The analysis of perturbed risk processes with Markovian arrivals

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Abstract

In this paper, we study the perturbed risk processes with Markovian arrivals. We present explicit formulas for the Laplace transform of the time to cross a certain level before ruin, the Laplace transform of the time of recovery and the distribution of the maximum severity of ruin, as well as the expected discounted dividends and the distribution of the total dividends prior to ruin for the risk model in the presence of a constant dividend barrier.

Keywords: Perturbed risk processes; Markovian arrival processes; First passage times; Time of recovery; Maximum severity of ruin; Dividend barrier

1 Introduction

Risk processes perturbed by diffusion have been studied extensively in the risk theory literature. For the perturbed classical risk processes, Dufresne and Gerber (1991) derived results for ruin probabilities, Gerber and Landry (1998) presented results for discounted penalty functions, Tsai and Willmot (2002) & Tsai (2003) analyzed the (discounted) jointed density of the surplus before ruin and the deficit at ruin, the distribution and moments of the deficit, and other quantities of interests. For

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the perturbed classical risk processes with barriers, Zhou (2004) derived the Laplace transform of the first passage time across a given level before ruin. Li (2006) derived the distribution of dividend payment and Frostig (2008) considered expected value of dividend payment and expected time to ruin. For perturbed Sparre–Andersen risk processes and the perturbed risk processes in a Markovian environment, Li and Garrido (2005) and Lu and Tsai (2007) derived explicit formulas for expected discounted penalty functions respectively. Recently, considering the perturbed risk process with Markovian arrival process (see for example, Neuts 1979 and Asmussen 2003), Badescu and Breuer (2008) derived the Laplace transform of the time to ruin based on a martingale approach introduced by Asmussen and Kella (2000).

In this paper, stimulated by Li (2008b) and as a continuation of Ren (2009), we study the Laplace transform of the time to cross a certain level before ruin, the Laplace transform of the time of recovery and the distribution of the maximum severity of ruin for perturbed risk processes with claims arriving according to a Markovian arrival process with representation, say, $\left(\vec{\boldsymbol{\gamma}}^\top, \mathbf{D}_0, \mathbf{D}_1\right)$. That is, we assume that claims occurs according to a background Markov process $J(t)$ with $m < \infty$ states, initial distribution $\vec{\boldsymbol{\gamma}}^\top$ and intensity matrix $\mathbf{D}_0 + \mathbf{D}_1$. The matrix $\mathbf{D}_0$ gives the intensity of state changes without claim arrivals and $\mathbf{D}_1$ the intensity of state changes with claim arrivals. As pointed out in p. 303 of Asmussen (2003), a state change with arrivals may be from state i to itself. Claims arriving with a transition from state i to state j in the process $J(t)$ is assumed to have probability density function p_{ij}. Furthermore, when $J(t) = i$, the premium rate is c_i and the risk process is perturbed by a Brownian motion with drift 0 and infinitesimal variance σ_i^2.

Then, given an initial level u of the surplus, the perturbed risk process that is

$$U(t) = u + \int_0^t c_{J(s)} ds - \sum_{k=1}^{N(t)} X_k + \int_0^t \sigma_{J(s)} dB(s), \quad t \geq 0, \tag{1.1}$$

where $\{N(t), t \geq 0\}$ counts the number of claims in time interval $(0, t]$, X_k represents the size of the kth claim and $B(t)$ is an independent standard Brownian motion.

Surprisingly, with a slight modification of the definition of the concepts such as the time of recovery to reflect the characteristics of a Brownian perturbation, the results developed here for quantities such as the probability that the surplus attains a certain level before ruin for a perturbed risk process ensembles in form those results developed for non–perturbed processes in Li (2008b).

Before introducing the main results, we note that the Markovian arrival process is very general. On the one hand, it may represent a renewal process where the interclaim times follow phase–type distributions, which are dense in the set of distributions with non-negative support. On the other hand, it allows for situations where interclaim times and/or claim size random variables are dependent. For example, with $\mathbf{D}_0 = -\lambda$ and $\mathbf{D}_1 = \lambda$, it reduces to a Poisson process with rate λ; with $\mathbf{D}_0 = \mathbf{B}$ and $\mathbf{D}_1 = \vec{\mathbf{b}}\,\vec{\boldsymbol{\alpha}}^\top$, it reduces to a renewal process with the interclaim times following a phase–type distribution with representation $(\vec{\boldsymbol{\alpha}}^\top, \mathbf{B}, \vec{\mathbf{b}})$, where $\vec{\mathbf{b}} = -\mathbf{B}\vec{\mathbf{e}}$ with $\vec{\mathbf{e}}$ being a column vector of one's,; with $\mathbf{D}_0 = \mathbf{Q} - diag(\lambda_i)$ and $\mathbf{D}_1 = diag(\lambda_i)$, where $diag(\lambda_i)$ denotes a diagonal matrix with λ_i on the diagonal, it reduces to a Markov Modulated Poisson Process with the claim rate being λ_i when an independent Markov process with infinitesimal generator $\mathbf{Q}$ is in state i.

2 The Laplace transform of the first passage time

For $b \geq u$, define

$$T_b = min\{t \geq 0 : U(t) = b\} \tag{2.1}$$

be the first time when the surplus reach level b and for $\delta \geq 0$ define

$$R_{ij}(u,b) = \mathbb{E}_i\left[e^{-\delta T_b} I(J(T_b) = j) | U(0) = u\right], \quad i,j = 1,2,\cdots,m, \tag{2.2}$$

to be the Laplace transform of the first passage time T_b and the state at the passage time is in j given initial state i and surplus level u.

Then, using arguments similar to Ng and Yang (2006) and Badescu (2008), we can show that for $i,j = 1,2,\cdots,m$,

$$\begin{aligned} R_{ij}(u) &= e^{-\delta dt}(1 + d_{0,ii}dt)\mathbb{E}[R_{ij}(u + c_i dt + \sigma_i B(dt))] \\ &+ e^{-\delta dt}\sum_{k\neq i} d_{0,ik}dt\mathbb{E}[R_{kj}(u + c_i dt + \sigma_i B(dt))] \\ &+ e^{-\delta dt}\sum_{k=1}^{m} d_{1,ik}dt\int_0^\infty R_{kj}(u-x,b)p_{ik}(x)dx + o(dt). \end{aligned} \tag{2.3}$$

Ignoring terms with order $o(dt)$, we can show that the matrix, $\mathbf{R}(u,b) = (R_{ij}(u,b))_{i,j=1}^m$, satisfies

$$\mathbf{0} = \boldsymbol{\Delta}_{\sigma^2/2}\mathbf{R}''(u,b) + \boldsymbol{\Delta}_c\mathbf{R}'(u,b) + (\mathbf{D}_0 - \delta\mathbf{I})\mathbf{R}(u,b) + \int_0^\infty \mathbf{p}(x)\mathbf{R}(u-x,b)dx, \tag{2.4}$$

where $\mathbf{0}$ is a $m \times m$ matrix with all elements being 0, $\boldsymbol{\Delta}_{\sigma^2/2} = \mathrm{diag}(\sigma_1^2/2, \ldots, \sigma_m^2/2)$, $\boldsymbol{\Delta}_c = \mathrm{diag}(c_1, \ldots, c_m)$, and $\{\mathbf{p}(x)\}_{ij} = d_{1,ij}\,p_{ij}(x)$.

It is obvious that for $a > 0$, $\mathbf{R}(u, a+b) = \mathbf{R}(u,b)\mathbf{R}(b, a+b)$. This together with the boundary conditions $\mathbf{R}(b,b) = \mathbf{I}$ imply that $\mathbf{R}(u,b)$ have the form

$$\mathbf{R}(u,b) = e^{-\mathbf{K}(b-u)} \tag{2.5}$$

for all $u < b$. Since $\lim_{b\to\infty}\mathbf{R}(u,b) = 0$, all eigenvalues of $\mathbf{K}$ must have positive real parts and thus the matrices $\mathbf{K}$ and $\mathbf{R}(u,b)$ are non–singular.

As in Li (2008a), substituting (2.5) into (2.4) and then canceling $\mathbf{R}(u,b)$ yields

$$\mathbf{0} = \mathbf{\Delta}_{\sigma^2/2}\mathbf{K}^2 + \mathbf{\Delta}_c\mathbf{K} + (-\delta\mathbf{I} + \mathbf{D}_0) + \int_0^\infty \mathbf{p}(x)e^{-\mathbf{K}x}dx. \qquad (2.6)$$

To solve the matrix equation above, let

$$\mathbf{L}_\delta(s) = \mathbf{\Delta}_{\sigma^2/2}s^2 + \mathbf{\Delta}_c s + (-\delta\mathbf{I} + \mathbf{D}_0) + \int_0^\infty \mathbf{p}(x)e^{-sx}dx. \qquad (2.7)$$

The equation

$$\det(\mathbf{L}_\delta(s)) = 0 \qquad (2.8)$$

is a generalization of Lundberg's fundamental equation. We next show that

Lemma 2.1 *Equation (2.8) has exactly m roots with positive real parts.*

Proof: We follow the ideas in Badescu and Lothar (2008) to carry out the prove. Let $\mathbf{\Delta}_d$ be a diagonal matrix with i-th element being the ith diagonal element of $\mathbf{D}_0$, That is $\{\Delta_d\}_{ii} = d_{ii}$. Write $\mathbf{L}_\delta(s) = \mathbf{B}(s) + \mathbf{A}(s)$, where

$$\mathbf{B}(s) = \mathbf{\Delta}_{\sigma^2/2}s^2 + \mathbf{\Delta}_c s + (-\delta\mathbf{I} + \mathbf{\Delta}_d), \qquad (2.9)$$

is a diagonal matrix and

$$\mathbf{A}(s) = -\Delta_d + \mathbf{D}_0 + \int_0^\infty \mathbf{p}(x)e^{-sx}dx, \qquad (2.10)$$

is indecomposable.

Since $\{\Delta_d\}_{ii} < 0$ for every $i = 1, \cdots, m$, it is easy to show that the diagonal matrix $\mathbf{B}(s)$ has exactly m roots with positive real part.

Consider a domain that is a half disk centered at 0, lying in the right half of the complex plane, and a sufficiently large radius ξ. If follows from a matrix generalization of Rouche's Theorem (De Smit 1983) that $\mathbf{L}_\delta(s) = \mathbf{B}(s) + \mathbf{A}(s)$ also has exactly m zeros with positive real part if we can show that

$$|\mathbf{B}(s)|_{ii} \geq \sum_{j=1}^{m} |\mathbf{A}(s)|_{ij}, \quad \text{for all } i = 1, 2, \cdots, m, \qquad (2.11)$$

on such a boundary.

For $\mathfrak{Re}\,\xi > 0$, this is true because for $\sum_{j=1}^m |A(s)_{ij}|$ is bounded. For $\xi = \imath\eta$ on the imaginary axis,

$$\begin{aligned} |\mathbf{B}(s)|_{ii} &= |-\frac{\sigma_i^2}{2}\eta^2 + c_i\imath\eta - \delta + D_{0,ii}| \\ &= \sqrt{(-\frac{\sigma_i^2}{2}\eta^2 - \delta + D_{0,ii})^2 + (c_i\eta)^2} \quad \geq |D_{0,ii}|, \text{for all } i = 1, 2, \cdots, m. \end{aligned} \qquad (2.12)$$

On the other hand,

$$
\begin{aligned}
|D_{0,ii}| &= \sum_{j=1}^{m} (|\mathbf{D}_0 - \mathbf{\Delta}_d + \mathbf{D}_1|)_{ij} \\
&= \sum_{j=1}^{m} (|\mathbf{D}_0 - \mathbf{\Delta}_d| + |\mathbf{D}_1|)_{ij} \\
&\geq \sum_{j=1}^{m} \left(|\mathbf{D}_0 - \mathbf{\Delta}_d| + | \int_0^\infty \mathbf{p}(x) e^{-\imath \eta x} dx | \right)_{ij} \\
&\geq \sum_{j=1}^{m} (|\mathbf{D}_0 - \mathbf{\Delta}_d + \int_0^\infty \mathbf{p}(x) e^{-\imath \eta x} dx|)_{ij} \\
&= \sum_{j=1}^{m} (|\mathbf{A}(s)|)_{ij}. \qquad (2.13)
\end{aligned}
$$

The first equality in (2.13) is due to the fact the row sum of the matrix $\mathbf{D}_0 + \mathbf{D}_1$ is zero. The second equality is true because all entries of the matrices $\mathbf{D}_0 - \mathbf{\Delta}_d$ and $\mathbf{D}_1$ are non–negative.

Combining (2.12) and (2.13) shows the validity of (2.11), so (2.8) has exactly m roots with positive real parts. In the sequel, we assume that they are distinct and have values $\rho_1, \cdots, \rho_m$. ■

For $i = 1, 2, \cdots, m$, let $\vec{\mathbf{h}}_i$ be an eigenvector of $\mathbf{L}_\delta(\rho_i)$ corresponding to the eigenvalue 0. Then

$$\vec{\mathbf{0}} = \mathbf{L}_\delta(\rho_i)\vec{\mathbf{h}}_i = \mathbf{\Delta}_{\sigma^2/2}(\rho_i^2 \vec{\mathbf{h}}_i) + \mathbf{\Delta}_c(\rho_i \vec{\mathbf{h}}_i) + (-\delta \mathbf{I} + \mathbf{D}_0)\vec{\mathbf{h}}_i + \int_0^\infty \mathbf{p}(x)(e^{-\rho_i x}\vec{\mathbf{h}}_i)dx, \qquad (2.14)$$

where $\vec{\mathbf{0}}$ is a $m \times 1$ vector with all elements being 0. Combining these m vector equations, we have the matrix equation

$$\mathbf{0} = \mathbf{\Delta}_{\sigma^2/2}\mathbf{H}\mathbf{\Delta}_\rho^2 + \mathbf{\Delta}_c \mathbf{H}\mathbf{\Delta}_\rho + (-\delta \mathbf{I} + \mathbf{D}_0)\mathbf{H} + \int_0^\infty \mathbf{p}(x)\mathbf{H}(e^{-\mathbf{\Delta}_\rho x})dx, \qquad (2.15)$$

where $\mathbf{H} = (\vec{\mathbf{h}}_1, \vec{\mathbf{h}}_2, \ldots, \vec{\mathbf{h}}_m)$. Right–multiplying both sides by $\mathbf{H}^{-1}$, we have

$$\mathbf{0} = \mathbf{\Delta}_{\sigma^2/2}\mathbf{H}\mathbf{\Delta}_\rho^2\mathbf{H}^{-1} + \mathbf{\Delta}_c \mathbf{H}\mathbf{\Delta}_\rho\mathbf{H}^{-1} + (-\delta \mathbf{I} + \mathbf{D}_0) + \int_0^\infty \mathbf{p}(x)\mathbf{H}(e^{-\mathbf{\Delta}_\rho x})\mathbf{H}^{-1}dx. \qquad (2.16)$$

Comparing (2.16) and (2.6), we have proved

Theorem 2.2 *The matrix* $\mathbf{K}$ *defined in equation (2.5) can be calculated by*

$$\mathbf{K} = \mathbf{H}\mathbf{\Delta}_\rho\mathbf{H}^{-1}. \qquad (2.17)$$

Remarks:

1. For a Markovian Brownian motion risk process, there is no jumps, so $\mathbf{D}_1 = 0$ and $\mathbf{D}_0 = \mathbf{D}$. Then, equation (2.6) becomes

$$\mathbf{0} = \mathbf{\Delta}_{\sigma^2/2}\mathbf{K}^2 + \mathbf{\Delta}_c\mathbf{K} - \delta\mathbf{I} + \mathbf{D} \tag{2.18}$$

   and the Generalized Lundberg equation is given by

$$0 = \det \mathbf{L}_\delta(s) = \det(\mathbf{\Delta}_{\sigma^2/2}s^2 + \mathbf{\Delta}_c s - \delta\mathbf{I} + \mathbf{D}). \tag{2.19}$$

   In the special case with $m = 1$ and so $\mathbf{B} = 0$, it has one positive root

$$\rho^+ = \frac{-c + \sqrt{c^2 + 2\sigma^2\delta}}{\sigma^2}$$

   and one negative root

$$\rho^- = \frac{-c - \sqrt{c^2 + 2\sigma^2\delta}}{\sigma^2}.$$

   Consequently, (2.5) becomes

$$R(u, b) = e^{-\rho^+(b-u)}, \tag{2.20}$$

   which is Equation (25) in Chapter 3 of Harrison (1985). We note that ρ^+ and ρ^- are denoted by r and s in Equations (2.14) and (2.15) in Gerber and Shiu (2004) respectively.

2. In the perturbed classical risk process where $N(t)$ is a Poisson process with rate λ, we have $\mathbf{D}_0 = -\lambda$ and $\mathbf{D}_1 = \lambda$. Then, Equation (2.6) becomes

$$0 = (\sigma^2/2)\rho^2 + c\rho + (-\delta - \lambda) + \int_0^\infty p(x)e^{-\rho x}dx, \tag{2.21}$$

   which is Equation (5) in Gerber and Landry (1998). So

$$R(u, b) = e^{-\rho(b-u)}. \tag{2.22}$$

3 The time of recovery

The concept of the time of recovery for non–perturbed risk processes was discussed in Gerber (1990), Egidio dos Reis (1993), Gerber and Shiu (1998) and Li (2008b). In this section, we extend the concept to perturbed risk processes. When perturbation causes ruin, since the surplus process "recovers" immediately, we simply define the time of recovery to be the time of ruin. It turns out in Section 4 and 5 that this extension allows us to generalize many results for non-perturbed risk processes to perturbed risk processes.

Since the risk process $U(t)$ has no upwards jumps. For any real number b we my define T_b to be the time when $U(t)$ first up–crosses level b. In particular, T_0 represents the time of recovery. Define

$$\begin{aligned}\psi_{ij}(u) &= \mathbb{E}_i[e^{-\delta T_0} I(T<\infty, J(T_0)=j)|U(0)=u] \\ &= \mathbb{E}_i[e^{-\delta T} I(T<\infty, J(T)=j, U(T)=0)|U(0)=u] \\ &+ \mathbb{E}_i[e^{-\delta T_0} I(T<\infty, J(T_0)=j, U(T)<0)|U(0)=u] \end{aligned} \tag{3.1}$$

to be the Laplace transform of the time of recovery at state j conditional on initial state i and surplus u.

Noticing that the first summand is just the Laplace transform of the distribution of the time to ruin due to diffusion, we will denote it as $\psi_{d,ij}(u)$ in the following. Conditional on the ruin time T, the deficit $U(T)$, and the state at ruin $J(T)$, using the law of iterated expectations, the second summand in (3.1) can be written as

$$\begin{aligned}&\mathbb{E}_i[e^{-\delta T} e^{-\delta(T_0-T)} I(T<\infty, U(T)<0, J(T_0)=j)|U(0)=u] \\ =\ &\mathbb{E}_i[e^{-\delta T} I(T<\infty, U(T)<0)\vec{\mathbf{e}}_{J(T)}^{\top} e^{\mathbf{K}U(T)}\vec{\mathbf{e}}_j|U(0)=u], \end{aligned} \tag{3.2}$$

where $\vec{\mathbf{e}}_j$ is a column vector with the j-th element being 1 and all other elements being zero, and the expectation on the right hand side is taken over the joint distribution of the triplet $(T, U(T), J(T))$. Substituting (3.2) into (3.1), we have

$$\begin{aligned}\psi_{ij}(u) &= \psi_{d,ij}(u) + \mathbb{E}_i[e^{-\delta T} I(T<\infty, U(T)<0)\vec{\mathbf{e}}_{J(T)}^{\top} e^{\mathbf{K}U(T)}\vec{\mathbf{e}}_j|U(0)=u] \\ &= \psi_{d,ij}(u) + \mathbb{E}_i[e^{-\delta T} I(T<\infty, U(T)<0)\vec{\mathbf{e}}_{J(T)}^{\top}\mathbf{H}e^{\mathbf{\Delta}_\rho U(T)}\mathbf{H}^{-1}\vec{\mathbf{e}}_j|U(0)=u]. \end{aligned} \tag{3.3}$$

Remarks:

1. For a Markovian Brownian motion risk process, the time of recovery is identical to the time of ruin. Because of symmetry, The Laplace transform of the time of recovery (ruin) may be obtained using formula (2.5) and (2.17) by replacing u by 0, b by u, ρ_i by the ith solution of equation (2.19) with c_i replaced by $-c_i$ for $i=1,\cdots,m$, and $\mathbf{H}$ by the collection of the corresponding eigenvectors. In the special case with $m=1$, we have

$$\psi(u) = e^{-\frac{c+\sqrt{c^2+2\sigma^2\delta}}{\sigma^2}u} = e^{\rho^- u}, \tag{3.4}$$

which is Equation (24) in Chapter 3 of Harrison (1985).

2. With $\mathbf{D}_0 = -\lambda$ and $\mathbf{D}_1 = \lambda$, equation (3.3) reduces to

$$\psi(u) \quad = \quad \psi_d(u) + \mathbb{E}[e^{-\delta T + \rho U(T)} I(T < \infty, U(T) < 0) | U(0) = u], \quad (3.5)$$

with the first summand being the Laplace transform of the time to ruin due to diffusion and the second summand being the expected discounted penalty at ruin with penalty function $w(x, y) = e^{-\rho y}$. Methods for calculating expected discounted penalty for perturbed classical risk model are available in for example, Gerber and Landry (1995), Tsai and Willmot (2002), Tsai (2003) and Ren (2005).

3. With $\mathbf{D}_0 = \mathbf{B}$ and $\mathbf{D}_1 = \vec{\mathbf{b}}\vec{\boldsymbol{\alpha}}^\top$, $U(t)$ reduces to a renewal process with the inter-claim times following a phase–type distribution with representation $(\vec{\boldsymbol{\alpha}}^\top, \mathbf{B}, \vec{\mathbf{b}})$. In this case, the distribution of the state at ruin, $J(T)$, is given by $\vec{\boldsymbol{\alpha}}^\top$ and is independent of those of the time of ruin T and the deficit at ruin, $U(T)$. So (3.3) reduces to

$$\begin{aligned} \psi_{ij}(u) &= \psi_{d,ij}(u) + \mathbb{E}_i[e^{-\delta T} I(T < \infty, U(T) < 0) \vec{\boldsymbol{\alpha}}^\top e^{\mathbf{K}U(T)} \vec{\mathbf{e}}_j | U(0) = u] \\ &= \psi_{d,ij}(u) + \mathbb{E}_i[e^{-\delta T} I(T < \infty, U(T) < 0) \vec{\boldsymbol{\alpha}}^\top \mathbf{H} e^{\boldsymbol{\Delta}_\rho U(T)} \mathbf{H}^{-1} \mathbf{e}_j | U(0) = u]. \end{aligned} \quad (3.6)$$

As for perturbed classical risk processes, the evaluation of (3.6) is closely related to the expected discounted penalty function for the perturbed Sparre–Andersen model. Formulas can be found, for example, in Li and Garrido (2005).

4. For a perturbed Markovain risk process, one may resort to Lu and Tsai (2007) for formulas to evaluate (3.3).

4 The probability that the surplus attains a certain level before ruin

The probability that the surplus attains a certain level before ruin for non–perturbed risk processes was studied in Dickson and Gray (1984), Gerber and Shiu (1998), and Li (2008). For perturbed classical risk processes, it was studied in Zhou (2004). This section extends some of their results to the perturbed risk processes with Markovian arrivals.

For $a < u < b$, let T_a and T_b be the time when the surplus process first up–crosses level a and b respectively. Similar to Section 6 of Gerber and Shiu (1998) and Section 4 of Li (2008b), let $\mathbf{A}(a, b|u) = \{A_{ij}(a, b|u)\}_{i,j=1}^m$ and $\mathbf{B}(a, b|u) = \{B_{ij}(a, b|u)\}_{i,j=1}^m$ with

$$A_{ij}(a, b|u) = \mathbb{E}_i[e^{-\delta T_a} (T_a < T_b, J(T_a) = j) | U(0) = u] \quad (4.1)$$

and

$$B_{ij}(a,b|u) = \mathbb{E}_i[e^{-\delta T_b}(T_b < T_a, J(T_b)=j)|U(0)=u]. \tag{4.2}$$

Obviously

$$\mathbf{A}(a,\infty|u) = \boldsymbol{\psi}(u-a) \tag{4.3}$$

and

$$\mathbf{B}(-\infty,b|u) = \mathbf{R}(u,b). \tag{4.4}$$

By considering on whether $T_a < T_b$ or not, we have

$$\mathbf{A}(a,\infty|u) = \mathbf{A}(a,b|u) + \mathbf{B}(a,b|u)\mathbf{A}(a,\infty|b), \tag{4.5}$$

and

$$\mathbf{B}(-\infty,b|u) = \mathbf{B}(a,b|u) + \mathbf{A}(a,b|u)\mathbf{B}(-\infty,b|a). \tag{4.6}$$

Letting $a=0$ in (4.5) and (4.6) yields

$$\boldsymbol{\psi}(u) \;=\; \mathbf{A}(0,b|u) + \mathbf{B}(0,b|u)\boldsymbol{\psi}(b), \tag{4.7}$$

and

$$\mathbf{R}(u,b) = \mathbf{B}(0,b|u) + \mathbf{A}(0,b|u)\mathbf{R}(0,b). \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\begin{aligned}\mathbf{A}(0,b|u) &= [\boldsymbol{\psi}(u) - \mathbf{R}(u,b)\boldsymbol{\psi}(b)][\mathbf{I} - \mathbf{R}(0,b)\boldsymbol{\psi}(b)]^{-1} \\ &= [\boldsymbol{\psi}(u) - e^{-\mathbf{K}(b-u)}\boldsymbol{\psi}(b)][\mathbf{I} - e^{-\mathbf{K}b}\boldsymbol{\psi}(b)]^{-1}\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\mathbf{B}(0,b|u) &= [\mathbf{R}(u,b) - \boldsymbol{\psi}(u)\mathbf{R}(0,b)][\mathbf{I} - \boldsymbol{\psi}(b)\mathbf{R}(0,b)]^{-1} \\ &= [e^{\mathbf{K}u} - \boldsymbol{\psi}(u)][e^{\mathbf{K}b} - \boldsymbol{\psi}(b)]^{-1}\end{aligned} \tag{4.10}$$

Remarks:

1. Surprisingly, With the definitions of matrices $\boldsymbol{\psi}$ and $\mathbf{R}$ incorporating diffusion, Equations (4.9) and (4.10) exactly resemble in form Equations (4.20) and (4.21) in Li (2008b) respectively.

2. For a Brownian motion risk process, because of (2.20) and (3.4), equations (4.9) and (4.10) reduces to

$$A(0,b|u) \;=\; \frac{e^{\rho^+ b}e^{\rho^- u} - e^{\rho^+ u}e^{\rho^- b}}{e^{\rho^+ b} - e^{\rho^- b}} \tag{4.11}$$

and

$$B(0,b|u) \;=\; \frac{e^{\rho^+ u} - e^{\rho^- u}}{e^{\rho^+ b} - e^{\rho^- b}}, \tag{4.12}$$

which are identical to Equations (20) and (19) in Chapter 3 of Harrison (1985). Equation (4.12) is also Equation (2.17) in Gerber and Shiu (2004).

3. For a perturbed classical risk process, another expression for $B(0,b|u)$ was obtained in Zhou (2004).

5 The distribution of the maximum severity of ruin

The distribution of the maximum severity of ruin for non–perturbed risk processes was studied in Picard (1994), Li and Dickson (2006), Li (2008b), and Li and Lu (2008). This section extends some of their results to perturbed risk processes. As in Section 3, when diffusion causes ruin, the surplus process returns to (up–crosses) level zero instantaneously, so we define the maximum severity of ruin only when a claim causes ruin. Particularly, let

$$M(u) = \sup\{|U(t)|, U(T) < 0, T \le t \le T_0\} \tag{5.1}$$

be the maximum severity of ruin when a claim causes ruin. Define $\mathbf{F}(z,u) = \{F_{ij}(z,u)\}_{i,j=1}^m$ with

$$F_{ij}(u,z) = \mathbb{P}_i(M(u) \le z, T < \infty, U(T) < 0, J(T) = j), \quad z > 0 \tag{5.2}$$

being the distribution of the maximum severity of ruin if ruin is caused by a claim and the state at ruin is j.

Before determining $\mathbf{F}(z,u)$, we need the following results.

Let $\mathbf{\Psi}(u)$ denote the probability of ruin with the state of recovery being j conditional on initial state i and surplus u. That is

$$\begin{aligned} \Psi_{ij}(u) &= \mathbb{P}_i[(T < \infty, J(T_0) = j)|U(0) = u] \\ &= \Psi_{d,ij}(u) + \mathbb{P}_i[I(T < \infty, J(T_0) = j, U(T) < 0)|U(0) = u], \end{aligned} \tag{5.3}$$

where $\Psi_{d,ij}(u) = \mathbb{P}_i[(T < \infty, J(T) = j, U(T) = 0)|U(0) = u]$ is the probability of ruin at state j due to diffusion conditional on initial state i and surplus u. Then $\mathbf{\Psi}(u)$ can be obtained by setting $\delta = 0$ in the matrix $\boldsymbol{\psi}(u)$.

For $b > u > 0$, define $\boldsymbol{\xi}(u,b) = \{\xi_{ij}(u,b)\}_{i,j=1}^m$ with

$$\xi_{ij}(u,b) = \mathbb{P}_i[(\sup_{0 \le t \le T} < b, T < \infty, J(T_0) = j)|U(0) = u] \tag{5.4}$$

being the probability that ruin occurs before the surplus reaches level b and the state at the time of recovery is j.

For $b \ge u > 0$, define $\boldsymbol{\chi}(u,b) = \{\chi_{ij}(u,b)\}_{i,j=1}^m$ with

$$\chi_{ij}(u,b) = \mathbb{P}_i[((T_b < T_0, J(T_b) = j)|U(0) = u] \tag{5.5}$$

being the probability that the surplus process reaches level b at state j before ruin. Noticing that $\boldsymbol{\chi}(u,b)$ may be obtained by setting $\delta = 0$ in the definition of $\mathbf{B}(0, b|u)$, by (4.10) we have

$$\boldsymbol{\chi}(u,b) \quad = \quad [e^{\mathbf{K}_0 u} - \boldsymbol{\Psi}(u)][e^{\mathbf{K}_0 b} - \boldsymbol{\Psi}(b)]^{-1}, \tag{5.6}$$

where $\mathbf{K}_0 = \lim_{\delta \to 0} \mathbf{K}$.

Considering whether the surplus process reaches level b or not before ruin, we have

$$\boldsymbol{\Psi}(u) = \boldsymbol{\xi}(u,b) + \boldsymbol{\chi}(u,b)\boldsymbol{\Psi}(b). \tag{5.7}$$

As a consequence of equations (5.6) and (5.7), both $\boldsymbol{\xi}(u,b)$ and $\boldsymbol{\chi}(u,b)$ can be expressed in terms of the matrix $\boldsymbol{\Psi}$.

We remark that from initial surplus level u, the probability of ruin before the surplus process hits level $b > u$ is given by $\vec{\boldsymbol{\gamma}}^\top \boldsymbol{\xi}(u,b)\vec{\mathbf{e}}$. Furthermore, the distribution of the maximum surplus before ruin is given by $\vec{\boldsymbol{\gamma}}^\top \boldsymbol{\chi}(u,b)\vec{\mathbf{e}}$.

Now we are ready to derive an expression for the distribution of the maximum severity of ruin. For $y > 0$, let $\mathbf{G}(y|u) = \{G_{ij}(y|u)\}_{i,j=1}^m$ with

$$G_{ij}(y|u) = \mathbb{P}_i(T < \infty, 0 < |U(T)| \leq y, J(T) = j|U(0) = u), y > 0 \tag{5.8}$$

being the probability that ruin occurs due to a claim, the deficit at ruin is at most y and the state at ruin is j conditional on initial state i and surplus u. In order for $M(u) \leq z$, it must be true that $|U(T)| = y$ for some $0 < y < z$ and that conditional on the size of the deficit y, the surplus process up–crosses level 0 before down–crosses level z. Integrating over y from 0 to z, we have

$$\mathbf{F}(u,z) = \int_0^z \mathbf{G}(y|u)\boldsymbol{\chi}(z-y,z)dy. \tag{5.9}$$

Substituting (5.6) into (5.9) yields

$$\mathbf{F}(u,z) = \int_0^z \mathbf{G}(y|u)[e^{\mathbf{K}_0(z-y)} - \boldsymbol{\Psi}(z-y)]dy[e^{\mathbf{K}_0 z} - \boldsymbol{\Psi}(z)]^{-1}. \tag{5.10}$$

To simplify (5.10) further, we notice that

$$\begin{aligned}
\boldsymbol{\Psi}(u+z) \quad &= \quad \boldsymbol{\Psi}_d(u)\boldsymbol{\Psi}(z) + \int_0^z \mathbf{G}(y|u)\boldsymbol{\Psi}(z-y)dy + \int_z^\infty \mathbf{G}(y|u)e^{\mathbf{K}_0(z-y)}dy \\
&= \quad \boldsymbol{\Psi}_d(u)\boldsymbol{\Psi}(z) - \int_0^z \mathbf{G}(y|u)[e^{\mathbf{K}_0(z-y)} - \boldsymbol{\Psi}(z-y)]dy \\
&\quad + \int_0^\infty \mathbf{G}(y|u)e^{\mathbf{K}_0(z-y)}dy.
\end{aligned} \tag{5.11}$$

where $\mathbf{\Psi}_d(u) = \{\Psi_{d,ij}(u)\}_{i,j=1}^m$. So

$$\int_0^z \mathbf{G}(y|u)[e^{\mathbf{K}_0(z-y)} - \mathbf{\Psi}(z-y)]dy = \mathbf{\Psi}_d(u)\mathbf{\Psi}(z) - \mathbf{\Psi}(u+z) + \int_0^\infty \mathbf{G}(y|u)e^{\mathbf{K}_0(z-y)}dy. \quad (5.12)$$

Now, to evaluate the integration in (5.12), we set $z = 0$ in (5.11) and obtain

$$\mathbf{\Psi}(u) = \mathbf{\Psi}_d(u)\mathbf{\Psi}(0) + \int_0^\infty \mathbf{G}(y|u)e^{-\mathbf{K}_0 y}dy. \quad (5.13)$$

Obviously, $\mathbf{\Psi}(0) = \mathbf{I}$ is an identity matrix, so

$$\int_0^\infty \mathbf{G}(y|u)e^{-\mathbf{K}_0 y}dy = \mathbf{\Psi}(u) - \mathbf{\Psi}_d(u). \quad (5.14)$$

With (5.12) and (5.14), equation (5.10) simplifies to

$$\mathbf{F}(u,z) = [\mathbf{\Psi}_d(u)\mathbf{\Psi}(z) - \mathbf{\Psi}(u+z) + e^{\mathbf{K}_0 z}(\mathbf{\Psi}(u) - \mathbf{\Psi}_d(u))][e^{\mathbf{K}_0 z} - \mathbf{\Psi}(z)]^{-1}. \quad (5.15)$$

Remarks:

1. Equation (5.27) of Li (2008b) gives the distribution of the severity of ruin conditional on ruin occurs for a Sparre–Andersen risk model with phase–type interclaim times . It may be obtained by setting $\mathbf{\Psi}_d(u) = 0$ in (5.15) and then dividing the result by the probability of ruin.

2. For the perturbed classical risk process, $\mathbf{K}_0 = 0$, $\mathbf{\Psi}(u)$ becomes the probability of ruin and $\mathbf{\Psi}_d(u)$ becomes the probability of ruin due to diffusion. Therefore, (5.15) reduces to

$$\mathbf{F}(u,z) = \frac{\Psi_d(u)\Psi(z) - \Psi(u+z) + \Psi_c(u)}{1 - \Psi(z)}, \quad (5.16)$$

   where $\Psi_c(u) = \Psi_(u) - \Psi_d(u)$ is the probability of ruin due to claims.

An illustration:

This illustration shows how diffusion affects the conditional distribution of the maximum severity of ruin (conditional on the occurrence of ruin in the non–perturbed case and on the occurrence of ruin due to a claim in the perturbed case). We consider a classical risk process with parameters $u = 1$, $\lambda = 1$, $c = 1.1$, and claim sizes follow exponential distribution with mean 1, and a perturbed classical risk process with the same parameters except that $\sigma > 0$. Figure 1 shows the conditional distributions of maximum severity of ruin with some different values of σ. It indicates that perturbation seems to enlarge the tail of the conditional distribution of the maximum severity of ruin.

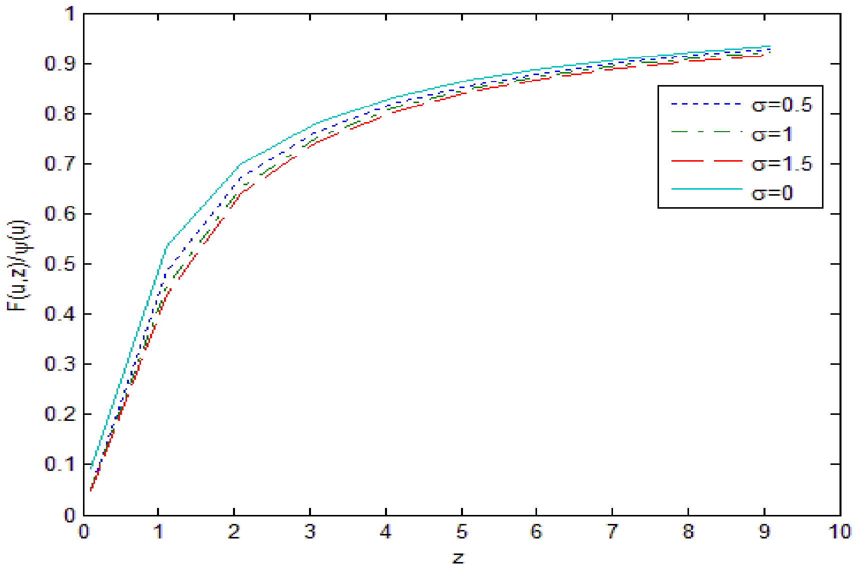

Figure 1: The Conditional Distribution of the Maximum Severity of Ruin

6 Dividend Problems

In this section we study the expected discounted aggregate claims and the distribution of the aggregate claims for the risk model in (1.1) in the presence of a constant dividend barrier.

6.1 The expected discounted aggregate dividends

Now we consider the surplus process (1.1) modified by the payment of dividends. When the surplus exceeds a constant barrier $b\,(\geq u)$, dividends are paid continuously so the surplus stays at the level b until it becomes less that b. Let $U_b(t)$ be the surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy and define $\bar{T} = \inf\{t \geq 0 : U_b(t) < 0\}$ to be the time of ruin. Let $\delta > 0$ be the force of interest for valuation and define

$$D_{u,b} = \int_0^{\bar{T}} e^{-\delta\, t} dD(t), \qquad 0 \leq u \leq b,$$

to be the present value of all dividends until time of ruin $\bar{T}$ given that the initial surplus is u, where $D(t)$ is the aggregate dividends paid by time t. Define

$$V_i(u;b) = \mathbb{E}_i\big[D_{u,\,b}|U_b(0) = u\big], \qquad 0 \leq u \leq b,\, i = 1, 2, \ldots, m,$$

to be the expected present value of the dividend payment before ruin given the initial state is i and the initial surplus is u.

Let $\vec{\mathbf{V}}(u;b) = \big(V_1(u;b), V_2(u;b), \ldots, V_m(u;b)\big)$ be an $m \times 1$ vector. Since no dividend are paid unless the surplus reaches the level b before ruin occurs, we have, for $0 \le u \le b$,

$$V_i(u;b) = \sum_{j=1}^{m} B_{ij}(0, b|u) V_j(b;b), \quad i = 1, 2, \ldots, m.$$

In matrix form,

$$\vec{\mathbf{V}}(u;b) = \mathbf{B}(0, b|u)\vec{\mathbf{V}}(b;b), \qquad 0 \le u \le b.$$

It follows from a similar heuristic reasoning as in Gerber et al. (2006), we can show that $\frac{\partial V_i(u;b)}{\partial u}(u;b)\big|_{u=b} = 1$, i.e., $\frac{\partial \vec{\mathbf{V}}(u;b)}{\partial u}\big|_{u=b} = \vec{\mathbf{e}}$, then $\vec{\mathbf{V}}(b;b) = \left[\frac{\partial \mathbf{B}(0,b|u)}{\partial u}\big|_{u=b}\right]^{-1} \vec{\mathbf{e}}$, and

$$\begin{aligned} \vec{\mathbf{V}}(u;b) &= \mathbf{B}(0, b|u) \left[\frac{\partial \mathbf{B}(0, b|u)}{\partial u}\big|_{u=b}\right]^{-1} \vec{\mathbf{e}} \\ &= \left[e^{\mathbf{K}u} - \boldsymbol{\psi}(u)\right] \left[\mathbf{K}e^{\mathbf{K}b} - \boldsymbol{\psi}'(b)\right]^{-1} \vec{\mathbf{e}}\,, \qquad 0 \le u \le b. \end{aligned} \tag{6.1}$$

In the perturbed classical risk process where $N(t)$ is a Poisson process with rate λ, we have $m = 1$, $\mathbf{D}_0 = -\lambda$ and $\mathbf{D}_1 = \lambda$. So (6.1) simplifies to

$$V(u;b) = \frac{e^{\rho u} - \psi(u)}{\rho e^{\rho b} - \psi'(b)}, \qquad 0 \le u \le b. \tag{6.2}$$

Eq. (6.2) is an alternative expression for (14) in Li (2006).

6.2 The distribution of the total dividend payments

In this section, we consider the particular case when $\delta = 0$. Define

$$V_{n,i}(u;b) = \mathbb{E}_i\left[D^n(\bar{T})\big|U_b(0) = u\right], \quad i = 1, 2, \ldots, m, 0 \le u \le b,$$

to be the n-the moment of the total dividend payment prior to the time of ruin given that the initial state is i and the initial surplus is u. Denote $\vec{\mathbf{V}}_n(u;b)$ as the column vector with the i-th element being $V_{n,i}(u;b)$. Since the dividends are only payable if the surplus attains level b prior to ruin, then

$$V_{n,i}(u;b) = \sum_{k=1}^{m} \chi_{ik}(u;b) V_{n,k}(b;b), \qquad 0 \le u \le b,$$

or in matrix form,

$$\vec{\mathbf{V}}_n(u;b) = \boldsymbol{\chi}(u;b)\vec{\mathbf{V}}_n(b;b).$$

It follows from the same arguments as in Li (2006) or Li and Lu (2007) that the vector $\vec{\mathbf{V}}_n(b;b)$ can be evaluated by the following boundary condition:

$$\vec{\mathbf{V}}_n'(b;b) = n\vec{\mathbf{V}}_{n-1}(b;b), \qquad n \in \mathbb{N}^+, \tag{6.3}$$

where $\vec{\mathbf{V}}_0(u;b) = \vec{\mathbf{e}}$. Then $\vec{\mathbf{V}}_n(b;b) = n[\boldsymbol{\chi}'(b;b)]^{-1}\vec{\mathbf{V}}_{n-1}(b;b)$, where $\boldsymbol{\chi}'(b;b) = \frac{\partial \boldsymbol{\chi}(u;b)}{\partial u}\big|_{u=b}$, and

$$\begin{aligned}
\vec{\mathbf{V}}_n(u;b) &= n\boldsymbol{\chi}(u;b)[\boldsymbol{\chi}'(b;b)]^{-1}\vec{\mathbf{V}}_{n-1}(b;b) \\
&= n!\boldsymbol{\chi}(u;b)[\boldsymbol{\chi}'(b;b)]^{-1}\{[\boldsymbol{\chi}'(b;b)]^{-1}\}^{n-1}\vec{\mathbf{e}} \\
&= n!\boldsymbol{\chi}(u;b)[\boldsymbol{\chi}'(b;b)]^{-n}\vec{\mathbf{e}}. \qquad (6.4)
\end{aligned}$$

Denote $\mathbf{W}(u;b) = \boldsymbol{\chi}(u;b)[\boldsymbol{\chi}'(b;b)]^{-1}$ for $0 \le u \le b$. Then

$$\mathbf{W}(b;b) = [\boldsymbol{\chi}'(b;b)]^{-1},$$

and it follows from (5.6) that

$$\mathbf{W}(u;b) = \left[e^{\mathbf{K}_0 u} - \boldsymbol{\Psi}(u)\right]\left[\mathbf{K}_0 e^{\mathbf{K}_0 b} - \boldsymbol{\Psi}'(b)\right]^{-1}. \tag{6.5}$$

Thus, (6.4) can be rewritten as

$$\vec{\mathbf{V}}_n(u;b) = n!\boldsymbol{\chi}(u;b)[\mathbf{W}(b;b)]^n\vec{\mathbf{e}}. \tag{6.6}$$

Further define

$$M_i(u,y;b) = \mathbb{E}_i\left[e^{yD(\bar{T})}\middle|U(0) = u\right], \quad i = 1,2,\ldots,m,$$

to be the moment generating function of $D(\bar{T})$ given that the initial state is i. Denote $\vec{\mathbf{M}}(u,y;b) = (M_1(u,y;b), M_2(u,y;b), \ldots, M_m(u,y;b))^\top$.

Taylor expansion gives

$$\begin{aligned}
\vec{\mathbf{M}}(u,y;b) &= \sum_{n=0}^{\infty}\frac{y^n}{n!}\vec{\mathbf{V}}_n(u;b) \\
&= \left\{\mathbf{I} + \boldsymbol{\chi}(u;b)\sum_{n=1}^{\infty} y^n[\mathbf{W}(b;b)]^n\right\}\vec{\mathbf{e}} \\
&= \left\{\mathbf{I} - \boldsymbol{\chi}(u;b) + \boldsymbol{\chi}(u;b)[\mathbf{I} - y\mathbf{W}(b;b)]^{-1}\right\}\vec{\mathbf{e}} \\
&= [\mathbf{I} - \boldsymbol{\chi}(u;b)]\vec{\mathbf{e}} \\
&\quad + \boldsymbol{\chi}(u;b)\left[[\mathbf{W}(b;b)]^{-1} - y\mathbf{I}\right]^{-1}[\mathbf{W}(b;b)]^{-1}\vec{\mathbf{e}}. \qquad (6.7)
\end{aligned}$$

Then

$$\begin{aligned} M_i(u,y;b) &= \vec{\mathbf{e}}_i^\top \vec{\mathbf{M}}(u,y;b) = 1 - \vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)\vec{\mathbf{e}} \\ &\quad + \vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)\vec{\mathbf{e}}\, \frac{\vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)}{\vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)\vec{\mathbf{e}}} \left[[\mathbf{W}(b;b)]^{-1} - y\mathbf{I}\right]^{-1} [\mathbf{W}(b;b)]^{-1}\vec{\mathbf{e}}\,. \end{aligned}$$

Inverting the moment generating function shows that the distribution of $D(\bar{T})$, given that the initial state is i, is a mixture of the degenerate distribution at 0 with weight $p_i = 1 - \vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)\vec{\mathbf{e}}$ and a continuous distribution with weight $q_i = \vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)\vec{\mathbf{e}}$ phase-type pdf

$$f_i(x) = \vec{\boldsymbol{\gamma}}_i^\top e^{\mathbf{T}x}\vec{\mathbf{t}},$$

where

$$\begin{aligned} \vec{\boldsymbol{\gamma}}_i^\top &= \frac{\vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)}{\vec{\mathbf{e}}_i^\top \boldsymbol{\chi}(u;b)\vec{\mathbf{e}}}, \\ \mathbf{T} &= -[\mathbf{W}(b;b)]^{-1} = -\left[\mathbf{K}_0 e^{\mathbf{K}_0 b} - \boldsymbol{\Psi}'(b)\right]\left[e^{\mathbf{K}_b} - \boldsymbol{\Psi}(b)\right]^{-1}, \\ \vec{\mathbf{t}} &= -\mathbf{T}\vec{\mathbf{e}}. \end{aligned}$$

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