A two-dimensional extension of Bougerol's identity in law for the exponential functional of Brownian motion

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1 Introduction and Main Results

(1.1) Let $(B_t, t \geq 0)$ denote a one-dimensional Brownian motion starting from 0, and, more generally, consider: $B_t^{(\nu)} \equiv B_t + \nu t$, $t \geq 0$, a Brownian motion with drift $\nu \in \mathbb{R}$. It has been recognized for quite some time, see, e.g. the compendium of articles about Brownian exponential functionals Yor [Y1], Dufresne [D1], that the law of

$$A_t^{(\nu)} = \int_0^t ds \exp(2B_s^{(\nu)})$$

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for fixed time t, is not so simple to express, even for $\nu = 0$ (in this case, we write A_t for $A_t^{(0)}$). Thanks to the Cameron-Martin relationship between the laws of $(B_u^{(\nu)}, u \leq t)$ and $(B_u, u \leq t)$, for any $t \geq 0$, $\nu \in \mathbb{R}$, the discussion of the law of $A_t^{(\nu)}$ is equivalent to that of the two-dimensional variable:

$$(A_t, B_t)$$

which is, thus, not so simple to express either.

(1.2) In this note, we shall focus on the case $\nu = 0$, and we first recall that the law of A_t, for fixed t, may be obtained, slightly indirectly, from Bougerol's identity in law:

$$\text{for fixed } t, \quad \sinh(B_t) \stackrel{\text{(law)}}{=} \beta(A_t),$$

where on the RHS $(\beta(u), u \geq 0)$ is a one-dimensional Brownian motion independent from $(B_s, s \geq 0)$. Our main result in this paper is:

Theorem 1. *For fixed t, the three following two-dimensional random variables are identically distributed:*

$$(1) \qquad (\sinh(B_t)\,,\ \sinh(L_t))$$

$$(2) \qquad (\beta(A_t)\,,\ \exp(-B_t)\lambda(A_t))$$

$$(3) \qquad (\exp(-B_t)\beta(A_t)\,,\ \lambda(A_t))$$

where $(L_t, t \geq 0)$ denotes the local time at 0 of $(B_t, t \geq 0)$, whereas (β, λ) consists of another one-dimensional Brownian motion $(\beta(u), u \geq 0)$, and its local time $(\lambda(u), u \geq 0)$ at level 0; the two Brownian motions B and β are assumed to be independent.

(1.3) We now make some comments about the result in Theorem 1:
(a) Bougerol's identity is the identity in law between the two first components of (1) and (2); one way to prove it is to start from the identity in law between the two first components of (2) and (3), which is obtained by time reversal at time t. This is explained and developed in Alili-Dufresne-Yor [ADY]; for ease of reference, see also p.200 of Yor [Y1], where the proof from [ADY] is reproduced. The end of the proof in [ADY] of Bougerol's identity in law consists in showing, mainly with the help of Itô's formula that the processes $(\sinh(B_t), t \geq 0)$ and $(\exp(-B_t)\beta_{A_t}, t \geq 0)$ are identically distributed.
(b) The novelty in the present paper is that we are now able to consider, jointly with the first components of (1)-(2)-(3), the second components, and

to show that, for fixed t, these two-dimensional vectors have the same distribution.
(c) Note that each of the first components of the three random vectors is symmetric in law, so that, in fact, it suffices to show the identity in law between:

$$(\sinh(|B_t|)\,,\ \sinh(L_t)) \tag{4}$$
$$(|\beta(A_t)|\,,\ \exp(-B_t)\lambda(A_t)) \tag{5}$$
$$(\exp(-B_t)|\beta(A_t)|\,,\ \lambda(A_t)). \tag{6}$$

(d) Now, it is worth noting that, if we write (X, Y) for any of the three random vectors in (4)-(5)-(6), then there is furthermore the symmetry property

$$(X, Y) \overset{\text{(law)}}{=} (Y, X). \tag{7}$$

This follows from the well-known fact that, for fixed t:

$$(|B_t|, L_t) \overset{\text{(law)}}{=} (L_t, |B_t|) \quad \text{since:} \quad (|B_t|, L_t) \overset{\text{(law)}}{=} R_t(U, 1-U),$$

with $R_t = |B_t^{(3)}|$, the Euclidean norm of a three-dimensional Brownian motion $(B_t^{(3)}, t \geq 0)$. This is a well-known consequence of the symmetry principle (or, at process level, of Pitman's theorem about BES(3)), which yields the following expression for the law of $(|B_t|, L_t)$:

$$P(|B_t| \in dx, L_t \in d\ell) = \frac{2dx\, d\ell}{\sqrt{2\pi t^3}}(x+\ell)\exp\left(-\frac{(x+\ell)^2}{2t}\right) dx\, d\ell, \qquad x, \ell \geq 0.$$

To see that (7) holds for (5) and (6), we also use:

$$(\beta(A_t), \exp(-B_t)\lambda(A_t)) \overset{\text{(law)}}{=} (\sqrt{A_t}|\beta_1|, \exp(-B_t)\sqrt{A_t}\lambda(1)) \tag{8}$$

(by independence and Brownian scaling).
Now, property (7) for (5) and (6) follows from the conjunction of:

- **i)** $(|\beta_1|, \lambda_1) \overset{\text{(law)}}{=} (\lambda_1, |\beta_1|)$, as just discussed above;
- **ii)** $(A_t, \exp(-2B_t)A_t) \overset{\text{(law)}}{=} (\exp(-2B_t)A_t, A_t)$,

which itself follows from the time reversal stability of the law of $(B_u, u \leq t)$, *i.e.*:

$$(B_t - B_{(t-u)}, u \leq t) \overset{\text{(law)}}{=} (B_u, u \leq t)\ ,$$

a key ingredient in the [ADY] proof of Bougerol's identity, as already mentioned.

(1.4) We now present the organisation of the remainder of the paper:

a) In Section 2, we take up the opportunity to give yet another proof of Bougerol's identity in law, which, this time, we view as the identity in law between the right-hand sides of (1) and (3) in Theorem 1, that is:

$$\text{for fixed } t, \quad \sinh(L_t) \overset{\text{(law)}}{=} \lambda(A_t). \tag{9}$$

Of course, since: $|B_t| \overset{\text{(law)}}{=} L_t$, for fixed t, (9) is easily shown to be equivalent to:

$$\sinh(B_t) \overset{\text{(law)}}{=} \beta(A_t), \tag{10}$$

the "traditional" form of stating Bougerol's identity in law. Here, in Section 2, we shall prove (9) "from scratch", in fact under the following "avatar form".

Theorem 2. *Let $H_u = \int_0^u \frac{dh}{R_h^2}$, $u \geq 0$, denote the "clock" associated with the two-dimensional Bessel process $(R_h, h \geq 0)$ starting from 1, and let $(\sigma_\theta, \theta \geq 0)$ denote a stable $(1/2)$ subordinator, independent from $(R_h, h \geq 0)$. Then, for fixed $\ell > 0$, one has:*

$$H_{\sigma_s} \overset{\text{(law)}}{=} \sigma_\ell\,, \ \textit{where } s = \sinh(\ell). \tag{11}$$

The identity (11) is rather spectacular in that the "trace" of H on the set of zeros of an independent Brownian motion is distributed (at least, as far as one-dimensional marginals are concerned) as σ_ℓ. A puzzling question which we have not been able to solve so far is whether the identity in law (11) or, equivalently (9), holds between processes.([1])

b) In Section 3, we give two proofs of Theorem 1. These two proofs are rather different, although, technically, they rely essentially on the same ingredients, which are recalled at the beginning of Section 3, and among which is the identity in law (see Yor [Y1], Paper♯6, p.94):

$$A_{S_p}^{(\nu)} \overset{\text{(law)}}{=} \frac{\beta_{1,a}}{2\gamma_b} \overset{\text{(law)}}{=} \frac{1 - U^{1/a}}{2\gamma_b}\,, \tag{12}$$

where:

[1]Added in November 2011: In fact, the possibility of the identity in law between processes was ruled out by J. Bertoin. Hence, we are now writing a three-author joint paper ([BDY]) on this topic.

- $A_t^{(\nu)} = \int_0^t ds \exp 2(B_s + \nu s)$,
- S_p denotes an exponential variable with parameter p, independent from B,
- $\beta_{u,v}$ is a beta variable, with parameters u and v,
- U is uniform on $[0,1]$,
- γ_b is a gamma variable with parameter b,
- the parameters a and b are given by: $a = \dfrac{\mu+\nu}{2}$, $b = \dfrac{\mu-\nu}{2}$, with $\mu = \sqrt{2p+\nu^2}$.

The first proof is "asymmetric" and hinges upon identities in law between beta and gamma variables, whilst in the second proof, we have privileged a symmetrical presentation which reflects the symmetry of the law of $(|B_t|, L_t)$, for fixed t. (For a "light" discussion on this topic, see [Y4].)

2 A Proof of Theorem 2

(2.1) Notation

For this proof, we shall use the notation about the laws $P_r^{(\nu)}$ of Bessel processes, with index ν, and starting position r as in [Y1, Y2]. In particular, the relationship, where $\mathcal{R}_t = \sigma\{R_s, s \le t\}$:

$$P_r^{(\nu)}{}_{|\mathcal{R}_t} = \left(\frac{R_t}{r}\right)^{\nu} \exp\left(-\frac{\nu^2}{2} H_t\right) \bullet P_{r|\mathcal{R}_t}^{(0)} \tag{13}$$

will play an important rôle.

We now need to bring into the picture an independent stable (1/2) variable σ_s in the picture, whose Laplace transform(2) taken at $(\lambda^2/2)$ is $\exp(-\lambda s)$, we do not change the notation $P_r^{(\nu)}$ (although we might use a symbol like $\widetilde{P}_r^{(\nu)}$ for the probability on a space enriched with σ_s, but we shall not do this). Besides (13), we shall also use the explicit form of the Laplace transform of R_σ^2 (for fixed σ) under $P_r^{(\nu)}$:

$$E_r^{(\nu)}(\exp(-uR_\sigma^2)) = \frac{1}{(1+2\sigma u)^{1+\nu}} \exp\left(-\frac{r^2 u}{(1+2u\sigma)}\right). \tag{14}$$

[2] We use λ, or $(\lambda^2/2)$, for the argument of the Laplace transform, as there is no risk of confusion with the local times in Theorem 1

(2.2) Beginning of the proof: a first reduction

Clearly, proving (11) is equivalent to showing:

$$E_1^{(0)}\left(\exp\left(-\frac{\lambda^2}{2}H_{\sigma_s}\right)\right) = \exp(-\lambda\ell), \qquad \lambda, \ell \geq 0,$$

where $\ell = a(s) = \log(s + \sqrt{1+s^2})$. Using (13) above, the previous identity is equivalent to:

$$E_1^{(\lambda)}\left(\frac{1}{(R_{\sigma_s})^\lambda}\right) = \exp(-\lambda\ell). \tag{15}$$

The elementary formula:

$$\frac{1}{\rho^\lambda} = \frac{1}{\Gamma\left(\frac{\lambda}{2}\right)}\int_0^\infty du\, u^{\frac{\lambda}{2}-1}\exp(-u\rho^2)$$

together with Fubini's theorem allow to write (15) in the equivalent form:

$$\frac{1}{\Gamma\left(\frac{\lambda}{2}\right)}\int_0^\infty du\, u^{\frac{\lambda}{2}-1}E_1^{(\lambda)}(\exp(-uR_{\sigma_s}^2)) = \exp(-\lambda\ell). \tag{16}$$

Another proof of (16)

To prove that

$$\frac{1}{\Gamma(\frac{\lambda}{2})}\int_0^\infty du\, u^{\frac{\lambda}{2}-1}E\left[\frac{1}{(1+2u\sigma_s)^{1+\lambda}}\exp\left(-\frac{u}{1+2u\sigma_s}\right)\right] \;=\; \exp(-\lambda\ell), \tag{16'}$$

we first let $v = 2u\sigma_s/(1+2u\sigma_s)$ to rewrite the left-hand side as

$$\frac{1}{\Gamma(\frac{\lambda}{2})}\int_0^1 dv\, v^{\frac{\lambda}{2}-1}(1-v)^{\frac{\lambda}{2}}E(2\sigma_s)^{-\frac{\lambda}{2}}e^{-\frac{v}{2\sigma_s}}.$$

We know that $s^2/2\sigma_s \overset{d}{=} \gamma_{\frac{1}{2}}$, and thus the last expression becomes

$$\frac{\Gamma(\frac{\lambda+1}{2})s^{-\lambda}}{\Gamma(\frac{\lambda}{2})\Gamma(\frac{1}{2})}\int_0^1 dv\, v^{\frac{\lambda}{2}-1}(1-v)^{\frac{\lambda}{2}}\left(1+\tfrac{v}{s^2}\right)^{-\frac{\lambda+1}{2}}$$

$$= \frac{\Gamma(\frac{\lambda}{2}+1)\Gamma(\frac{\lambda+1}{2})s^{-\lambda}}{\Gamma(\lambda+1)\Gamma(\frac{1}{2})}\,{}_2F_1(\tfrac{\lambda}{2}, \tfrac{\lambda+1}{2}, \lambda+1; -s^{-2}).$$

Using formula (9.8.3) on p.259 of [Leb] and the duplication formula for the gamma function, this is equal to

$$\frac{\Gamma(\frac{\lambda}{2}+1)\Gamma(\frac{\lambda+1}{2})s^{-\lambda}}{\Gamma(\lambda+1)\Gamma(\frac{1}{2})}\left(\frac{1+\sqrt{1+s^{-2}}}{2}\right)^{-\lambda} \;=\; (s+\sqrt{s^2+1})^{-\lambda}$$

$$= \exp(-\lambda \arg\sinh(s)).$$

With the help of (14), we see that (16) is equivalent to:

$$\frac{1}{\Gamma\left(\frac{\lambda}{2}\right)}\int_0^\infty du\, u^{\frac{\lambda}{2}-1} E\left[\frac{1}{(1+2u\sigma_s)^{1+\lambda}}\exp\left(-\frac{u}{(1+2u\sigma_s)}\right)\right] = \exp(-\lambda\ell).$$

Using the classical result: $P(\sigma_s \in dt) = \frac{se^{-\frac{s^2}{2t}}}{\sqrt{2\pi t^3}}dt$, the previous identity is equivalent to:

(17)

$$\int_0^\infty du\, u^{\frac{\lambda}{2}-1}\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\frac{se^{-\frac{s^2}{2t}}}{(1+2ut)^{1+\lambda}}\exp\left(-\frac{u}{1+2ut}\right) = \Gamma\left(\frac{\lambda}{2}\right)\exp(-\lambda\ell).$$

We now consider each of the sides of (17) as a Mellin transform in λ; indeed:

- the LHS of (17) is:

$$\int_0^\infty \frac{du}{u}\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\frac{se^{-\frac{s^2}{2t}}}{(1+2ut)}\left(\frac{\sqrt{u}}{1+2ut}\right)^\lambda \exp\left(-\frac{u}{1+2ut}\right);$$

- the RHS of (17) is, after some elementary change of variables:

$$2\int_0^\infty \frac{dz}{z} z^\lambda \exp(-(ze^\ell)^2).$$

Thus, due to the injectivity of the Mellin transform, (17) holds iff for any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \geq 0$, one has:

$$\int_0^\infty \frac{du}{u}\int_0^\infty \frac{dt}{\sqrt{2\pi t^3}}\frac{se^{-\frac{s^2}{2t}}}{(1+2ut)}\varphi\left(\frac{\sqrt{u}}{1+2ut}\right)\exp\left(-\frac{u}{1+2ut}\right)$$

$$= 2\int_0^\infty \frac{dz}{z}\varphi(z)\exp(-(ze^\ell)^2). \qquad (18)$$

(2.3) Proof of (18)

On the LHS we make the change of variables: $\left\{t \to z = \frac{\sqrt{u}}{1+2ut}\right\}$, which yields:

$$\frac{dz}{z} = \frac{(2u)dt}{(1+2ut)}; \quad \frac{1}{2t} = \frac{uz}{\sqrt{u}-z}, \quad \text{hence } t = \frac{1}{2u}\left(\frac{\sqrt{u}}{z}-1\right).$$

This allows to transform (18) into:

$$\int_0^\infty \frac{du}{\sqrt{2u}} \int_0^{\sqrt{u}} \frac{dz}{z}\varphi(z) \frac{e^{-z\sqrt{u}}}{\sqrt{2\pi\left(\frac{\sqrt{u}-z}{z}\right)^3}} s e^{-s^2\left(\frac{uz}{\sqrt{u}-z}\right)}$$

$$= 2\int_0^\infty \frac{dz}{z}\varphi(z)\exp(-(ze^\ell)^2).$$

Since this equality must be true for all functions φ, it is equivalent to:

$$\int_{z^2}^\infty \frac{du}{\sqrt{2u}} \frac{z^{3/2}}{\sqrt{2\pi(u-z)^3}} s\exp\left(-s^2\left(\frac{uz}{\sqrt{u}-z}\right)\right)\exp(-z\sqrt{t})$$

$$= 2\exp(-z^2e^{2\ell}). \tag{19}$$

We make the elementary change of variables: $u = v^2$, then: $v = w + z$; (19) is then transformed into:

$$e^{-z^2}\int_0^\infty \frac{z^{3/2}e^{-zw}}{\sqrt{\pi w^3}} s \underbrace{\exp\left(-s^2\left(\frac{(w+z)^2 z}{w}\right)\right)} = 2\exp(-z^2e^{2\ell}). \tag{20}$$

The underlined exponential writes:

$$\exp\left(-s^2 z\left\{\frac{w^2+2wz+z^2}{w}\right\}\right) = \exp(-2s^2z^2)\exp\left(-s^2z\left(w+\frac{z^2}{w}\right)\right).$$

Thus, there appears, outside the integral on the LHS of (20), the quantity:

$$\exp(-z^2 - 2s^2z^2),$$

which we send to the RHS. Thus, we obtain the new identity:

$$\int_0^\infty \frac{dw z^{3/2}}{\sqrt{\pi w^3}}\exp(-zw)s\exp\left(-(s^2z)\left(w+\frac{z^2}{w}\right)\right)$$

$$= 2\exp(z^2[2s^2 - e^{2\ell} + 1]). \tag{21}$$

The exponent on the RHS is:

$$2s^2 - e^{2\ell} + 1 = \frac{1}{2}(e^\ell - e^{-\ell})^2 - e^{2\ell} + 1 = -\sinh(2\ell).$$

On the other hand, making on the LHS of (21) the change of variables: $w = zh$, we obtain:

$$\int_0^\infty dh(zs)\frac{e^{-z^2h}}{\sqrt{\pi h^3}}\exp\left(-(s^2z^2)\left(h+\frac{1}{h}\right)\right) = 2\exp(-z^2\sinh(2\ell)). \tag{22}$$

Letting $h = 2k$ on the LHS of (22), we obtain:

$$\int_0^\infty \frac{(dk)(zs)}{\sqrt{2\pi k^3}} \exp(-2z^2 k) \exp(-s^2 z^2 (2k)) \exp\left(-\frac{s^2 z^2}{2k}\right)$$

$$= \exp(-z^2 \sinh(2\ell)). \tag{23}$$

This identity may be rewritten "purely" in terms of $s = \sinh(\ell)$ (recall that $\cosh(\ell) = \sqrt{1+s^2}$). Thus, we need to show:

(24)
$$\int_0^\infty \frac{(dk)(zs)}{\sqrt{2\pi k^3}} \exp(-2z^2 k(s^2+1)) \exp\left(-\frac{s^2 z^2}{2k}\right) = \exp(-z^2(2s)\sqrt{1+s^2}).$$

Now, the LHS equals, with $a = zs$:

$$E\left[\exp\left(-\frac{4z^2\cosh^2(\ell)}{2}\sigma_a\right)\right],$$

which is (well known to be) equal to:

$$\exp(-2z(\cosh\ell)a) = \exp(-2z^2 s\sqrt{1+s^2}).$$

Thus, (24) has finally been proven; hence, we have proven (8) (that is, Theorem 2) along lines very similar to the proof of (25) in [Y1].

3 Two Proofs of Theorem 1

(3.1) Some basic ingredients and facts
In this subsection, for $\theta > 0$, S_θ denotes an exponentially distributed random variable with parameter θ, independent from the underlying Brownian motion $(B_t, t \geq 0)$. Then:
a) for $t \geq 0$, denote $g_t = \sup\{u < t : B_u = 0\}$ the last zero of B before t. Then, the processes $(B_u, u \leq g_{S_\theta})$ and $(B_{g_{S_\theta}+u}, u \leq S_\theta - g_{S_\theta})$ are independent. As a consequence, $L_{S_\theta} (\equiv L_{g_{S_\theta}})$ and B_{S_θ} are independent; furthermore, the two variables

$$L_{S_\theta} \quad \text{and} \quad |B_{S_\theta}|$$

are identically distributed on $\mathbb{R}_+$, with common density:

$$\sqrt{2\theta}\exp(-\sqrt{2\theta}u)$$

Proof : $\quad P(L_{S_\theta} \geq \ell) = P(S_\theta \geq \tau_\ell) = E[\exp(-\theta\tau_\ell)] = \exp(-\ell\sqrt{2\theta}),$

while the identity in law: $L_t \overset{\text{(law)}}{=} |B_t|$, for fixed t, implies: $L_{S_\theta} \overset{\text{(law)}}{=} |B_{S_\theta}|$.
b) The main result in Yor [Y1], p. 94, (see e.g. [D1, Y3] for a summary) is:

$$A_{S_\theta}^{(\nu)} \overset{\text{(law)}}{=} \frac{\beta_{1,a}}{2\gamma_b}, \tag{25}$$

where

- $\beta_{u,v}$ is a beta variable, with parameters u and v,
- γ_b is a gamma variable, with parameter b,

The variables $\beta_{1,a}$ and γ_b are independent, and:

$$\begin{aligned} a &\equiv a(\nu,\theta) = \frac{1}{2}(\nu + \sqrt{2\theta+\nu^2}) \\ b &\equiv b(\nu,\theta) = \frac{1}{2}(-\nu + \sqrt{2\theta+\nu^2}). \end{aligned}$$

In particular, this yields an expression for the Mellin transform of the law of $A_{S_\theta}^{(\nu)}$ in terms of the gamma function:

$$E[(A_{S_\theta}^{(\nu)})^p] = 2^{-p}\,\frac{\Gamma(1+p)\Gamma(1+a)\Gamma(b-p)}{\Gamma(1+p+a)\Gamma(b)}. \tag{26}$$

c) Finally, we note the elementary fact:

$$P(S_\theta \in ds\,;\ \exp(-\eta S_\theta)) = \frac{\theta}{\theta+\eta} P(S_{\theta+\eta} \in ds), \tag{27}$$

which will be useful later.

(3.2) <u>An asymmetric proof of Theorem 1</u>
(i) In order to prove the identity in law between (1) and (2) (see Theorem 1), or, rather, between (4) and (5), it is clearly equivalent to prove that, for $S \equiv S_1$ a standard exponential variable:

$$\left(\sqrt{2S}\sinh(|B_t|)\,,\ \frac{\sinh(|B_t|)}{\sinh(L_t)}\right) \overset{\text{(law)}}{=} \left(\sqrt{2S}|\beta_{A_t}|\,,\ \frac{|\beta_{A_t}|}{\lambda_{A_t}}\exp(B_t)\right). \tag{28}$$

Now, the RHS of (28) is equal, in law, to:

$$\left(\sqrt{2S}\sqrt{A_t}|\beta_1|\,,\ \frac{|\beta_1|}{\lambda_1}\exp(B_t)\right) \overset{\text{(law)}}{=} \left(\sqrt{2A_t}|\beta_S|\,,\ \frac{|\beta_S|}{\lambda_S}\exp(B_t)\right).$$

In order to work easily with the latter expression, we shall use that, from **(3.1)** a):

$$\left(\sqrt{2}|\beta_S|\,,\ \frac{|\beta_S|}{\lambda_S}\right) \stackrel{\text{(law)}}{=} \left(\mathbf{e}\,,\ \frac{\mathbf{e}}{\mathbf{e}'}\right),$$

where $\mathbf{e}$ and $\mathbf{e}'$ are two standard, independent exponentials. Thus, the proof of Theorem 1 is now reduced to showing the identity in law:

$$\left(\sqrt{2\mathbf{e}}\sinh(|B_t|)\,,\ \frac{\sinh(|B_t|)}{\sinh(L_t)}\right) \stackrel{\text{(law)}}{=} \left(\mathbf{e}\sqrt{A_t}\,,\ \exp(B_t)\frac{\mathbf{e}}{\mathbf{e}'}\right). \tag{29}$$

(ii) With the help of the "classical" Bougerol identity in law, and using again **(3.1)** a), but this time for B, we see that proving (29) is equivalent to proving, for any $\theta > 0$, the identity in law:

$$\left(\sqrt{2\mathbf{e}}|N|\sqrt{A_{S_\theta}}\,,\ \frac{|N|\sqrt{A_{S_\theta}}}{|N'|\sqrt{A'_{S'_\theta}}}\right) \stackrel{\text{(law)}}{=} \left(\mathbf{e}\sqrt{A_{S_\theta}}\,,\ \exp(B_{S_\theta})\frac{\mathbf{e}}{\mathbf{e}'}\right), \tag{30}$$

where N and N' are two independent Gaussian variables with mean 0, variance 1, and independent from all other random elements.
(iii) In fact, we shall show (30) via the following (equivalent) identity: for any $f : \mathbb{R}_+ \to \mathbb{R}_+$, Borel, and any $\nu \in (-1, 1)$,
(31)

$$E[f(\sqrt{2\mathbf{e}}|N|\sqrt{A_{S_\theta}})(|N|\sqrt{A_{S_\theta}})^\nu]C_{\nu,\lambda} = E\left[f(\mathbf{e}\sqrt{A_{S_\theta}})\exp(\nu B_{S_\theta})\left(\frac{\mathbf{e}}{\mathbf{e}'}\right)^\nu\right],$$

where:

$$C_{\nu,\lambda} = E[(|N|\sqrt{A_{S_\theta}})^{-\nu}].$$

(To go from (30) to (31), we have composed the first components of the 2 sides in (30), and raised the second components to the power ν.)
(iv) In the sequel, universal constants $C'_{\nu,\lambda}$, $C''_{\nu,\lambda}$ depending on ν and θ shall keep appearing; however, we do not need to compute them precisely, since they only appear as normalizing constants.
For instance, the RHS of (31) is (here, we use both Cameron-Martin and the fact c) in Subsection (**3.1**)) equal to:

$$E\left[f\left(\gamma_{1+\nu}\sqrt{A^{(\nu)}_{S_{\theta-\frac{\nu^2}{2}}}}\right)\right]C'_{\nu,\lambda}\,, \tag{32}$$

while using the result on the distribution of $A^{(\nu)}_{S_\eta}$, for general ν and η, the expression (32) is found to be equal to:

$$E\left[f\left(\gamma_{1+\nu}\left(\frac{\beta_{1,\frac{1}{2}(\sqrt{2\theta}+\nu)}}{2\gamma'_{\frac{1}{2}(\sqrt{2\theta}-\nu)}}\right)^{1/2}\right)\right]C'_{\nu,\lambda}. \tag{33}$$

On the other hand, the LHS of (31) is equal to:

$$E\left[f\left(\sqrt{2\mathbf{e}}\sqrt{2\gamma_{\frac{1+\nu}{2}}}\left(\frac{\beta_{1+\frac{\nu}{2},\frac{1}{2}(\sqrt{2\theta})}}{2\gamma'_{\frac{1}{2}(\sqrt{2\theta}-\nu)}}\right)^{1/2}\right)\right]C''_{\nu,\lambda}. \tag{34}$$

(v) Thus, comparing (33) and (34), the theorem will be proven once we have shown:

$$2\left(\mathbf{e}\gamma_{\frac{1+\nu}{2}}\right)^{1/2}(\beta_{1+\frac{\nu}{2},\frac{1}{2}(\sqrt{2\theta})})^{1/2} \overset{\text{(law)}}{=} \gamma_{1+\nu}(\beta_{1,\frac{1}{2}(\sqrt{2\theta}+\nu)})^{1/2}$$

or, equivalently, squaring both sides:

$$4\mathbf{e}\gamma_{\frac{1+\nu}{2}}\beta_{1+\frac{\nu}{2},\frac{1}{2}(\sqrt{2\theta})} \overset{\text{(law)}}{=} \gamma^2_{1+\nu}\beta_{1,\frac{1}{2}(\sqrt{2\theta}+\nu)}. \tag{35}$$

In order to prove this last identity in law, we multiply both sides by an independent $\gamma_{1+\frac{1}{2}(\nu+\sqrt{2\theta})}$ variable. From the beta-gamma algebra, we now see that proving the identity in law (35) is equivalent to prove:

$$4\mathbf{e}\gamma_{\frac{1+\nu}{2}}\gamma_{1+\frac{\nu}{2}} \overset{\text{(law)}}{=} \gamma^2_{1+\nu}\mathbf{e},$$

which, itself is equivalent to:

$$4\gamma_{\frac{1+\nu}{2}}\gamma_{1+\frac{\nu}{2}} \overset{\text{(law)}}{=} \gamma^2_{1+\nu}. \tag{36}$$

As explained in Chaumont-Yor ([CY], Exercise 4.5) and in the Appendix of [D2], the identity in law (36) may be thought of as a probabilistic translation of the duplication formula for the gamma function. Theorem 1 has now been completely proven.

(3.3) A symmetric proof of Theorem 1

(3.3.1) This second proof of Theorem 1 consists in showing:

$$(\sinh(|B_t|), \sinh(L_t)) \overset{\text{(law)}}{=} (\exp(-B_t)\sqrt{A_t}|\beta_1|, \sqrt{A_t}\lambda_1). \tag{37}$$

Multiplying both sides by an independent $\sqrt{2\mathbf{e}}$, it is equivalent to show:

$$\sqrt{2\mathbf{e}}(\sinh(|B_t|), \sinh(L_t)) \overset{\text{(law)}}{=} (\exp(-B_t)\sqrt{A_t}\mathbf{e}, \sqrt{A_t}\mathbf{e}'). \tag{38}$$

Furthermore, by a Laplace transform argument, proving (38) is equivalent to proving that for any independent exponential time S_θ:

$$\sqrt{2\mathbf{e}}(\sinh(|B_{S_\theta}|), \sinh(L_{S_\theta})) \overset{\text{(law)}}{=} (\exp(-B_{S_\theta})\sqrt{A_{S_\theta}}\mathbf{e}, \sqrt{A_{S_\theta}}\mathbf{e}'). \tag{39}$$

Now, the variables $|B_{S_\theta}|$ and L_{S_θ} are iid (see Subsection 3.1), and:

$$\sinh(|B_{S_\theta}|) \stackrel{\text{(law)}}{=} \sinh(L_{S_\theta}) \stackrel{\text{(law)}}{=} |N|\sqrt{A_{S_\theta}},$$

where we have made use of Bougerol's (one-dimensional, original) identity! Thus, the LHS of (39) is distributed as

$$\sqrt{2\mathbf{e}}(|N|\sqrt{A_{S_\theta}}, |N'|\sqrt{A'_{S'_\theta}}),$$

primes indicating an independent copy of a non-primed quantity. Hence, proving (39) has been now shown to be equivalent to the proof of:

$$\sqrt{2\mathbf{e}}(|N|\sqrt{A_{S_\theta}}, |N'|\sqrt{A'_{S'_\theta}}) \stackrel{\text{(law)}}{=} (\exp(-B_{S_\theta})\sqrt{A_{S_\theta}}\mathbf{e}, \sqrt{A_{S_\theta}}\mathbf{e}'). \tag{40}$$

It is this identity in law which we shall prove by showing the identity of the mixed ($2c$; $2d$) moments of both sides.
We now present the remaining identity (still to be shown, to complete this second proof) in the form of a Proposition.

Proposition 3. *The identity in law (40) is equivalent to the following identity: for every $c, d \geq 0$,*

$$\Gamma(1+c+d)E[A^c_{S_\theta}]E[A^d_{S_\theta}] = \Gamma(1+c)\Gamma(1+d)E[(A_{S_\theta})^{c+d}\exp(-2cB_{S_\theta})]. \tag{41}$$

Remark: The LHS of (41) obviously defines a symmetric function of (c, d); thus, we may check that the RHS of (41) is also symmetric in (c, d). Indeed, this easily follows from the stability of Brownian motion by time reversal, *i.e.*: $(\widehat{B}_u = B_t - B_{(t-u)}, u \leq t)$, is distributed as $(B_u, u \leq t)$.

Proof of Proposition 3: The ($2c$; $2d$) mixed moment of the LHS of (40) is equal to:

$$2^{c+d}\Gamma(1+c+d)E(|N|^{2c})E(|N|^{2d})E[(A_{S_\theta})^c]E[(A_{S_\theta})^d].$$

Now, recall that $N^2 \stackrel{\text{(law)}}{=} 2\gamma_{1/2}$, hence:

$$E(|N|^{2c}) = 2^c E[(\gamma_{1/2})^c] = 2^c \frac{1}{\Gamma\left(\frac{1}{2}\right)}\Gamma\left(\frac{1}{2}+c\right).$$

On the other hand, the ($2c$, $2d$) mixed moment of the RHS of (40) is (applying the Cameron-Martin formula, (27) and finally (26)):

$$\begin{aligned}
&\Gamma(1+2c)\Gamma(1+2d)E[(A_{S_\theta})^{c+d}e^{-2cB_{S_\theta}}] \\
&\quad = \Gamma(1+2c)\Gamma(1+2d)E[(A^{(-2c)}_{S_\theta})^{c+d}e^{2c^2 B_{S_\theta}}] \\
&\quad = \Gamma(1+2c)\Gamma(1+2d)E(A^{(-2c)}_{S_{\theta-2c^2}})^{c+d} \\
&\quad = \frac{2^{-c-d}\theta}{\theta-2c^2}\Gamma(1+2c)\Gamma(1+2d)\frac{\Gamma(1+c+d)\Gamma(1-c+\frac{1}{2}\sqrt{2\theta})\Gamma(-d+\frac{1}{2}\sqrt{2\theta})}{\Gamma(1+d+\frac{1}{2}\sqrt{2\theta})\Gamma(c+\frac{1}{2}\sqrt{2\theta})}.
\end{aligned}$$

The LHS of (40) is similarly found to be:

$$2^{c+d}\Gamma(1+c+d)\frac{\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2}+d)}{\Gamma(\frac{1}{2})^2}\frac{\Gamma(1+c)\Gamma(1+\frac{1}{2}\sqrt{2\theta})\Gamma(-c+\frac{1}{2}\sqrt{2\theta})}{\Gamma(1+c+\frac{1}{2}\sqrt{2\theta})\Gamma(\frac{1}{2}\sqrt{2\theta})}$$
$$\times\frac{\Gamma(1+d)\Gamma(1+\frac{1}{2}\sqrt{2\theta})\Gamma(-d+\frac{1}{2}\sqrt{2\theta})}{\Gamma(1+d+\frac{1}{2}\sqrt{2\theta})\Gamma(\frac{1}{2}\sqrt{2\theta})}.$$

The equality between the RHS ad LHS of (40) is obtained from two elementary properties of the gamma function, namely the recurrence $\Gamma(x+1) = x\Gamma(x)$, and the duplication formula

$$\Gamma(1+2\alpha) = 2^{2\alpha}\Gamma(1+\alpha)\frac{\Gamma\left(\frac{1}{2}+\alpha\right)}{\sqrt{\pi}},$$

which we apply for $\alpha = c$ and $\alpha = d$. More details are given in [BDY]. ■

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