ACHIEVING HIGHER ORDER CONVERGENCE FOR THE PRICES OF EUROPEAN OPTIONS IN BINOMIAL TREES

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Abstract. A new family of binomial trees as approximations to the Black–Scholes model is introduced. For this class of trees, the existence of complete asymptotic expansions for the prices of vanilla European options is demonstrated and the first three terms are explicitly computed. As special cases, a tree with third order convergence is constructed and the conjecture of Leisen and Reimer that their tree has second order convergence is proven.

1. Introduction

Binomial trees are a popular method of pricing stock options. Yet many fundamental questions remain to be answered. In particular, in recent years there has been detailed study of the asymptotic behaviour of prices on certain trees but convergence beyond first order has not previously been demonstrated rigorously for any tree even for European options. In this paper, we introduce a new class of trees which generalizes the tree of Leisen and Reimer, show that their pricing errors have smooth asymptotic expansions for European options, and explicitly compute the first three terms. This allows us to prove the 1996 conjecture of Leisen and Reimer [13] that their tree has second order convergence for European options. We are also able to construct a tree with third order convergence. In addition, we give an existence proof for trees with infinite order of convergence, and give a numerical construction for a tree with approximately fourth order of convergence.

Binomial trees were introduced by Cox-Ross-Rubinstein to study the American put option, and they used a central limit theorem argument to show that their tree converged, but did not study the rate nor smoothness of convergence. Diener and Diener, [6], and Walsh, [19],

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independently studied the convergence of variants of the CRR [5] tree. They proved that the lead term of the convergence was of order -1 in the number of steps for vanilla call and put options but only of order $-1/2$ for digital options. They also showed that the lead error term was oscillatory with the oscillations being driven by the distance in log-space between the strike and neighbouring nodes in the tree.

This suggests that if one modifies the tree so that this quantity does not change then the oscillations should disappear, and indeed this had previously been tried by Tian [18] with numerical demonstrations that convergence was smooth. Chang and Palmer, [4], studied Tian's tree in depth and were able to show that convergence was of the form

$$A/n + o(n^{-1}),$$

even for digital options when the strike was placed on the geometric mean of two nodes.

In a similar vein, Joshi [11] studies a tree with odd numbers of steps in which the strike has been placed on the geometric mean of the central nodes on the final step. It is proven there that in that case a complete asymptotic expansion for the price with errors in powers of n^{-1} exists. This was shown to hold even for digital options. Our approach here is to generalize that work by turning one of the main results of that paper which is that the probability of an up-move has a nice asymptotic expansion into a hypothesis, and then studying the class of trees which results. This is assumed to hold when each of stock and bond is numeraire. The resulting class of trees is shown to encompass the tree of Leisen and Reimer [13] which allows a detailed analysis of that tree's convergence.

In particular, we are able to show that for all trees in the class the prices of vanilla European options, including calls, puts, digitals, and asset or nothing options, have complete asymptotic expansions. We are also able to compute the first three terms in the asymptotic expansion explicitly. This allows us to choose the coefficients in such a way as to make the price correct to third order. We also observe that the coefficients implied by the Leisen–Reimer tree give second order convergence providing a proof of their 1996 conjecture that their tree has second order convergence for European options.

A natural question is "how high can you go?" i.e. can one construct trees with even higher orders of convergence. We answer this in the affirmative, and give an existence proof for trees with infinite order of convergence.

The approach we adopt in this paper is to divide the pricing into the price of a digital option and an asset or nothing option. By using a different numeraire for each, one then is reduced to computing the asymptotics of binomial probabilities. These binomial probabilities are then transformed into integrals, and the discrete parameter replaced by a continuous one. The analysis of this integral is then carried out by careful manipulation of Taylor expansions.

There are by now many papers on the topic of binomial trees of which the Leisen and Reimer tree [13] has been the most successful. Their approach is to say that since a binomial tree is an attempt to approximate a normal density with a discrete binomial one, a fruitful approach should be to take known approximations of binomials by normals and invert them. They also change viewpoint in that they start by specifying probabilities and use these to specify node locations rather than the conventional approach of doing the opposite. We will also do this here. They also demonstrated that the trees of Tian [17], Jarrow–Rudd [8] and Cox–Ross–Rubinstein [5] only had first order convergence.

An obvious question is to ask what results have been proven for American options. Amin and Khanna [1] first proved convergence in that case. Jiang and Dai [9] and Qian et al [15] have proven uniform convergence. For the CRR model, Lamberton [12] showed that the error was bounded above by $Cn^{-3/4}$. For the Leisen–Reimer tree, Leisen [14] has shown that American puts have first order convergence, whilst the models of Jarrow and Rudd [8] and Tian [18] have convergence between order 1/2 and 1. However, there are no results establishing the form of the lead order term, and indeed it is likely to be oscillatory for all existing trees. Broadie and Detemple, [3], suggest using the Black–Scholes formula for the final step to damp the oscillations and then apply Richardson extrapolation; they achieve interesting speed ups, but their work is purely numerical.

We discuss the structure of the paper. In section 2, we recall basic definitions and define what we mean by a recombining tree. We then define our new class of trees in section 3. We state our main results and show how they follow from a technical result in section 4. We give some numerical illustration to our results in section 5. We develop the asymptotics of binomial coefficients necessary for our proofs in section 6. We prove the main technical results in sections 7 and 8. We show how our results apply to the Leisen–Reimer tree in section 9. We briefly discuss the case of the American put option in section 10, and conclude in section 11. In order to make it easy for the reader to implement the tree, we present C++ code to compute the probabilities and node locations for the third order tree in the appendix.

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2. Defining recombining trees

This paper is solely focused on the case of recombining trees in the Black–Scholes model. We therefore commence by defining those terms. We work in the Black-Scholes model, and have, as usual, the parameters, r the risk-free rate, T expiry time, σ the volatility, and K the strike. We denote the spot price at time zero by S, and at time t by S_t. The riskless bond has value e^{rt} at time t. The process followed by the stock before discretization in the risk-neutral measure (i.e. the martingale measure if the bond is numeraire,) is

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{2.1}$$

and if we take the stock as numeraire, we get in the pricing measure (henceforth, called the *stock measure*)

$$dS_t = (r + \sigma^2)S_t dt + \sigma S_t dW_t. \tag{2.2}$$

See, for example [10], for further details.

A family of recombining binomial trees is a sequence of discrete processes, X_i^n, defined at discrete times which converges to the process in the Black–Scholes model an interval $[0, T]$. The crucial properties are

- The process X_i^n is defined at the times Ti/n for $i = 0, 1, \ldots, n$.
- The processes are Markovian.
- The value of X_i^n takes precisely $i+1$ values with non-zero probability.
- The distribution of X_{i+1}^n/X_i^n is a two point distribution independent of X_i, and is the same for each i.
- There exists a choice of probability equivalent to the given distribution such that
$$\mathbb{E}(X_{i+1}^n/X_i^n) = e^{rT/n}.$$

In simpler terms, for a given n, the tree is defined by the choice of up and down moves which will be the same in multiplicative terms at every step. So an up move followed by a down move is the same as a down move followed by an up move. The values of up, U_n, and down, D_n, need to be chosen so that

$$D_n < e^{rT/n} < U_n,$$

so that

$$p_n = \frac{e^{rT/n} - D_n}{U_n - D_n}$$

is between zero and one.

3. The new class of trees

In [11] and [13], families of trees with nice convergence properties were introduced. Our objective is to introduce a class of trees which includes both these trees together with enough flexibility to eliminate error terms. The fundamental properties of those trees is that they are defined for odd numbers of steps, the probability of an up move has a nice asymptotic expansion in both the risk-neutral and stock measures, and they are approximately centred on the strike of the option. In particular, precisely half the nodes are above the strike. They are also recombining trees with all nodes essentially the same in terms of relative moves in log-space.

We define

$$d_j = \frac{(r + (-1)^{j-1}\frac{1}{2}\sigma^2)T + \log(S/K)}{\sigma\sqrt{T}}. \tag{3.1}$$

We let p^1_{2k+1} be the probability of an up-move in the stock measure with $2k+1$ steps, and p^2_{2k+1}, correspondingly for the bond measure.

Definition 3.1. *We shall say that a family of recombining binomial trees has smooth asymptotics centred on K, if*

- *it is defined for odd numbers of steps,*
- *the probabilities have asymptotic expansions*

$$p^j_{2k+1} \sim \frac{1}{2} + \frac{d_j}{2\sqrt{2}}\frac{1}{\sqrt{k}} + \sum_{m=1}^{\infty} a^j_m k^{-\frac{1}{2}-m}.$$

Our point of view in this paper, following [13], is that when specifying a tree one should prescribe the probabilities and use these to deduce the locations of up and down moves rather than doing the more natural opposite process. In this case, this means that one should specify the terms in the asymptotic expansion.

That the adjusted tree studied in [11] is within this class follows trivially from results in that paper. We show in Section 9 that the Leisen–Reimer tree is also in this class, and compute the first two terms explicitly.

We recall the relationships between node locations and probabilities from [13]. Let

$$r_n = \exp\left(r\frac{T}{n}\right), \tag{3.2}$$

we then have

$$U_{2k+1} = r_{2k+1}\frac{p^1_{2k+1}}{p^2_{2k+1}}, \tag{3.3}$$

$$D_{2k+1} = \frac{r_{2k+1} - p^2_{2k+1}U_{2k+1}}{1 - p^2_{2k+1}}, \tag{3.4}$$

We need to justify the term "centred on K." Note that K enters through d_1 and d_2. We show that for large numbers of steps that precisely half the nodes are above the strike:

Proposition 3.1. *If a family of recombining trees has smooth asymptotics centred on K then the central node on layer $2k$ in a tree with $2k+1$ steps is equal to K up to an error of order $1/k$.*

Proof. We need to examine $(U_{2k+1}D_{2k+1})^k$. It follows quickly from above that

$$U_{2k+1}D_{2k+1} = r^2_{2k+1}\frac{p^1_{2k+1}(1-p^1_{2k+1})}{p^2_{2k+1}(1-p^2_{2k+1})}. \tag{3.5}$$

From the asymptotic expansions, this implies

$$U_{2k+1}D_{2k+1} = r^2_{2k+1}\frac{\frac{1}{4} - \frac{d_1^2}{8k}}{\frac{1}{4} - \frac{d_2^2}{8k}} + \mathcal{O}(k^{-2}), \tag{3.6}$$

$$= r^2_{2k+1}\left(1 - \frac{d_1^2}{2k} + \frac{d_2^2}{2k}\right) + \mathcal{O}(k^{-2}). \tag{3.7}$$

Now

$$d_1^2 - d_2^2 = 2\log(S) - 2\log(S) + 2rT,$$

so on taking logs, we obtain

$$\log(U_{2k+1}D_{2k+1}) = \frac{2rT}{2k+1} - \frac{rT}{k} + \frac{1}{k}(\log(K) - \log(S)) + \mathcal{O}(k^{-2}).$$

After k pairs of up-down moves, we therefore obtain

$$\log(K) - \log(S) + \mathcal{O}(k^{-1}),$$

which proves the result since the initial location was $\log S$. □

Since the nodes in the final layer will be found by moving up or down fron the central node in the previous layer, and the size of move is order $k^{-1/2}$, we have

Corollary 3.1. *In the final layer, half the nodes will be above the strike for large k.*

4. Smooth asymptotics

We wish to show that a complete asymptotic expansion for the price of a vanilla option exists for these trees. This was shown in [11] for a certain sub-class of our trees and inspecting the proof there shows that it also holds for the entire class. However, for completeness and because we wish to extend the argument in different ways, we reproduce the argument. We can decompose the pay-off of a call option to be

$$(S_T - K)_+ = S_T I_{S_T > K} - K I_{S_T > K}$$

and price each of the two parts with a different numeraire to get

$$C(S_0, T) = S_0 \mathbb{P}_S(S_T > K) - K e^{-rT} \mathbb{P}_B(S_T > K).$$

with the subscripts S and B denote probabilities in the stock and bond measure respectively. See [2] or [10] for further discussion. This means that it is sufficient to consider the convergence of the two probabilities, $\mathbb{P}(S_T > K)$, and $\mathbb{P}_B(S_T > K)$.

In financial terms, we have decomposed the pay-off into the sum of a digital option and an asset or nothing option. If we can prove all our results for each of these options, then the call option follows immediately. Note also that since our trees are martingale trees, we will always price the forward contract correctly by construction, and the error for a put option will then be the same as for a call option with the same strike, by put-call parity.

The probability of $S_T > K$ is simply the probability that more than half the moves are upwards for a centred tree for large n. Thus we need to compute the asymptotics of the probability of this event.

Our key technical result is

Theorem 4.1. *If*

$$p_{2k+1} \sim \frac{1}{2} + \frac{1}{\sqrt{k}} \sum_{m=0}^{\infty} \alpha_m k^{-m}$$

then the probability of $k+1$ *up-moves is equal to*

$$N(2\sqrt{2}\alpha_0) + \frac{1}{k} e_1 + \frac{1}{k^2} e_2 + \dots,$$

with

$$e_1 = \frac{2}{\sqrt{\pi}} e^{-4\alpha_0^2} \left(\alpha_0^3 + \frac{3}{8}\alpha_0 + \alpha_1 \right),$$

and

$$e_2 = \frac{2}{\sqrt{\pi}} e^{-4\alpha_0^2} (\frac{3}{8}\left(\alpha_0^3 + \alpha_1\right) + \frac{13}{6}\alpha_0^5 + 3\alpha_0^2\alpha_1$$
$$+ \alpha_2 - 4\alpha_0(\alpha_0^3 + \alpha_1)^2 - \frac{7}{128}\alpha_0 - 3(\alpha_0^5 + \alpha_0^2\alpha_1) - \frac{25}{48}\alpha_0^3).$$

If p_{2k+1} and q_{2k+1} have the same α_m for $m < r$, and their α_r terms differ by β_r then the difference in probabilities of $k+1$ up moves is

$$e^{-4\alpha_0^2} \frac{2}{\sqrt{\pi}} \beta_r k^{-r} + \mathcal{O}(k^{-r-1}).$$

We have two immediate corollaries of this result

Corollary 4.1. *If*

$$\begin{aligned} \alpha_0 =& \frac{d_j}{2\sqrt{2}}, \\ \alpha_1 =& -\alpha_0^3 - \frac{3}{8}\alpha_0, \\ \alpha_2 =& \frac{5}{6}\alpha_0^5 + \frac{13}{12}\alpha_0^3 + \frac{25}{128}\alpha_0, \end{aligned}$$

then the probability of at least $k+1$ up-moves is

$$N(d_j) + \mathcal{O}(k^{-3}).$$

Thus for a given K it is possible to construct a family of trees centred on K such that prices of the asset or nothing option, the digital option, the call option and the put option, all struck at K have third order convergence. Conversely, if a tree is centred on K and has second order convergence then α_1 has the value stated, if, in addition, it has third order convergence then α_2 has the value stated.

The proof of this corollary is simply to pick the value of α_1 that makes the first error term disappear, and then to pick the value of α_2 that makes the second term disappear. We do this in each of stock and bond measures picking α_0 each time so that the tree is centred on K.

Our second corollary is

Corollary 4.2. *It is possible to construct a family of trees such that the price of the asset or nothing option, the digital option, the call option and the put option struck at K are correct to infinite order.*

Proof. We first construct the tree to third order as above. Now suppose we have constructed the tree to order N then for each of the asset or nothing and the digital option, the error is of the form

$$e_N k^{-N} + \mathcal{O}(k^{-N-1}).$$

From our theorem, if we change the up-probability by $\beta k^{-1/2-N}$ then the error changes by

$$e^{-d_j^2/2}\frac{2}{\sqrt{\pi}}\beta k^{-N}.$$

So taking the obvious choice of β in each of stock and bond measures makes the tree converge to one higher order. Repeating, we can achieve arbitrarily high order of convergence. For infinite order of convergence, we invoke the theorem of Borel that an asymptotic series can be summed in an asymptotic sense to get probabilities that cancel errors of all order. This is sometimes call the Borel–Ritt lemma, and is Theorem 1.2.6 in [7]. $\square$

Note that we have not given a constructive proof of this result. However, repeating the arguments in this paper whilst retaining enough terms would allow an explicit construction to arbitrarily high order. We leave this to other researchers.

5. Numerical work

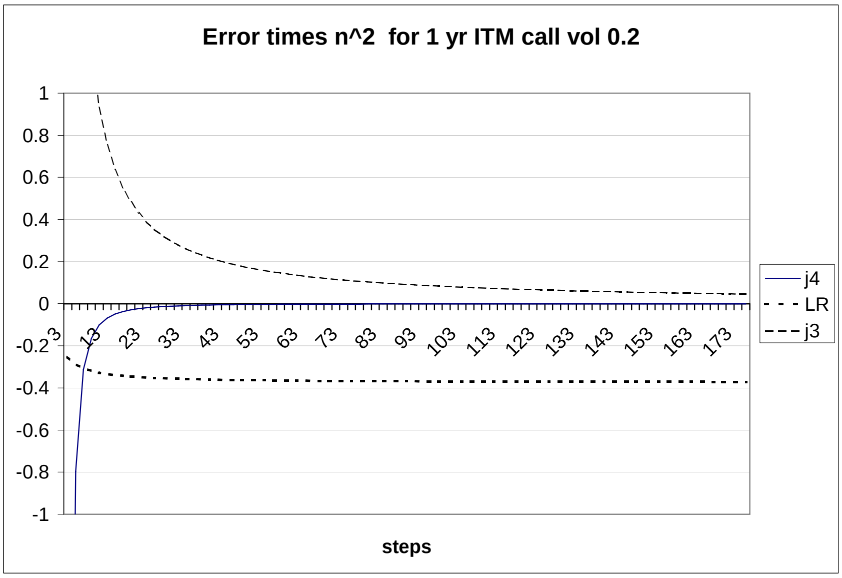

Figure 5.1. The error times number of steps-squared for a call option in the Black–Scholes model as a function of the number of steps.

Whilst the primary focus of this work is theoretical, it is interesting to see how much of a difference these changes make to convergence in an illustrative case. In addition, in this section we suggest a choice for the next order of coefficients.

Given the results for the first two error terms, we can conjecture that if a tree is convergent to order k and all higher terms in the asymptotic expansion for the probabilities are zero, then the lead error is of the form

$$e^{-4\alpha^2}\frac{2}{\sqrt{\pi}}q_k(\alpha),$$

with $\alpha = d_j/\sqrt{8}$, and q_k an odd polynomial of order $2k+1$ in α. If one computes the error for many values of α one can fit a polynomial to estimate q_k, and then approximately cancel the error.

We carried out this approach for the third error term and obtained

$$\alpha_3 = -0.1025\alpha_0 - 0.9285\alpha_0^3 - 1.43\alpha_0^5 - 0.5\alpha_0^7. \tag{5.1}$$

One should realize that these parameters are only an approximation and will have the effect of making the third order lead term very small but not zero. We illustrate the convergence for spot 105, strike 100, $r = d = 0$, $\sigma = 0.2$, expiry 1 in figure 5.1. We denote the Leisen–Reimer (C) tree from [13] by LR. The tree with the first three terms specified, we denote by $j3$ and with the fourth estimated term by $j4$. The error is multiplied by the number of steps squared. We see that the Leisen–Reimer tree has second order convergence as expected. The other two trees have higher order convergence with $j4$ doing substantially better than $j3$. Even with only 21 steps, the error is approximately 0.5 bps versus a price of 10.9.

6. Asymptotics of binomial coefficients

We will factor the probability of being in-the-money (i.e. of being above the strike) into a probability-independent coefficient and an integral. In this section, we analyze the properties of that coefficient.

We will use Stirling's formula. This states that a gamma function has a complete asymptotic expansion, and the first few terms are

$$\Gamma(z) \sim e^{-z}z^{z-\frac{1}{2}}\sqrt{2\pi}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \mathcal{O}(z^{-3})\right). \tag{6.1}$$

Since we have $\Gamma(n+1) = n!$, we can deduce properties of binomial coefficients. Our principal result is

Theorem 6.1. *The expression*

$$f(N) = 2^{-2N} \begin{pmatrix} 2N+1 \\ N+1 \end{pmatrix} \frac{N+1}{N^{\frac{1}{2}}}$$

has a complete asymptotic expansion in powers of N^{-1} for large N and we have

$$f(N) = \frac{2}{\sqrt{\pi}} \left(1 + \frac{3}{8}\frac{1}{N} - \frac{7}{128}\frac{1}{N^2} \right) + \mathcal{O}(N^{-3}).$$

Proof. Let the part of the asymptotic expansion in powers of $1/N$ for $\Gamma(N)$ (i.e. the part inside parentheses) be denoted by $g(N)$. We can write

$$f(N)\frac{N^{\frac{1}{2}}}{N+1} = 2^{-2N}\frac{\Gamma(2N+2)}{\Gamma(N+2)\Gamma(N+1)}, \tag{6.2}$$

$$= 2^{-2N}\frac{(2N+2)^{2N+3/2}}{(N+2)^{N+3/2}(N+1)^{N+1/2}}\frac{\sqrt{2\pi}}{2\pi}e\frac{g(2N+2)}{g(N+2)g(N+1)}, \tag{6.3}$$

$$= \frac{(N+1)^{2N+3/2}}{(N+2)^{N+3/2}(N+1)^{N+1/2}}\frac{2e}{\sqrt{\pi}}\frac{g(2N+2)}{g(N+2)g(N+1)}. \tag{6.4}$$

So we have

$$f(N) = \frac{(N+1)^{N+2}}{(N+2)^{N+3/2}N^{1/2}}\frac{2}{\sqrt{\pi}}e\frac{g(2N+2)}{g(N+2)g(N+1)}, \tag{6.5}$$

$$= \frac{(1+1/N)^{N+2}}{(1+2/N)^{N+3/2}}\frac{2}{\sqrt{\pi}}e\frac{g(2N+2)}{g(N+2)g(N+1)}. \tag{6.6}$$

To ease notation, in the rest of the proof we write $x = 1/N$, recalling that an asymptotic expansion at infinity and a Taylor expansion at zero are the same thing via this change of coordinates. The existence of the asymptotic expansion is now clear, it remains to compute the first three terms.

In what follows, we will use the fact that if a and b are smooth then

$$e^{a(x)+x^k b(x)} = e^{a(x)} + \mathcal{O}(x^k).$$

This is clear from the power series for e^x. We need to compute the series for $(1+x)^{1/x+2}$. We have using the expansion for log

$$
\begin{aligned}
(1+x)^{1/x} &= \exp\left(\frac{1}{x}\log(1+x)\right), && (6.7)\\
&= \exp\left(\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)+\mathcal{O}(x^3)\right), && (6.8)\\
&= \exp\left(1-\frac{x}{2}+\frac{x^2}{3}+\mathcal{O}(x^3)\right), && (6.9)\\
&= e\exp\left(-\frac{x}{2}+\frac{x^2}{3}\right)+\mathcal{O}(x^3), && (6.10)\\
&= e\left(1+\left(-\frac{x}{2}+\frac{x^2}{3}\right)+\frac{1}{2}\left(-\frac{x}{2}\right)^2\right)+\mathcal{O}(x^3), && (6.11)\\
&= e\left(1-\frac{1}{2}x+\frac{11}{24}x^2\right)+\mathcal{O}(x^3). && (6.12)
\end{aligned}
$$

Multiplying this by $(1+x)^2$, we get

$$(1+x)^{1/x+2} = e\left(1+\frac{3}{2}x+\frac{11}{24}x^2\right)+\mathcal{O}(x^3). \tag{6.13}$$

A similar computation yields

$$(1+2x)^{-(1/x+3/2)} = e^{-2}\left(1-x+\frac{5}{6}x^2\right)+\mathcal{O}(x^3). \tag{6.14}$$

Multiplying together we obtain for the first part (i.e. excluding the terms involving g)

$$\frac{2}{\sqrt{\pi}}\left(1+\frac{1}{2}x-\frac{5}{24}x^2\right)+\mathcal{O}(x^3).$$

Translating the part involving g into x, it becomes

$$\frac{1+\frac{1}{24}x+\left(\frac{1}{4\times 288}-\frac{1}{24}\right)x^2}{\left(1+\frac{1}{12}x+\left(\frac{1}{288}-\frac{1}{12}\right)x^2\right)\left(1+\frac{1}{12}x+\left(\frac{1}{288}-\frac{1}{6}\right)x^2\right)}.$$

Using the fact that for any α, β, γ,

$$(1+\alpha x+\beta x^2)^{-1} = 1-\alpha x+(\alpha^2-\beta)x^2+\mathcal{O}(x^3),$$

this is equal to

$$1-\frac{1}{8}x+\left(\frac{1}{288\times 4}+\frac{1}{24}+\frac{1}{144}+\frac{1}{6}\right)x^2+\mathcal{O}(x^3).$$

Multiplying everything out, we obtain

$$\frac{2}{\sqrt{\pi}}\left(1+\frac{3}{8}x-\frac{7}{128}x^2\right),$$

as desired. □

7. The proof

We now tackle the proof of Theorem 4.1. Let

$$\Phi(n,k,p)=\sum_{j=k}^{n}\binom{n}{j}p^j(1-p)^j, \tag{7.1}$$

denote the incomplete binomial sum. We have to show that if p_{2k+1} is the sequence of probabilities from a centred binomial tree, then a complete asymptotic expansion exists for $\Phi(2k+1,k+1,p_{2k+1})$ and we need to compute the first few terms. We defer the results of the effect of a perturbation to the next section. We will work modulo $\mathcal{O}(k^{-3})$ throughout, however, one should realize that this is for convenience and one could retain as many terms as desired and that this really represents a complete asymptotic expansion with lead order k^{-3}. In any case, the existence of a complete expansion follows from a translation of the proof in [11], and so our focus is on explicitly computing the expansion.

As in [6] and [11], we will use

$$\sum_{j=k}^{n}\binom{n}{j}p^j(1-p)^j=k\binom{n}{k}\int_0^p y^{k-1}(1-y)^{n-k}dy. \tag{7.2}$$

and we can therefore write

$$\Phi(2k+1,k+1,p_{2k+1})=(k+1)\binom{2k+1}{k+1}\int_0^{p_{2k+1}} y^k(1-y)^k dy. \tag{7.3}$$

Note the nice feature that the integrand is symmetric around $y=\frac{1}{2}$. Following the argument in [11], this can be rewritten as

$$\Phi(2k+1,k+1,p_{2k+1})=\frac{1}{2}+(k+1)\binom{2k+1}{k+1}\int_0^{p_{2k+1}-\frac{1}{2}}\left(\frac{1}{2}+y\right)^k\left(\frac{1}{2}-y\right)^k dy. \tag{7.4}$$

We factorize the second term as

$$2^{-2k}\frac{k+1}{\sqrt{k}}\binom{2k+1}{k+1},$$

and

$$\sqrt{k}\int\limits_{0}^{p_{2k+1}-\frac{1}{2}}(1-4y^2)^k dy.$$

The first of these terms has been analyzed in Theorem 6.1. We now need to analyze the second term, and then multiply the two expansions together.

It will be easier conceptually and notationally to deal with a Taylor series at zero and a continuous parameter. We therefore set $x = k^{-1}$ and study the limit as x goes to zero from above. The upper limit of the integral is now

$$x^{1/2}(\alpha_0 + x\alpha_1 + x^2\alpha_2 + \mathcal{O}(x^3)).$$

Our integral is therefore to third order equal to

$$x^{-1/2}\int\limits_{0}^{x^{1/2}(\alpha_0+x\alpha_1+x^2\alpha_2)}(1-4y^2)^{1/x}dy = \int\limits_{0}^{\alpha_0+x\alpha_1+x^2\alpha_2}(1-4xz^2)^{1/x}dz, \tag{7.5}$$

with the variable change $y = x^{1/2}z$. This can be written as

$$\int\limits_{0}^{\alpha_0+x\alpha_1+x^2\alpha_2} e^{\frac{1}{x}\log(1-4xz^2)}dz.$$

We perform another change or variables

$$w = \sqrt{\frac{1-e^{-4xz^2}}{4x}}, \tag{7.6}$$

$$z = \sqrt{-\frac{\log(1-4xw^2)}{4x}}. \tag{7.7}$$

Note that x is small so these transformations will be invertible. The transformation has been chosen to make the exponent equal to $-4w^2$. Taylor expanding and computing terms, we get

$$z = w(1 - xw^2 + \frac{5}{6}x^2w^4) + \mathcal{O}(x^3), \tag{7.8}$$

$$w = z(1 + xz^2 + \frac{13}{6}x^2z^4) + \mathcal{O}(x^3) \tag{7.9}$$

We have

$$dz = dw - 3xw^2dw + \frac{25}{6}x^2w^4dw + \mathcal{O}(x^3).$$

The upper limit of the integral becomes

$$\alpha_0 + (\alpha_0^3 + \alpha_1)x + \left(\frac{13}{6}\alpha_0^5 + 3\alpha_0^2\alpha_1 + \alpha_2\right)x^2 + \mathcal{O}(x^3).$$

Our integral therefore becomes up to third order a sum of three terms:

$$\int_0^{\alpha_0+(\alpha_0^3+\alpha_1)x+\left(\frac{13}{6}\alpha_0^5+3\alpha_0^2\alpha_1+\alpha_2\right)x^2} e^{-4w^2}dw$$

$$- 3x \int_0^{\alpha_0+(\alpha_0^3+\alpha_1)x} w^2e^{-4w^2}dw + \frac{25}{6}x^2\int_0^{\alpha_0} w^4e^{-4w^2}dw.$$

The first term is directly evaluable in terms of the cumulative normal function and is equal to

$$\frac{\sqrt{\pi}}{2}\left[N\left(2\sqrt{2}\left(\alpha_0 + (\alpha_0^3 + \alpha_1)x + \left(\frac{13}{6}\alpha_0^5 + 3\alpha_0^2\alpha_1 + \alpha_2\right)x^2\right)\right) - \frac{1}{2}\right].$$

Taylor expanding about $x = 0$, we get

$$\frac{\sqrt{\pi}}{2}\Big[N(2\sqrt{2}\alpha_0) + N'(2\sqrt{2}\alpha_0)2\sqrt{2}(\alpha_0^3 + \alpha_1)x$$
$$+ N'(2\sqrt{2}\alpha_0)2\sqrt{2}\left(\frac{13}{6}\alpha_0^5 + 3\alpha_0^2\alpha_1 + \alpha_2\right)x^2$$
$$+ N''(2\sqrt{2}\alpha_0)(2\sqrt{2})^2\frac{(\alpha_0^3 + \alpha_1)^2}{2}x^2 - \frac{1}{2}\Big].$$

This is enough to see that the lead term is $N(d_j)$, since $\alpha_0 = d_j/(2\sqrt{2})$ and the lead term of the Stirling's formula part is $2/\sqrt{\pi}$.

Note that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, so

$$N'(2\sqrt{2}\alpha_0) = \frac{1}{\sqrt{2\pi}}e^{-d_j^2/2},$$

and similarly for N''.

To evaluate the second and third terms, we use integration by parts to obtain

$$\int w^2e^{-4w^2}dw = -\frac{w}{8}e^{-4w^2} + \frac{1}{8}\int e^{-4w^2}dw, \tag{7.10}$$

and

$$\int w^4e^{-4w^2}dw = -\frac{w^3}{8}e^{-4w^2} - \frac{3}{64}we^{-4w^2} + \frac{3}{64}\int e^{-4w^2}dw. \tag{7.11}$$

For the second term we obtain a sum of two terms. The first arising from the integral of e^{-4w^2} is

$$-\frac{3}{8}\frac{\sqrt{\pi}}{2}\left[N(2\sqrt{2}\alpha_0)-\frac{1}{2}+N'(2\sqrt{2}\alpha_0)2\sqrt{2}(\alpha_0^3+\alpha_1)x\right],$$

and the second is

$$\frac{3}{8}x(\alpha_0+(\alpha_0^3+\alpha_1)x)e^{-4(\alpha_0^3+(\alpha_0+\alpha_1)x)^2}.$$

We need to expand the exponential and after doing so and multiplying, we get

$$e^{-4\alpha_0^2}\left[\alpha_0 x+(\alpha_0+\alpha_1-8\alpha_0^5-8\alpha_0^2\alpha_1)x^2\right].$$

The third term evaluates to

$$x^2\frac{25}{16}\frac{\sqrt{\pi}}{2}\left[-\frac{1}{3}\alpha_0^3e^{-4\alpha_0^3}-\frac{1}{8}\alpha_0e^{-4\alpha_0^3}+\frac{1}{8}\left(N(2\sqrt{2}\alpha_0)-\frac{1}{2}\right)\right].$$

These terms are summed and multiplied by the expansion from Stirling's formula; the result is then immediate. A slightly surprising aspect of the formula is that the cumulative normal only appears in the zeroth order term. For the higher order terms, its appearances cancel. We conjecture that this is the case for all orders but do not yet have a proof.

8. Perturbations

Suppose we add a term $k^{-n-\frac{1}{2}}\beta$ to the asymptotic expansion for k we want to show that the error expansion changes by

$$\frac{2}{\sqrt{\pi}}e^{-4\alpha_0^2}\beta x^k+\mathcal{O}(x^{k+1}),$$

in order to complete the proof of Theorem 4.1.

The key point here is that the perturbation will change the upper limit of the integral defining the asymptotics:

$$\int_0^{p(x)} e^{\frac{1}{x}\log(1-4xz^2)}dz.$$

by $x^k\beta+\mathcal{O}(x^{k+1})$. When we perform our change of variables, the only term which will change to order $k+1$ will be the the first one that is the integral of e^{-4w^2}. In addition, the upper limit of the variable changed integral will change only by βx^k to order $k+1$.

The effect of the change will therefore be the same as when changing the second order term (except for the power) when doing the original case and the result follows.

9. The Leisen–Reimer tree

Leisen and Reimer [13] defined three trees. They proceeded by specifying the probabilities of up and down moves as $h^{-1}(d_j)$ with h depending upon n, and h is chosen to be the inverse of a normal approximation to the binomial distribution. They study three choices. The first of these is observed to have numerical convergence of order -1 and the other two of order -2. Here we concentrate on the third choice, called "C," since it seems to be the one in most common use, e.g. [16]. From Corollary 4.1, if the tree is in our class then second order convergence will hold if and only if

$$\alpha_1 = -\frac{3}{8}\alpha_0 - \alpha_0^3.$$

We must therefore compute the first part of the asymptotic expansion.

The value of the probability for n steps is $h^{-1}(d_j)$, where

$$h^{-1}(z) = 0.5 \mp \left[0.25 - 0.25\exp\left\{-\left(\frac{z}{n+\frac{1}{3}+\frac{0.1}{n+1}}\right)^2\left(n+\frac{1}{6}\right)\right\}\right]^{\frac{1}{2}} \tag{9.1}$$

Here, we have $n = 2k+1$ and we want the asymptotics in terms of k.

Observing that it is the plus branch that is relevant here (see e.g. [16],) and simplifying we have

$$h^{-1}(z) = 0.5 + 0.5\left[1 - \exp\left\{-\left(\frac{z}{n+\frac{1}{3}+\frac{0.1}{n+1}}\right)^2\left(n+\frac{1}{6}\right)\right\}\right]^{\frac{1}{2}}. \tag{9.2}$$

Expanding the exponential and taking the square root, we get

$$h^{-1}(z) = \frac{1}{2} + \frac{1}{2}\frac{z}{\sqrt{n}} - \frac{z}{8n^{3/2}} - \frac{z^3}{8n^{3/2}} + \mathcal{O}(n^{-5/2}), \tag{9.3}$$

and it is clear that an expansion in odd powers of $n^{-1/2}$ exists.

We need to set $n = 2k+1$, we have

$$\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{2k+1}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{k}}\left(1 - \frac{1}{4}\frac{1}{k}\right) + \mathcal{O}(k^{-5/2}).$$

Substituting, we obtain

$$h^{-1}(z) = \frac{1}{2} + \frac{z}{2\sqrt{2}}\frac{1}{k^{1/2}} - \left(\frac{3}{16\sqrt{2}}z + \frac{1}{16\sqrt{2}}z^3\right) + \mathcal{O}(k^{-5/2}). \tag{9.4}$$

Putting $z = d_j$, we see that the tree is in the centred class and that

$$\alpha_1 = -\alpha_0^3 - \frac{3}{8}\alpha_0,$$

as required, and we are done.

10. American put options

Although the focus of this paper is on European options, the reader will inevitably ask the question of what difference the changes make to the pricing of American options. From our analysis of the Leisen–Reimer tree, we see that our probabilities only differ from theirs at order $k^{-5/2}$. The convergence of the Leisen–Reimer tree is first order for American put options (even after Richardson extrapolation [13], [14], [16].) The effect of this is that for any reasonable number of steps, the pricing error for an American option is much larger than the difference in pricing errors for the two trees, and we can expect their prices to agree to second order for American puts whilst being only correct to first order. The upshot is that whilst the new tree is worthwhile for American options, being if anything slightly simpler than the Leisen–Reimer tree, it will not make a noticeable difference to convergence speed.

11. Conclusion

We have presented a detailed analysis of the convergence of a class of trees for pricing vanilla European options. This has enabled us to prove that the Leisen–Reimer tree has second order convergence, and to construct a new tree that has third order convergence. The problem of achieving smooth and higher order convergence for American options still remains open.

Appendix A. Code

In order to make it easy for the reader who wishes to implement the tree, we present brief C++ code that computes the probability of an up move in the risk-measure, the multiplier for an up move, and the multiplier for a down move. Since these are the same for every node, this function would only be called once per tree. Note that the computation is actually slightly simpler than that for the Leisen–Reimer tree. A full implementation is contained in release 0.8 of the QuantLib open-source project obtainable from `www.quantlib.org`.

```
double ComputeP4(double k, double dj)
```

```
{
        double alpha = dj/(sqrt(8.0));
        double alpha2 = alpha*alpha;
        double alpha3 = alpha*alpha2;
        double alpha5 = alpha3*alpha2;
        double alpha7 = alpha5*alpha2;

        double beta = -0.375*alpha-alpha3;

        double gamma =  (5.0/6.0)*alpha5 + (13.0/12.0)*alpha3
                         +(25.0/128.0)*alpha;
        double delta =  -0.1025 *alpha- 0.9285 *alpha3
                        -1.43 *alpha5 -0.5 *alpha7;
        double p =0.5;
        double rootk= sqrt(k);
        p+= alpha/rootk;
        p+= beta /(k*rootk);
        p+= gamma/(k*k*rootk);
        // delete next line to get results for j three tree
        p+= delta/(k*k*k*rootk);

        return p;
}

void GetTreeValues(double Spot, double Strike, double vol,
                   double Expiry, double r,
                   unsigned long NumberSteps,
                   double& Probability,
                   double& UpMultiplier,
                   double& DownMultiplier)
{
        double d2 = ((r-0.5*vol*vol)*Expiry
                     + log(Spot/Strike))/(vol*sqrt(Expiry));
        double d1 = ((r+0.5*vol*vol)*Expiry
                     + log(Spot/Strike))/(vol*sqrt(Expiry));


        if (NumberSteps % 2 ==0)
                throw("we can only do odd numbers of steps");

        double Dt =Expiry/NumberSteps;
```

```
        unsigned long nUL = NumberSteps/2UL;
        double n = static_cast<double>(nUL);

        Probability = ComputeP4(n, d2);
        double Probability2 =  ComputeP4(n,d1);
        double rn =exp(r*Dt);

        UpMultiplier = rn*Probability2/Probability;
        DownMultiplier=(rn-Probability*UpMultiplier)
                      /(1-Probability);
        return;

}
```

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