MONTE CARLO MARKET GREEKS IN THE DISPLACED DIFFUSION LIBOR MARKET MODEL

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Abstract. The problem of developing sensitivities of exotic interest rates derivatives to the observed implied volatilities of caps and swaptions is considered. It is shown how to compute these from sensitivities to model volatilities in the displaced diffusion LIBOR market model. The example of a cancellable inverse floater is considered.

JEL codes: C15, G13
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1. Introduction

The LIBOR market model (lmm) is a standard model for pricing exotic interest rate derivatives, particularly for those with early termination features, and Monte Carlo simulation is the most common method of its implementation in practice. Consequently, there have been significant advances in efficient Monte Carlo implementation of the LMM, and the calculation of corresponding Monte Carlo sensitivities, see for example Joshi (2003), Glasserman and Zhao (1999) and Giles and Glasserman (2006). Much of the work on sensitivities, however, focuses on *model* rather than *market* sensitivities. That is, existing methods consider sensitivities to quantities such as initial LIBORs and LIBOR volatilities rather than sensitivities to quantities such as futures/swap rates and cap/swaption volatilities. Yet it is the latter that are of greater importance to traders and risk managers, since they are interested in price responses to directly observed market movements rather than responses to parameters resulting from, and depending on, calibration methods employed.

In this paper, we introduce for the displaced diffusion lmm a Monte Carlo based method for computing market sensitivities. Since market-quoted quantities affect prices only *indirectly* through the calibrated model parameters, the key problem lies in *translating* movements in market-quoted quantities to corresponding movements in model parameters. In general, simply "bumping" a market-quoted quantity and recalibrating to obtain the "bumped" model parameters is undesirable since practitioners often use calibrations

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based on semi-global optimizations,[1] and such approaches result in unpredictable movements to model parameters that, in turn, result in unreliable sensitivities.

There appears to have been very little discussion of this problem in the literature. Piterbarg (2004) discusses the use of volatility scenarios. Brace (2007) looks at the problem of computing swaption vegas using matrix inversion.

The difficulty in the problem lies in the fact that when we compute the sensitivity to a given swaption volatility, we wish to perturb in such a way that the other calibrated instruments' prices remain invariant. A swaption vega is thus not just an attribute of the calibration and the swaption but also of the set of swaptions and caps chosen. This reflects the fact from elementary calculus that a partial derivative is only defined in terms of what else is held constant.

The approach introduced in this paper takes as given a *base* calibration, representing model parameters resulting from the calibration of implementor's choice, and defines for each market- quoted quantity a new calibration that is in some sense a *minimal perturbation* of the base calibration. The key point to note is that the new parameters are computed *relative* to base parameters, by perturbation, and not completely afresh from a bump-and-recalibrate type procedure. This results in a more efficient calculation of "bumped" model parameters, and more reliable price sensitivities.

We stress that our approach does not require recalibration for each bump of a market-quoted quantity to determine the corresponding perturbations to model parameters. Moreover, the resulting perturbations to model parameters are minimal in a sense and hence introduce least amount of noise, resulting in stable sensitivities.

Whilst in this paper we focus solely on the displaced diffusion LIBOR market model, the fast adjoint method has been developed for other market models, see Joshi and Yang (2009*a*) and Joshi and Yang (2009*b*), and it is clear that similar arguments will apply to those cases.

The remainder of this paper is organized as follows. In Section 2, we review the displaced diffusion LIBOR market model and how to compute model vegas. We define and show how to compute market vegas in Section 3. We consider the example of a cancellable inverse floater in Section 4. In Appendix A, we develop the necessary analytic formulas.

[1]For example, the method proposed in Pedersen (1998).

2. Preliminaries

In this section we give a brief review of the displaced diffusion LMM, and adapt to this model the pathwise Greeks approach introduced in Giles and Glasserman (2006).

2.1. **Displaced Diffusion LIBOR Market Model.**

Fix $n \in \mathbb{N}$, an increasing sequence $0 < T_0 < T_1 < \cdots < T_n \in \mathbb{R}$, and let $\tau_k = T_{k+1} - T_k$, for $0 \leq k \leq n-1$. Denote by $f_k(t)$ the LIBOR rate over the interval (T_k, T_{k+1}) for $0 \leq t \leq T_k$, and define

$$\eta(t) = \min\{0 \leq k \leq n \colon T_k > t\}, \tag{2.1}$$

so that $\eta(t)$ is the index of the *nearest* maturing LIBOR rate at time t. Let $\delta_k \in \mathbb{R}_+$ for $0 \leq k < n$, and define $\bar{f}_k(t) = f_k(t) + \delta_k$. The dynamics of the displaced LIBOR rates, $\bar{f}_k(t)$ are assumed to satisfy the sdes

$$d\bar{f}_k(t) = \bar{f}_k(t)\lambda_k(t)dt + \bar{f}_k(t)\gamma_k(t)^t dw(t), \tag{2.2}$$

under the *spot LIBOR measure*, $\mathbb{P}$, where $w(t)$ is a standard d-dimensional $\mathbb{P}$-Wiener process, $\gamma_k \colon \mathbb{R}_+ \to \mathbb{R}^d$ is the deterministic volatility for $\bar{f}_k$, and

$$\lambda_k(t) = \sum_{i=\eta(t)}^{k} \frac{\tau_i \bar{f}_i(t)\gamma_i(t)^t\gamma_k(t)}{1 + \tau_i f_i(t)}. \tag{2.3}$$

Let $0 < t_0 < t_1 < \cdots < t_N$ be the simulation times for Monte Carlo, where without loss of generality $\{T_0, T_1, \ldots, T_n\} \subset \{t_0, t_1, \ldots, t_N\}$, and let $\Delta_i = t_{i+1} - t_i$ for $0 \leq i \leq N-1$. Then under the log-Euler discretization scheme, $\bar{f}_k$ in (2.2) are approximated by

$$\bar{f}_k(t_{i+1}) \approx \bar{f}_k(t_i)\, e^{\left(\lambda_k(t_i) - \frac{1}{2}|\gamma_k(t_i)|^2\right)\Delta_i + \gamma_k(t_i)\cdot\xi_i\sqrt{\Delta_i}}, \tag{2.4}$$

where ξ_i is a d-dimensional standard normal random variate. We will take the terms $\gamma_k(t)$ to be piecewise constant and not to vary within a step; any calibration can be rewritten in this form.

For future reference, we define

$$\boldsymbol{\gamma}_i = (\gamma_1(t_i), \gamma_2(t_i), \ldots, \gamma_n(t_i))^t\sqrt{\Delta_i}, \tag{2.5}$$

so that the k-th row of $\boldsymbol{\gamma}_i$ corresponds to the integrated "volatility" term for the k-th LIBOR rate in (2.2) at time step t_i.

We refer the reader to Brace (2007) for further details on the displaced diffusion LIBOR market model where it is called "shifted BGM."

2.2. **Giles and Glasserman (2006) Greeks.** For notational convenience, define $f_k(t) = f_k(T_k)$ for $t \geq T_k$, and let

$$\bar{\boldsymbol{f}}_i = (\bar{f}_1(t_i), \bar{f}_2(t_i), \dots, \bar{f}_n(t_i))^t. \tag{2.6}$$

In view of the Euler scheme in (2.4), we can regard $\bar{\boldsymbol{f}}_{i+1}$ as a function of $\bar{\boldsymbol{f}}_i$ and write

$$\bar{\boldsymbol{f}}_{i+1} = \Phi_i(\bar{\boldsymbol{f}}_i), \tag{2.7}$$

where $\Phi_i \colon \mathbb{R}^n \to \mathbb{R}^n$ and $\Phi_{i,k}(\bar{\boldsymbol{f}}_i) = \bar{f}_k(t_{i+1})$ is the k-th component of Φ_i.

Now, consider an interest rate dependent cash flow X occurring at time t_l. For example, X may be one of the coupon payments associated with a structured product. Then to compute the price sensitivities for X with respect to a *model* parameter, say α, which only affects step r, repeated application of the chain rule gives

$$\frac{\partial X}{\partial \alpha} = \frac{\partial X}{\partial \bar{\boldsymbol{f}}_l} \frac{\partial \bar{\boldsymbol{f}}_l}{\partial \bar{\boldsymbol{f}}_{l-1}} \cdots \frac{\partial \bar{\boldsymbol{f}}_{r+1}}{\partial \alpha}. \tag{2.8}$$

A key observation in Giles and Glasserman (2006) was that by computing the above product *backwards* rather than forwards, each step involves a matrix-by-vector multiplication rather than matrix-by-matrix multiplication. This results in a reduction of the order of computation from n^3 to n^2 for each step. In the case of a reduced-factor model, they managed to go further by using the special structure of the model and reduced the computational order to nd.

Giles and Glasserman only considered the case of log-normal rates; however, their results were accelerated and extended to the case of displaced rates by Denson and Joshi (2009). This means that we can assume that we have available to us a methodology for rapidly computing model Greeks. In particular, if D is the price of our exotic interest rate derivative today, then we will assume, in what follows, that we know

$$\frac{\partial D}{\partial \gamma_{i,jk}}$$

for all i, j and k. Our objective is to express market vegas as linear combination of these *model* or *elementary* vegas.

The standard path-wise method only applies to Lipschitz continuous payoffs. However, it is possible to extend it to the case where a jump discontinuity occurs across a hypersurface, see Chan and Joshi (2009).

3. Calibration and Minimal Perturbations

Note from (2.4) that an Euler discretization scheme for a displaced LMM is completely specified by the initial LIBORs, $\boldsymbol{f}_0$, the displacements and the pseudo-root matrices, $\boldsymbol{\gamma}_i$, at each simulation time. In practice, the initial LIBORs are usually obtained by bootstrapping market quoted rates of various maturities, whilst the pseudo-roots are obtained by first fitting the LIBOR volatilities to quoted cap and/or swaption volatilities and then possibly performing factor reduction. In view of this, we will refer to a set of initial LIBORs and pseudo-root matrices thus obtained as a *base calibration*, $\mathcal{C}_0$, of displaced LMM. Note that we do not insist on the particular bootstrapping method, nor do we insist on the particular LIBOR volatility calibration, for the base calibration. The perturbed calibrations we define below will be *relative* to the base calibration that the implementor provides.

From now on, we focus on the Euler discretization scheme (2.6) for the displaced diffusion LMM with n underlying LIBOR rates and driven by a d-dimensional Wiener process. Moreover, to ease notational burden we assume that $n = N + 1$ and $T_i = t_i$, for all $0 \leq i < n$, so that LIBOR rate expiry times and simulation times coincide. The pseudo-roots, $\boldsymbol{\gamma}_i$, will then be $n \times d$ dimensional matrices with each row corresponding to a LIBOR rate. However, since expired LIBORs have zero volatility, the first i rows of $\boldsymbol{\gamma}_i$ will be zero, and the space of pseudo-root calibrations will hence consist of approximately $r = \frac{1}{2}n^2 d$ (possible) non-zero entries[2] making up the n pseudo-root matrices.

Although it is not essential for what follows, we remark that it is actually the covariance matrices that determine a calibration rather than the pseudo-square roots in that two different pseudo-square roots may give the same covariance matrix and therefore the same prices to all instruments. For example, if O is an orthogonal matrix and A is a pseudo-root matrix then

$$AA^t = AO(AO)^t.$$

Our motivation for working with pseudo-roots is that the space of d-factor pseudo-root matrices is trivially invariant under all perturbations. This is in marked contrast to covariance matrices where a small perturbation may lead to an increase in factors and/or a failure to be positive definite.

Since to us a calibration is a vector of pseudo-root matrices, we can, after rearrangement, regard it as an element of Euclidean space, $\mathbb{R}^r$, for some large r. We denote by v_i, the non-zero pseudo-root entries regarded as elements of $\mathbb{R}^r$, ordered for example by time, row and column. Since a calibration is determined by forwards as well as pseudo-root matrices, our space of calibrations is then $\mathbb{R}^r \times \mathbb{R}^n$. Let $X_1, X_2, \ldots, X_m$ be the market-quoted implied volatilities from which the base calibration is determined. Within possible LIBOR market models, the terms X_i are then functions on $\mathbb{R}^r \times \mathbb{R}^n$.

[2] It may be the case that the chosen base calibration imposes restrictions that reduce the number of non-zero pseudo-root entries.

In what follows, our objective is to compute vegas and the forward rates will be fixed. We will therefore treat the forward-rates parametrically and generally ignore them notationally. Now that we have identified calibrations as elements of $\mathbb{R}^r$, we have a natural distance between them which is simply the Euclidean distance. With this metric, two close together calibrations will give similar forward rate evolutions on every path of a simulation. An alternative measure of calibration distance would be to take the distance between covariance matrices: however, this would have the disadvantage that close together calibrations would not need to have similar evolutions on a path-wise basis.

For our instruments of interest, swaptions and caps, the implied volatility can be computed approximately via an analytic formula. The terms X_i are therefore given as computable functions on $\mathbb{R}^r \times \mathbb{R}^n$ and we can compute their gradients as functions on $\mathbb{R}^r$,

$$\boldsymbol{w}_i = \nabla_v X_i = \left(\frac{\partial X_i}{\partial v_1}, \frac{\partial X_i}{\partial v_2}, \dots, \frac{\partial X_i}{\partial v_r} \right)^t . \tag{3.1}$$

Denote by W_X the subspace of $\mathbb{R}^r$ spanned by these gradients so that

$$W_X = \langle \boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_r \rangle_{\mathbb{R}}. \tag{3.2}$$

Now, for computing price sensitivities, we wish to bump X_i to $X_i + \epsilon$, for some $\epsilon \in \mathbb{R}_+$, say 1 basis point, whilst holding X_j for $j \neq i$ fixed, and to construct a corresponding calibration. As explained earlier, computing afresh the base calibration for the bumped market is unreliable. Instead, we propose an alternative calibration, $\mathcal{C}_i$, that is consistent with the bump *and* as close as possible to the base calibration $\mathcal{C}_0$. Intuitively, rather than considering the bumped market completely independently of the base market and performing recalibration, we take as given the base calibration and look for the nearest perturbation that is consistent with the bump. We now give a precise definition of $\mathcal{C}_i$.

Assume that we are given a base calibration $\mathcal{C}_0$ and let v_i, where $1 \leq i \leq r$, be the non-zero pseudo-root entries ordered in some way as described above. Define $\boldsymbol{v} = (v_1, v_2, \dots, v_r)^t$, and recall that X_i can be regarded as a function of $\boldsymbol{v}$. Provided that the X_i thus considered are sufficiently smooth, (which is certainly the case for the Hull–White approximation,) we have

$$X_i(\boldsymbol{v} + \boldsymbol{h}) = X_i(\boldsymbol{v}) + \langle \boldsymbol{w}_i, \boldsymbol{h} \rangle + O(|\boldsymbol{h}|^2), \tag{3.3}$$

where $\boldsymbol{w}_i$ are as defined in (3.1) and $\boldsymbol{h} \in \mathbb{R}^r$ represents a *perturbation* from the base pseudo-root entries $\boldsymbol{v}$. For the bump $X_i \mapsto X_i + \epsilon$, the perturbation, $\boldsymbol{h}$, must satisfy $X_i(\boldsymbol{v} + \boldsymbol{h}) \approx X_i(\boldsymbol{v}) + \epsilon$, whence

$$\langle \boldsymbol{w}_i, \boldsymbol{h} \rangle = \epsilon. \tag{3.4}$$

Moreover, since the perturbation must leave the remaining X_j fixed, we require $X_j(\boldsymbol{v}+\boldsymbol{h}) \approx X_j(\boldsymbol{v})$, or equivalently

$$\langle \boldsymbol{w}_j, \boldsymbol{h} \rangle = 0 \tag{3.5}$$

for $j \neq i$.

Since the dimensionality of the space of calibrations is usually much greater than the number of market instruments being calibrated to, these equations are not sufficient to determine a unique perturbation $\boldsymbol{h}$. For this reason, we impose the additional condition that the perturbation is *minimal*, in the sense that the perturbed calibration is as close to $\mathcal{C}_0$ as possible. Of course, the notion of closeness is norm dependent and we have chosen the L^2-norm which is convenient both numerically and geometrically. Hence we impose a final constraint on the perturbation, $\boldsymbol{h}$,

$$\langle \boldsymbol{h}, \boldsymbol{h} \rangle \to \text{minimized.} \tag{3.6}$$

We have given a mathematical description of $\boldsymbol{h}$, but we still need an algorithm to find it. To explain how the perturbation may be computed in practice, we give a geometric construction. Let

$$H_i = \langle \boldsymbol{w}_j \mid j \neq i \rangle_{\mathbb{R}} \subset \mathbb{R}^r, \tag{3.7}$$

and let $H_i^{\perp} \subset \mathbb{R}^r$ be the subspace orthogonal to H_i.

We can now rewrite the three conditions determining the *minimal* perturbation, $\boldsymbol{h}_i$, corresponding to X_i as follows:

$$\boldsymbol{h}_i \in H_i^{\perp}, \quad \langle \boldsymbol{w}_i, \boldsymbol{h}_i \rangle = \epsilon, \quad \text{and} \quad \langle \boldsymbol{h}_i, \boldsymbol{h}_i \rangle \text{ minimal.} \tag{3.8}$$

There is a simple geometric solution to this problem: $\boldsymbol{h}_i$ lies in the linear span of the projection of $\boldsymbol{w}_i$ onto $H_i^{\perp}$. We therefore project and then linearly scale to achieve $\langle \boldsymbol{w}_i, \boldsymbol{h}_i \rangle = \epsilon$. This procedure will only fail if the projection of $\boldsymbol{w}_i$ onto $H_i^{\perp}$ is zero. However, this is to be expected, since in that case X_i is locally (i.e. to first order) a linear combination of the remaining X_j, and it is not possible to perturb X_i while keeping X_j, for $j \neq i$, fixed. In practice, such redundant calibrating instruments should be discarded.

So given a base calibration $\mathcal{C}_0$ consisting of $\boldsymbol{f}_0$ and $\boldsymbol{v}$, we can compute for each X_i the corresponding perturbation, $\boldsymbol{h}_i$, and obtain the calibration, $\mathcal{C}_i$ consisting of $\boldsymbol{f}_0$ and $\boldsymbol{v} + \boldsymbol{h}_i$.

This allows us to write the vega with respect to instrument X_i as a linear combination of the model vegas. Our market vega is now simply

$$\frac{\partial D}{\partial X_i} = \frac{1}{\epsilon} \sum_{j=1}^{r} h_{i,j} \frac{\partial D}{\partial v_j},$$

where the terms $\frac{\partial D}{\partial v_j}$ are the model vegas from Section 2.2 with the indices relabelled.

Rates 5%	**Rates** 6%	**Rates** 7%	**Rates** 8%	**Rates** 9%
1.42×10^{-4}	1.76×10^{-5}	2.21×10^{-6}	1.58×10^{-6}	-1.85×10^{-7}
3.68×10^{-4}	1.13×10^{-4}	2.72×10^{-5}	5.65×10^{-6}	3.42×10^{-6}
4.47×10^{-4}	2.04×10^{-4}	7.52×10^{-5}	2.14×10^{-5}	5.20×10^{-6}
4.16×10^{-4}	2.73×10^{-4}	1.32×10^{-4}	5.78×10^{-5}	9.31×10^{-6}
3.79×10^{-4}	2.98×10^{-4}	1.78×10^{-4}	7.79×10^{-5}	2.91×10^{-5}
3.44×10^{-4}	3.08×10^{-4}	1.96×10^{-4}	1.09×10^{-4}	5.59×10^{-5}
2.89×10^{-4}	2.96×10^{-4}	2.10×10^{-4}	1.40×10^{-4}	7.07×10^{-5}
2.67×10^{-4}	2.88×10^{-4}	2.18×10^{-4}	1.46×10^{-4}	8.60×10^{-5}
1.58×10^{-4}	2.46×10^{-4}	2.37×10^{-4}	1.75×10^{-4}	9.37×10^{-5}
1.72×10^{-4}	2.20×10^{-4}	2.03×10^{-4}	1.50×10^{-4}	1.08×10^{-4}
1.50×10^{-4}	2.11×10^{-4}	1.85×10^{-4}	1.61×10^{-4}	1.18×10^{-4}
1.23×10^{-4}	1.80×10^{-4}	1.82×10^{-4}	1.58×10^{-4}	1.19×10^{-4}
1.13×10^{-4}	1.50×10^{-4}	1.53×10^{-4}	1.43×10^{-4}	1.18×10^{-4}
8.15×10^{-5}	1.22×10^{-4}	1.49×10^{-4}	1.31×10^{-4}	1.19×10^{-4}
6.28×10^{-5}	1.07×10^{-4}	1.06×10^{-4}	1.19×10^{-4}	1.06×10^{-4}
2.73×10^{-5}	6.61×10^{-5}	1.05×10^{-4}	1.10×10^{-4}	8.34×10^{-5}
4.88×10^{-6}	4.27×10^{-5}	7.13×10^{-5}	9.78×10^{-5}	9.31×10^{-5}
-2.80×10^{-5}	-8.92×10^{-6}	4.76×10^{-5}	1.29×10^{-4}	2.11×10^{-4}
-3.17×10^{-5}	-2.45×10^{-5}	-3.82×10^{-5}	-6.23×10^{-5}	-2.93×10^{-5}
-3.92×10^{-6}	-3.23×10^{-5}	-9.56×10^{-5}	-1.91×10^{-4}	-2.98×10^{-4}

TABLE 1. Vegas for a cancellable inverse floater with respect to co-terminal swaptions only

By construction, we have that up to order $|\boldsymbol{h}|^2$, the vega of an instrument X_i with respect to itself according to this definition agrees with the usual definition, and the vega of each X_i with respect to X_j for $j \neq i$ is zero.

4. Numerical illustrations

Now that we have defined the market vegas for exotic interest rate derivatives, it is interesting to examine their values for a particular case. We study a cancellable inverse floater swap. For this product, the issuer pays a floorlet on an inverse floating LIBOR rate and receives a floating LIBOR rate. The issuer is also able to cancel future payments on each of the payment dates. The product is therefore naturally exposed to both caplet/floorlet volatility via the individual payments, and swaption volatility via the right to cancel the swap. The natural hedges are therefore caplets on the individual rates and the *co-terminal* swaptions with the same underlying rates and expiry date as the contract.

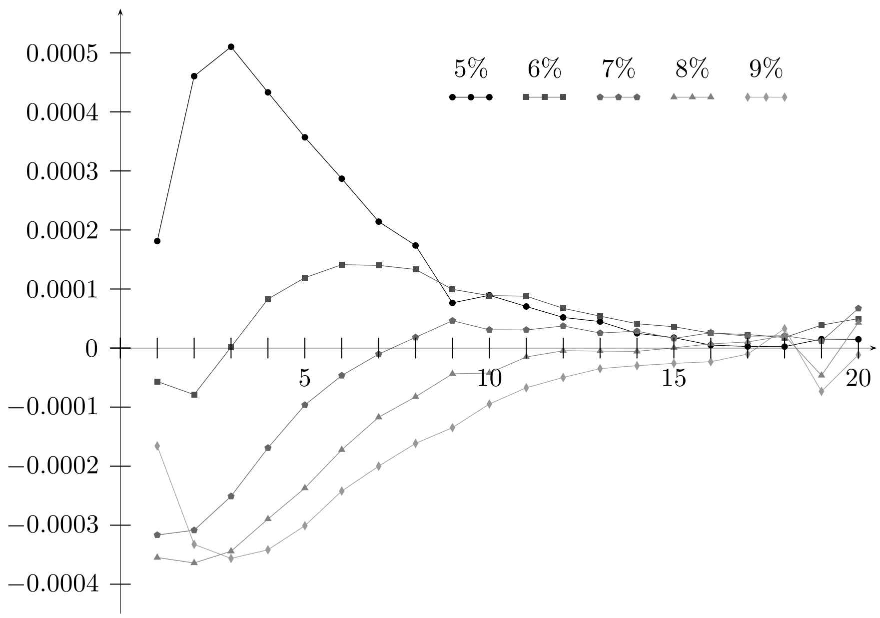

FIGURE 1. Vegas for a cancellable inverse floater with respect to co-terminal swaptions only

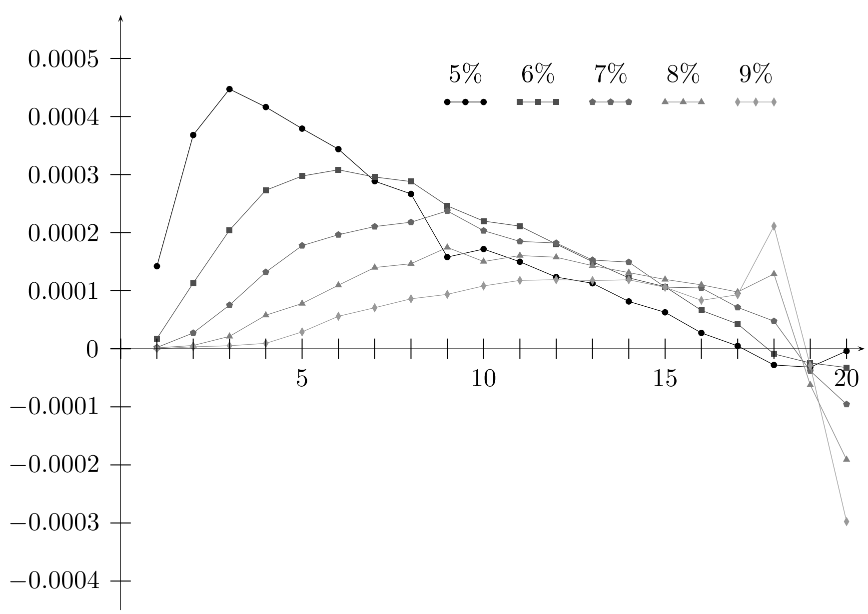

FIGURE 2. Vegas for a cancellable inverse floater with respect to co-terminal swaptions holding caplets fixed

Rates 5%	**Rates** 6%	**Rates** 7%	**Rates** 8%	**Rates** 9%
1.81×10^{-4}	-5.67×10^{-5}	-3.17×10^{-4}	-3.55×10^{-4}	-1.66×10^{-4}
4.61×10^{-4}	-7.89×10^{-5}	-3.09×10^{-4}	-3.64×10^{-4}	-3.33×10^{-4}
5.10×10^{-4}	1.56×10^{-6}	-2.51×10^{-4}	-3.44×10^{-4}	-3.56×10^{-4}
4.33×10^{-4}	8.31×10^{-5}	-1.69×10^{-4}	-2.90×10^{-4}	-3.42×10^{-4}
3.57×10^{-4}	1.19×10^{-4}	-9.65×10^{-5}	-2.37×10^{-4}	-3.01×10^{-4}
2.87×10^{-4}	1.41×10^{-4}	-4.65×10^{-5}	-1.72×10^{-4}	-2.4×10^{-4}
2.14×10^{-4}	1.40×10^{-4}	-1.03×10^{-5}	-1.17×10^{-4}	-2.00×10^{-4}
1.74×10^{-4}	1.33×10^{-4}	1.82×10^{-5}	-8.25×10^{-5}	-1.62×10^{-4}
7.65×10^{-5}	9.95×10^{-5}	4.64×10^{-5}	-4.37×10^{-5}	-1.35×10^{-4}
8.94×10^{-5}	8.88×10^{-5}	3.09×10^{-5}	-4.23×10^{-5}	-9.48×10^{-5}
7.04×10^{-5}	8.78×10^{-5}	3.07×10^{-5}	-1.52×10^{-5}	-6.73×10^{-5}
5.18×10^{-5}	6.73×10^{-5}	3.73×10^{-5}	-4.59×10^{-6}	-4.98×10^{-5}
4.48×10^{-5}	5.40×10^{-5}	2.54×10^{-5}	-5.17×10^{-6}	-3.49×10^{-5}
2.50×10^{-5}	4.12×10^{-5}	2.86×10^{-5}	-5.60×10^{-6}	-2.98×10^{-5}
1.76×10^{-5}	3.61×10^{-5}	1.63×10^{-5}	4.11×10^{-7}	-2.60×10^{-5}
5.00×10^{-6}	2.53×10^{-5}	2.59×10^{-5}	7.04×10^{-6}	-2.31×10^{-5}
2.63×10^{-6}	2.29×10^{-5}	2.01×10^{-5}	1.03×10^{-5}	-1.02×10^{-5}
2.40×10^{-6}	1.78×10^{-5}	2.08×10^{-5}	2.08×10^{-5}	3.29×10^{-5}
1.51×10^{-5}	3.89×10^{-5}	1.17×10^{-5}	-4.63×10^{-5}	-7.33×10^{-5}
1.47×10^{-5}	4.98×10^{-5}	6.72×10^{-5}	4.32×10^{-5}	-1.15×10^{-5}

TABLE 2. Vegas for a cancellable inverse floater with respect to co-terminal swaptions holding caplets fixed

To facilitate reproduction of our results, we take a simple base calibration and product. We have

$$T_j = 0.5(j+1), \quad \text{for} \quad j = 0, \ldots, 20.$$

The issuer pays

$$(K - 2f_k(T_k))_+\tau - f_k(T_k)\tau$$

at time T_{k+1} where $\tau = 0.5$. We value the product from the issuer's perspective.

We take all rates to have 0.0 displacement and constant volatility at level 18%. Before factor reduction, the correlation between rate i and rate j is

$$L + (1-L)e^{-\beta|T_i - T_j|},$$

with $L = 0.5$ and $\beta = 0.2$. We reduce to five factors by setting all eigenvalues after the first 5 to zero, and then rescaling to obtain a correlation matrix.

We consider two sets of possible vega instruments. For set A, we take the at-the-money co-terminal swaptions with underlying times T_j. For set B, we take the set A together with the at-the-money caplets on the individual

Rates 5%	**Rates** 6%	**Rates** 7%	**Rates** 8%	**Rates** 9%
2.07×10^{-5}	-3.58×10^{-5}	-1.61×10^{-4}	-1.75×10^{-4}	-7.48×10^{-5}
5.20×10^{-5}	-9.72×10^{-5}	-2.34×10^{-4}	-2.53×10^{-4}	-1.69×10^{-4}
6.09×10^{-5}	-1.37×10^{-4}	-2.77×10^{-4}	-3.02×10^{-4}	-2.35×10^{-4}
5.47×10^{-5}	-1.61×10^{-4}	-3.04×10^{-4}	-3.34×10^{-4}	-2.81×10^{-4}
4.10×10^{-5}	-1.78×10^{-4}	-3.20×10^{-4}	-3.56×10^{-4}	-3.14×10^{-4}
2.24×10^{-5}	-1.91×10^{-4}	-3.30×10^{-4}	-3.69×10^{-4}	-3.36×10^{-4}
3.34×10^{-6}	-2.01×10^{-4}	-3.36×10^{-4}	-3.78×10^{-4}	-3.52×10^{-4}
-1.70×10^{-5}	-2.10×10^{-4}	-3.39×10^{-4}	-3.81×10^{-4}	-3.63×10^{-4}
-3.24×10^{-5}	-2.18×10^{-4}	-3.41×10^{-4}	-3.84×10^{-4}	-3.69×10^{-4}
-4.67×10^{-5}	-2.22×10^{-4}	-3.40×10^{-4}	-3.82×10^{-4}	-3.71×10^{-4}
-5.94×10^{-5}	-2.26×10^{-4}	-3.38×10^{-4}	-3.79×10^{-4}	-3.71×10^{-4}
-6.97×10^{-5}	-2.28×10^{-4}	-3.35×10^{-4}	-3.75×10^{-4}	-3.69×10^{-4}
-7.87×10^{-5}	-2.27×10^{-4}	-3.29×10^{-4}	-3.69×10^{-4}	-3.64×10^{-4}
-8.51×10^{-5}	-2.23×10^{-4}	-3.23×10^{-4}	-3.62×10^{-4}	-3.59×10^{-4}
-8.89×10^{-5}	-2.18×10^{-4}	-3.11×10^{-4}	-3.53×10^{-4}	-3.53×10^{-4}
-8.69×10^{-5}	-2.07×10^{-4}	-3.00×10^{-4}	-3.43×10^{-4}	-3.44×10^{-4}
-7.87×10^{-5}	-1.90×10^{-4}	-2.82×10^{-4}	-3.31×10^{-4}	-3.36×10^{-4}
-5.72×10^{-5}	-1.55×10^{-4}	-2.56×10^{-4}	-3.30×10^{-4}	-3.62×10^{-4}
-2.32×10^{-5}	-9.75×10^{-5}	-1.91×10^{-4}	-2.74×10^{-4}	-3.35×10^{-4}

TABLE 3. Vegas for a cancellable inverse floater with respect to caplets holding co-terminal swaptions fixed

forward rates. Since the last caplet and co-terminal swaption are the same, we have the last caplet twice but only consider it once when computing. This means that the cardinality of A is 20 and that of B is 39.

In order to minimize noise, we run $2^{20} - 1$ Sobol paths using a log-Euler approximation with one step per reset using the spot measure. We compute the exercise strategy using the extension to the least-squares method of Carrière (1996) and Longstaff and Schwartz (2001), discussed in Amin (2003) and Joshi (2006). We hold the exercise strategy fixed when computing Greeks as is usually done. We use quadratic polynomials in the first forward rate, the contiguous co-terminal swap-rate and the final zero-coupon bond as basis functions. We use 65,536 training paths. The duality gaps are developed using 256 inner and outer paths with the Joshi (2006) method.

We use five different flat yield curves varying the level of rates. In case j the initial forward rates are $(4 + j)\%$. The discount factor for time 0.5 is 0.95 in all cases. The lower bound estimates are

$$0.0926, 0.2113, 0.3256, 0.4192.$$

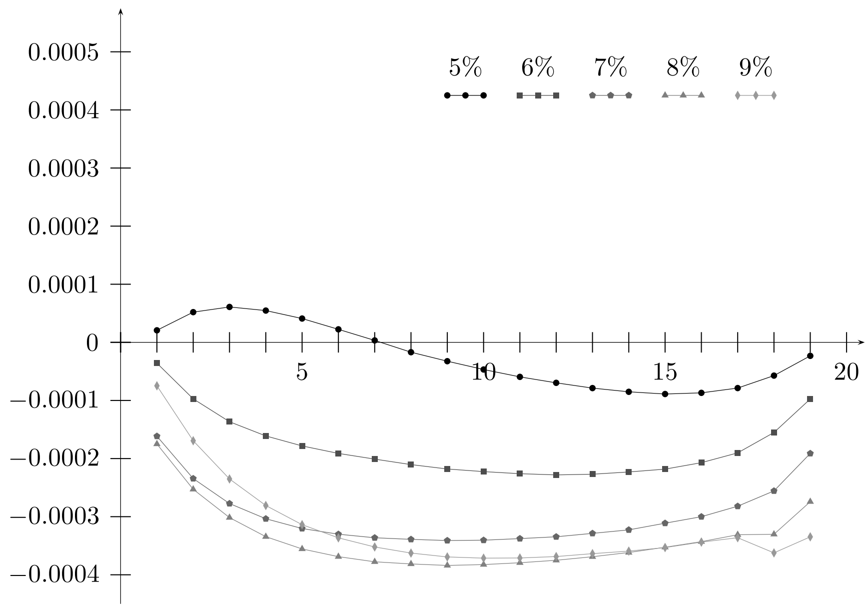

FIGURE 3. Vegas for a cancellable inverse floater with respect to caplets holding co-terminal swaptions fixed

The duality gaps to the upper bounds are

$$0.0064, 0.0024, 0.0012, 0.0006.$$

For set A, we present the results in Table 1 and Figure 1. For set B, we present the swaption vegas in Table 2 and Figure 2, and the caplet vegas in Table 3 and Figure 3. The value of ϵ used in the calculation of vegas is 0.01. Thus we are showing the effect of a one-percent change in the implied volatility.

We see some striking differences in the swaption vegas according to whether the caplets are held fixed. When caplets are fixed, the swaption vegas are almost all positive reflecting the increased value of the cancellability. The caplet vegas are generally negative on the other hand reflecting the fact that the product pays floorlets. When we consider the swaptions alone, these two effects merge and the swaption vegas are often negative.

5. CONCLUSION

We have developed a new methodology for computing sensitivities to market-quoted implied volatilities in the displaced diffusion LIBOR market model. We have seen that the swaption Greeks of a cancellable inverse floater vary greatly according to whether the caplets are held constant.

Appendix A. Analytic formulas and approximations

We require the gradient of the implied volatility as a function of the pseudo-root elements. In this appendix, we develop formulas for its entries. For swaptions, we must use an analytic approximation to the implied volatility and differentiate that. For caps, there is the different complication that the implied volatility is found by first pricing all underlying caplets and then finding the single constant volatility that matches their total price. We also have the issue that for displaced diffusion models the implied volatility is generally defined as the input to a log-normal (i.e. zero displacement) model that matches the price correctly, whereas the pseudo-root elements are defined for displaced log rates. So even for caplets the computation is not trivial.

We first discuss the algorithm for caps. (Caplets are a special case.) Let a cap C have underlying rates $f_\alpha, \dots, f_\beta$ with $\alpha \leq \beta$. The displaced implied volatility of one of the caplets, C_m, comprising C is given by the simple formula

$$\hat{\sigma}_m = \sqrt{\frac{1}{T_m} \sum_{i \leq m, f} \gamma_{i,mf}^2}.$$

We thus have

$$\frac{\partial \hat{\sigma}_m}{\partial \gamma_{i,mf}} = \frac{\gamma_{i,mf}}{T_m \hat{\sigma}_m},$$

for $i \leq m$. Note that $\gamma_{i,mf} = 0$ for $i > m$. Each pseudo-root element will underlie at most one of the caplets, so the derivative of the price of C with respect to $\gamma_{i,mf}$ is equal to that of C_m. Thus

$$\frac{\partial C}{\partial \gamma_{i,mf}} = \frac{\partial C_m}{\partial \gamma_{i,mf}} = \mathrm{VegaD}(C_m) \frac{\gamma_{i,mf}}{T_m \hat{\sigma}_m} \tag{A.1}$$

where VegaD denotes the sensitivity with respect to volatility in the displaced Black formula, and is given by

$$\sqrt{T_m}(f_m + \delta_m) N'(d_1)$$

where N is the cumulative normal function, and

$$d_1 = \frac{\log\left(\frac{f_m + \delta_m}{K + \delta_m}\right) + 0.5 \hat{\sigma}_m^2 T_m}{\hat{\sigma}_m \sqrt{T_m}},$$

where K is the strike of the cap.

We thus have the gradient of the price as a function of the underlying pseudo-root elements. However, it is the gradient of the (log-normal) implied volatility that is required. Let $\hat{\sigma}_C$ be the cap's implied volatility (computed via a numerical root search). We have

$$\frac{\partial C}{\partial \gamma_{i,mf}} = \frac{dC}{d\hat{\sigma}_C}\frac{\partial \hat{\sigma}_C}{\partial \gamma_{i,mf}}.$$

Since $\frac{dC}{d\hat{\sigma}_C}$ is computable as a sum of caplet vegas, rearranging, we obtain a simple expression for $\frac{\partial \hat{\sigma}_C}{\partial \gamma_{i,mf}}$ as desired.

For swaptions, we use a modification of the Hull and White (2000) analytic approximation. The modification is required in that we need the log-normal implied volatility in a displaced diffusion model. We proceed via coefficient freezing. We do not consider drifts in the derivation since in the pricing measure obtained by taking the annuity as numeraire, the swap-rate will be a martingale. We consider a swaption on a rate $\mathrm{SR}_{\alpha,\beta}$ with underlying rates $f_\alpha, \dots, f_\beta$. We can write up to drift terms,

$$d \log \mathrm{SR}_{\alpha,\beta} = \sum_{i=\alpha}^{\beta} \frac{\partial \log \mathrm{SR}_{\alpha,\beta}}{\partial \log(f_i + \delta_i)} d\log(f_i + \delta_i). \tag{A.2}$$

Expanding the logs, we obtain

$$d \log \mathrm{SR}_{\alpha,\beta} = \sum_{i=\alpha}^{\beta} \frac{f_i + \delta_i}{\mathrm{SR}_{\alpha,\beta}} \frac{\partial \mathrm{SR}_{\alpha,\beta}}{\partial (f_i + \delta_i)} d\log(f_i + \delta_i). \tag{A.3}$$

Clearly,

$$\frac{\partial \mathrm{SR}_{\alpha,\beta}}{\partial (f_i + \delta_i)} = \frac{\partial \mathrm{SR}_{\alpha,\beta}}{\partial f_i}.$$

So we have approximately,

$$d \log \mathrm{SR}_{\alpha,\beta} = \sum_{i=\alpha}^{\beta} \frac{f_i + \delta_i}{\mathrm{SR}_{\alpha,\beta}} \frac{\partial \mathrm{SR}_{\alpha,\beta}}{\partial f_i} \sum_f \gamma_{if}(t) dw^f(t).$$

Let

$$Z_i = \frac{f_i(0) + \delta_i}{\mathrm{SR}_{\alpha,\beta}(0)} \frac{\partial \mathrm{SR}_{\alpha,\beta}}{\partial f_i}(0),$$

and our coefficient-frozen approximation is then

$$d \log \mathrm{SR}_{\alpha,\beta} = \sum_{i=\alpha}^{\beta} \sum_{f=1}^{d} Z_i \gamma_{if}(t) dw^f(t).$$

From this we obtain an expression for the total variance, $V_{\alpha,\beta}$, of $\log \mathrm{SR}_{\alpha,\beta}$ as a function of $\gamma_{k,if}$ which can be differentiated:

$$V_{\alpha,\beta} = \sum_{k \le \alpha} \sum_{f=1}^{d} \left(\sum_{i=\alpha}^{\beta} Z_i \gamma_{k,if} \right)^2 .$$

The derivative of

$$\hat{\sigma}_{\alpha,\beta} = \sqrt{\frac{1}{T_\alpha} V_{\alpha,\beta}}$$

with respect to any element $\gamma_{k,if}$ is now straight-forward to compute. We note that expressions for the terms $\frac{\partial \mathrm{SR}_{\alpha,\beta}}{\partial f_i}$ can be found in Jäckel and Rebonato (2003), and that these do not involve $\gamma_{k,if}$.

Whilst we have used the Hull–White approximation, here, we note that our main construction does not depend on this choice, and any analytic approximation that can be straight-forwardly differentiated could be used.

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