substitution $y = c(1 - e^{-\delta x})/\delta$. Hence, equation (4.2) implies that equation (4.1) holds, and so equation (2.2) yields equation (4.3). □

Next, we apply Lemma 3.1 and Theorem 3.1 to the compound Poisson risk model modified by interest.

Lemma 4.2 *There exists a unique positive number, κ_2, such that*

$$\left[E\left(\exp\left\{\kappa_2 Y\right\}\right)\right]^{-1} = \frac{\lambda}{c} \int_0^\infty \frac{e^{-\kappa_2 y}}{(1 + \delta y/c)^{\lambda/\delta + 1}}\, dy . \tag{4.5}$$

Theorem 4.2 *Let κ_2 be as in Lemma 4.2. Then for any $u \geq 0$,*

$$\psi_\delta^*(u) \;\leq\; \beta^* E\left(e^{\kappa_2 Y}\right) \frac{\lambda}{c}\, e^{-\kappa_2 u} \int_0^\infty \frac{e^{-\kappa_2 y(1+\delta u/c)}}{(1+\delta y/c)^{\lambda/\delta+1}}\, dy\,, \tag{4.6}$$

where

$$(\beta^*)^{-1} = \inf_{t\geq 0} \frac{\int_t^\infty e^{\kappa_2 y} dF(y)}{e^{\kappa_2 t}\bar{F}(t)}\,. \tag{4.7}$$

In particular, if F is NWUC, then for any $u \geq 0$,

$$\psi_\delta^*(u) \;\leq\; \frac{\lambda}{c} e^{-\kappa_2 u} \int_0^\infty \frac{e^{-\kappa_2\, y(1+\delta u/c)}}{(1+\delta y/c)^{\lambda/\delta+1}}\, dy\,. \tag{4.8}$$

Proof. In this case, we have

$$\begin{aligned} E\left(\exp\left\{-\kappa_2 c\, \bar{s}_{\overline{X}|}^{(\delta)}\right\}\right) &= \int_0^\infty \exp\left\{-\kappa_2 c(e^{\delta x}-1)/\delta\right\} \lambda e^{-\lambda x}\, dx \\ &= \frac{\lambda}{c} \int_0^\infty \frac{e^{-\kappa_2 y}}{(1+\delta y/c)^{\lambda/\delta+1}}\, dy \end{aligned} \tag{4.9}$$

where equation (4.9) follows from the substitution

$$y = c(e^{\delta x}-1)/\delta\,. \tag{4.10}$$

Hence, for this model equation (3.1) can be expressed as (4.5). Moreover,

$$\begin{aligned} E\left[\exp\left\{-\kappa_2(ue^{\delta X} + c\,\bar{s}_{\overline{X}|}^{(\delta)})\right\}\right] &= \int_0^\infty \exp\left\{-\kappa_2\left(ue^{\delta x} + c(e^{\delta x}-1)/\delta\right)\right\} \lambda e^{-\lambda x}\, dx \\ &= \frac{\lambda}{c} e^{-\kappa_2 u} \int_0^\infty \frac{e^{-\kappa_2 y(1+\delta u/c)}}{(1+\delta y/c)^{\lambda/\delta+1}}\, dy \end{aligned} \tag{4.11}$$

where equation (4.11) follows from the substitution given by equation (4.10).

Thus, equations (3.2) and (4.11) yield equation (4.6), and equations (3.4) and (4.11) lead to equation (4.8). □

Remark 4.1 If the individual claim amount distribution belongs to the class of NWUC distributions, which includes the class of DFR (decreasing failure rate) distributions, then equation (4.8) applies. Otherwise, we can replace β^* by 1 in equation (4.6) for other cases since $\left[E\left(e^{\kappa_2 Y}\right)\right]^{-1} \leq \beta^* \leq 1$.

Sundt and Teugels (1995, 1997) give a Lundberg type upper bound for $\psi_\delta^*(u)$ in terms of an adjustment coefficient function $\xi_\delta(u)$; see equation (7) of Sundt and Teugels (1997). However, $\xi_\delta(u)$ itself is the solution of a differential equation for the Laplace transform of an auxiliary function of $\psi_\delta^*(u)$. Hence $\xi_\delta(u)$ is not calculable since $\psi_\delta^*(u)$ is not available.

Since

$$\int_0^\infty \frac{e^{-\kappa_2 y(1+u\delta/c)}}{(1+\delta y/c)^{\lambda/\delta+1}}\,dy \;\le\; \int_0^\infty e^{-\kappa_2 y(1+\delta u/c)}dy \;=\; 1/\kappa_2(1+\delta u/c),$$

by Theorem 4.2 we have the following simplified upper bounds for $\psi_\delta^*(u)$, which imply that the upper bounds in Theorem 4.2 go to zero much more quickly than the exponential upper bound $e^{-\kappa_2 u}$ does.

Corollary 4.1 *Under the conditions of Theorem 4.2, for any $u \geq 0$,*

$$\psi_\delta^*(u) \;\le\; \beta^* E\left(e^{\kappa_2 Y}\right) \frac{\lambda}{c\kappa_2} \frac{e^{-\kappa_2 u}}{1+\delta u/c}. \tag{4.12}$$

In particular, if F is NWUC, then for any $u \geq 0$,

$$\psi_\delta^*(u) \;\le\; \frac{\lambda}{c\kappa_2} \frac{e^{-\kappa_2 u}}{1+\delta u/c}. \tag{4.13}$$

□

We note, however, that for small values of u, the bounds in Corollary 4.1 can easily exceed 1.

We now give some numerical examples to illustrate the application of the bounds in Theorems 4.1 and 4.2.

Example 4.1 Let Y have an exponential distribution with

$$F(y) = 1 - e^{-y/\mu}, \quad y \geq 0, \ \mu > 0\,.$$

In this case, an explicit formula for $\psi_\delta^*(u)$ is available, namely

$$\psi_\delta^*(u) = \frac{\Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\mu} + \frac{u}{\mu}\right)}{\Gamma\left(\frac{\lambda}{\delta}, \frac{c}{\delta\mu}\right) + \frac{\delta}{\lambda}\left(\frac{c}{\lambda\mu}\right)^{\lambda/\delta} e^{-c/\delta\mu}} \tag{4.14}$$

where $\Gamma(\alpha, z) = \int_z^\infty y^{\alpha-1}e^{-y}dy, \alpha > 0, z \geq 0$ is the incomplete gamma function. See, for example, Gerber (1979).

We set $c = 110, \lambda = 100$ and $\mu = 1$ so that $E(Y) = Var(Y) = 1$. We consider three different values of δ: 0.01, 0.05 and 0.1. We first calculate the adjustment coefficients κ_1 and κ_2, shown in Table 1, then compare upper bounds with exact values in Tables 2-4, where

'Exact' means exact value calculated from (4.14), 'Recursion' means the upper bound (4.8) derived by the recursive method, 'Martingale' means the upper bound (4.3) derived by the martingale method, and 'Lundberg' means Lundberg's upper bound $e^{-\kappa_0 u}$, where κ_0 is the adjustment coefficient in the compound Poisson risk model (without interest), which satisfies

$$E\left[\exp\left\{-\kappa_0(cX - Y)\right\}\right] = 1.$$

It is easily verified that $\kappa_0 = 1/11$. It can been seen from Tables 2-4 that the upper bounds derived by the recursive method are tighter than both those derived by the martingale method and by Lundberg's upper bound, and are fairly close to the exact values for the two smaller values of δ. In each case, a simple upper bound is $\psi_0^*(u) = (1-\kappa_0)e^{-\kappa_0 u}$, i.e. the ultimate ruin probability when $\delta = 0$. This gives much tighter bounds than those derived by the martingale method, but these bounds are not as tight as those derived by the recursive method, and the difference between these two bounds increases as δ increases.□

Example 4.2 Let Y have the gamma density

$$f(y) = \frac{\gamma^\alpha y^{\alpha-1}}{\Gamma(\alpha)} e^{-\gamma y}, \quad y \geq 0, \tag{4.15}$$

where $\gamma > 0$ and $0 < \alpha < 1$, so that the distribution has a DFR. In this case,

$$M_Y(t) = \left(\frac{\gamma}{\gamma - t}\right)^\alpha, \quad t < \gamma. \tag{4.16}$$

We set $\delta = 0.1$, $\alpha = \gamma = 0.75$, $c = 110$, and $\lambda = 100$. We note that $E(Y) = \alpha/\gamma = 1$ as is the case in Example 4.1, but $Var(Y) = \alpha/\gamma^2 = 4/3$ is greater than that in Example 4.1. Thus, we expect that the ruin probabilities, and hence the upper bounds, in this example will be greater than those in Example 4.1. An explicit formula for $\psi_\delta^*(u)$ is not available in this case. However, we find that $\kappa_0 = 0.07757$, $\kappa_1 = 0.07764$, and $\kappa_2 = 0.07828$, and Table 5 shows that the upper bounds in this case are greater than those in Table 4. □

Example 4.3 Let Y have the gamma density of (4.15), but with $\alpha > 1$, so that the distribution has an increasing failure rate. We set $\delta = 0.1$, $\alpha = \gamma = 1.25$, $c = 110$, and $\lambda = 100$. Then $E(Y) = \alpha/\gamma = 1$ as is the case in Examples 4.1 and 4.2, but $Var(Y) = \alpha/\gamma^2 = 0.8$ is smaller than in Examples 4.1 and 4.2. Thus, we expect that the ruin probabilities, and hence the upper bounds, in this example will be less than in Examples 4.1 and 4.2. In this example, equations (4.3) and (4.6) apply to $\psi_\delta^*(u)$ with $\beta^* = 1$ in equation (4.6). We find that $\kappa_0 = 0.10137$, $\kappa_1 = 0.10146$, and $\kappa_2 = 0.10228$, and the values in Table 6 confirm the above comments. □

We again observe in Examples 4.2 and 4.3 that the upper bounds obtained by the martingale method are not a great improvement on the Lundberg bound. Indeed, in each case the values of κ_0 and κ_1 are very similar. In each of Examples 4.2 and 4.3, we also calculated a tight numerical upper bound for $\psi_0^*(u)$. This is shown under "Numerical" in Tables 5 and 6. These values were calculated using the algorithm described as "Method 1" in Dickson *et al* (1995, Section 3.1), with a scaling factor of 100, a value which gives tight numerical bounds. We observe that in Table 5, the numerical bound is tighter for lower values of u, whereas in Table 6, the recursive method gives the tightest bounds. In general, the tightest upper bound will not always be given by the recursive method. We note that in each of Examples 4.1-4.3 the value of κ_2 is greater than the value of κ_0. It can be shown that this is always the case, and in the more general Sparre Andersen model R_2 is always greater than R_0.

In conclusion, the results in this paper give analytical upper bounds for the ruin probability in the Sparre Andersen model with interest, and yield applications to the compound Poisson model. All of our numerical investigations showed that upper bounds derived by the recursive method are tighter than those derived by the martingale method.

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Table 1: Adjustment coefficients in Example 4.1

δ	κ_1	κ_2
0.01	0.09092	0.09100
0.05	0.09096	0.09133
0.10	0.09100	0.09174

Table 2: Upper bounds in Example 4.1 when $\delta = 0.01$

u	Exact	Recursion	Martingale	Lundberg
0	0.9082	0.9090	1.0000	1.0000
10	0.3609	0.3659	0.4028	0.4029
20	0.1422	0.1473	0.1623	0.1623
30	0.0556	0.0593	0.0654	0.0654
40	0.0216	0.0239	0.0263	0.0263
50	0.0083	0.0096	0.0106	0.0106

Table 3: Upper bounds in Example 4.1 when $\delta = 0.05$

u	Exact	Recursion	Martingale	Lundberg
0	0.9049	0.9087	1.0000	1.0000
10	0.3415	0.3644	0.4027	0.4029
20	0.1239	0.1461	0.1622	0.1623
30	0.0433	0.0586	0.0653	0.0654
40	0.0145	0.0235	0.0263	0.0263
50	0.0047	0.0094	0.0106	0.0106

Table 4: Upper bounds in Example 4.1 when $\delta = 0.1$

u	Exact	Recursion	Martingale	Lundberg
0	0.9014	0.9083	1.0000	1.0000
10	0.3209	0.3626	0.4025	0.4029
20	0.1060	0.1448	0.1620	0.1623
30	0.0325	0.0578	0.0652	0.0654
40	0.0092	0.0231	0.0263	0.0263
50	0.0024	0.0092	0.0106	0.0106

Table 5: Upper bounds in Example 4.2 when $\delta = 0.1$

u	Recursion	Martingale	Lundberg	Numerical
0	0.9207	1.0000	1.0000	0.9091
10	0.4205	0.4601	0.4604	0.4178
20	0.1921	0.2117	0.2120	0.1929
30	0.0878	0.0974	0.0976	0.0891
40	0.0401	0.0448	0.0449	0.0411
50	0.0183	0.0206	0.0207	0.0190

Table 6: Upper bounds in Example 4.3 when $\delta = 0.1$

u	Recursion	Martingale	Lundberg	Numerical
0	0.8988	1.0000	1.0000	0.9091
10	0.3229	0.3626	0.3629	0.3328
20	0.1160	0.1314	0.1317	0.1214
30	0.0417	0.0477	0.0478	0.0443
40	0.0150	0.0173	0.0173	0.0162
50	0.0054	0.0063	0.0063	0.0059

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