Fourier inversion formulas in option pricing and insurance

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Abstract

Several authors have used Fourier inversion to compute prices of puts and calls, some using Parseval's theorem. The expected value of $\max(S-K,0)$ also arises in excess-of-loss or stop-loss insurance, and we show that Fourier methods may be used to compute them. In this paper, we take the idea of using Parseval's theorem further: (1) formulas requiring weaker assumptions; (2) relationship with classical inversion theorems for probability distributions; (3) formulas for payoffs which occur in insurance. Numerical examples are provided.

1 Introduction

Lewis (2001) gives formulas which price options without having first to find the distribution of the underlying, by applying Parseval's theorem. Borovkov & Novikov (2002) do not explicitly name Parseval's theorem, but some of their option pricing formulas can be obtained using Parseval's theorem. All that is needed in those papers is the characteristic function (= Fourier transform) of the distribution of the logarithm of the underlying and the Fourier transform of the payoff function. Fourier methods are applied to option pricing by several other authors, for instance Bakshi & Madan (2000), Carr & Madan (1999), Heston (1993), Lee (2004), Raible (2000).

In insurance, the payoff

$$(S-K)_+ \;=\; \max(S-K,0)$$

also occurs in excess-of-loss or stop-loss contracts, so Parseval's theorem might also be used to calculate pure premiums. This paper explores the computation of both option prices and insurance premiums via Parseval's theorem in a unified setting.

The mathematical problem is the same in insurance as in option pricing, that is, the computation of $\mathbb{E}\, g(S)$ for some function g. The difference is that in many cases option pricing models focus on the logarithm of S (the "log-price"), while insurance applications are usually phrased in terms of the distribution of S itself. For instance, the Black-Scholes formula for a call option is the expectation of the payoff $(S-K)_+$, where $\log S$ has a normal distribution; more recent models also specify the distribution of the log-price, rather than S itself. The consequence is that the Fourier transform which is likely to be known is that of $X = \log S$. This explains the particular form of the formulas in Lewis (2001) and Borovkov & Novikov (2002).

By contrast, in insurance applications the distribution of S is often (though not always) one for which the Fourier transform $\mathbb{E}\exp(iuS)$ is known. This is why we will identify two different classes of inversion formulas: (1) those where the Fourier transform of $\log(S)$ is known (and thus appears in the inversion formula), and (2) those where the Fourier transform of S appears. The first kind of inversion formula will be referred to as "Mellin-type", since it is the Mellin transform $\mathbb{E}\exp(iu\log(S)) = \mathbb{E}\, S^{iu}$ which is used, and the other kind will be called "Fourier-type". The formulas in Lewis (2001) and in Borovkov & Novikov (2002) are thus all of Mellin type, while the insurance examples in Section 4 below are all of Fourier type. We do not suggest that this classification is essential, or that it neatly differentiates option pricing from insurance, but we found it useful in presenting a unified view of the applications of Parseval's theorem to option pricing and insurance.

Section 2 states the particular form of the Parseval theorem we will use, and recalls two standard theorems of probability theory which are directly related to the pricing formulas which follow. Section 3 gives the main results of the paper. Lewis (2001) gives formulas which require the finiteness of

$\mathbb{E}\, S^{\alpha}$ for α in some interval $[a, b]$, with $a < 0$, $b > 1$ (see his Theorem 3.2); Borovkov & Novikov (2002) make similar assumptions. This is good enough in many cases, but not always feasible. We give general formulas which do not require this type of assumption (this is where our formulas are reminiscent of the classical inversion formulas for distribution functions). Moreover, we do not assume that the underlying has a probability density function, as many authors have done. Section 4 gives some numerical applications. The appendices contain some background on Fourier transforms and some of the proofs.

Notation. We denote $F_X(x) = \mathbb{P}\{X \le x\}$ the distribution function of X, and μ_X the measure on $\mathbb{R}$ induced by F_X, that is, $\mu_X(B) = \mathbb{P}\{X \in B\}$, B a Borel subset of $\mathbb{R}$. The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$, $f \in L^1$, is denoted

$$\hat{f}(u) \;=\; \int_{\mathbb{R}} e^{iux} f(x)\, dx.$$

The Fourier transform of a signed measure μ with finite total mass $|\mu| < \infty$ is written

$$\hat{\mu}(u) \;=\; \int_{\mathbb{R}} e^{iux}\, \mu(dx).$$

If $\mu = \mu_X$, then this is plainly the characteristic function of the distribution of X,

$$\hat{\mu}_X(u) \;=\; \mathbb{E}\, e^{iuX}.$$

Integrals of functions h which are not in L^1 occur as inverse Fourier transforms. These are called "principal value" integrals, and are denoted

$$PV\!\int_{-\infty}^{\infty} h(x)\, dx \;=\; \lim_{M\to\infty} \int_{-M}^{M} h(x)\, dx.$$

2 Preliminaries

2.1 Parseval's theorem

The problem considered in this paper is to compute

$$\mathbb{E} g(X) \;=\; \int g(x)\, d\mu_X(x)$$

using Parseval's theorem. This theorem requires that the function g be integrable over $\mathbb{R}$, which is not always the case in applications. To remedy this situation, we multiply the payoff function g by a damping factor $r(x)$ to turn it into a new function g^r that is in L^1. Let

$$g^r(x) \;=\; r(x)g(x), \quad d\mu^r_X(x) \;=\; \frac{1}{r(x)} d\mu_X(x). \tag{2.1}$$

Then

$$\mathbb{E}g(X) \;=\; \int g^r(x)\, d\mu^r_X(x).$$

We consider two specific cases: (1) exponential damping factors $r(x) = e^{\alpha x}$, α some real constant; (2) polynomial damping factors $r(x) = (1+cx)^{-b}$, $c > 0$, $b \in \{1, 2, \dots\}$. (Lewis (2001) only used exponential damping factors. Of course other choices for $r(\cdot)$ are possible.)

For instance, the call and put payoff functions

$$g_1(x) \;=\; (e^x - K)_+, \quad g_2(x) \;=\; (K - e^x)_+ \tag{2.2}$$

are not integrable over $\mathbb{R}$, and so the usual form of Parseval's Theorem (Theorem A2, Appendix A) is not directly applicable. However, when using $r(x) = e^{\alpha x}$ for some $\alpha < -1$, $g_1(x)$ is replaced with

$$g^r_1(x) \;=\; (e^{(\alpha+1)x} - Ke^{\alpha x})I_{\{x>K\}},$$

which is integrable.

Parseval's theorem is found by first noting that the function

$$G(y) \;=\; \int g^r(x-y)\, d\mu^r_X(x), \tag{2.3}$$

is a convolution, and then concluding that

$$\widehat{G}(u) \;=\; \widehat{g^r}(-u)\widehat{\mu^r_X}(u).$$

By Theorem A.1 (Appendix A), this implies

$$\frac{1}{2}[G(0+) + G(0-)] \;=\; \frac{1}{2\pi} PV\!\int \widehat{g^r}(-u)\widehat{\mu^r_X}(u)\, du. \tag{2.4}$$

There are conditions for this equation to hold, which will be discussed presently. We first look at whether the left-hand side of the last equation may be replaced with $G(0) = \mathbb{E}g(X)$. This is of importance, because there are cases where the function G is not continuous at the origin. For instance, consider the simplistic case where $r(x) = e^{-x}$ and

$$g(x) \;=\; I_{\{x>0\}}, \quad X \;=\; 0 \text{ or } 1 \text{ with probability } \tfrac{1}{2}.$$

Then $G(0+) = \frac{1}{2}$ and $G(0-) = 1$, which means that Eq.(2.4) returns $\frac{3}{4}$, not the correct value $\mathbb{E}g(X) = \frac{1}{2}$. It can be seen that the reason for this is that there is a probability mass at $x = 0$, which also happens to be a discontinuity point of g.

The following lemma (proof in Appendix B) gives conditions under which G is continuous at the origin, and will be sufficient for our purposes.

Lemma 2.1 *Assume* $|\mu_X^r| = \mathbb{E}|r(X)^{-1}| < \infty$.
(a) Suppose that there are $a_1 < a_2 < \cdots < a_n$ *such that*

(i) $\mathbb{P}\{X = a_j\} \;=\; 0$ *for* $j = 1, \ldots, n$,

(ii) g^r *is uniformly bounded and piecewise continuous over* $(-\infty, a_1)$, (a_1, a_2), $\cdots$, (a_{n-1}, a_n), (a_n, ∞), *and has finite limits* $g(a_j-), g(a_j+)$.

Then $G(y)$ *(Eq.(2.3)) is continuous at* $y = 0$.

(b) If $\mathbb{E}\, X_+ < \infty$, *then* $G(y) = \mathbb{E}(X - K - y)_+$ *is continuous at* $y = 0$. *If* $X \geq 0$ *and* $K > 0$, *then* $G(y) = \mathbb{E}(K + y - X)_+$ *is continuous at* $y = 0$.

The next theorem is the theoretical foundation of the rest of the paper; it is a direct consequence of Theorem A.2.

Theorem 2.1 *Let* X *be a random variable, and suppose (2.1) holds. Assume that*

(a) $|\mu_X^r| < \infty$,

(b) $g^r \in L^1$,

(c) the function G *defined in Eq.(2.3) is continuous at the origin and satisfies condition (b) of Theorem A.1.*

Then

$$\mathbb{E}\, g(X) \;=\; \frac{1}{2\pi} PV \int \widehat{g^r}(-u)\widehat{\mu_X^r}(u)\, du.$$

We give a sufficient condition for part (b) of Theorem A.1 to hold (proof in Appendix B).

Lemma 2.2 *Condition (b) of Theorem A1 is satisfied if g^r has bounded variation over $\mathbb{R}$. This is in particular true if g^r is uniformly bounded, piecewise differentialble and such that, ignoring a finite number of discontinuities,*

$$\int_{-\infty}^{\infty} \left| \frac{dg^r(x)}{dx} \right| dx \; < \; \infty.$$

For any function φ and any constant α, we denote

$$\varphi^{(\alpha)}(x) \; = \; e^{\alpha x}\varphi(x), \qquad x \in \mathbb{R}.$$

The Fourier transform of $\varphi^{(\alpha)}$ will be denoted $\widehat{\varphi^{(\alpha)}}$:

$$\widehat{\varphi^{(\alpha)}}(u) \; = \; \int_{-\infty}^{\infty} \phi(x) e^{\alpha x} e^{iux}\, dx \; = \; \hat{\phi}(u - i\alpha).$$

For a signed measure μ on $\mathbb{R}$ and $\alpha \in \mathbb{R}$, define a new signed measure $\mu^{(\alpha)}$ by

$$\mu^{(\alpha)}(dx) \; = \; e^{\alpha x}\mu(dx).$$

Then the Fourier transform of $\mu_X^{(\alpha)}$ is

$$\widehat{\mu_X^{(\alpha)}}(u) \; = \; \mathbb{E}\, e^{(iu+\alpha)X} \; = \; \hat{\mu}_X(u - i\alpha).$$

An application of Theorem 2.1 with $r(x) = e^{-\alpha x}$ then gives

$$\begin{aligned} \mathbb{E}\, g(X) \; &= \; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \widehat{g^{(-\alpha)}}(-u) \; \widehat{\mu_X^{(\alpha)}}(u)\, du \\ &= \; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}(-u + i\alpha)\hat{\mu}_X(u - i\alpha)\, du. \end{aligned}$$

By comparison, the direct application of Parseval's theorem (without a damping factor) would give

$$\frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}(-u)\hat{\mu}_X(u)\, du.$$

Hence, the damping factor $e^{-\alpha x}$ changes the path of integration in the complex plane, by translating it by $-i\alpha$ units.

In the case of polynomial damping factors, for $\beta \in \{1, 2, \dots\}$ and $c > 0$ we let

$$g^{[-\beta]}(x) \;=\; (1+cx)^{-\beta} g(x), \qquad d\mu_X^{[\beta]}(x) \;=\; (1+cx)^{\beta} d\mu_X(x).$$

In the cases we consider, the Fourier transform of $g^{[-\beta]}$ may be expressed in terms of special functions. Because β is a positive integer, the Fourier transform of $\mu_X^{[\beta]}(x)$ is a linear combination of $\hat{\mu}_X$ and its derivatives.

2.2 Two classical theorems

We state two standard theorems which are intimately related to the option or stop-loss formulas which follow. Each expresses the distribution function of a random variable as a Fourier inversion integral. The best known proofs of these results (see Lucaks, 1970, p.31, and Kendall & Stuart, 1977, p.97) rely on Dirichlet integrals, but they can also be proved using Parseval's theorem.

Theorem 2.2 *If a and $a+h$ are continuity points of F_X, then*

$$F_X(a+h) - F_X(a) \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \frac{1 - e^{-iuh}}{iu} e^{-iua} \hat{\mu}_X(u)\, du.$$

Theorem 2.3 *If F_X is continuous at $x = b$, then*

$$F_X(b) \;=\; \frac{1}{2} + \frac{1}{2\pi} PV \int_0^{\infty} \frac{1}{iu} [e^{iub} \hat{\mu}_X(-u) - e^{-iub} \hat{\mu}_X(u)]\, du.$$

In option pricing, Theorem 2.3 leads to the well-known formula

$$\mathbb{E}(e^X - K)_+ \;=\; \mathbb{E}(e^X)\Pi_1 - K\Pi_2,$$

where

$$\Pi_1 \;=\; \mathbb{E}\left[e^X \mathbf{1}_{\{e^X > K\}}\right] / \mathbb{E}(e^X) \;=\; \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{K^{-iu} \hat{\mu}_X(u-i)}{iu \hat{\mu}_X(-i)}\right] du$$

$$\Pi_2 \;=\; \mathbb{P}\{e^X > K\}.$$

2.3 Mellin-type and Fourier-type formulas

Lewis (2001) and other authors consider payoffs which are explicit functions of e^X, such as the usual call and put payoffs g_1 and g_2 in (2.2). This is because most financial models are expressed in terms of the log-price. For instance, a formula for $\mathbb{E}(S-K)_+$ is obtained in terms of

$$\mathbb{E}e^{iu\log S} \;=\; \mathbb{E}S^{iu}. \tag{2.5}$$

The insurance applications considered in Section 4, however, lead to expressions of the type $\mathbb{E}(S-K)_+$, but the inversion formulas are in terms of the Fourier transform $\mathbb{E}(e^{iuS})$.

The expression in Eq.(2.5) is known as the Mellin transform of the distribution of S. In order to distinguish these two situations, we will call "Mellin-type" the formulas where $\mathbb{E}\,S^{iu}$ appears, and "Fourier-type" those where $\mathbb{E}(e^{iuS})$ appears.

3 Inversion formulas

In this section, formulas are derived for the expectations of the payoffs g_1 and g_2 in (2.2). In each case, Parseval's theorem yields an inversion integral along the line $u - i\alpha$ in the complex plane, if α can be found such that (i) $g^{(-\alpha)}$ is in L^1 and (ii) $\mathbb{E}\exp(\alpha X)$ is finite. It is not always possible to find such α, depending on the function g considered and also the distribution of X. For this reason, we derive general formulas which do not assume that such $\alpha \neq 0$ exists. The idea is to truncate the distribution of X in such a way that Parseval's theorem applies for some $\alpha \neq 0$, next to let α tend to 0, and, finally, to remove the truncation of the distribution of X.

An important point to keep in mind in what follows is that if there is $\alpha > 0$ such that $\mathbb{E}\exp(\alpha X) < \infty$, then necessarily $\mathbb{E}\exp(\alpha' X) < \infty$ for $0 < \alpha' < \alpha$ (the same applies for $\alpha < 0$). The set of α such that $g^{(-\alpha)} \in L^1$, when not empty, is either an interval or a single point. Hence, the set of α such that both $\mathbb{E}\exp(\alpha X) < \infty$ and $g^{(-\alpha)} \in L^1$ is either empty or an interval (possibly reduced to a single point). This has numerical implications, since the observed accuracy of the integral formula often varies with α within the allowed interval.

3.1 Mellin-type formulas

The proof of the next theorem can be found in Appendix B. Part (b) gives formulas which do not require damping factors and therefore apply in all cases.

Theorem 3.1 *Let $S \geq 0$, $K > 0$ and*

$$h(u) \;=\; \frac{K^{-iu+1}}{iu(iu-1)}\,\mathbb{E}(S^{iu}).$$

(a) If there exists $\alpha < 0$ such that $\mathbb{E}(S^{\alpha}) < \infty$, then

$$\mathbb{E}(K-S)_+ \;=\; K\mathbb{P}\{S=0\} + \frac{1}{2\pi} PV \int_{-\infty}^{\infty} h(u-i\alpha)\,du.$$

If, moreover, $\mathbb{E}(S) < \infty$, then

$$\mathbb{E}(S-K)_+ \;=\; \mathbb{E}\,S - K\mathbb{P}\{S>0\} + \frac{1}{2\pi} PV \int_{-\infty}^{\infty} h(u-i\alpha)\,du.$$

(b) In all cases,

$$\mathbb{E}(K-S)_+ \;=\; \frac{K}{2}[1+\mathbb{P}\{S=0\}] + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}[h(u)]\,du.$$

If $\mathbb{E}(S) < \infty$,

$$\mathbb{E}(S-K)_+ \;=\; \mathbb{E}\,S - \frac{K}{2}[1+\mathbb{P}\{S=0\}] + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}[h(u)]\,du.$$

These formulas extend those given for calls and puts in Lewis (2001) and Borovkov & Novikov (2002).

3.1.1 Example: S has a discrete distribution

First, suppose that $X \equiv x_0$. This means that $\hat{\mu}_X(u) = e^{iux_0}$, and

$$h(u) \;=\; -\frac{e^c e^{iu(x_0-c)}}{iu(1-iu)},$$

where $c = \ln K$. Then

$$\mathrm{Re}[h(u)] \;=\; -\frac{e^c \cos(ux_c)}{1+u^2} - \frac{e^c \sin(ux_c)}{u(1+u^2)}, \qquad x_c = x_0 - c.$$

Now

$$\frac{1}{\pi}\int_0^\infty \frac{\cos(ux_c)}{1+u^2}\,du \;=\; \frac{1}{2}\int_{-\infty}^\infty \frac{e^{iux_c}}{\pi(1+u^2)}\,du \;=\; \frac{1}{2}e^{-|x_c|}$$

(this is $\frac{1}{2}$ times the characteristic function of the Cauchy distribution). Also, letting $\operatorname{sign} x_c = I_{\{x>0\}} - I_{\{x<0\}}$,

$$\begin{aligned}\int_0^\infty \frac{\sin(ux_c)}{u(1+u^2)}\,du \;&=\; \int_0^\infty \int_0^{x_c} \frac{\cos(uy)}{1+u^2}\,dy\,du \\ &=\; \int_0^{x_c} \frac{1}{2}e^{-|y|}\,dy \;=\; (\operatorname{sign} x_c)\frac{1}{2}(1-e^{-|x_c|}).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{e^c}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[h(u)]\,du \;&=\; \frac{e^c}{2} - \frac{e^c}{2}[e^{-|x_c|} + (\operatorname{sign} x_c)(1-e^{-|x_c|})] \\ &=\; \frac{e^c}{2}[1-(\operatorname{sign} x_c)](1-e^{-|x_c|}) \;=\; (e^c - e^{x_0})_+.\end{aligned}$$

Since a discrete distribution is a linear combination of degenerate distributions, this shows that the formula is correct for discrete random variables $S > 0$. It is true for any $S \geq 0$, because if $\mathbb{P}\{S = 0\} = 1$, then

$$\mathbb{E}(K - S)_+ = K,$$

while

$$K + \frac{1}{2\pi}\int_0^\infty \frac{e^c}{iu}\left[e^{iuc}\frac{\mathbb{E}(0^{-iu})}{1+iu} - e^{-iuc}\frac{\mathbb{E}(0^{iu})}{1-iu}\right] du \;=\; K.$$

3.2 Fourier-type formulas

We now look at formulas for the payoffs

$$g_3(x) \;=\; (x-K)_+, \qquad g_4(x) \;=\; (K-x)_+ I_{\{0\leq x\leq K\}}$$

in terms of $\hat{\mu}_X(u) \;=\; \mathbb{E}(e^{iuX})$. Exponential damping factors $e^{\alpha x}$ can be used just as in the previous section, but we show that one may also use polynomial damping factors.

3.2.1 Exponential damping factors

Theorem 3.2 *(a) If there exists* $\alpha > 0$ *such that* $\mathbb{E}(e^{\alpha X}) < \infty$*, then* $\mathbb{E}(X - K)_+ < \infty$ *for* $K \in \mathbb{R}$ *and*

$$\mathbb{E}(X-K)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}_3(-u+i\alpha)\hat{\mu}_X(u-i\alpha)\,du,$$

where

$$\hat{g}_3(z) = -\frac{e^{izK}}{z^2}.$$

(b) Let $X \geq 0$*. For any* $\alpha \in \mathbb{R}$ *such that* $\mathbb{E}(e^{\alpha X}) < \infty$ *(including* $\alpha = 0$*) and* $K \geq 0$*,*

$$\mathbb{E}(K-X)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}_4(-u+i\alpha)\hat{\mu}_X(u-i\alpha)\,du,$$

where

$$\hat{g}_4(z) = \frac{1}{z^2}(1 + izK - e^{izK}).$$

Proof. Part (a) is a direct application of Parseval's theorem and Lemma 2.1, given that if $\mathrm{Im}(z) > 0$, then for any K,

$$\hat{g}_3(z) = \int_K^{\infty} (x-K)e^{izx}\,dx = e^{izK}\int_0^{\infty} ye^{izy}\,dy = -\frac{e^{izK}}{z^2}.$$

For part (b), it is clear that $g_4^{(-\alpha)} I_{[0,K]} \in L^1$ for any $\alpha \in \mathbb{R}$; also, the condition in Lemma 2.1(b) and Lemma 2.2 are satisfied. Provided $\hat{\mu}_X(-i\alpha) < \infty$, we can thus apply Parseval's theorem, with

$$\hat{g}_4(z) = \int_0^{\infty} (g_3(x) + K - x)e^{izx}\,dx = \frac{1}{z^2}(1 + izK - e^{izK}). \qquad \blacksquare$$

In Section 3.2.3, a variation on part (a) of this theorem is given which does not require that $\mathbb{E}(e^{\alpha X}) < \infty$ for any $\alpha > 0$.

3.2.2 Polynomial damping factors

An alternative to the formulas in Theorem 3.2 is to use a polynomial damping factor. For $\beta \in \{1, 2, \ldots\}$ and $c > 0$, let

$$g^{[-\beta]}(x) \;=\; (1+cx)^{-\beta} g(x), \qquad d\mu_X^{[\beta]}(x) \;=\; (1+cx)^{\beta} d\mu_X(x).$$

For the payoff $g_3(x) = (x-K)_+$, given $\beta \geq 2$,

$$\widehat{g_3^{[-\beta]}}(u) \;=\; \int_{\mathbb{R}} \frac{e^{iux}(x-K)_+}{(1+cx)^{\beta}}\, dx \;=\; \frac{e^{iuK}}{c^2(1+cK)^{\beta-2}} \Psi(2, 3-\beta; -iu(1+cK)/c), \tag{3.1}$$

where

$$\Psi(\alpha, \gamma; z) \;=\; \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \frac{e^{-zt} t^{\alpha-1}}{(1+t)^{\alpha-\gamma+1}}\, dt, \qquad \alpha > 0,$$

is the confluent hypergeometric function of the second kind. (The integral formula above holds (i) for $\mathrm{Re}(z) > 0$ and also (ii) for $\mathrm{Re}(z) = 0, \mathrm{Im}(z) \neq 0$ if $\gamma \leq 1$; for more details, see Lebedev, 1972, Chapter 9.)

The function Ψ in (3.1) may be expressed in terms of the incomplete gamma function; since

$$\Psi(2, 3-\beta; z) \;=\; \Psi(1, 3-\beta; z) - \Psi(1, 2-\beta; z),$$

formula (3.1) may be written in terms of

$$\Psi(1, \gamma; z) \;=\; z^{1-\gamma} e^{z} \Gamma(\gamma - 1; z), \tag{3.2}$$

where

$$\Gamma(a; z_0) \;=\; \int_{z_0}^{\infty} x^{a-1} e^{-x}\, dx, \qquad |\arg(z_0)| < \pi.$$

Because β is a positive integer, an alternative is to use integration by parts to show that, if $n = 0, 1, 2, 3, \ldots$,

$$\Psi(1, -n; z) \;=\; \sum_{j=0}^{n} \frac{(-z)^j}{(n+1-j)_{j+1}} + \frac{(-z)^{n+1}}{(n+1)!} \Psi(1, 1; z), \qquad \mathrm{Re}(z) \geq 0.$$

The remaining hypergeometric function $\Psi(1,1;z)$ may in turn be written as an incomplete gamma function (see (3.2)), or else as

$$\Psi(1,1;z) \;=\; -e^z E_1(-z), \qquad E_1(z) \;=\; -\int_{-z}^{\infty} \frac{e^{-t}}{t}\,dt, \qquad |\arg(-z)| < \pi.$$

Here $E_1(\cdot)$ is the exponential integral function. For more details on the special functions above, see Abramowitz and Stegun (1970) or Lebedev (1972).

Now, let us turn to the payoff $g_4(x) = (K-x)_+$. In order for the Fourier transform of

$$g_4^{[-\beta]}(x) \;=\; (1+cx)^{-\beta}(K-x)_+$$

to be defined, we assume $X \geq 0$. The Fourier transform of $g_4^{[-\beta]}$ may be found in the obvious way: since

$$g_4^{[-\beta]}(x) \;=\; g_3^{[-\beta]}(x) + (1+cx)^{-\beta}(K-x),$$

we get, for $\beta \geq 2$,

$$\begin{aligned}
\widehat{g_4^{[-\beta]}}(u) \;&=\; \widehat{g_3^{[-\beta]}}(u) + \int_0^{\infty} \frac{e^{iux}(K-x)}{(1+cx)^{\beta}}\,dx \\
&=\; \widehat{g_3^{[-\beta]}}(u) + \left(\frac{K}{c} + \frac{1}{c^2}\right)\Psi(1, 2-\beta; -iu/c) - \frac{1}{c^2}\Psi(1, 3-\beta; -iu/c).
\end{aligned}$$

The case $\beta = 1$ is different:

$$\widehat{g_4^{[-1]}}(u) \;=\; \frac{cK+1}{c^2}\left[\Psi(1,1;-iu/c) - e^{iuK}\Psi(1,1;-iu(1+cK)/c)\right] - \frac{1}{iuc}(e^{iuK}-1).$$

As to the Fourier transform of $\mu_X^{[\beta]}(x)$,

$$\widehat{\mu_X^{[\beta]}}(u) \;=\; \sum_{k=0}^{\beta}\binom{\beta}{k}c^k\int_{\mathbb{R}} x^k e^{iux}\,d\mu_X(x) \;=\; \sum_{k=0}^{\beta}\binom{\beta}{k}(-ci)^k\frac{\partial^k}{\partial u^k}\hat{\mu}_X(u).$$

Recall that if $\mathbb{E}(|X|^k) < \infty$, then $\frac{\partial^k}{\partial u^k}\hat{\mu}_X(u)$ exists for all $u \in \mathbb{R}$. We have the following result.

Theorem 3.3 *Let $K \geq 0$ and $c > 0$.*
(a) If $\beta \in \{2, 3, \dots\}$, $\mathbb{E}|X|^\beta < \infty$, then

$$\mathbb{E}(X - K)_+ \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \widehat{g_3^{[-\beta]}}(u) \left[\sum_{k=0}^{\beta} \binom{\beta}{k} \beta^k (-ci)^k \frac{\partial^k}{\partial u^k} \hat{\mu}_X(u) \right] du.$$

(b) If $\beta \in \{0, 1, 2, 3, \dots\}$, $X \geq 0$ and $\mathbb{E}\, X^\beta < \infty$, then

$$\mathbb{E}(K - X)_+ \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \widehat{g_4^{[-\beta]}}(u) \left[\sum_{k=0}^{\beta} \binom{\beta}{k} \beta^k (-ci)^k \frac{\partial^k}{\partial u^k} \hat{\mu}_X(u) \right] du.$$

3.2.3 Formulas without damping factors

We first show how an inversion formula can be found for $\mathbb{E}(X - K)_+$ as a direct application of Theorem 2.3, when $X \geq 0$. For all X,

$$\mathbb{E}(X - K)_+ \;=\; \int_K^{\infty} \mathbb{P}(X > y)\, dy.$$

If $X \geq 0$, define a new distribution (sometimes called the "ladder height" distribution associated with X) with density

$$f_{\overline{X}}(x) \;=\; \frac{\mathbb{P}(X > x)}{\mathbb{E}\, X} \mathbf{1}_{\{x>0\}}.$$

Then

$$\mathbb{E}(X - K)_+ \;=\; (\mathbb{E}\, X)\, \mathbb{P}(\overline{X} > K).$$

An easy calculation yields

$$\hat{\mu}_{\overline{X}}(u) \;=\; \frac{\hat{\mu}_X(u) - 1}{iu\mathbb{E}\, X}.$$

Theorem 2.3 says that

$$\mathbb{P}\{\overline{X} > K\} \;=\; \frac{1}{2} + \frac{1}{\pi} PV \int_0^{\infty} \mathrm{Re} \left[\frac{e^{-iuK} \hat{\mu}_{\overline{X}}(u)}{iu} \right] du,$$

which implies that if $X \geq 0$, $\mathbb{E}(X) < \infty$, then for any $K \geq 0$

$$\mathbb{E}(X - K)_+ = \quad \frac{\mathbb{E}\, X}{2} + \frac{1}{\pi} PV \int_0^{\infty} \mathrm{Re} \left[\frac{e^{-iuK}(1 - \hat{\mu}_X(u))}{u^2} \right] du. \qquad (3.3)$$

It is possible to apply the idea once more: consider a new random variable $\overline{\overline{\mathrm{X}}}$ with density

$$\begin{aligned}
f_{\overline{\overline{\mathrm{X}}}}(x) &= \frac{\mathbb{P}(\overline{X} > x)}{\mathbb{E}\,\overline{X}}\mathbf{1}_{\{x>0\}} = \frac{\mathbb{E}(X-x)_+}{\mathbb{E}(X^2)/2} \\
\hat{\mu}_{\overline{\overline{\mathrm{X}}}}(u) &= \frac{\hat{\mu}_{\overline{X}}(u)-1}{iu\mathbb{E}\,\overline{X}} = \frac{\hat{\mu}_{\overline{X}}(u)-1-iu\mathbb{E}\,X}{(iu)^2\mathbb{E}(X^2)/2}.
\end{aligned}$$

This requires $\mathbb{E}\,X^2 < \infty$. The function $f_{\overline{\overline{\mathrm{X}}}}$ is integrable and differentiable; therefore Theorem A.1 implies that if $X \geq 0$, $\mathbb{E}(X) < \infty$, then for any $K > 0$,

$$\mathbb{E}(X-K)_+ = \frac{1}{\pi}PV\!\int_0^\infty \mathrm{Re}\left[\frac{e^{-iuK}[1+iu\mathbb{E}(X)-\hat{\mu}_X(u)]}{u^2}\right] du. \tag{3.4}$$

To remove the assumption that $\mathbb{E}(X^2) < \infty$, observe that the right-hand sides of (3.3) and (3.4) differ by

$$\frac{\mathbb{E}\,X}{2} - \frac{\mathbb{E}(X)}{\pi}\lim_{M\to\infty}\int_0^M \mathrm{Re}\left[\frac{i\,e^{-iuK}}{u}\right] du = 0.$$

Next, compare (3.3) with part (a) of Theorem 3.2. The differences are that Theorem 3.2(a) requires the additional assumption that $\mathbb{E}(e^{\alpha X}) < \infty$ for some $\alpha > 0$, but does not assume that $X \geq 0$. We now extend those formulas to cases where $\mathbb{E}(e^{\alpha X})$ may be infinite for all $\alpha > 0$, and where X may take positive and negative values.

Suppose $\mathbb{E}(e^{\alpha X}) < \infty$ for some $\alpha > 0$. Then Theorem 3.2 involves

$$\begin{aligned}
&\int_{-M}^{M} \hat{g}_3(-u+i\alpha)\hat{\mu}_X(u-i\alpha)\,du \\
&\qquad = -\int_{-M-i\alpha}^{M-i\alpha} \frac{e^{-izK}}{z^2}\hat{\mu}_X(z)\,dz \\
&\qquad = -\int_{-M-i\alpha}^{M-i\alpha} \frac{e^{-izK}}{z^2}(\hat{\mu}_X(z)-1)\,dz - \int_{-M-i\alpha}^{M-i\alpha}\frac{e^{-izK}}{z^2}\,dz.
\end{aligned}$$

As $M\to\infty$, the last integral tends to 0 if $K \geq 0$, and to $2\pi K$ if $K < 0$ (use residues). The path of integration in the remaining integral can be pushed up to the real axis, yielding (3.3) when M tends to infinity (the pole at the origin leaves $\pi E(X)$).

We have thus proved the following result. (In parts (a) and (b) the integral is easily seen to converge absolutely; part (b) follows upon writing $(K - X)_+ = [(-K) - (-X)]_+$.)

Theorem 3.4 *(a) If* $\mathbb{E}(X_+) < \infty$*, then for any* $K \in \mathbb{R}$*,*

$$\mathbb{E}(X - K)_+ \;=\; \frac{\mathbb{E}\,X}{2} + (-K)_+ + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iuK}(1 - \hat{\mu}_X(u))}{u^2}\right] du.$$

(b) If $\mathbb{E}[(-X)_+] < \infty$*, then for any* $K \in \mathbb{R}$*,*

$$\mathbb{E}(K - X)_+ \;=\; K_+ - \frac{\mathbb{E}\,X}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{iuK}(1 - \hat{\mu}_X(-u))}{u^2}\right] du.$$

(c) If $X \geq 0$*,* $\mathbb{E}(X) < \infty$*, then for any* $K > 0$*,*

$$\mathbb{E}(X - K)_+ \;=\; \frac{1}{\pi} PV\!\!\int_0^\infty \mathrm{Re}\left[\frac{e^{-iuK}[1 + iu\mathbb{E}(X) - \hat{\mu}_X(u)]}{u^2}\right] du.$$

Note that parts (a) and (b) imply the known formula (*e.g.* Sato, 1999, p.29)

$$\mathbb{E}(|X|) \;=\; \mathbb{E}(X_+) + \mathbb{E}[(-X)_+] \;=\; \frac{1}{\pi}\int_{-\infty}^\infty \frac{\mathrm{Re}[1 - \hat{\mu}_X(u)]}{u^2} du.$$

4 Examples

In the first two examples (compound Poisson/exponential, generalized Pareto) there are closed form expressions for the expected payoffs as well as for Fourier and Mellin transforms. It is therefore possible to test the inversion formulas derived above against the exact expected payoffs. In the other examples (compound Poisson/Pareto, compound Poisson/Pareto plus α-stable), there are no closed form expressions for the expected payoffs we consider, and simulation is used to assess the performance of the Fourier inversion formulas.

4.1 Compound Poisson/exponential distribution

In this example, the explicit distribution is known, as well as both the Fourier and Mellin transforms. We will show that this distribution is intimately related to the hypergeometric functions ${}_0F_1$ and ${}_1F_1$. Recall that

$$ {}_0F_1(c;z) \;=\; \sum_{m=0}^{\infty} \frac{z^m}{m!(c)_m}, \qquad z \in \mathbb{C}, \quad -c \notin \mathbb{N}, $$

where $(c)_0 = 1$, $(c)_m = c(c+1)\cdots c(c+m-1)$, and that

$$ {}_1F_1(a,c;z) \;=\; \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!}, \quad z \in \mathbb{C}, \quad -c \notin \mathbb{N}. $$

The latter is known as the confluent hypergeometric function of the first kind. It is known that (Lebedev, 1972, p.267)

$$ e^{-z}\,{}_1F_1(a,c;z) \;=\; {}_1F_1(c-a,c;-z). $$

Suppose that

$$ S \;=\; \sum_{k=1}^{N} X_k, \qquad X_k \sim \exp(1), \qquad N \sim \text{Poisson}(\lambda). $$

First, the characteristic function of S is

$$ \mathbb{E}(e^{iuS}) \;=\; \exp\left[\lambda\left(\frac{1}{1-iu} - 1\right)\right] \;=\; {}_1F_1(1,1;\lambda iu/(1-iu)). $$

Next, we may calculate the density of the distribution explicitly for $x > 0$:

$$ \begin{aligned} \frac{\partial}{\partial x}\mathbb{P}\{S \le x\} \;&=\; \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\frac{x^{n-1}e^{-x}}{(n-1)!} \\ &=\; \lambda e^{-\lambda-x}\sum_{m=0}^{\infty}\frac{(\lambda x)^m}{m!(m+1)!} \;=\; \lambda e^{-\lambda-x}{}_0F_1(2;\lambda x). \end{aligned} $$

Other authors have expressed this in terms of Bessel functions, but one might argue that hypergeometric functions are more natural here. Next, turn to expectations of payoffs g_3 and g_4: if $K>0$,

$$\begin{aligned}\mathbb{E}(S-K)_+ &= \lambda e^{-\lambda}\int_K^\infty (x-K)e^{-x}{}_0F_1(2;\lambda x)\,dx\\ \mathbb{E}(K-S)_+ &= Ke^{-\lambda}+\lambda e^{-\lambda}\int_0^K (K-x)e^{-x}{}_0F_1(2;\lambda x)\,dx.\end{aligned}$$

Finally, the Mellin transform of S is, for $r>0$,

$$\begin{aligned}\mathbb{E}(S^r) &= \sum_{n=1}^\infty \frac{\lambda^n e^{-\lambda}}{n!}\mathbb{E}(X_1+\cdots+X_n)^r\\ &= \sum_{n=1}^\infty \frac{\lambda^n e^{-\lambda}}{n!}\int_0^\infty \frac{x^{r+n-1}e^{-x}}{\Gamma(n)}\,dx\\ &= \lambda e^{-\lambda}\sum_{m=0}^\infty \frac{\lambda^m}{(m+1)!}\frac{\Gamma(m+1+r)}{m!}\\ &= \lambda e^{-\lambda}\Gamma(1+r)\sum_{m=0}^\infty \frac{\Gamma(m+1+r)}{\Gamma(1+r)}\frac{\Gamma(2)}{\Gamma(m+2)}\frac{\lambda^m}{m!}\\ &= \lambda e^{-\lambda}\Gamma(1+r)\,{}_1F_1(1+r,2;\lambda). \qquad (4.1)\end{aligned}$$

We can thus write (for $\Re(r)>-1$)

$$\mathbb{E}(S^r) = \lambda\Gamma(1+r)\,{}_1F_1(1-r,2;-\lambda). \qquad (4.2)$$

Observe that the integral moments $\mathbb{E}(S^k)$, $k=1,2,\ldots$, form an infinite series in (4.1), but that they are a finite one in (4.2): if $k=0,1,\ldots$,

$$\mathbb{E}(S^k) = \lambda\Gamma(1+k)\,{}_1F_1(1-k,2;-\lambda) = \lambda k!\sum_{j=0}^{k-1}\frac{(1-k)_j}{(j+1)!}\frac{(-\lambda)^j}{j!}.$$

We computed $\mathbb{E}(1-S)_+$, $\lambda=1$, by conditioning on N and also with the Mellin inversion formula (results not shown). The latter was quicker, the results identical.

4.2 Generalized Pareto

In this case there are closed form expressions, in terms of special functions, for the expected payoffs $\mathbb{E}(K-X)_+$ and $\mathbb{E}(X-K)_+$, as well as for the Fourier and Mellin transforms of X.

For $a > 0$, let

$$B(a,b;y) \;=\; \int_0^y x^{a-1}(1-x)^{b-1}\,dx, \qquad 0 \le y \le 1.$$

This is the incomplete beta function. For $a, b > 0$, $B(a,b;1) = B(a,b)$ is the beta function.

If a, b and $\theta > 0$, we write $X \sim$ **Generalized Pareto**(a, b, θ) if the density function of X is

$$\frac{1}{B(a,b)}\frac{\theta^a\, x^{b-1}}{(\theta+x)^{a+b}}\, I_{\{x>0\}}. \tag{4.3}$$

Note that the usual 2–parameter **Pareto**(a,θ) is thus **Generalized Pareto**$(a, 1, \theta)$. Here, letting $y = x/(\theta + x)$ yields

$$\begin{aligned}
B(a,b)\,\mathbb{E}(K-X)_+ &= \int_0^K (K-x)\frac{\theta^a\, x^{b-1}}{(\theta+x)^{a+b}}\,dx \\
&= K\int_{\theta/(K+\theta)}^1 y^{a-1}\,(1-y)^{b-1}\,dy - \theta\int_{\theta/(K+\theta)}^1 y^{a-2}\,(1-y)^{b}\,dy \\
&= K\int_0^{K/(K+\theta)} u^{b-1}\,(1-u)^{a-1}\,du \\
&\quad\; - \theta\int_0^{K/(K+\theta)} u^{b}(1-u)^{a-2}\,du \\
&= KB\Big(b, a; \tfrac{K}{K+\theta}\Big) - \theta B\Big(b+1, a-1; \tfrac{K}{K+\theta}\Big).
\end{aligned}$$

Similarly, if $a > 1$,

$$B(a,b)\,\mathbb{E}(X-K)_+ = \theta B\left(a-1, b+1; \tfrac{\theta}{K+\theta}\right) - KB\left(a, b; \tfrac{\theta}{K+\theta}\right).$$

The Mellin transform of the **Generalized Pareto**(a, b, θ) distribution is

$$\mathbb{E}(X^{iu}) \;=\; \theta^{iu}\frac{B(a-iu, b+iu)}{B(a,b)} \;=\; \theta^{iu}\frac{\Gamma(a-iu)\Gamma(b+iu)}{\Gamma(a)\Gamma(b)}\,,$$

while its Fourier transform is

$$\mathbb{E}(e^{iuX}) \;=\; \frac{1}{B(a,b)}\int_0^\infty \frac{e^{iu\theta x}x^{b-1}}{(1+x)^{a+b}}\,dx \;=\; \frac{\Gamma(a+b)}{\Gamma(a)}\Psi(b, 1-a; -iu\theta).$$

As an illustration, suppose one wishes to compute the excess-of-loss premium $\mathbb{E}(X-K)_+$ if $X \sim \textbf{Pareto}(5,1)$, using a polynomial damping factor. For $\beta = 3$ and $c = 1$,

$$\begin{aligned}
\widehat{g^{[-3]}}(u) &= \left[1-(K+1)\frac{iu}{2}\right]\left[\frac{e^{iuK}}{(1+K)}+iue^{-iu}E_1(-iu(K+1))\right]-\frac{e^{iuK}}{2(1+K)}\\
\widehat{\mu_X^{[3]}}(u) &= \frac{5}{2}[1+iu-u^2e^{-iu}E_1(-iu)].
\end{aligned}$$

The excess-of-loss premium

$$\mathbb{E}(X-K)_+ = \int_{-\infty}^{\infty}\widehat{g^{[-3]}}(u)\widehat{\mu_X^{[3]}}(u)\,du$$

can then be obtained by numerical integration.

4.3 Compound Poisson/generalized Pareto

In this and the next example the only explicit expressions for the stop–loss premiums are the Fourier inversion formulas; the numerical stop–loss (SL) premiums so obtained are compared to simulation results.

The i.i.d. random variables X_j represent individual claim amounts (or "severities"). The compound Poisson variable S represents the aggregate claims over some time interval:

$$S = \sum_{j=1}^{N} X_j,$$

where $N \sim \textbf{Poisson}(\lambda)$. N is assumed independent of the $\{X_j\}_{j\geq 1}$.

Suppose we want to compute an SL premium for the aggregate claims variable S:

$$\mathbb{E}(S-K)_+ = \int_0^{\infty} g(x)\,d\mu_S(x) = \int_0^{\infty} g^{[-\beta]}(x)\,d\mu_S^{[\beta]}(x),$$

where $g(x) = (x-K)_+$. By Theorem 3.3, for an integer $\beta \geq 2$ the SL premium is also equal to

$$\mathbb{E}(S-K)_+ = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{g^{[-\beta]}}(-u)\widehat{\mu_S^{[\beta]}}(u)\,du, \tag{4.4}$$

where

$$\widehat{\mu_S^{[\beta]}}(u) \;=\; \sum_{j=0}^{\beta} \binom{\beta}{j} (-i)^j \frac{d^j}{du^j} \hat{\mu}_S(u), \qquad u \in \mathbb{R}. \tag{4.5}$$

Since S is compound Poisson, $\hat{\mu}_S(u) = e^{\lambda[\hat{\mu}_X(u)-1]}$. For instance, if $\beta = 3$ then

$$\begin{aligned}\widehat{\mu_S^{[3]}}(u) &= \hat{\mu}_S(u) - 3i\hat{\mu}_S'(u) - 3\hat{\mu}_S''(u) + i\hat{\mu}_S'''(u) \\ &= e^{\lambda[\hat{\mu}_X(u)-1]} \left\{1 - 3i\lambda\hat{\mu}_X'(u) - 3[\lambda\hat{\mu}_X''(u) + \lambda^2(\hat{\mu}_X'(u))^2] \right. \\ &\quad \left. + i[\lambda\hat{\mu}_X'''(u) + 3\lambda^2\hat{\mu}_X'(u)\hat{\mu}_X''(u) + \lambda^3(\hat{\mu}_X'(u))^3]\right\}.\end{aligned}$$

If the $\{X_j\}_{j\geq 1}$ have a **Generalized Pareto**(a, b, θ) distribution (see (4.3)), then

$$\frac{d^j}{du^j}\hat{\mu}_X(u) \;=\; \int_0^\infty e^{iux} \frac{\theta^a}{B(a,b)} \frac{(ix)^j\, x^{b-1}}{(\theta+x)^{a+b}} dx, \quad u \in \mathbb{R}, \quad 0 \leq j < a.$$

In general these may be written in terms of the special function Ψ; if a and b are integers, then the derivatives may be expressed in terms of the exponential integral function as in Section 3.2.2. Table 1 lists the SL premiums obtained from (4.4) by numerical integration. Different values of the Poisson parameter λ and of the retention limit K are used, but the Pareto parameters are fixed at $a = 5$, $b = 3$ and $\theta = 1$. These are compared with simulated SL premiums based on 1,000,000 replications (for $K = 0$ the exact value $\mathbb{E}(S) = \lambda b/(a-1)$ is reported). Adding and subtracting the values between parenthesis to the simulated premiums yields 95% (asymptotic) confidence intervals, giving a measure of the simulation accuracy.

The Fourier premiums were computed using Romberg's method, coded in Maple or Matlab. We see that the Fourier SL premiums are in close agreement with the simulated premiums. The computing time is of the order of 1 or 2 seconds for each value of K and seems independent of the choice of parameters.

Table 2 lists additional SL premiums, also computed from (4.4). Here both the Poisson parameter λ and the Pareto parameter a vary, but the other two Pareto parameters are fixed at $b = 3$ and $\theta = 1$, in such a way that the expected value of S remains equal to 1. These illustrate the speed at which these SL premiums tend to 0 as the retention K increases.

Table 1: **SL premiums - compound Poisson/gen. Pareto**[$\lambda; (a = 5,\ b = 3,\ \theta = 1)$]

	$\lambda = 1$		$\lambda = 2$		$\lambda = 3$	
K	Simulated	Fourier	Simulated	Fourier	Simulated	Fourier
0	0.7493 ($\pm$0.00196)	0.75	1.4993 ($\pm$0.00277)	1.5	2.2476 ($\pm$0.00339)	2.25
0.25	0.5959 ($\pm$0.00183)	0.5962	1.2865 ($\pm$0.00270)	1.2871	2.0119 ($\pm$0.00335)	2.0140
0.5	0.4657 ($\pm$0.00169)	0.4660	1.0915 ($\pm$0.00261)	1.0914	1.7863 ($\pm$0.00330)	1.7882
1	0.2821 ($\pm$0.00141)	0.2822	0.7702 ($\pm$0.00235)	0.7702	1.3804 ($\pm$0.00313)	1.3825

Table 2: **SL premiums - compound Poisson/gen. Pareto**[$\lambda; (a,\ b = 3,\ \theta = 1)$]

	$\lambda = 1,\ a = 4$	$\lambda = 2,\ a = 7$	$\lambda = 3,\ a = 10$
K	Fourier	Fourier	Fourier
1	0.4924	0.3448	0.2774
1.5	0.3451	0.1857	0.1202
2	0.2442	0.0968	0.0474
2.5	0.1745	0.0493	0.0174
3	0.1260	0.0248	0.0060

4.4 Compound Poisson/gen. Pareto plus α–stable

Dufresne & Gerber (1991) considered a risk process made up of the sum of a compound Poisson process and a Brownian motion. Furrer (1998) looked at a risk model where an α-stable process is added to the compound Poisson process. The motivation for those models is that the added Brownian motion or stable process accounts for very large claims or for larger exogenous random perturbations, such as changes in portfolio composition or in investment income.

In this section, we therefore look at how stop-loss premiums can be computed if aggregate claims are represented by

$$Z = S + J = \sum_{j=1}^{N} X_j + J, \tag{4.6}$$

where S is compound Poisson and J has an α-stable distribution. The claims have a generalized Pareto distribution, and S and J are assumed independent. There is of course no explicit expression for the distribution function or density of Z, so simulation and Fourier inversion are the only possible ways of computing SL premiums.

If J has an α-stable distribution, then

$$\hat{\mu}_J(u) = e^{-\Psi_J(u)},$$

where

$$\Psi_J(u) = p^\alpha |u|^\alpha [1 - i\rho \,\mathrm{sign}(u) \tan(\alpha\pi/2)] + i\gamma u$$

if $0 < \alpha < 1$ or $1 < \alpha \leq 2$, or

$$\Psi_J(u) = p|u|(1 + i\rho \,\mathrm{sign}(u) \log |u|) + i\gamma u$$

if $\alpha = 1$. The case $\alpha = 2$ corresponds to the normal distibution; p is a scale parameter, and $\rho \in [-1, 1]$ relates to the symmetry (or lack thereof) of the distribution. When $0 < \alpha < 2$, the distribution is symmetric if $\rho = 0$, it is concentrated on $(0, \infty)$ if $\rho = 1$, and concentrated on $(-\infty, 0)$ if $\rho = -1$. (For more details, see Sato (1999) or Samorodnitsky and Taqqu (1994).)

It is known that when $0 < \alpha < 2$ the right tail of the α-stable distribution with characteristic function given above behaves as follows (Samorodnitsky and Taqqu, 1994, p.16):

$$\lim_{y\to\infty} y^\alpha \mathbb{P}(X > y) \;=\; C_\alpha \frac{1+\rho}{2} p^\alpha.$$

Hence, excluding the case $\rho = -1$, SL premiums $\mathbb{E}(Z - K)_+$ can be finite only if $\alpha > 1$. We will assume $\rho > -1$ and $\alpha > 1$. The previous equation also implies that the expected value of an α-stable distribution is finite if, and only if, $\alpha > 1$, and that the second moment of an α-stable distribution is infinite for all $\alpha < 2$.

We know the Fourier transforms of S and J, so the formulas in Sections 3.2.2 and 3.2.3 may be used to compute SL premiums. However, the restrictions on moments excludes polynomial damping factors if $\alpha < 2$, since the minimum possible value of β is 2 (Theorem 3.3(a)). In the case where J is concentrated on $(0, \infty)$ it is nevertheless possible to apply Theorem 3.3(b) with $\beta = 0$ or 1, since

$$\mathbb{E}(Z - K)_+ \;=\; \mathbb{E}(K - Z)_+ + \mathbb{E}(Z) - K.$$

However, the formula in Theorem 3.4(a) always applies here, since the sole assumption is $\mathbb{E}|Z| < \infty$.

We give a numerical example with $\alpha = 2$, so that J has a normal distribution (which we assume zero-mean) and

$$\widehat{\mu}_Z(u) = \exp\left\{\lambda\left[\widehat{\mu}_X(u) - 1\right] - \frac{\sigma^2 u^2}{2}\right\}.$$

If enough moments of the Pareto claims distribution exist then it is possible to apply Theorem 3.3(a).

Table 3: **SL premiums - compound Poisson/gen. Pareto**$[\lambda; (a = 5,\ b = 3,\ \theta = 1)]$ + **normal**$(0, 1)$

	$\lambda = 1$		$\lambda = 2$		$\lambda = 3$	
K	Simulated	Fourier	Simulated	Fourier	Simulated	Fourier
0	0.7535 (±0.00277)	0.75	1.4975 (±0.00339)	1.5	2.2489 (±0.00391)	2.25
0.25	0.8124 (±0.00215)	0.8098	1.4238 (±0.00298)	1.4252	2.0968 (±0.00366)	2.0971
0.5	0.6675 (±0.00200)	0.6653	1.2401 (±0.00286)	1.2413	1.8869 (±0.00357)	1.8872
1	0.4364 (±0.00170)	0.4348	0.9195 (±0.00259)	0.9203	1.5013 (±0.00336)	1.5016

Table 3 illustrates the calculations for different values of the Poisson parameter λ and of the retention limit K, for fixed Pareto parameters $a = 5$, $b = 3$, $\theta = 1$ and a $\mathbf{N}(0, 1)$ perturbation. A polynomial damping factor is used, with $\beta = 3$ and $c = 1$. The 1,000,000 simulated SL premiums and their 95% (asymptotic) confidence interval widths are given for comparison (again here $\mathbb{E}(S) = \lambda b/(a - 1)$ is the exact value for $K = 0$).

The effect of the normal perturbation is clear. The extra variability is controlled by the parameter σ^2. Table 4 illustrates its effect for a smaller variance of $\sigma^2 = 1/2$.

Table 4: **SL premiums - compound Poisson/gen. Pareto**$[\lambda; (a = 5,\ b = 3,\ \theta = 1)]$ + **normal**$(0, 1/2)$

	$\lambda = 1$		$\lambda = 2$		$\lambda = 3$	
K	Simulated	Fourier	Simulated	Fourier	Simulated	Fourier
0	0.7508 (±0.00240)	0.75	1.4998 (±0.00310)	1.5	2.2454 (±0.00366)	2.25
0.25	0.7232 (±0.00197)	0.7224	1.3646 (±0.00284)	1.3643	2.0537 (±0.00351)	2.0577
0.5	0.5780 (±0.00184)	0.5775	1.1749 (±0.00273)	1.1746	1.8373 (±0.00344)	1.8412
1	0.3594 (±0.00155)	0.3593	0.8509 (±0.00247)	0.8504	1.4424 (±0.00324)	1.4460

5 Conclusion

Hopefully these illustrations convincingly show that Parseval's theorem is applicable to the computation of stop–loss premiums. The method allows for quite general aggregate claims models, including compound Poisson distributions perturbed by some other Lévy process. The only requirement is that the characteristic functions, or Mellin transforms, of the distibutions involved be known.

On the down side, the integrals that have to be computed are often true principal value integrals (*i.e.* that do not converge absolutely) involving circular functions. There are cases where the function to be integrated has to be carefully studied to make the numerical integration work (this was not the case in the numerical examples given in this paper).

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A Parseval's theorem

One form of the Parseval identity is (Malliavin, 1995, p.134) is: if $h, \hat{h} \in Ł^1$ and μ is a signed measure with $|\mu| < \infty$, then

$$\int_{-\infty}^{\infty} h(x)\, \mu(dx) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(u)\hat{\mu}(-u)\, du.$$

This is a fairly restrictive result, because the damped functions we consider often do not satisfy the condition $\hat{h} \in L^1$; for instance, the function $h(x) = e^{-x} I_{(0,\infty)}(x)$ is in L^1 but its transform, $\hat{h}(u) = 1/(1 - iu)$, is not. A less restrictive inversion theorem is thus required for the applications considered in this paper. We need the following classical inversion theorem.

Theorem A.1 (Apostol, 1974, p.324) *Suppose h is a real function which satisfies the following conditions:*

(a) $h \in L^1$ and

(b) either (b1) or (b2) holds:

(b1) $h(x+)$ and $h(x-)$ both exist and the integrals below are finite for some $\epsilon > 0$:

$$\int_0^\epsilon \frac{h(x+t) - h(x+)}{t}\, dt, \qquad \int_{-\epsilon}^0 \frac{h(x-t) - h(x-)}{t}\, dt;$$

(b2) $h(x)$ has bounded variation in some open neighborhood of x. (This implies that $h(x+)$ and $h(x-)$ both exist. A sufficient condition for a function to have bounded variation is having a derivative.)

Then

$$\frac{1}{2}[h(x+) + h(x-)] \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} e^{-iux} \hat{h}(u)\, du.$$

We are now able to derive the Parseval identity we need. Let μ be a signed measure with $|\mu| < \infty$ and $h \in L^1$, and suppose that the convolution

$$y \;\mapsto\; (\tau_\mu h)(y) \;=\; \int_{-\infty}^{\infty} h(y-x)\, \mu(dx)$$

satisfies assumption (b) of Theorem A.1, and that moreover $(\tau_\mu h)(y)$ is continuous at $y = 0$. It is known that $h \in L^1$ implies $\tau_\mu h \in L^1$ (Malliavin, 1995, p.114), and Theorem A.1 yields

$$(\tau_\mu h)(0) \;=\; \int_{-\infty}^{\infty} h(-x)\, \mu(dx) \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{h}(u) \hat{\mu}(u)\, du.$$

To obtain Parseval's theorem, replace $h(x)$ with $g(-x)$, after noting that

$$\int_{-\infty}^{\infty} e^{iux} g(-x)\, dx \;=\; \hat{g}(-u),$$

to get

$$\int_{-\infty}^{\infty} g(x)\, \mu(dx) \;=\; \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}(-u) \hat{\mu}(u)\, du. \tag{A.1}$$

We have thus proved:

Theorem A.2 *Let μ be be a signed measure with $|\mu| < \infty$. Suppose that (i) $g \in L^1$, (ii) the function*

$$y \mapsto \int_{-\infty}^{\infty} g(x-y)\, \mu(dx)$$

is continuous at $y = 0$ and (iii) g satisfies condition (b) of Theorem A.1. Then A.1 holds.

B Proofs of theorems

Proof of Lemma 2.1. (a) Write

$$\int g^r(x-y)\,d\mu_X^r(x) \;=\; \sum_{k=1}^{n+1}\int_{I_k} g^r(x-y)\,d\mu_X^r(x),$$

where $\{I_k\}$ are the indicator functions of the intervals $(-\infty, a_1)$, (a_1, a_2), ... , (a_{n-1}, a_n), (a_n, ∞). Then, as $y \to 0$,

$$g^r(x-y)I_k(x) \to g^r(x)I_k(x)$$

for all x. Since g^r is uniformly bounded and $|\mu_X^r| < \infty$,

$$\int_{I_k} g^r(x-y)\,d\mu^r(x) \;\to\; \int_{I_k} g^r(x)\,d\mu^r(x)$$

by dominated convergence, which yields the result.

To prove part (b), observe that $G(y)$ is finite because $(X-K-y)_+ \le X_+ + (y+K)_-$, and the result follows from

$$\mathbb{E}(X-K-y)_+ \;=\; \int_{K+y}^{\infty} \mathbb{P}(X > x)\,dx.$$

In the other case $G(y) = \mathbb{E}(K+y-X)_+$ is always finite, and continuity follows by dominated convergence. ∎

Proof of Lemma 2.2. Let V be the variation of g^r over $\mathbb{R}$. If $\{y_j\}$ is an increasing sequence, then

$$\begin{aligned}\sum_j |\Delta G(y_j)| \;&\le\; \int \sum_j |g^r(y_j+x) - g^r(y_{j-1}+x)|\,d\mu_X^r(x)\\ &\le\; \int V\,d\mu_X^r(x) \;=\; V|\mu_X^r| \;<\; \infty.\end{aligned}$$

It is well known that the total variation of a function is bounded above by the integral of the absolute value of the derivative, see for example Apostol (1974, p.128). ∎

Proof of Theorem 3.1. (a) Let $K = e^c$ and $S = e^X$. First, assume that $\mathbb{P}\{S = 0\} = 0$. If $g(x) = (e^c - e^x)_+$ and $z \in \mathbb{C}$, then

$$\begin{aligned}
\hat{g}(z) &= \int_{-\infty}^{c} e^{izx}(e^c - e^x)\,dx \\
&= \lim_{M\to\infty} \int_{-M}^{c} (e^{izx+c} - e^{(iz+1)x})\,dx \\
&= \lim_{M\to\infty} \left\{ e^c \left[\frac{e^{izx}}{iz}\Big|_{-M}^{c} \right] - \left[\frac{e^{(iz+1)x}}{(iz+1)}\Big|_{-M}^{c} \right] \right\} \\
&= e^{(iz+1)c}\left(\frac{1}{iz} - \frac{1}{iz+1}\right) + \lim_{M\to\infty}\left(\frac{1}{iz+1}e^{-(iz+1)M} - \frac{1}{iz}e^{-izM+c}\right).
\end{aligned}$$

The limit exists, and equals 0, if and only if, $\mathrm{Im}(z) < 0$. Hence,

$$\hat{g}(z) \;=\; \frac{e^{(iz+1)c}}{iz(iz+1)}, \qquad \mathrm{Im}(z) < 0.$$

Let $h(z) = \hat{g}(-z)\hat{\mu}_X(z)$. We need to restrict z to $\mathrm{Im}(z) > 0$ for $\hat{g}(-z)$ to exist, and, therefore, we need to assume that $\mathbb{E}(e^{\alpha X})$ exists for some $\alpha < 0$. This proves the first formula in (a), if $\mathbb{P}\{S = 0\} = 0$.

If $\mathbb{P}\{S = 0\} > 0$, then define a new variable S^* with distribution

$$\mathbb{P}\{S^* \in A\} \;=\; \frac{\mathbb{P}\{S \in A, S > 0\}}{\mathbb{P}\{S > 0\}} \;=\; \mathbb{P}(S \in A \,|\, S > 0).$$

Then

$$\mathbb{E}(K - S)_+ \;=\; K\mathbb{P}\{S = 0\} + \mathbb{P}\{S > 0\}\mathbb{E}(K - S^*)_+$$

which yields the result, since

$$\mu_{S^*}(u) \;=\; \frac{\mu_S(u)}{\mathbb{P}\{S > 0\}}.$$

The second formula in (a) follows from the usual relationship $y_+ - (-y)_+ = y$, $y = S - K$.

Part (b) is obtained by first assuming that $\mathbb{P}\{S = 0\} = 0$ and that there exists $\alpha < 0$ such that $\mathbb{E}(S^\alpha) < \infty$. The first formula in part (a) then holds. The function $h(z)$ is analytic in the upper complex plane, except for a simple pole at the origin. We have

$$\mathbb{E}(K - S)_+ \;=\; \frac{1}{2\pi} PV\!\int_{-\infty-i\alpha}^{\infty-i\alpha} h(z)\,dz,$$

where the path of integration is the line $\{z \mid \mathrm{Im}(z) = -\alpha\}$. For $0 < \epsilon < M$, define a closed path of integration $C_{M,\epsilon}$ as in Figure 1. The integral of $h(z)$ along $C_{M,\epsilon}$ is 0.

It is easy to see that

$$\lim_{M\to\infty} \int_{-M}^{-M-i\alpha} h(z)\,dz \;=\; \lim_{M\to\infty} \int_{M}^{M-i\alpha} h(z)\,dz \;=\; 0$$

and so

$$PV \int_{-\infty}^{\infty} h(u - i\alpha)\,du \;=\; \int_{R_\epsilon} h(z)\,dz + \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) h(z)\,dz,$$

where R_ϵ is the half–circle around the origin in Figure 1. We find

$$\begin{aligned} \lim_{\epsilon\to 0+} \int_{R_\epsilon} h(z)\,dz &= \lim_{\epsilon\to 0+} \int_{\pi}^{0} h(\epsilon e^{i\theta}) \epsilon i e^{i\theta} d\theta \\ &= \lim_{\epsilon\to 0+} \int_{\pi}^{0} \frac{e^{-i\epsilon e^{i\theta}+c}}{(-i\epsilon e^{i\theta})(1 - i\epsilon e^{i\theta})} i\epsilon e^{i\theta}\,d\theta \\ &= K\pi. \end{aligned}$$

Hence,

$$\mathbb{E}(e^c - S)_+ \;=\; \frac{K}{2} + \frac{1}{2\pi} \int_0^\infty [h(u) + h(-u)]\,du.$$

Since $h(u) + h(-u) = 2\mathrm{Re}[h(u)]$, we thus have

$$\mathbb{E}(K - S)_+ \;=\; \frac{K}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}[\hat{g}(-u)\hat{\mu}_X(u)]\,du. \tag{B.1}$$

This formula was obtained under the assumption that there exists $\alpha < 0$ such that $\mathbb{E}(S^\alpha) < \infty$. If this is not the case, then consider

$$X^a \;=\; X \vee (-a),$$

for $a > 0$. As $a \to \infty$, $\mathbb{E}(e^{iuX^a}) \to \mathbb{E}(e^{iuX})$ uniformly in $u \in \mathbb{R}$. Since

$$|\hat{g}(u)| \;\sim\; \frac{e^c}{u^2} \quad \text{as } |u| \to \infty,$$

we find that

$$\mathbb{E}(K - e^X)_+ \;=\; \lim_{a\to\infty} \mathbb{E}(K - e^{X^a})_+ \;=\; \frac{K}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}[\hat{g}(-u)\hat{\mu}_X(u)]\,du$$

by dominated convergence. Finally, (B.1) holds for all $S > 0$.

The last formula may be proved another way. The function $h(z)$ may be rewritten as

$$e^c \frac{e^{-izc}}{-iz} \frac{\hat{\mu}_X(z)}{1 - iz}.$$

Note that $\hat{\mu}_X(z)/(1 - iz)$ is the Fourier transform of the convolution of an exponential density with the law of X. This suggests proceeding as follows:

$$\mathbb{E}(e^c - e^X)_+ \;=\; e^c \int_{-\infty}^{c} (1 - e^{x-c})\, d\mu_X(x) \;=\; e^c \mathbb{P}\{X + G \leq c\},$$

if $G \sim \exp(1)$ is independent of X. By Theorem 2.3,

$$\begin{aligned} e^c \mathbb{P}\{X + G \leq c\} &= \frac{e^c}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^c}{iu} \left[e^{iuc} \frac{\hat{\mu}_X(-u)}{1 + iu} - e^{-iuc} \frac{\hat{\mu}_X(u)}{1 - iu} \right] du \\ &= \frac{K}{2} + \frac{1}{2\pi} \int_0^\infty [h(u) + h(-u)]\, du, \end{aligned}$$

which is the same as (B.1). Finally, the formulas in (b) are found by taking into account the cases where $\mathbb{P}\{S = 0\} > 0$, as in the proof of (a). ■

Figure 1