COBWEB THEOREMS WITH PRODUCTION LAGS AND PRICE FORECASTING

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ABSTRACT. The classical cobweb theorem is extended to include production lags and price forecasts. Price forecasting based on a longer period has a stabilizing effect on prices. Longer production lags do not necessarily lead to unstable prices; very long lags lead to cycles of constant amplitude. The classical cobweb requires elasticity of demand to be greater than that of supply; this is not necessarily the case in a more general setting, price forcasting has a stabilizing effect. Random shocks are also considered.

Keywords: Cobweb theorem; production lags; stable markets; price fluctuations

1. INTRODUCTION

"But surely the cob-web cycle is an oversimplification of reality" (Samuelson [26], p.4). Many other famous and less famous economists must have expressed the same opinion over the past decades. We propose to bring the cobweb model at least a little closer to reality by introducing production lags and price forecasts into it. The way equilibrium is reached in a theoretical model should then be better understood. In particular, we ask whether production lags cause intstability of prices, and whether the classical condition for a cobweb to lead to equilibrium (that elasticity of demand be greater than elasticity of supply) still holds in a more general model[1].

There is a considerable literature on business or economic cycles. The cobweb theorem has a long history, see Ezekiel [11]. We follow Chapter 2 of van Doorn [8] in stating its classical form. The following assumptions are made:

- (A1) supply depends only on the price forecast;
- (A2) actual market price adjusts to demand, so as to eliminate excess demand instantaneously in the trading period;
- (A3) price forecast equals most recent observed price, and
- (A4) there are no inventories, and neither buyers nor sellers have an incentive to speculate.

Let P_t be the market price for a unit of commodity at time t. The quantity demanded at period t, denoted by Q_t^d is given by

$$Q_t^d = a_0 - a_1 P_t = D(P_t),$$

while the quantity supplied is

$$Q_t^s = b_0 + b_1 \hat{P}_t = S(\hat{P}_t),$$

where $\hat{P}_t$ is the price forecast (the result of forecasting at time $t-1$). The conditions $a_1, b_1 > 0$ ensure that quantity demanded decreases and quantity supplied increases as functions of price. The assumptions stated above mean that

$$D(P_t) = S(\hat{P}_t) \ \text{ and } \ \hat{P}_t = P_{t-1}.$$

Making the substitutions, it is seen that the price sequence follows

$$P_t = -\frac{b_1}{a_1} P_{t-1} + \frac{a_0 - b_0}{a_1}. \tag{1}$$

[1] We take the demand elasticity to be a positive (absolute) value

In the classical cobweb model the solution to (1) is then

$$P_t = (P_0 - P^*)\left(-\frac{b_1}{a_1}\right)^t + P^*, \tag{2}$$

where P^* is the *equilibrium* solution to (1), that is,

$$P^* = -\frac{b_1}{a_1}P^* + \frac{a_0 - b_0}{a_1}.$$

The sequence defined in (2) converges to P^* if and only if $b_1 < a_1$. In words, the condition for convergence is that minus the slope of demand as a function of price be larger than the slope of supply as a funtion of price. Usually, economic modelling assumes that the market is initially in equilibrium at P^*, and that an exogenous disturbance results in $P_0 \neq P^*$. An important question is whether such disturbances persist or die out; this can be answered by studying the conditions for convergence of P_t to P^*.

We have in mind the prices of metals, for instance copper, which have greatly fluctuated and shown some appearance of cycles over time. An important aspect of mining is the lag between the time the decision to increase or decrease production is made and the time the decision actually takes effect in the market. It takes several years for a planned new mine to start producing, and this has made some believe that the lags themselves may be a main cause of price fluctuations. In this paper, we study models that include production lags as well as price forecasting, the latter based on current and past prices. Stability means that prices converge to an equilibrium as time passes. Random disturbances will also be included; in those models "stability" will mean that prices have a limit probability distribution (in other words that the price process has a stationary limit).

Stability is not synonymous with absence of fluctuations, but it is a property of markets in which price fluctuations tend to dampen over time. By varying the parameters, one can get an idea of what may generate fluctuations. Is it production lags? Is it how prices are forecast? We look at those questions in the following sections.

Given prices $P_t, P_{t-1}, \ldots$, let $\hat{P}_{t+\ell}$ be the price forecast used by producers to establish production at time $t + \ell$. (A more explicit notation would be $\hat{P}_{t,t+\ell}$, but we will use the simpler $\hat{P}_{t+\ell}$, as the lag ℓ will be fixed.) The classical cobweb theorem has a lag of one time unit.

The market clearing condition is now

$$D(P_{t+\ell}) = S(\hat{P}_{t+\ell}). \tag{3}$$

This leads us to study the dynamical system

$$P_{t+\ell} \;=\; D^{-1} \circ S(\hat{P}_{t+\ell}). \tag{4}$$

Important questions are under what conditions this recursion has an equilibrium point, and if so whether P_t converges to it when $t \to \infty$. Once again, such convergence does not exclude fluctuations, but it does say that perturbations are damped by the system over time.

The classical description of the cobweb theorem (such as the one we gave above) assumes that the supply and demand functions are linear. We will assume that the demand and supply functions are respectively

$$D(p) \;=\; p^{-d}, \quad S(p) \;=\; p^s \;, \quad d, s > 0. \tag{5}$$

One advantage of these functions is that price and production always remain positive, while linear functions may lead to negative prices (see (2)). Tractability is achieved by using the logarithm of prices, as will be shown below. It will be seen that a very important quantity is the ratio of the elasticities of supply and demand, which we denote $c = s/d$.

Let $\ell \in \{1, 2, \dots\}$ be the production lag. We retain assumptions A1, A2 and A4, but replace (A3) with

(A3′) the price forecast $\hat{P}_{t+\ell}$ is a weighted geometric average of $(P_{t-m}, \dots, P_t)$ for some $m \in \{0, 1, 2, \dots\}$.

(*N.B.* In the sequel "ℓ" will always stand for lag, and m for memory.)

This means

$$\log \hat{P}_{t+\ell} = \sum_{j=0}^{m} \alpha_j \log P_{t-j},$$

where the weights α_j add up to 1. Letting $\pi_t = \log P_t$, $\hat{\pi}_t = \log \hat{P}_t$, the log-price forecast is given by

$$\hat{\pi}_{t+\ell} = \sum_{j=0}^{m} \alpha_j \, \pi_{t-j} \,, \quad \text{where} \sum_{j=0}^{m} \alpha_j = 1. \tag{6}$$

A moving average model has all weights non-negative. Several forecasting schemes have been identified in the literature, though always for $\ell = 1$ (see [29] for a brief description). *Static expectations* refers to $\hat{\pi}_{t+1} = \pi_t$, or $m = 0$; *extrapolative expectations* means $m = 1$ and $\alpha_0 > 1$, as

$$\alpha_0 \pi_t + \alpha_1 \pi_{t-1} = \pi_t + (\alpha_0 1)\pi_t + (1 - \alpha_0)\pi_{t-1} = \pi_t + (\alpha_0 - 1)(\pi_t - \imath_{t-1}).$$

Adaptive expectations refers to

$$\hat{\pi}_t = \lambda \hat{\pi}_{t-1} + (1 - \lambda)\pi_t.$$

This is the limit case when $m \to \infty$ and α_m is proportional to λ^m, as we explain in Section 2.7. We call this *exponential smoothing*. Note that all these schemes are applied to log-price, this is what makes the model tractable.

Van Doorn [8] attributes to Hicks the use of the logarithm of the price rather than the price itself, in the context of a one-lag model or one with "distributed lags", but the study of such systems is not carried out mathematically in [8].

Under (5), the market clearing equation (3) reads

$$P_{t+\ell} = (\hat{P}_{t+\ell})^{-s/d}. \tag{7}$$

From (6) the sequence π_t satisfies

$$\pi_{t+\ell} = -c \sum_{j=0}^{m} \alpha_j \pi_{t-j} \,, \quad c = s/d. \tag{8}$$

Since $c \sum_j \alpha_j > 0$, the unique equilibrium point is $\pi^* = 0$ or, equivalently, $P^* = 1$. The solution π_t of (8) has the general form (see [12])

$$\pi_t = \sum_{j=1}^{\ell+m} b_j \, x_j^t,$$

where $(x_1, \dots, x_{\ell+m})$ are the zeroes of the *characteristic polynomial*

$$h_{\ell,m}(x) = x^{\ell+m} + c \sum_{j=0}^{m} \alpha_j x^{m-j}. \tag{9}$$

The constants $b_j; j = 1, \dots, \ell + m$ may be found from the initial conditions for π_t.

Definition 1. *The system (8) is said to be* stable *if* $\lim_{t\to\infty} \pi_t = \pi^* = 0$, *given any initial conditions. Otherwise it is* unstable.

If the characteristic polynomial (9) has complex zeroes then π_t has oscillatory components, which means that the sequence may fluctuate around π^* even though it eventually converges to π^*. This will happen frequently in our examples.The magnitude of the zeroes will determine whether the system is stable or not; if all the zeroes of the characteristic polynomial have norm (modulus) strictly less than one then the system is stable; if at least one zero has norm greater than or equal to one then the system is unstable. If the zero or zeroes with largest norm have norm precisely equal to 1 then there will be oscillations with constant amplitude, at least for some initial conditions.

The classical form of the cobweb model has $\ell = 1$, $m = 0$ and linear supply and demand functions. In our setting, the cobweb model with $\ell = 1$ and $m = 0$ becomes

$$P_t = (\hat{P}_t)^{-s/d} = (P_{t-1})^{-s/d},$$

which gives

$$\pi_t = -\left(\frac{s}{d}\right)\pi_{t-1} = \left(-\frac{s}{d}\right)^t \pi_0.$$

In the classical cobweb model the market is stable if, and only if, $s < d$; in other words, stability occurs if and only if, the elasticity of supply is smaller than the elasticity of demand. When this is the case, $\pi_t \to \pi^* = 0$. This necessary and sufficient condition for stability, $s < d$, is remarkably simple and easy to interpret. We will see that when lags and price forecasts are introduced the conditions for stablility are no longer so simple.

Chiarella [6] studies a system where expected prices follow adaptive expectations, when the demand curve is linear, while the supply curve is non-linear (with a single point of inflexion, convex to the left, and concave to the right, “a fairly general non-linear S-shaped supply function”, [6], p.383). He then shows that the system is either (1) stable, (2) unstable but cyclical, or (3) chaotic. These are very interesting results, but we follow a different route.

The paper is organised as follows. Section 2 studies the deterministic models in some detail, mostly numerically, although some simple results are proved mathematically. This section avoids the generality of Section 3 but focuses instead on experiments that lead to interesting patterns and related questions, which will be studied more deeply in subsequent sections.

In Section 3 we derive general results on deterministic models trying to answer the questions raised in Section 2. The models are represented by linear difference equations; stability is determined by the study of the roots of the characteristic polynomial (9) of those difference equations. An important tool is Rouché’s Theorem, see below. Our results seem to contradict the view that production lags, by themselves, cause instabilities.

In Section 4 we incorporate randomness into (8), in the form of additive and multiplicative noise. This leads to the question of stability of products of random matrices, a topic that so far belonged more in physical chaos theory than in economics.

Notation. The set of complex numbers (or “complex plane”) is denoted $\mathbb{C}$. The norm (or modulus, or absolute value) of $z = x + iy$ (x, y real) is $|z| = \sqrt{x^2 + y^2}$, its conjugate is $\overline{z} = x - iy$. The circle with centre z and radius ρ in $\mathbb{C}$ is denoted $C_{z,\rho}$; the open disk (or “ball”) with centre z and radius ρ is denoted $B_{z,\rho}$; the closed disk is denoted $\overline{B}_{z,\rho}$.

We will use Rouché’s Theorem from complex analysis: if ϕ, ψ are analytic on and inside a closed contour L, and $|\phi(z)| > |\psi(z)|$ for $z \in L$, then ϕ and $\phi + \psi$ have the same number of zeroes inside L. Here is a first application: whatever the averaging period m and the delay ℓ, there will be instability if the ratio of elasticities $c = s/d$ is large enough.

Theorem 1. *Let $\ell \geq 1$, $m \geq 0$. There exists $0 < c_0 < \infty$ such that for all $c \geq c_0$ the system defined by (8) is unstable.*

Proof. Let

$$g(z) \;=\; \sum_{j=0}^{m} \alpha_j z^{m-j},$$

and thus $h_{\ell,m}(z) = z^{\ell+m} + cg(z)$. We show that at least one zero of

$$\frac{z^{\ell+m}}{c} + g(z)$$

is outside $C_{0,\rho}$, for some $\rho \geq 1$. There exists $\rho \geq 1$ such that $|g(z)| \geq \epsilon > 0$ for all $z \in C_{0,\rho}$. Therefore

$$|g(z)| \;>\; \frac{|z|^{\ell+m}}{c}$$

on $C_{0,\rho}$ for all c large enough. Apply Rouché's theorem with $\phi(z) = g(z)$ and $\psi(z) = z^{\ell+m}/c$. Then $\phi(z) + \psi(z)$ has the same number of zeroes inside $C_{0,\rho}$ as $g(z)$, that is, at most m. That leaves at least ℓ zeroes on or outside $C_{0,\rho}$, implying instability. □

Remark. A superficially more general version of our model would have supply and demand functions

$$D(p) \;=\; k_d\, p^{-d}, \quad S(p) \;=\; k_s\, p^s\,, \quad d, s > 0. \tag{10}$$

Consider the change of variables

$$p \;=\; \tau\tilde{p},\; \tau \;=\; \left(\frac{k_d}{k_s}\right)^{\frac{1}{d+s}},\; D(p) \;=\; \sigma\tilde{D}(\tilde{p}),\; S(p) \;=\; \sigma\tilde{S}(\tilde{p}), \sigma \;=\; k_d^{\frac{s}{d+s}}\, k_s^{\frac{d}{d+s}}; \tag{11}$$

this is a change in currency together with a change in units. It can be verified that

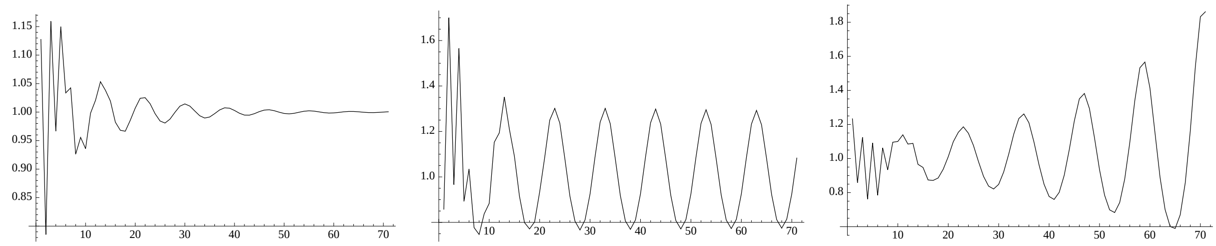
$$\tilde{D}(\tilde{p}) \;=\; \tilde{p}^{-d},\; \tilde{S}(\tilde{p}) =\; \tilde{p}^s.$$

There is thus no greater generality in (10) than in (5).

2. Deterministic Models: Numerical Examples

In this section we present numerical experiments that illustrate the influence of the parameters α, c, ℓ and m on stability. The patterns observed here will motivate the more general (and mathematical) analysis in Section 3. In all the examples we choose erratic initial conditions $\pi_t = (-10)^t \sin(1/(t+3)), t = 0, \ldots, \ell+m$.

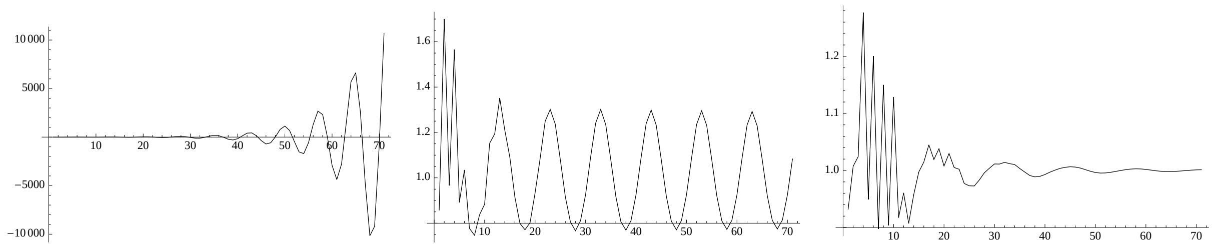
Figure 1 shows the price π_t as a function of time t when $c = 1.7, m = 5, \alpha = (.2, .2, .2, .2, .1, .1)$, and the delay ℓ takes one of the values $\ell = 2, 3, 4$. Notice that $c = 1.7$ yields instabilities with $\ell = 1$ (this is the classical cobweb theorem). However, for $\ell = 2$ the system is stable, for $\ell = 3$ it is nearly periodic with constant amplitude, and for $\ell = 4$ it is unstable. For larger lags $\ell \geq 5$ we also observed instability. (However, there is an important comment on this example at the end of Subsection 3.2.)



(a) $\ell = 2$ (b) $\ell = 3$ (c) $\ell = 4$

FIGURE 1. As production lag ℓ increases, behaviour changes from stable to unstable.

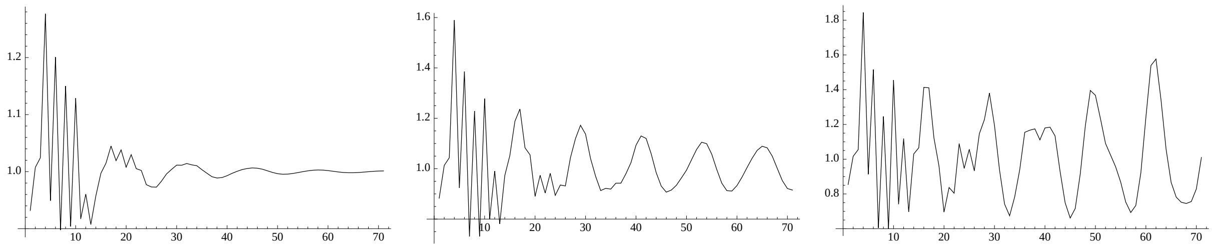
Figure 2 shows the effect of the forecasting period m on the price behaviour, for $c = 1.7$ and $\ell = 3$. The first plot shows the logarithm of price π_t for $m = 1, \alpha = (.7, .3)$ (the price itself soon reaches values larger than 10^6). The next plot is the behaviour of π_t when $m = 5$ ($\alpha = (.2, .2, .2, .2, .1, .1)$) and the last one when $m = 7$ ($\alpha = (.2, .1, .1, .1, .1, .1, .1, .1)$). These plots hint at a stabilising effect of increasing m, and are consistent with other experiments we made.



(a) $m = 1$ (b) $m = 5$ (c) $m = 7$

FIGURE 2. As m increases, behaviour changes from unstable to stable.

In Figure 3 the parameter c is varied, while $\ell = 3$, $m = 7$ and $\alpha = (.2, .1, .1, .1, .1, .1, .1, .1)$. There is apparent stability, except that for the largest value of c there are oscillations of more or less constant amplitude. The last plot, where $c = 3.8$ shows nearly cyclical behaviour. Other experiments (not shown) with larger values of c caused the prices to diverge. Recall that in the classical cobweb stability occurs only if $c < 1$.



(a) $c = 1.7$ (b) $c = 3.0$ (c) $c = 3.8$

FIGURE 3. As c increases, behavior changes from stable to unstable.

The rest of this section presents detailed discussions of particular cases, that sometimes show intriguing patterns which, to our knowledge, have not been noted in the context of economic cycles.

2.1. The case $\mathbf{m = 0}$.

Suppose $m = 0$ and $\ell \in \{1, 2, 3, \dots\}$. The price sequence satisfies

$$\pi_{t+\ell} = -c\pi_t.$$

(Once again recall that the classical cobweb theorem has $m = 0$ and $\ell = 1$.) The solution can be written as

$$\pi_{k\ell - j} = (-c)^k \pi_{-j}\,, \quad k \in \{1, 2, \dots\}, \quad j \in \{0, \dots, \ell - 1\}.$$

Here, there are ℓ price dynamics that work "in parallel", *i.e.* they are not coupled. Each initial condition π_{-j} determines $\pi_{\ell-j}, \pi_{2\ell-j}$ and so on. If $0 < c < 1$ then there are damped oscillations that tend to zero as $k \to \infty$. If $c = 1$ then oscillations of same amplitude and period 2ℓ persist endlessly, and if $c > 1$ then the log-prices alternate in sign but increase geometrically in size as time goes by. The role of the ratio of elasticities is clear in this case.

2.2. **The case $\boldsymbol{\ell} = \mathbf{1}$, $\mathbf{m} = \mathbf{1}$.**

Theorem 2. *Let $\ell = m = 1$. Then the sequence π_t is stable if, and only if,* $1 - 1/c < \alpha_0 < 1/(2c) + 1/2$.

Proof. The solution π_t of (8) is stable if, and only if, both zeroes of (9) have norm less than 1. The characteristic polynomial (9) is

$$h_{1,1}(x) = x^2 + c\alpha_0 x + c\alpha_1 \quad \text{where} \quad \alpha_1 = 1 - \alpha_0.$$

From Theorem 4.2 of [12], stability for this second order equation is equivalent to the following three conditions holding simultaneously:

$$\begin{aligned} 1 + c\alpha_0 + c(1-\alpha_0) &> 0 \\ 1 - c\alpha_0 + c(1-\alpha_0) &> 0 \\ 1 - c(1-\alpha_0) &> 0. \end{aligned}$$

Given that $c > 0$, these are in turn equivalent to $1 - 1/c < \alpha_0 < 1/(2c) + 1/2$. □

Here the system is unstable for $c \geq 3$, because the condition in Theorem 2 cannot then be satisfied. When $\ell = 1, m = 0$, the model is stable only when $c < 1$, which says that the elasticity of demand is larger than the elasticity of supply. By contrast, when $m = \ell = 1$ stability can be achieved for any c smaller than 3, which means that the elasticity of supply only needs to be smaller than three times the elasticity of demand. It is somewhat surprising that increasing the forecasting period m from 0 to 1 has such a significant effect. Observe that the only α_0 that produces stability for all $c < 3$ is $2/3$. There is no obvious reason why it should be $\alpha_0 = 2/3$ that makes this region largest; this corresponds to assigning twice as much weight to the most recent price as the previous one. Note also that if $0 < c < 1$ then $1/(2c) + 1/2 > 1$, meaning that for those values of c the sequence is stable in particular for $\alpha_0 \in (1, 1/(2c) + 1/2)$, which corresponds to establishing the price forecast by extrapolating the two most recent prices. For example, if $c = 1/2, \log P_t = 1, \log P_{t-1} = 0, \alpha_0 = 5/4$ then the forecast is $\log \hat{P}_{t+1} = 5/4$; the price sequence is stable in this case, even though *a priori* one might think that extrapolating the most recent prices would be a destabilizing policy. In the economics literature, the expression "extrapolative expectations" refers to forecasting based on the last two prices. This idea was first studied mathematically in a macroeconomic model of inventories by Metzler [19]. Metzler studies a somewhat different problem, but the algebra is similar to our case $\ell = 1, m = 1$. (Metzler and others believed that extrapolation was a cause of instability). Turnovsky [29] mentions the destabilizing effect of extrapolation (*i.e.* $\alpha_0 > 1$). Extrapolative expectations is also called a "myopic" forecast by other authors, for instance Wheaton [32] claims that this is a cause of real estate oscillations.

2.3. **The cases $\mathbf{m} = \mathbf{1}, \boldsymbol{\ell} \geq \mathbf{2}, \boldsymbol{\alpha_0} = \mathbf{0}$ or $\mathbf{1}$.** When $m = 1$ and $\ell \geq 2$, equation (8) cannot be solved exactly, except for the two special cases $\alpha_0 = 0$ and $\alpha_0 = 1$, which are tractable. In the first case, the characteristic polynomial is

$$h_{\ell,m}(x) = x^{\ell+1} + c$$

which has zeros x_j with norm

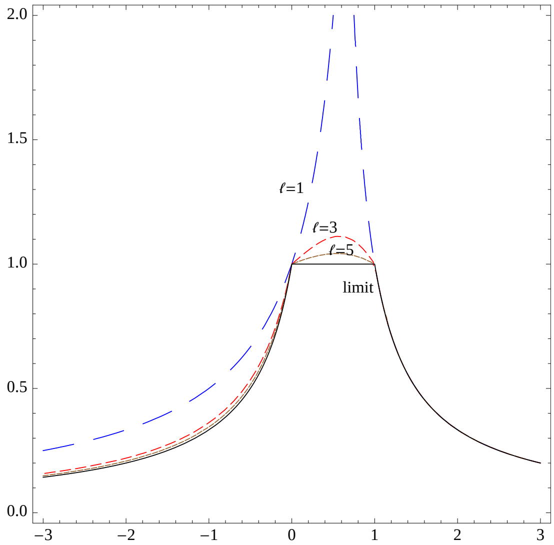
$$|x_j| = c^{1/(\ell+1)}, \quad j = 1, \ldots, \ell + 1.$$

Hence the condition $c < 1$ is a necessary and sufficient condition for stability when $\alpha_0 = 0$. In the case $\alpha_0 = 1$ the zeros of the characteristic polynomial

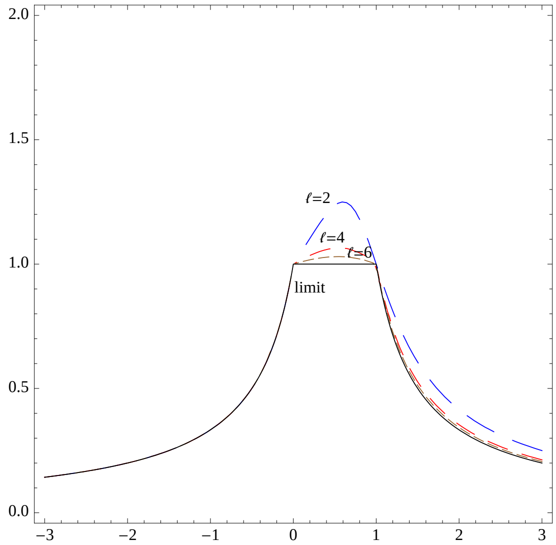
$$h_{\ell,m}(x) = x^{\ell+1} + cx$$

are 0 and $x_j = c^{\frac{1}{\ell}} e^{\frac{2\pi(j-1)}{\ell}}$, $j = 1, \ldots, \ell$, and once again $c < 1$ is a necessary and sufficient condition for stability.

2.4. **The cases $\mathbf{m = 1, \boldsymbol{\ell} \geq 2, \boldsymbol{\alpha}_0}$ arbitrary.** The graphs in Figure 4 show the region of stability for $m = 1$ and ℓ between 1 and 100, computed using *Mathematica*®. The stability region is the set of (α_0, c) that lead to a stable price sequence; the curves on the graphs are the upper boundaries of the stability regions. We have observed numerically that the stability region shrinks to some limit set as ℓ increases, though the shape of the upper boundary is different for even and odd values of ℓ.



(a) ℓ odd.



(b) ℓ even.

FIGURE 4. $m = 1$. The region of stability is the area below each of the curves shown.

The stability region is largest when $\ell = 1$; the upper boundaries for $\ell \geq 5$ are indistinguishable from the limit when α_0 is outside $(0, 1)$. The limit as ℓ tends to infinity of the upper boundary of the stability region coincides (as far as we can tell numerically) with the curve

$$c = (|\alpha_0| + |1 - \alpha_0|)^{-1}$$

(solid line). We give a partial justification in Section 3.

2.5. **The case $\mathbf{1 < m < \infty}$: equal weights $\boldsymbol{\alpha}_\mathbf{j}$.** In our next numerical experiments the weights are equal, *i.e.* $\alpha_j = 1/(m + 1), j = 0, \dots, m$. Hence,

$$h_{\ell,m}(x) = x^{\ell+m} + \frac{c}{m+1} \sum_{j=0}^{m} x^{m-j}. \tag{12}$$

Figure 5 plots the supremum of the values of c that preserve stability, that is, the values of

$$c^*(\ell, m) = \inf_{c>0} \left(\max_{j \in \{1,\dots,m+\ell\}} |x_j(c)| = 1 \right),$$

where $\{x_j(c)\}$ are the zeroes of (12). The horizontal axis shows the values of the averaging period m, and the four dotted lines correspond to $\ell = 1, 2, 3, 4$. The dotted lines visually appear to be linear functions of m, with a slope that decreases as the lag ℓ increases. A closer look at the actual values of $c^*(\ell, m)$ shows that for fixed $\ell = 2, 3, 4$ the functions are not precisely linear in m, but for $\ell = 1$ the slope is indeed constant.

We are able to prove that if $\ell = 1$ then $c^*(\ell, m) = m + 1$, see Section 3.3.

Figure 6 shows the zeroes of the characteristic polynomial $h_{\ell,m}(x)$ in the complex plane for $m = 6$ and various values of ℓ. Each number "ℓ" on a plot represents the location of a zero of $h_{\ell,m}(x)$. It is seen that, at least in those cases, the zeroes move towards the unit circle as ℓ increases, and, furthermore, that the zeroes are approximately uniformly spread around the

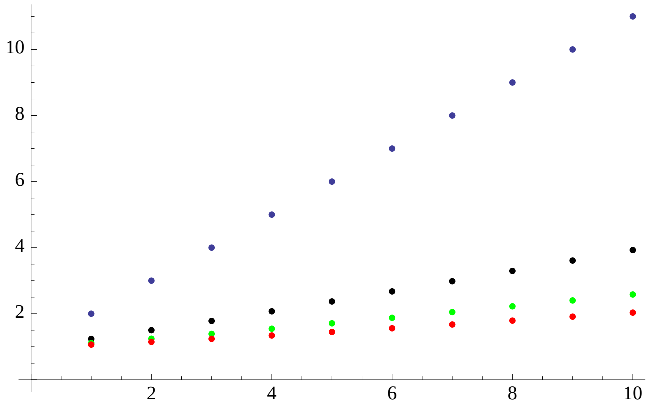


FIGURE 5. Critical value of the ratio of elasticities c as a function of m. The top line corresponds to $\ell = 1$, the ones below correspond to $\ell = 2, 3, 4$ (in that order).

circle. Intuitively this means that the behaviour of prices tends to oscillations when ℓ increases. We now provide an incomplete justification for this. The characteristic polynomial is

$$h_{\ell,m}(z) \;=\; z^{\ell+m} + cg(z), \qquad g(z) \;=\; \frac{1}{m+1}\sum_{j=0}^{m} z^{m-j}.$$

Letting $w = z^{\ell+m}$, this may be rewritten as $H_{\ell,m}(w) = w + cg(w^{\frac{1}{\ell+m}})$.

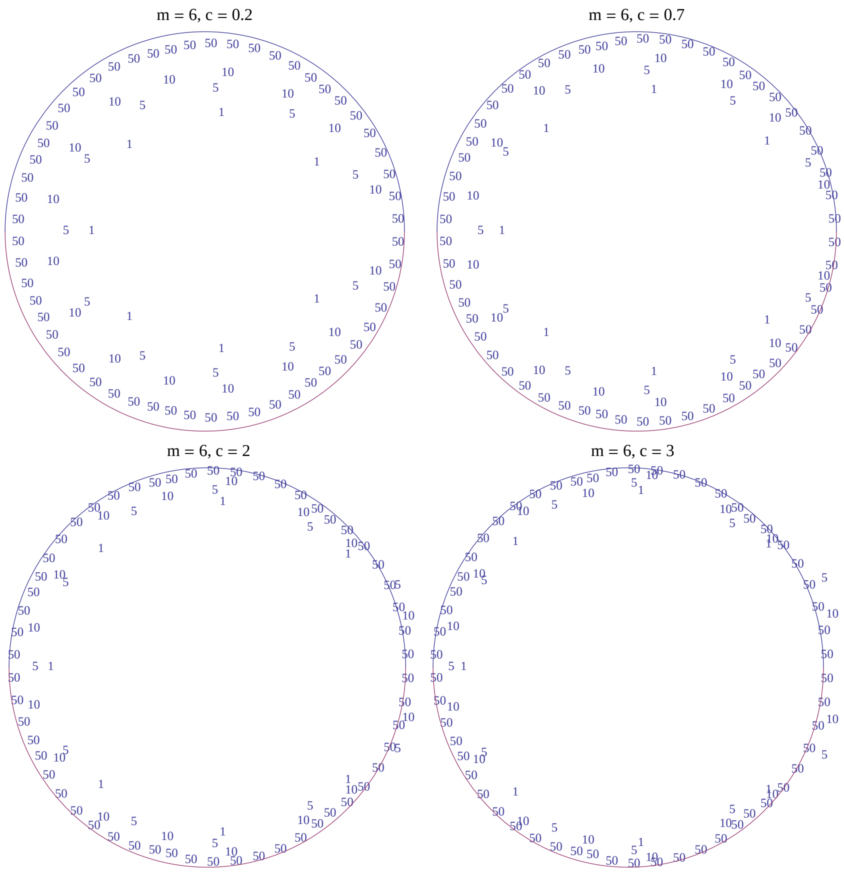


FIGURE 6. Zeroes of the characteristic polynomial $h_{\ell,m}(x)$ for different values of c, with $m = 6$, in the complex plane. Each zero is indicated by the number "ℓ". The circle has centre 0 and radius 1 ($C_{0,1}$).

We now use Rouché's Theorem with $\phi(z) = cg(z), \psi(z) = z^{\ell+m}$ and $C = C_{0,\rho}$, for $0 < \rho < 1$. The zeroes of ϕ are all on the unit circle, and thus $|\phi(z)| \geq \epsilon > 0$ for $z \in C_{0,\rho}$. Hence, for ℓ larger than some ℓ_0 the inequality $|\phi(z)| > |\psi(z)|$ is verified for $z \in C_{0,\rho}$; this implies that the zeroes of $h_{\ell,m}$ are all outside $B_{0,\rho}$, for any $0 < \rho < 1$. Now $w^{\frac{1}{\ell+m}} \to 1$ as $\ell \to \infty$, and we are left with

$$\lim_{\ell\to\infty} H_{\ell,m}(w) \;=\; w + c.$$

Finally, reverting to $w = z^{\ell+m}$ then says that the zeroes of $h_{\ell,m}$ are, approximately, the solutions of

$$z^{\ell+m} \;=\; -c.$$

The zeroes are then approximately equal to

$$c^{\frac{1}{\ell+m}} e^{i\frac{2\pi j}{\ell+m}}, \qquad j = 1, \dots, \ell+m.$$

Although not rigorously derived, this yields good approximations of the arguments $2\pi j/(\ell+m)$, but not always a good one for the norms of the zeroes of $h_{\ell,m}$. For the latter it is better to rely on the fact that the norm of the product of the zeroes of $h_{\ell,m}$ is $|h_{\ell,m}(0)| = c/(m+1)$, which yields the improved approximation

$$\left(\frac{c}{m+1}\right)^{\frac{1}{\ell+m}} e^{i\frac{2\pi j}{\ell+m}}, \qquad j = 1, \dots, \ell+m. \tag{13}$$

As an example, consider the first graph in Figure 6, with equals weights $\alpha_j = 1/(m+1)$, $c = .2$ and $m = 6$. For $\ell = 5$ the exact zeroes are $r_j e^{i\theta_j}$, with

$$\begin{array}{llll}
r_1 = 0.694659, & \theta_1 = 3.14159, & & \\
r_2 = 0.688829, & \theta_2 = -2.59503, & r_3 = 0.688829, & \theta_3 = 2.59503, \\
r_4 = 0.700281, & \theta_4 = -2.03232, & r_5 = 0.700281, & \theta_5 = 2.03232, \\
r_6 = 0.716104, & \theta_6 = -1.49399, & r_7 = 0.716104, & \theta_7 = 1.49399, \\
r_8 = 0.730140, & \theta_8 = -0.92760, & r_9 = 0.730140, & \theta_9 = 0.92760, \\
r_{10} = 0.804108, & \theta_{10} = -0.35203, & r_{11} = 0.804108, & \theta_{11} = 0.35203,
\end{array}$$

while the approximations are $re^{i\theta_j}$, where $r = 0.723819$ and the θ_j are

$$3.14159, \quad \pm 2.57039, \quad \pm 1.9992, \quad \pm 1.428, \quad \pm 0.856798, \quad \pm 0.285599.$$

If (13) were the true zeroes of $h_{\ell,m}$ then the solutions

$$\pi_t \;= \sum_{j=1} b_j x_j^t$$

would have period $\ell + m$. In the mining area, many believe that that the observed price cycles correspond to the production lag ℓ. We see that this is approximately the case in our model, but only when m is small.

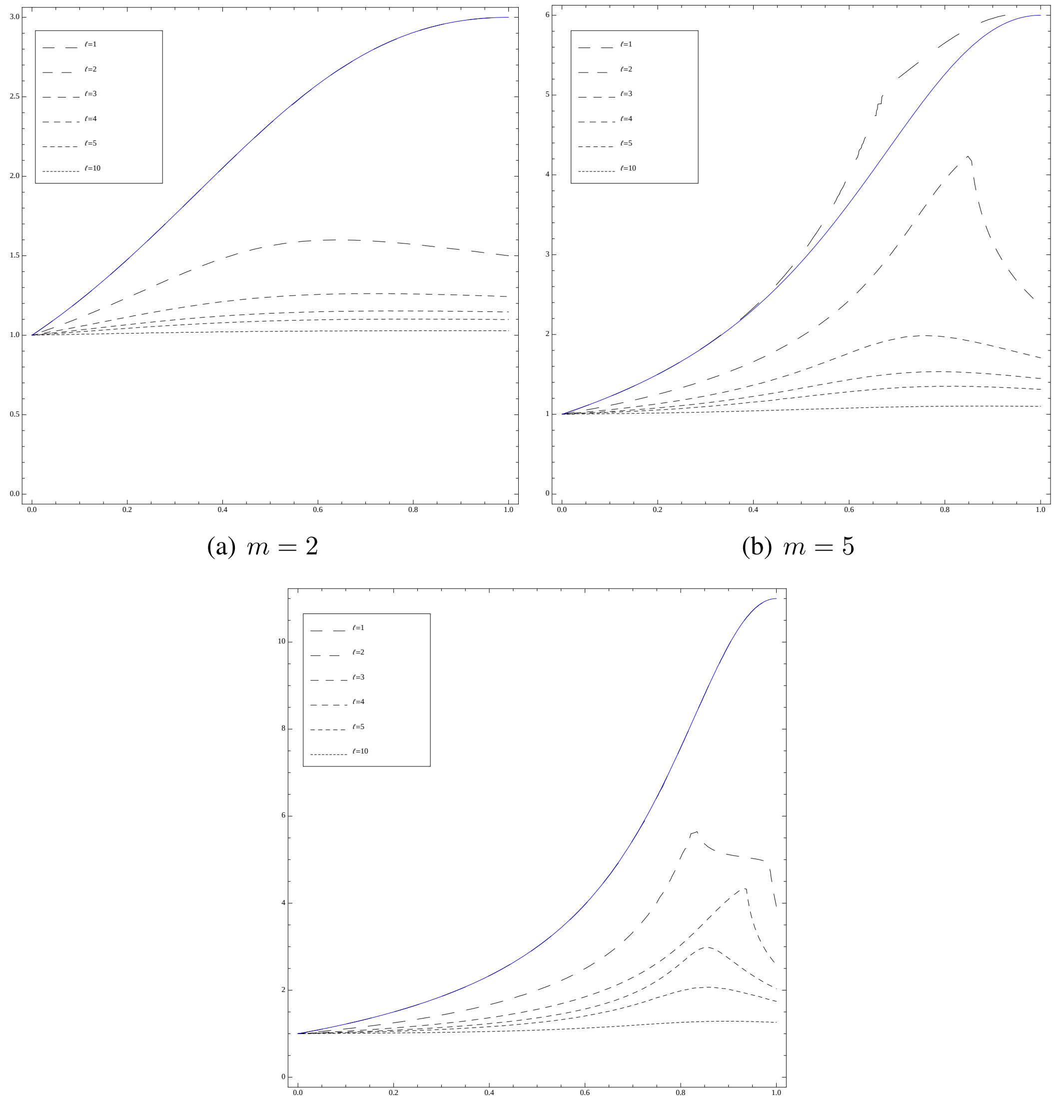
2.6. **Geometric weights.** Geometric weights are used in many forecasting models. A parameter $\lambda > 0$ is chosen and the weights α_j follow the geometric progression

$$\alpha_j = \frac{\lambda^j(1-\lambda)}{(1-\lambda^{m+1})}, \quad j = 0, \dots, m. \tag{14}$$

Figure 7 shows plots of the critical boundary value c^* as a function of λ for different values of ℓ, m. In all cases it appears that the stability region decreases to $c^* = 1$ as ℓ increases. The solid line is the function

$$\tilde{c}(\lambda) \stackrel{\text{def}}{=} \frac{(1+\lambda)(1-\lambda^{m+1})}{(1-\lambda)(1+\lambda^{m+1})},$$

which is the boundary for the region for $\ell = 1$ and even values of m, as we will prove in Section 3.



(a) $m = 2$ (b) $m = 5$

(c) $m = 10$

FIGURE 7. Stability regions for geometric weights are shown for different production lags ℓ.

2.7. **Exponential smoothing.** If $0 < \lambda < 1$ and we formally let $m \to \infty$ in (14), we get

$$\hat{\pi}_{t+\ell} \; = \; (1-\lambda) \sum_{j=0}^{\infty} \lambda^j \pi_{t-j}.$$

Rewriting the same for $\hat{\pi}_{t+\ell-1}$ and eliminating $\pi_{t-1}, \pi_{t-2}, \ldots$, we then find that

$$\hat{\pi}_{t+\ell} \; = \; \lambda \hat{\pi}_{t+\ell-1} + (1-\lambda)\pi_t.$$

This says that the forecast made at time t for the price at time $t+\ell$ is a weighted average of last period's forecast and the most recent price, using a fixed proportion λ. This procedure is mentioned in [8], page 24; it is sometimes called "exponential smoothing" or "adaptive expectations". From (7), $\pi_{t+\ell} \; = \; -c\hat{\pi}_{t+\ell}$ and thus

$$\pi_{t+\ell} \; = \; \lambda \pi_{t+\ell-1} - c(1-\lambda)\pi_t \tag{15}$$

The case $\ell = 1$ is remarkably simple:

$$\pi_{t+1} \; = \; [(1+c)\lambda - c]\pi_t.$$

By setting

$$\lambda \; = \; \frac{c}{1+c} \; = \; \frac{s}{d+s},$$

one gets $\pi_{t+1} = 0$, *i.e.* there is convergence to the equilibrium price in just one time step.

There is an explicit result when $\ell = 2$, reminiscent of the case $\ell = 1, m = 1$ (Theorem 2).

Theorem 3. *If (15) holds with $c > 0$, then*

(a) if $\ell = 1$, then the sequence π_t is stable if, and only if, $(c-1)/(c+1) < \lambda < 1$;

(b) if $\ell = 2$, then the sequence π_t is stable if, and only if, $1 - 1/c < \lambda < 1$.

Proof. For part (a), the condition is

$$-1 < (1+c)\lambda - c < 1 \qquad \text{or} \qquad \frac{c-1}{c+1} < \lambda < 1.$$

For part (b), the zeroes of the characteristic polynomial

$$x^2 - \lambda x + c(1-\lambda)$$

have norm less than 1 if, and only if ([12], p.172),

$$1 + \lambda + c(1-\lambda) > 0, \quad 1 - \lambda + c(1-\lambda) > 0, \quad 1 - c(1-\lambda) > 0.$$

These are equivalent to $1 - 1/c < \lambda < 1$. □

When $\ell > 2$ the characteristic polynomial has degree three or more, and an exact analysis of the roots is not possible. Figure 8 shows the boundary of the stability region as a function of λ. The solid lines are the functions $(1+\lambda)/(1-\lambda)$ and $1/(1-\lambda)$, and coincide numerically with $\ell = 1, 2$ respectively, as expected.

In all the experiments we made, for any c there is an interval $I(c) = (\lambda(c), 1)$ such that $\lambda \in I(c)$ implies stability. We have not been able to prove mathematically that this is always the case when $\ell \geq 3$.

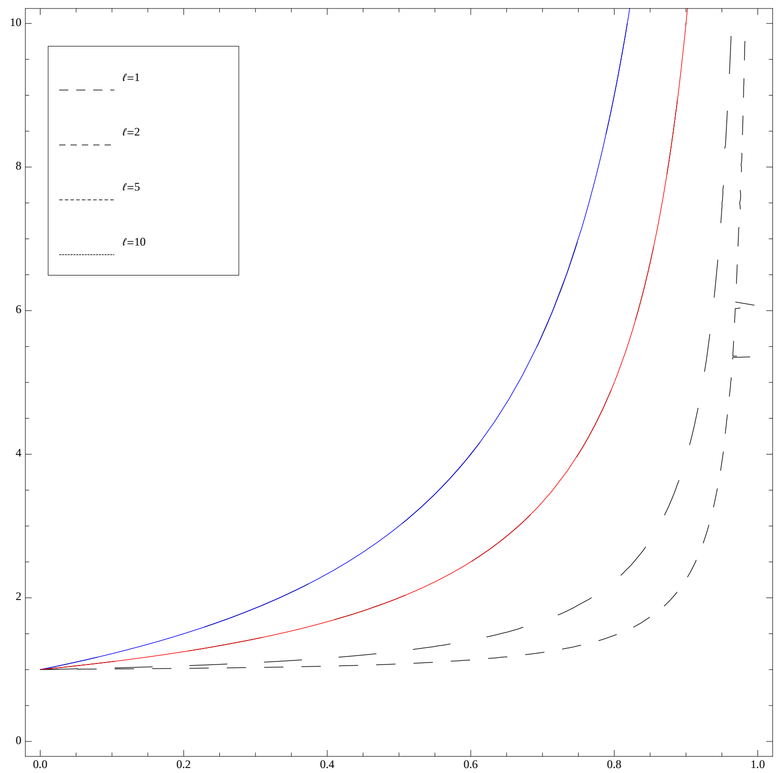


FIGURE 8. Stability region for exponential smoothing.

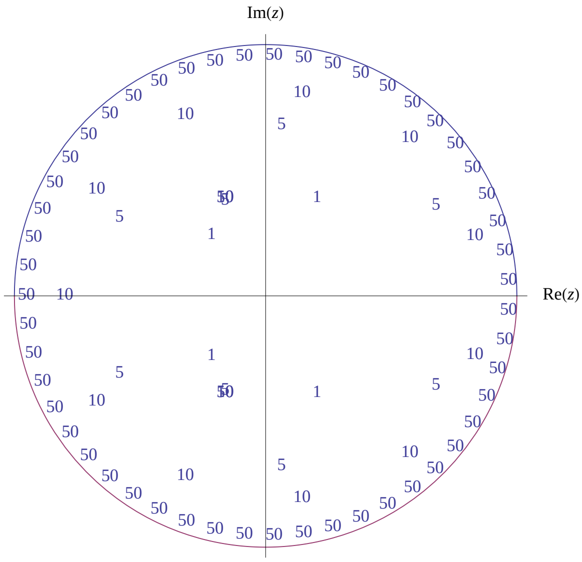


FIGURE 9. Zeroes of the characteristic polynomial $h_{\ell,m}(x)$ for non-equal weights, in the complex plane. Each zero is indicated by the number "ℓ". The circle has centre 0 and radius 1 ($C_{0,1}$).

2.8. **Arbitrary weights.** Figure 9 shows the locations of the zeroes in the complex plane in one case where the weights α_j are not equal: $\alpha = (0.1, 0.6, 0.2, 0.1)$, $c = 1$. The characteristic polynomial

$$h_{\ell,m}(z) \;=\; z^{\ell+3} + 0.2(0.1z^3 + 0.6z^2 + 0.2z + 0.1) \;= z^{\ell+3} + 0.2g(z)$$

has all its zeroes inside the unit circle $C_{0,1}$ for all ℓ. This is a simple consequence of the reverse triangle inequality: if $|z| \geq 1$ then

$$|h_{\ell,m}(z)| \;\geq\; |z^{\ell+3}| - 0.2|0.1z^3 + 0.6z^2 + 0.2z + 0.1| \;\geq\; |z|^{\ell+3} - 0.2|z|^3.$$

The last expression cannot be zero if $|z| \geq 1$, for any $\ell = 1, 2, \ldots$. We note that there are now $\ell + m - 2$ zeroes spread in a circular fashion, getting closer to $C_{0,1}$ as ℓ increases, as there are $\ell + m$ zeroes in total and two zeroes that remain near $-0.16 \pm 0.38i$. It will be shown in Section 3 that inside the unit circle the zeroes of $h_{\ell,m}$ have limits as ℓ tends to infinity, and that they are precisely the zeroes of $g(z) = \sum_{j=0}^{m} \alpha_j z^{m-j}$ (as defined in the proof of Theorem 1).

3. DETERMINISTIC MODELS: GENERAL RESULTS

Let us recall from (3) that for a lag ℓ and an averaging period $m + 1$, the market clearing condition

$$P^{-d}_{t+\ell} \;=\; \hat{P}^{s}_{t+\ell},$$

yields (8), that is

$$\pi_{t+\ell} \;=\; -\frac{s}{d}\sum_{j=0}^{m} \alpha_j \pi_{t-j}.$$

Writing $c = s/d$ as before, the characteristic polynomial (9) is

$$h_{\ell,m} = x^{\ell+m} + c\sum_{j=0}^{m} \alpha_j x^{m-j},$$

and the solution of (8) is

$$\pi_t \;=\; \sum_{k=1}^{\ell+m} b_k\, x_k^t, \tag{16}$$

where $\{x_k\}$ are the zeroes of the characteristic polynomial and $\{b_k\}$ are constants. The long term behaviour of π_t is determined by the x_j with the maximum norm, among those j such that $b_j \neq 0$. Analytic expressions for the zeroes of polynomials are not available for $l + m > 2$, but we will derive results that narrow down the region where the zeroes are located.

In the literature, some have suggested that production lags themselves are the cause of fluctuating prices. For example, in [23], p.276, Phillips mentions that "the regulation of a system can be improved if the lengths of the time delay operating around the main control loop are reduced." Sterman ([27], Chapter 20) writes: "markets with negative feedbacks through which price seeks to equilibrate supply and demand often involve long time delays which lead to oscillation." Our results do support this view, but maybe not in the way one might have expected, in the sense that we do not find that longer lags necessarily lead to instability. What we find is that as the lag ℓ increases the maximum of the norms of the roots x_k tends to one. This means oscillations of constant amplitude, whether the system is stable or unstable for small production lags. Thus, longer lags have a stabilizing effect on unstable systems. This is the conclusion one draws from the general results in Subsections 3.1 and 3.2. Subsection 3.3 studies the case where the weights $\{\alpha_j\}$ are constant or form a geometric progression.

3.1. **General result on the location of the roots and stability.**

Theorem 4. *(a) Suppose $\rho \geq 1$. If $c\sum_{j=0}^{m} |\alpha_j| < \rho^\ell$ then the zeroes of $h_{\ell,m}$ are all less than ρ in norm, i.e. $|x_j| < \rho$. In particular, if*

$$c\sum_{j=0}^{m} |\alpha_j| < 1 \tag{17}$$

then the system is stable.

(b) Suppose $\rho \leq 1$. If $c\sum_{j=0}^{m} |\alpha_j| < \rho^{\ell+m}$ then the zeroes of $h_{\ell,m}$ are all less than ρ in norm.

Proof. Let $\rho \geq 1$ and $|z| \geq \rho$. Then, from the triangle inequality,

$$|h_{\ell,m}(z)| \;\geq\; |z|^{\ell+m} - c\sum_{j=0}^{m} |\alpha_j||z|^{m-j} \;\geq\; |z|^{\ell+m} - c|z|^m \sum_{j=0}^{m} |\alpha_j|.$$

If $c\sum_{j=0}^{m} |\alpha_j| < \rho^\ell$ then the last expression is positive. The zeroes of $h_{\ell,m}$ must then all be in $C_{0,\rho}$.

If $\rho \leq 1$ and $|z| \geq \rho$ then the result follows from

$$|h_{\ell,m}(z)| \;\geq\; |z|^{\ell+m} - c\sum_{j=0}^{m} |\alpha_j| \;\geq\; |\rho|^{\ell+m} - c\sum_{j=0}^{m} |\alpha_j|. \qquad \square$$

Observe that (17) is sufficient for stability, but not necessary. For instance, in the case $m = 1$ depicted in Figure 4 it is seen that when $0 < \alpha_0 < 1$ the system is stable for some $c > 1 = \alpha_0 + \alpha_1$.

Part (a) of the theorem implies that for any $\rho > 1$ and for ℓ greater than some ℓ_0, the zeroes of the the characteristic polynomial are all inside $C_{0,\rho}$. This means that for systems that are unstable for some ℓ there are larger ℓ's such that the maximum norm of the zeroes of the characteristic polynomial is close to 1; in other words, an unstable system eventually becomes less unstable as ℓ increases.

As an application of Theorem 4, let us now return to the cases $m = 1, \ell \geq 1, \alpha_0$ arbitrary, that we looked at in Section 2 (cf. Figure 4).

Suppose ℓ is odd, and fix $\alpha_0 \geq 1$. If we set $c = (|\alpha_0| + |1 - \alpha_0|)^{-1} = 1/(2\alpha_0 - 1)$ then a zero of $h_{\ell,1}(x)$ is $x = -1$, and the system is unstable. However, Theorem 4 says that if $c < (|\alpha_0| + |1 - \alpha_0|)^{-1}$ then the system is stable. Hence, the stability region for ℓ odd, $\alpha_0 \geq 1$ consists of the points (α_0, c) with $c < (|\alpha_0| + |1 - \alpha_0|)^{-1}$. The situation is similar when ℓ is even and $\alpha_0 \leq 0$; then $c = (|\alpha_0| + |1 - \alpha_0|)^{-1} = 1/(1 - 2\alpha_0)$ leads to a zero at $x = -1$ again, while Theorem 4 gives stability when $c < 1/(1 - 2\alpha_0)$. Hence, the stability region for ℓ even, $\alpha_0 \leq 0$ consists of the points (α_0, c) with $c < (|\alpha_0| + |1 - \alpha_0|)^{-1}$.

3.2. **Limiting behaviour with increasing production lags.** The next result shows that when the production lag ℓ increases without bound, all the zeroes of the characteristic polynomial $h_{\ell,m}(x)$ are arbitrarily close to the unit circle, with the possible exception of up to m zeroes inside the unit circle. This means, loosely speaking, that longer production lags lead to oscillations of constant amplitude, and not to oscillations of increasing amplitude. A system that has oscillations of increasing amplitude will be made less unstable for production lags that are long enough. This partly contradicts the view that long production lags in themselves cause erratic price behaviour.

We will use the following classical result from [25], page 261.

Theorem (Hurwitz) *Let G be a non-empty connected open set in the complex plane. Suppose $\phi, \phi_n, n \geq 1$ are analytic functions on G, and that ϕ_n converges uniformly on compacts to ϕ in G. Let U be a bounded open set of G with $\overline{U} \subset G$ such that f has no zero on ∂U. Then there is an index $n_U \in \mathbb{N}$ such that for each $n \geq n_U$ the functions ϕ and ϕ_n have the same number of zeroes in $\overline{U}$.*

Theorem 5. *Define*

$$g(z) = \sum_{j=0}^{m} \alpha_j z^{m-j}. \tag{18}$$

If g has k zeroes inside the unit circle $C_{0,1}$ label them $r_1, \ldots, r_k$. Let $\rho_1 \in (0, 1)$ be such that C_{0,ρ_1} includes $r_1, \ldots, r_k$ in its interior. Let $\rho_2 > 1$. Then there exists $\ell_0 < \infty$ such that for any $\ell \geq \ell_0$, $h_{\ell,m}(\cdot)$ has exactly the same number of zeroes in C_{0,ρ_1} as g_m and no zero outside C_{0,ρ_2}. In addition, there are sequences $r_{1,\ell}, \ldots, r_{k,\ell}$ such that

- $\lim_{\ell \to \infty} r_{i,\ell} = r_i,$ *for each $1 \leq i \leq k$, and*
- *For every $\ell \geq \ell_0$, $h_{\ell,m}(r_{i,\ell}) = g(r_i) = 0$.*

Proof. First, $h_{\ell,m}(z)$ converges uniformly on compact sets (in $B_{0,1}$) to $cg(z)$. Take ρ_1 as defined above, and apply Hurwitz's Theorem for $G = C_{0,\rho_1}$ to obtain that $h_{\ell,m}$ has the same number of zeroes as g inside C_{0,ρ_1} for all $\ell \geq \ell_0$. To obtain the limiting results for each of the zeroes, apply Hurwitz Theorem to a sequence of balls around each zero r_i with decreasing radius.

Second, for any $\rho_2 > 1$ there is ℓ_1 such that

$$c \sum_{j=0}^{m} |\alpha_j| \;\leq\; \rho_2^{\ell}, \quad \ell \geq \ell_1,$$

and thus by part (a) of Theorem 4 all the zeroes of $h_{\ell,m}$ are inside C_{0,ρ_2} when $\ell \geq \ell_1$. □

Theorem 5 explains the behaviour illustrated in Figures 6 and 9, namely that as ℓ increases, most or all the zeroes approach the boundary of the unit circle. It does not, however, explain why the zeroes are placed almost uniformly around the circle in the limit. The difference between

Figures 6 and 9 is that in the latter the polynomial $g(z)$ has zeroes inside the unit circle. As the theorem says, those zeroes remain there in the limit.

This leads us to reconsider the first example in Section 2. There it appeared that increasing ℓ led to instability (see Figure 1). When $\ell = 4$ (last plot) the largest norm among the roots of the characteristic polynomial is 1.028, which explains the increasing amplitude of the oscillations. However, with larger ℓ that largest norm increases a bit more but then gradually decreases towards 1, for instance for $\ell = 10$ the largest norm is 1.038, for $\ell = 20$ it is 1.023, and for $\ell = 40$ it is 1.013.

3.3. **Constant or geometric weights.**

Theorem 6. *Consider the model for the log price (8) with constant weights. If* $\ell = 1$ *then* $c^*(\ell, m) = m + 1$.

Proof. Write $\beta = c/(m+1)$ and define

$$\tilde{h}(z) \;=\; (1-z)h_{1,m}(z) \;=\; (1-z)z^{m+1} + \beta(1 - z^{m+1}).$$

The zeroes of $\tilde{h}$ are precisely those of $h_{1,m}$ together with the number 1. Any zero of $\tilde{h}$ satisfies

$$z^{m+1}(z - 1 + \beta) \;=\; \beta. \tag{19}$$

First, consider the case $\beta > 1$, which is the same as $c > m+1$. Then the norm of the left-hand side of (19) is

$$|z^{m+2} + (\beta - 1)z^{m+1}| \;\le\; |z|^{m+2} + (\beta - 1)|z|^{m+1}.$$

If $|z| < 1$, then this is no larger than $|z|^{m+1}\beta < \beta$, which is a contradiction. Thus, if $c > m+1$ then $h_{1,m}$ has no zero inside the unit circle, and the system is unstable.

Next, suppose $\beta = 1$, or $c = m+1$. Then (19) becomes $z^{m+2} = 1$, which has $m+2$ zeroes, all of norm 1, and thus $h_{1,m}$ has all its zeroes on the unit circle; thus $c^*(\ell, m) \le m+1$.

Finally, suppose that $0 < \beta < 1$ and that we restrict our search for zeroes to $|z| = 1$ (*i.e.* to the unit circle). Then (19) implies

$$|z - (1-\beta)| \;=\; \beta,$$

which has the unique solution $z = 1$ on the unit circle; it is readily checked that this is not a zero of $h_{1,m}$ and we conclude that $h_{1,m}$ has no zero z with norm equal to 1. If $|z| > 1$, we get

$$|z - (1-\beta)| \;=\; \frac{\beta}{|z|^{m+1}},$$

which has no solution because

$$|z - (1-\beta)| \;\ge |z| - (1-\beta) \;>\; \beta \;>\; \frac{\beta}{|z|^{m+1}}.$$

Hence, if $0 < \beta < 1$ then all the zeroes of $h_{1,m}$ are inside the unit circle; thus $c^*(\ell, m) \ge m+1$ for any $0 < \beta < 1$.

From all the above, we conclude that $c^*(1, m) = m + 1$ for $m \ge 0$. $\square$

Theorem 7. *Consider the price dynamics in (8) with geometric weights (14),* $0 < \lambda < 1$, *and let* $\ell = 1$. *Then the system is stable if*

$$0 \;<\; c \;< \tilde{c}(\lambda) \stackrel{\text{def}}{=} \frac{(1+\lambda)(1-\lambda^{m+1})}{(1-\lambda)(1+\lambda^{m+1})}.$$

Hence, $\tilde{c}(\lambda) \le c^*$.

Proof. Define

$$\sigma(\lambda) = \frac{1-\lambda^{m+1}}{1-\lambda} \quad c' = \frac{c}{\sigma(\lambda)}. \tag{20}$$

The characteristic polynomial is

$$h_{\ell,m}(z) \;=\; z^{\ell+m} + c'\lambda^m \sum_{j=0}^{m} (z/\lambda)^{m-j} \;=\; z^{\ell+m} + c'\left(\frac{z^{m+1}-\lambda^{m+1}}{z-\lambda}\right). \tag{21}$$

Let $y = z/\lambda$, then the zeroes of $h_{\ell,m}(z)$ are in a one-to-one correspondence with the zeroes of

$$y^{\ell+m} + \frac{c'}{\lambda^\ell}\left(\frac{1-y^{m+1}}{1-y}\right). \tag{22}$$

More specifically, the system (8) will be stable if, and only if, the zeroes of (22) are in $B_{1/\lambda}$. Multiply (22) by $\lambda^\ell(1-y)$ to get

$$\begin{aligned}\tilde{h}(y) &\stackrel{\text{def}}{=} \lambda^\ell(1-y)y^{\ell+m} + c'(1-y^{m+1}) \\ &= -\lambda^\ell y^{\ell+m+1} + \left(\lambda^\ell y^{\ell-1} - c'\right) y^{m+1} + c'.\end{aligned}$$

Except for $y = 1$, the zeroes of this polynomial are those of (22). Let

$$\begin{aligned}\Phi(y) &= -\lambda^\ell y^{\ell+m+1} \\ \Psi(y) &= \left(\lambda^\ell y^{\ell-1} - c'\right) y^{m+1} + c'.\end{aligned}$$

Let $\ell = 1$ and $c < \tilde{c}(\lambda)$. We will show that if $|y| = 1/\lambda$, then $|\Phi(y)| > |\Psi(y)|$ (see below for the proof). Applying Rouché's Theorem, this in turn will imply that all the zeroes of $\tilde{h}(y)$ are in $B_{1/\lambda}$.

We need to show that if $0 < \lambda < 1$, $|y| = 1/\lambda$ and $0 < c < \tilde{c}(\lambda)$, then

$$|c' + (\lambda - c')y^{m+1}| < \lambda^{-m-1}.$$

From the triangle inequality

$$|c' + (\lambda - c')y^{m+1}| \le c' + |\lambda - c'|\,|y^{m+1}| = c' + |\lambda - c'|\lambda^{-m-1}.$$

If $0 < c' < \lambda$ then

$$c' + |\lambda - c'|\lambda^{-m-1} < \lambda^{-m-1} \iff \lambda^{m+1}c' + \lambda - c' < 1 \iff (\lambda^{m+1}-1)c' < 1-\lambda,$$

which is true for all $0 < \lambda < 1$. If $c' \ge \lambda$ then

$$\begin{aligned} c' + |\lambda - c'|\lambda^{-m-1} < \lambda^{-m-1} &\iff \lambda^{m+1}c' + c' - \lambda < 1 \\ &\iff c' < \frac{1+\lambda}{1+\lambda^{m+1}} \\ &\iff c < \frac{(1+\lambda)(1-\lambda^{m+1})}{(1-\lambda)(1+\lambda^{m+1})} = \tilde{c}(\lambda). \end{aligned}$$

□

We note that the case of equal weights corresponds to $\lambda = 1$. The limit as $\lambda \to 1$ of $\tilde{c}(\lambda)$ is evaluated straightforwardly using l'Hôpital's rule, and it recovers the bound $m+1$ of Theorem 6.

Theorem 8. *If ℓ is odd and m even then $c^* \le \tilde{c}(\lambda)$. If $\ell = 1$ and m is even then $c^* = \tilde{c}(\lambda)$.*

Proof. We show that for $c = \tilde{c}(\lambda)$, $z = -1$ is always a zero of $h_{\ell,m}(z)$ when ℓ is odd and m even.

Replacing c with $\tilde{c}(\lambda)$ in (21)

$$h_{\ell,m}(z) = z^{\ell+m} + \frac{1+\lambda}{1+\lambda^{m+1}}\left(\frac{z^{m+1}-\lambda^{m+1}}{z-\lambda}\right),$$

so that, evaluating at $z = -1$,

$$h_{\ell,m}(-1) = (-1)^{\ell+m} + \frac{1+\lambda}{1+\lambda^{m+1}}\left(\frac{(-1)^{m+1}-\lambda^{m+1}}{-1-\lambda}\right) = -1+1 = 0.$$

If $\ell = 1$ and m is even then the above and Theorem 7 imply that $c^* = \tilde{c}(\lambda)$. □

4. Random disturbances

Pryor and Solomon [24] introduce randomness in observed prices in a cobweb model, and then study the average length of a cycle. Samuelson [26] imagines that producers might adjust their production according to expected price, and talks of introducing randomness in the price process, but does not develop those ideas. Turnovsky [29] studies stochastic stability for the cobweb model with linear supply and demand functions and $\ell = 1$, for forecast prices following either the weighted average model with $m = 1$, or adaptive expectations. Neither of those authors include production lags, as we do below.

In this section we introduce additive and multiplicative random disturbances in the logprice process; not surprisingly the additive ones are a relatively straightforward extension of the deterministic model studied above. Disturbances to the supply function mean multiplicative errors, which lead to a rather more involved analysis. There is a parallel with the approach used by Chiarella [6], since we end up computing Lyapunov exponents, which also relate to chaos. In both models it is the variability of elasticity of supply that is the origin of chaotic behaviour; in our model elasticity s changes randomly over time, while in Chiarella's case there a deterministic S-shaped supply curve.

The system (7) has demand and supply curves that are fixed through time. We now introduce time-varying supply curves. We leave demand fixed, since in the case of copper it appears that supply is much less predictable than demand. In the words of Dunsby [9], p.157: "Much of the short-term volatility in prices resulting from physical supply-demand imbalances (e.g., ignoring purely financial sources of volatility) derives from supply shocks. Demand tends to grow more steadily". The reasons given by Dunsby include technology, investment, wars, strikes, natural disasters, and declining yields.

Starting from (3),

$$D(P_{t+\ell}) = S(\hat{P}_{t+\ell}), \tag{23}$$

we let $D(p) = k_d p^{-d}$ as before, but write

$$S_t(p) \;=\; k_{s,t} p^{s_t}.$$

Next, we successively get

$$\begin{aligned} k_d(P_{t+\ell})^{-d} \;&=\; k_{s,t+\ell}\left(\exp\sum_{j=0}^{m}\alpha_j \log P_{t-j}\right)^{s_{t+\ell}} \\ P_{t+\ell} \;&=\; \left(\frac{k_{s,t+\ell}}{k_d}\right)^{-\frac{1}{d}} \exp\left(-\frac{s_{t+\ell}}{d}\sum_{j=0}^{m}\alpha_j \log P_{t-j}\right) \\ \pi_{t+\ell} \;&=\; -c_{t+\ell}\sum_{j=0}^{m}\alpha_j \pi_{t-j} + \epsilon_{t+\ell}, \end{aligned} \tag{24}$$

where

$$c_{t+\ell} \;=\; s_{t+\ell}/d \qquad \text{and} \qquad \epsilon_{t+\ell} \;=\; -\frac{1}{d}\frac{k_{s,t+\ell}}{k_d}.$$

In order to study the effect of varying supply, we let both c_t and ϵ_t be random (always assuming that $c_t > 0$). To keep matters simple we assume that $\{(c_t, \epsilon_t), t \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random vectors. Our first task is to find the expectation of π_t; simply take expectations on both sides of (24); if both $\mathbb{E}c_t$ and $\mathbb{E}\epsilon_t$ exist, then the expectation of π_t satisfies the recurrence

$$\mathbb{E}\pi_{t+\ell} \;=\; -(\mathbb{E}c_{t+\ell})\sum_{j=0}^{m}\alpha_j\pi_{t-j} + \mathbb{E}\epsilon_{t+\ell}. \tag{25}$$

This is the same system we studied before in the deterministic case. Although this is not mandatory, in order to simplify the algebra we will make the same change of units we made in Section 1, replacing the constants c, k_s, s with $\mathbb{E}c_t, \mathbb{E}k_{s,t}, \mathbb{E}s_t$ in (11). We then have, for the rest of this section,

$$\mathbb{E}\epsilon_t \;=\; 0, \qquad \mathbb{E}\pi_{t+\ell} \;=\; -(\mathbb{E}c_{t+\ell})\sum_{j=0}^{m}\alpha_j\pi_{t-j}.$$

The expected value of ϵ_t is zero, and thus the equilibrium value of $\mathbb{E}\pi_t$ is also zero.

Convergence of the expected value of π_t to zero does not imply that the sequence π_t has a limit distribution, or a finite variance, as t tends to infinity. In this model we will say that the sequence π_t is *stable* if it has a limit distribution as t tends to infinity, for any set of initial condtions $\pi_0, \pi_{-1}, \ldots, \pi_{t-m-\ell}$ (the latter are not random).

When $c_t = c$ is deterministic the sequence $\{\pi_t\}$ in (24) is an autoregressive process of order $\ell + m$, and there are well-known conditions for its stability. Observe that regarding the distribution of (c_t, ϵ_t) we are assuming nothing besides independence over time (c_t and ϵ_t may be dependent). When c_t is not deterministic the process $\{\pi_t\}$ is called a *random coefficient autoregressive process*.

We will study the problem of stability from two different points of view. The first one is the existence of the limit distribution, using results for the theory of products of random matrices. The second one will assume that (c_t, ϵ_t) have finite second moments, and we will look for conditions under which the second moment of π_t remains finite as t tends to infinity; this will also imply that the sequence has a limit distribution.

Turnovsky [29] uses a stochastic Lyapunov function to find sufficient (though not necessary) conditions for convergence with probability one pf P_t to some value P^*. There are significant differences between his approach and ours. First, Turnovsky needs the variance of the disturbances to tend to zero as the price approaches P^*, while we let the disturbances have constant variance; second, Turnovsky considers a more complex noise process, with correlation across time; finally, Turnovsky was writing before the work of Kesten and others, especially Vervaat, had become known. We find it counterintuitive to twist the model in the way Turnovsky [29] does to obtain convergence to a specific constant price; rather, random shocks lead either to instability or to a limit distribution, naturally excluding convergence to a specific price P^*,

4.1. **Existence of limit distribution under the weakest conditions.** The more technical discussion below is best introduced by describing the simpler case $\ell = 1, m = 0$:

$$\pi_{t+1} \;=\; -c_{t+1}\pi_t + \epsilon_{t+1}.$$

(This is a random version of the classical cobweb model.) The existence and properties of the limit distribution of π_t in this case was studied in great detail by Vervaat in [31]. Iterating the equation yields

$$\pi_t \;=\; \epsilon_t - c_t\epsilon_{t-1} + c_tc_{t-1}\epsilon_{t-2} + \cdots + (-1)^t c_tc_{t-1}\cdots c_1\pi_0.$$

In order to determine whether this sequence has a limit distribution, Vervaat first reverses the order of the subscripts, which does not alter the probability distribution, since the sequence

$\{(c_t, \epsilon_t)\}$ is asumed i.i.d. More specifically, denoting equality in distribution by "$\stackrel{\mathrm{d}}{=}$",

$$\pi_t \stackrel{\mathrm{d}}{=} \sum_{n=1}^{t} (-1)^{n-1} c_1 c_2 \cdots c_{n-1} \epsilon_n + (-1)^t c_1 c_2 \cdots c_t \pi_0. \tag{26}$$

He then uses the n-th root test for series:

$$\text{if} \qquad \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \;<\; 1 \qquad \text{then} \qquad \sum_{n\geq 1} |a_n| \;<\; \infty.$$

This is applied to the "time-reversed" series we just described:

$$\text{if} \quad \limsup_{t\to\infty} |c_1 c_2 \cdots c_t \epsilon_t|^{\frac{1}{t}} \;<\; 1 \quad \text{a.s.} \quad \text{then} \quad \sum_{t\geq 1} |c_1 c_2 \cdots c_t \epsilon_t| \;<\; \infty \quad \text{a.s..}$$

(Here "a.s." stands for "almost surely", which means the same as "with probability one".) Next consider $c_1 c_2 \cdots c_t$ and ϵ_t separately, recalling that $c_t > 0$. Since

$$(c_1 c_2 \cdots c_t)^{\frac{1}{t}} \;=\; \exp\left(\tfrac{1}{t} \sum_{k=1}^{t} \log c_k\right),$$

it is then obvious that if $\mathbb{E} \log c_1 < 0$ then, by the Law of Large Numbers,

$$\lim_{t\to\infty} \tfrac{1}{t} \sum_{k=1}^{t} \log c_k \;=\; \mathbb{E} \log c_1,$$

and thus

$$\lim_{t\to\infty} (c_1 c_2 \cdots c_t)^{\frac{1}{t}} \;=\; \lim_{t\to\infty} \exp\left(\frac{1}{t} \sum_{k=1}^{t} \log c_k\right) \;<\; 1.$$

If $\mathbb{E} \log |\epsilon_1|$ is finite, then

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} \log |\epsilon_k| \;\to\; \mathbb{E} \log |\epsilon_1|,$$

and thus

$$\lim_{t\to\infty} \log |\epsilon_t|^{\frac{1}{t}} \;=\; 0.$$

Finally,

$$\limsup_{t\to\infty} |c_1 c_2 \cdots c_t \epsilon_t|^{\frac{1}{t}} \;<\; 1$$

under the assumptions $\mathbb{E} \log c_1 \;<\; 0$, $\mathbb{E} \log |\epsilon_1| \;<\; \infty$, and thus the right-hand side of (26) has a.s. a finite limit. Note that $\mathbb{E} \log |c_1| < 0$ implies that $c_1 c_2 \cdots c_t$ tends to zero with probability one as t tends to infinity. The assumption regarding the distribution of ϵ_1 can be weakened by noting that values of $|\epsilon_1|$ smaller than 1 cannot cause divergence of the sum, and so requiring $\mathbb{E} \log^+ |\epsilon_1| < \infty$ is sufficient, if $\log^+ x = \max(\log x, 0)$.

There are results of the same nature as the ones above in the more general case where $\ell \geq 1$ and $m \geq 0$ are arbitrary in (24), but they are not as straighforward, even though the randomness in the system is generated by the same pair (c_t, ϵ_t). The process $\{\pi_t\}$ is in general not Markovian, and it is useful to obtain a Markovian representation for it by defining

$$X_t \;=\; (\pi_t, \ldots, \pi_{t-\ell-m+1})^\mathsf{T}, \quad B_t \;=\; (\epsilon_t, 0, \ldots, 0)^\mathsf{T}$$

$$A_t \;=\; \begin{pmatrix} \overbrace{0 \quad 0 \quad \cdots \quad 0}^{\ell-1} & -c_t\alpha_0 & \cdots & -c_t\alpha_{m-1} & -c_t\alpha_m \\ 1 \quad 0 & & & & \\ \quad\;\; 1 & & \mathbf{0} & & \\ \mathbf{0} & \ddots & & & \\ & & & 1 & 0 \end{pmatrix}.$$

Here A_t is $(\ell+m)\times(\ell+m)$, and B_t, X_t are $(\ell+m)\times 1$. The first line of A_t has $\ell-1$ leading zeros, followed by $-c\alpha_0, -c\alpha_1, \ldots, -c\alpha_m$, and a subdiagonal of 1's; the other elements of A_t are 0. The process $\{X_t\}$ is defined recursively as

$$X_t \;=\; A_t X_{t-1} + B_t, \qquad t = 1, 2, \ldots \tag{27}$$

This process is Markovian, because the sequence $\{(A_t, B_t)\}$ is i.i.d.

The adaptive expectations model with random disturbances becomes

$$\pi_{t+\ell} = \lambda\pi_t - c_{t+\ell}(1-\lambda)\pi_t + \epsilon_{t+\ell}.$$

To obtain a Markovian representation, set

$$X_t \;=\; (\pi_t, \ldots, \pi_{t-\ell+1})^\mathsf{T}, \quad B_t \;=\; (\epsilon_t, 0, \ldots, 0)^\mathsf{T}.$$

Then the matrix A_t has the form

$$A_t \;=\; \begin{pmatrix} \overbrace{\begin{matrix}\lambda & 0 & \cdots & 0\end{matrix}}^{\ell-1} & -c_t(1-\lambda) \\ \begin{matrix}1 & 0 & & \end{matrix} & 0 \\ \begin{matrix} & 1 & & \end{matrix} & 0 \\ \begin{matrix}\mathbf{0} & & & 1\end{matrix} & 0 \end{pmatrix}.$$

Here A_t is $\ell\times\ell$, and B_t, X_t are $\ell\times 1$. The process $\{X_t\}$ is defined recursively as before, by (27).

We will use the Euclidian vector norm $|\cdot|_e$ and a matrix norm $\|\cdot\|$ that is compatible with it, in the sense that

$$|Mx|_e \;\le\; \|M\|\cdot|x|_e \tag{28}$$

(see Chapter 5 of [14]). The notation $|A|$ refers to the matrix of the absolute values of the elements of A.

We now consider system (27) in some generality, with A_t an $N\times N$ random matrix (not necessarily of the form specified above). Conditions for the stability of (27) cannot be obtained as simply as in the one-dimensional case. This is essentially because the logarithm and exponential of matrices do not have the same properties as the corresponding functions of real numbers; in particular, for matrices M_1 and M_2 it is general not the case that

$$e^{M_1+M_2} \;=\; e^{M_1}e^{M_2}.$$

In the one-dimensional case the condition $\mathbb{E}\log|A_1| \;<\; 0$ implies that $A_1\cdots A_n$ tends to 0 geometrically; in (27) the corresponding condition is

$$\gamma(\{A_n\}) = \;\inf\{\tfrac{1}{n}\mathbb{E}\log\|A_n\cdots A_1\|, n\in\mathbb{N}\} < 0. \tag{29}$$

This is called the *top Lyapunov exponent* of the matrices $\{A_1, A_2, \ldots\}$. Some of the results we will use go back to Furstenberg and Kesten [15]. It is known ([17]) that if $\{A_n, n\ge 1\}$ is a stationary process and $\mathbb{E}\log^+\|A_1\| < \infty$, then $\gamma(\{A_n\}) \in [-\infty, \infty)$, and, moreover,

$$\gamma\left(\{A_n\}\right) = \lim_{n\to\infty}\tfrac{1}{n}\log\|A_n\cdots A_1\|,\; n\in\mathbb{N}.$$

Part (a) of the following theorem was proved in one dimension by Brandt [5] and extended to the vector case by Bougerol and Picard [4]. We have added part (b) for clarity (it is proved in the same way as part (a)).

Theorem 9. *(a) Let $\{(A_n, B_n), n\in\mathbb{Z}\}$ be a strictly stationary ergodic process such that both $\mathbb{E}(\log^+\|A_1\|)$ and $\mathbb{E}(\log^+\|B_1\|)$ are finite. Suppose that the top Lyapunov exponent γ defined by (29) is strictly negative. Then, for all $n\in\mathbb{Z}$, the series*

$$X_n \;=\; \sum_{k=0}^{\infty} A_n\cdots A_{n-k+1}B_{n-k}$$

converges a.s., and the process $\{X_n, n \in \mathbb{Z}\}$ is the unique strictly stationary solution of (27).

(b) Under the same conditions the process defined by (27) for $t \geq 1$ has a finite limit distribution as $t \to \infty$, and this limit is the same irrespective of the initial condition X_0.

There is no general formula to compute $\gamma(\{A_n\})$ given the distribution of $\{A_n\}$. We will give some properties of the top Lyapunov exponent in the next subsection, and then show numerical examples.

4.2. **Existence of limit distribution under first and second moment conditions.** Sufficient conditions for stability will now be given in terms of the first and second moments of (A_1, B_1). These are stronger conditions than the ones in Bougerol and Picard [4] (see Theorem 9), but they are easier to verify. We use results from Conlisk [7] that lead to sufficient conditions for stability of (27). See also [22] for similar results about a more general model. But first we give some relationships between spectral radius and Lyapunov exponent. The following results are required for our analysis; they may or may not be known, but we were unable to find proofs for all of them in the literature.

The direct (or Kronecker) product $A \otimes B$ of matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{k\ell})_{p\times q}$ is the $mp \times nq$ matrix

$$\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

We also use the vec operation, which stacks the columns of a matrix one on top of the other, the first column at the top. The main property of that operation is

$$\mathrm{vec}(ABC) \;=\; (C^{\mathsf{T}} \otimes A)\mathrm{vec}B.$$

Theorem 10. *(a) Suppose the i.i.d. matrices $\{A_n, n \geq 1\}$ satisfy $\mathbb{E}\|A_1\| < \infty$. Then*

$$\gamma(\{A_n\}) \;\leq\; \gamma(\{|A_n|\}) \;\leq\; \log \rho(\mathbb{E}|A_1|).$$

If A_1 is deterministic then the second inequality is an equality.

(b) Suppose the matrices $\{A_n, n \geq 1\}$ are i.i.d. and have finite second moments. Then

$$\gamma(\{A_n\}) \;\leq\; \tfrac{1}{2}\log \rho(\mathbb{E}(A_1 \otimes A_1)).$$

When A_1 is deterministic the two sides are equal.

(c) For arbitrary A_1, if $\mathbb{E}(A_1 \otimes A_1)$ is finite then $\mathbb{E}(A_1)$ is also finite, and moreover

$$\rho(\mathbb{E}(A_1))^2 \;\leq\; \rho(\mathbb{E}(A_1 \otimes A_1)). \tag{30}$$

Proof. The condition $\mathbb{E}\|A_1\|$ implies that $\mathbb{E}|A_1|$ is a finite matrix, because $|A_1(i,j)| \leq |A_1 e_j|_e \leq \|A_1\|$, where e_j is the unit vector with 1 in the j-th position and zeroes in the others; it also follows that $\mathbb{E}\log^+ \|A_1\| < \infty$.

(a) Justification for the second inequality may be found in the proof of Theorem 2 of [16], p.378), where non-negative A_n are considered. Turn to the first inequality, $\gamma(\{A_n\}) \leq \gamma(\{|A_n|\})$. It is known that $A_n \cdots A_1 \to 0$ if, and only if, $\gamma(\{A_n\}) < 0$ (by Lemma 3.4 in [4]). Assume that $g = \gamma(\{|A_n|\}) \in \mathbb{R}$ and define

$$C_n \;=\; e^{-g-\delta}A_n, \quad n \geq 1,$$

for some $\delta > 0$. Then

$$|C_n \cdots C_1| \;\leq\; |C_n| \cdots |C_1|$$

and

$$\frac{1}{n}\log\| \, |C_n|\cdots|C_1|\, \| \;=\; -g-\delta+\frac{1}{n}\log\| \, |A_n|\cdots|A_1|\, \| \;\to\; -\delta \;<\; 0,$$

which implies that $C_n\cdots C_1$ tends to 0. Thus

$$0 \;>\; \gamma(\{C_n\}) \;=\; -g-\delta+\gamma(\{A_n\}),$$

for any $\delta > 0$, and it follows that $\gamma(\{A_n\}) \le g = \gamma(\{|A_n|\})$. There remains the case $\gamma(\{|A_n|\}) = -\infty$; this is seen to be equivalent to

$$\limsup \frac{1}{n}\log\| \, |e^M A_n|\cdots|e^M A_1|\, \| \;<\; 0$$

for all $M > 0$. This plainly implies that the same holds if $\{|A_n|\}$ is replaced with $\{A_n\}$. (The last assertion follows from the fact that $\gamma(\{|A_n|\}) = -\infty$ is equivalent to

$$\limsup_n \frac{1}{n}\log\| \, |A_n|\cdots|A_1|\, \| \;<\; -M$$

for each $M > 0$, which is the same as

$$e^{Mn}|A_n|\cdots|A_1| \;\to\; 0$$

as n tends to infinity for all $M > 0$; this in turn implies

$$e^{Mn}|A_n\cdots A_1| \;\to\; 0,$$

which entails $\gamma(\{A_n\}) < -M$ for all $M > 0$.)

(b) Fix any $x \in \mathbb{R}^N$, and let

$$Z_n \;=\; A_n\cdots A_1 x.$$

Then $Z_n = A_n Z_{n-1}$ and, letting $V_n = \mathbb{E}\,\mathrm{vec}(Z_n Z_n^{\mathsf{T}})$,

$$Z_n Z_{n-1}^{\mathsf{T}} \;=\; A_n Z_{n-1} Z_{n-1}^{\mathsf{T}} \quad\Longrightarrow\quad V_n \;=\; \mathbb{E}(A_1\otimes A_1)V_{n-1}.$$

If $\rho(\mathbb{E}(A_1\otimes A_1)) < 1$ then V_n tends to 0 at a geometric rate, *i.e.* for all n large enough there is $K < \infty$ such that all the elements of V_n are smaller than or equal to $K\rho_1^n$, where $\rho(\mathbb{E}(A_1\otimes A_1) < \rho_1 < 1$. Hence, for $\delta > 0$

$$\mathbb{P}(|Z_n(j)| > \delta) \;\le\; \frac{\mathbb{E}Z_n(j)^2}{\delta^2} \;\le\; \frac{K\rho_1^n}{\delta^2}.$$

Apply the Borel-Cantelli lemma: if $\{E_n\}$ is a sequence of events, then

$$\sum_n \mathbb{P}(E_n) \;<\; \infty$$

implies that $\mathbb{P}(E_n \text{ infinitely often}) = 0$. Let

$$E_n \;=\; \{|Z_n(j)| > \delta\}.$$

Then $\sum_n \mathbb{P}E_n < \infty$, implying that $\mathbb{P}(\limsup|Z_n(j)| > \delta) = 0$. We conclude that if $\rho(\mathbb{E}(A_1\otimes A_1) < 1$ then Z_n tends to zero a.s.. This holds for every $x \in \mathbb{R}^N$, and thus $\rho(\mathbb{E}(A_1\otimes A_1) < 1$ implies $A_n\cdots A_1$ tends to 0, and in turn $\gamma(\{A_n\}) < 0$.

Consider an arbitrary i.i.d. sequence $\{A_n\}$ with finite second moments, and note that $\rho(\mathbb{E}((kA_1)\otimes(kA_1)) = k^2\rho(\mathbb{E}(A_1\otimes A_1)$ for any $k > 0$. If $\delta > 0$ and

$$k_\delta \;=\; e^{-\delta}[\rho(\mathbb{E}(A_1\otimes A_1)]^{-\frac{1}{2}},$$

then $\rho(\mathbb{E}((k_\delta A_1)\otimes(k_\delta A_1)) = e^{-2\delta}$, and thus $\gamma(\{k_\delta A_n\}) < 0$, implying that

$$\gamma(\{A_n\}) \;<\; \delta + \frac{1}{2}\log\rho(\mathbb{E}(A_1\otimes A_1))$$

for each $\delta > 0$, and the inequality in part (b) follows. If A_1 is deterministic then $\gamma = \log\rho(A_1) = \frac{1}{2}\log\rho(\mathbb{E}(A_1\otimes A_1))$, since $\rho(A_1\otimes A_1) = \rho(A_1)^2$.

(c) Among the elements of the matrix $\mathbb{E}(A_1\otimes A_1)$ there is $\mathbb{E}(A_1(i,j)^2)$, and if this is finite then $\mathbb{E}A_1(i,j)$ is also finite.

In [7] there is a proof that if $\rho(\mathbb{E}(A_1 \otimes A_1)) < 1$, then $\rho(\mathbb{E}(A_1)) < 1$ as well, from which we could derive (30). That proof is not very intuitive, however, and we propose a more direct argument. Let $x \in \mathbb{C}^N$ and $\{A_n\}$ an i.i.d. sequence of $N \times N$ matrices. For any complex Y it is elementary that $\mathbb{E}|Y|^2 \geq |\mathbb{E}Y|^2$. If $M = A_n \cdots A_1$ and $Y = x^\mathsf{T} M x$ then

$$\begin{aligned}
\mathbb{E}|Y|^2 &= \mathbb{E}(x^\mathsf{T} M x \overline{x}^\mathsf{T} M^\mathsf{T} \overline{x}) \\
&= \mathbb{E}[(\overline{x}^\mathsf{T} \otimes x^\mathsf{T}) \operatorname{vec}(M x \overline{x}^\mathsf{T} M^\mathsf{T})] \\
&= (\overline{x}^\mathsf{T} \otimes x^\mathsf{T}) \mathbb{E}(M \otimes M) \operatorname{vec}(x \overline{x}^\mathsf{T}).
\end{aligned}$$

Now, since $(N_1 N_2) \otimes (N_1 N_2) = (N_1 \otimes N_1)(N_2 \otimes N_2)$,

$$\mathbb{E}(M \otimes M) \;=\; \mathbb{E}[(A_n \cdots A_1) \otimes (A_n \cdots A_1)] \;=\; \mathbb{E} \prod_{j=1}^{n} (A_{n-j+1} \otimes A_{n-j+1}) \;=\; [\mathbb{E}(A_1 \otimes A_1)]^n,$$

and thus

$$\mathbb{E}|Y|^2 \;=(\overline{x}^\mathsf{T} \otimes x^\mathsf{T})[\mathbb{E}(A_1 \otimes A_1)]^n \operatorname{vec}(x \overline{x}^\mathsf{T}) \;\geq\; |\mathbb{E}Y|^2 \;=\; |\mathbb{E}x^\mathsf{T} M x|^2 \;=\; |x^\mathsf{T}(\mathbb{E}A_1)^n x|^2.$$

If x is a non-zero eigenvector of $\mathbb{E}(A_1)$ and λ the corresponding eigenvalue, then

$$(\overline{x}^\mathsf{T} \otimes x^\mathsf{T})[\mathbb{E}(A_1 \otimes A_1)]^n \operatorname{vec}(x \overline{x}^\mathsf{T}) \;\geq\; |\lambda|^{2n} |x|_e^4.$$

Divide both sides by $\sigma^n > \rho(\mathbb{E}(A_1 \otimes A_1))^n$ and then let n tend to infinity, to find $0 \leq |\lambda|^2/\sigma < 1$, or $|\lambda|^2 < \sigma$. This is true for all eigenvalues of $\mathbb{E}A_1$ and all $\sigma > \rho(\mathbb{E}(A_1 \otimes A_1))$, which gives $\rho(\mathbb{E}(A_1))^2 < \sigma$, and finishes the proof. □

Note that it is not always true that $\gamma(\{A_n\}) \leq \log \rho(\mathbb{E}A_1)$ for matrices $\{A_n\}$ that have both positive and negative entries; this happens in the second numerical example below.

Theorem 11. *(a) If*

$$\rho(\mathbb{E}|A_1|) < 1, \qquad \mathbb{E}|B|_e \;<\; \infty$$

then (27) is stable. Moreover, $\mathbb{E}X_t$ *is finite and satisfies*

$$\mathbb{E}X_t \;=\; \mathbb{E}(A_1)\mathbb{E}X_{t-1} + \mathbb{E}B_t. \tag{31}$$

$$\lim_{t\to\infty} \mathbb{E}X_t \;=\; (I_{\ell+m,\ell+m} - \mathbb{E}(A_1))^{-1}\mathbb{E}B_1 \;=\; 0. \tag{32}$$

(b) (Conlisk [7]*) Suppose* $\{(A_n, B_n)\}$ *are i.i.d. and have finite second moments. Then a sufficient condition for the system (27) to be stable is*

$$\rho(\mathbb{E}(A_1 \otimes A_1)) \;<\; 1.$$

When this is the case, the first and second moments of X_t *are finite, the first moments satisfy* (32) *and second moments satisfy*

$$\begin{aligned}
\operatorname{vec}\mathbb{E}(X_t X_t^\mathsf{T}) &= \mathbb{E}(A_n \otimes A_t) \operatorname{vec}\mathbb{E}(X_{t-1}X_{t-1}^\mathsf{T}) + [\mathbb{E}(B_t \otimes A_t) + \mathbb{E}(A_t \otimes B_t)] \operatorname{vec}\mathbb{E}X_{t-1} + \operatorname{vec}\mathbb{E}(B_t B_t^\mathsf{T}) \\
\lim_{t\to\infty} \operatorname{vec}\mathbb{E}(X_t X_t^\mathsf{T}) &= (I_{(\ell+m)^2,(\ell+m)^2} - \mathbb{E}(A_1 \otimes A_1))^{-1} \operatorname{vec}\mathbb{E}(B_1 B_1^\mathsf{T}).
\end{aligned}$$

Part (b) of the theorem shows that the second-order conditions

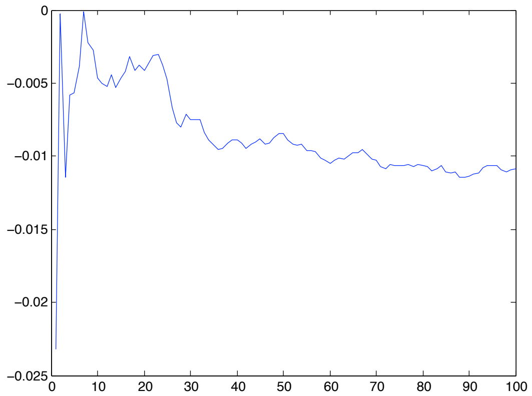
$$\log \rho(\mathbb{E}(A_1 \otimes A_1)) \;<\; 1, \qquad \mathbb{E}|\epsilon_1^2| \;<\; \infty,$$

ensure that Theorem 9 applies. Part (a) shows that the corresponding first-order conditions have the same consequence.

4.3. **Numerical Experiments.** In this section we will show plots of

$$\gamma_n = \frac{1}{n} \log \|A_n \cdots A_1\|$$

as n grows. This process is used to estimate the Lyapunov exponent $\gamma(\{A_n\})$. In all our experiments, we used simulation techniques for variance reduction and numerical stability methods in order to increase accuracy of the estimation. We devised statistical tests in order to determine the stopping criterion and we used asymptotic normality in order to produce statistical confidence intervals. The variables c_t are i.i.d. and have a scaled beta distribution. The support of this distribution is bounded and it is relatively easy to generate in simulations. The details of the numerical methodology are beyond the scope of this paper and will be reported elsewhere.



$m = 4, \ell = 2$
$\mathbb{E}(c_t) = 2$
$\log \rho(\mathbb{E}(A_t)) = -.0064$
$\frac{1}{2} \log \rho(\mathbb{E}(A_t \otimes A_t)) = 0.278$
$\hat{\gamma} \in (-0.0115, -0.0058)$

FIGURE 10. Top Lyapunov exponent estimation: a stable system.

Figure 10 shows the estimation process $\{\gamma_n\}$ for a stable system. Here c_t is a scaled **Beta**(2,5) distribution over the interval (0,7); its mean is 2 and variance is 1.25. The other parameters are shown. In this case, the deterministic system driven by $\mathbb{E}(A_t)$ has a negative spectral radius (*i.e.* the "average" process (25) is stable). The bound $\frac{1}{2} \log \rho(\mathbb{E}(A_t \otimes A_t))$ is also indicated. The confidence interval for the estimate shows that it is likely that $\hat{\gamma} < \log \rho(\mathbb{E}(A_t))$.

The Lyapunov exponent and stability of the system do not depend on the particular zero-mean iid sequence $\{\epsilon_t\}$ (provided it has finite mean). Figure 11 shows a realisation of the price sequence

$$\pi_{t+\ell} \;=\; -c_{t+\ell} \sum_{j=0}^{m} \alpha_j \pi_{t-j} + \epsilon_{t+\ell}, \quad \epsilon \sim \mathcal{N}(0, \sigma^2).$$

for system in Figure 10. As would be expected, increasing σ^2 has the effect of increasing the variance of the price (the two plots have different scales).

Our next experiment shows an unstable system. Figure 12 shows the plot of the estimator process γ_n. The parameters are shown on the right. In this case the confidence interval indicates that $\hat{\gamma} > \log \rho(\mathbb{E}(A_t))$. Here the distribution of c_t is a scaled **Beta**(2,5) over the inteval (0,6.3), with variance 1.0125.

$m = 6, \ell = 4$
$\mathbb{E}(c_t) = 1.8$
$\log \rho(\mathbb{E}(A_t)) = -.0019$
$\frac{1}{2} \log \rho(\mathbb{E}(A_t \otimes A_t)) = 0.1967$
$\hat{\gamma} \in (0.0155, 0.0191)$

In unstable cases the price sequence behaves more erratically, as would be expected.

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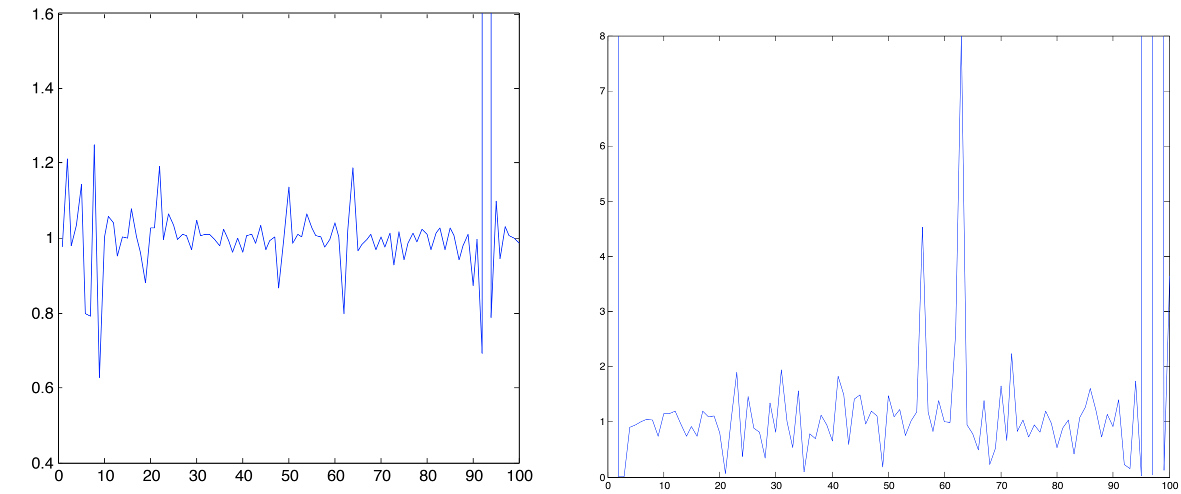

(a) $\sigma = 0.01$ (b) $\sigma = 0.1$

FIGURE 11. Price sequence for stable system.

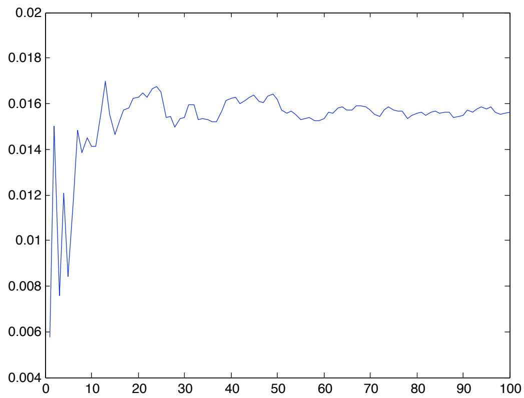

FIGURE 12. Top Lyapunov exponent estimation: an unstable system

6. CONCLUSION

We have formulated a more general version of the cobweb model, that includes production lags and explicit forecasting of prices. Power demand and supply functions, together with a focus on log-prices, lead to more or less tractable difference equations for the price. The classical cobweb theorem is shown to have extensions in those situations. Complex analysis helps to understand what happens when parameters are changed. We have studied the effect of price forecasting on stability; when the averaging period m is greater than one, stability requires less stringent conditions on the elasticities than in the classical cobweb theorem. Increasing the production lag ℓ may or may not lead to instability, but letting ℓ tend to infinity leads to cycles of constant amplitude. The random case is expressed as a bilinear model, and connects this problem with recent work on chaos (Lyapunov exponent of random matrices). In this respect we have provided some results and proofs that may be new.

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