ANALYZING THE BIAS IN THE PRIMAL-DUAL UPPER BOUND METHOD FOR EARLY EXERCISABLE DERIVATIVES: BOUNDS, ESTIMATION AND REMOVAL

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Abstract. We analyze the primal-dual upper bound method and prove that its bias is inversely proportional to the number of paths in sub-simulations for a large class of cases. We develop a methodology for estimating and reducing the bias. We present numerical results showing that the new technique is indeed effective.

1. Introduction

The pricing of early exercisable derivatives using Monte Carlo techniques is a classic hard problem in financial engineering. It is important since once the underlying dimension of the problem exceeds 5 other techniques are simply too slow. Much progress on the problem has been made in recent years. For a long time, the main focus was on how to estimate exercise strategies and these then led to unbiased estimates of lower bounds. However, without a corresponding upper bound it is hard to assess whether the lower bound is tight. A second strand of research has therefore developed which is focussed on upper bounds. One particularly popular technique is the primal-dual method of Andersen and Broadie (2004). However, it uses sub-simulations and tends to be used for model testing rather than live pricing because of its slowness (Brace, 2007). The sub-simulations cause an upwards bias of size long believed to be inversely proportional to the numbers of paths used (Kolodko & Schoenmakers, 2004). We prove that this is indeed true in reasonable cases, and we introduce a new method of decreasing the bias via a simple correction on each path. This greatly decreases the number of sub-simulation paths required to achieve a given level of accuracy.

The Andersen–Broadie method (AB method) is known as a primal-dual method because it takes as input a primal-method-developed lower bound exercise strategy and computes the difference between the lower bound implied by that strategy and an upper bound which is called the "duality gap." The method builds on work of Rogers (2002) and Haugh and Kogan (2004) who established that by using notions of the cost of hedging an option it was possible to develop an upper bound. This followed from earlier work by Davis and Karatzas (1994).

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Consider a Bermudan option with deflated pay-off Z_j which can be exercised at times $j = 1, 2, \ldots, N$. We price in some martingale measure. The essential observation is that one if takes a martingale, M_t, of zero inital value then its expectation at a finite stopping time is zero. So if our deflated pay-off is Z_t and τ denotes a stopping time then

$$\sup_{\tau} \mathbb{E}(Z_\tau) = \sup_{\tau} \mathbb{E}(Z_\tau - M_\tau) \leq \sup_{r} \mathbb{E}(Z_r - M_r),$$

where r is an arbitrary random time. The advantage of the final expression is that there is a clear best random time: take the maximum point on the path. So we get

$$\sup_{\tau} \mathbb{E}(Z_\tau) \leq \mathbb{E}(\max(Z_j - M_j)).$$

The crucial theorem is that the correct choice of M_j yields equality. This correct choice is to take the product itself with the optimal exercise strategy minus the correct amount of numeraire bonds to make M have zero initial value. If a point of exercise is reached then the product is exercised and the product with one less date purchased. The surplus cash which must be non-negative is used to buy numeraire bonds. Application of the method then becomes a question of how to approximate this optimal M_j.

The AB method is to use the product itself with a sub-optimal exercise strategy. Such a strategy will already have been developed when computing a lower bound so no further optimizations are required. The main difficulty lies in the fact that the value of the product is required along the path and this necessitates the use of sub-simulations. It also introduces an additional upwards bias. Whilst the true deflated value of the sub-optimally exercised product is a martingale, once we add the Monte Carlo errors this is no longer true, and that also complicates the theory.

The focus of this paper is the upwards bias caused by the sub-simulations. We prove that under some reasonable assumptions that the bias is indeed inversely proportional to the number of paths, n, in the sub-simulations. Secondly, we look at how the bias can be estimated and removed via Gaussian approximations. This is done both by using analytic formulas for expected maxima of small numbers of Gaussians, and by using a further Monte Carlo simulation for large numbers of them. We test the bias-removal algorithm for long-dated cancellable swaps and see that is effective, with the number of sub-paths required for a given level of error being greatly reduced.

There have been various papers on accelerating upper bound methods. Broadie and Cao (2008) and M. S. Joshi (2007) showed independently that it was not necessary to consider points out-of-the-money thereby reducing the number of sub-simulations required. Belomestny, Bender, and Schoenmakers (2009) introduced an approach based on estimating the martingale increments by regression thereby avoiding sub-simulations. Whilst this method is appealing, it seems hard to get it to provide useful bounds (Andersen & Piterbarg, 2010). M. Joshi and Tang (2014) develop a sub-simulation-free method based on delta hedging with the deltas estimated by regression of Greeks

computed via adjoint differentiation. Whilst this method is effective, it does not seem to be quite as effective as the AB method, and so it is still desirable to work on improvements.

Belomestny, Schoenmakers, and Dickmann (2013) introduce a new "multi-level" method. They adapt the AB method to make the approximate hedge a martingale and construct a sequence of martingales. They then use this sequence to make a new dual estimator of the upper bound. They also show that their modified AB method which takes two sub-simulations a step rather than one has bias proportional to $1/n$. They demonstrate that their new dual method is effective for a Bermudan swaption with 10 exercise dates and a flat yield curve. They defer to future work analysis of its effectiveness in harder cases.

An alternative approach to upper bounds is the multiplicative approach developed by Jamshidian (2004). This has not been as popular as the additive approach; possibly, this is because, as N. Chen and Glasserman (2007) observe, it has higher variance in its raw form.

The structure of this paper is as follows. In Section 2, we give the details of the AB method. We look at the bias when errors are Gaussian in Section 3. We extend to non-Gaussian errors in Section 4. The estimation and removal of Gaussian errors using analytic formulas is addressed in Section 5. We present numerical results for these in Section 6. We look at a Monte Carlo approach to bias estimation in Section 7 and conclude in Section 8.

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2. The Andersen–Broadie method

In this section, we review the Andersen–Broadie method in more detail and establish notation. The essential idea of their method is take the product itself with a sub-optimal exercise strategy as the hedge. They only considered the case of a product that pays only at the time of exercise. However, the method can be applied to a product that pays a stream of cash-flows until exercise. This extension was discussed in M. S. Joshi (2006).

We present the theory without the cash-flows before exercise first. We establish notation. We let (Z_j), $j = 0, ..., \mathcal{J}$ be a non-negative adapted process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The holder of the Bermudan derivative receives the deflated pay-off, Z_j, at time j if they exercise at time j. We shall assume that the derivative cannot be exercised at time zero, and must be exercised at time $\mathcal{J}$ if it has not been previously exercised. This simplifies the mathematics in that we do not have to deal with the case of non-exercise and an exercise out of the money at maturity simply yields a zero cash flow.

The Rogers/Haugh–Kogan method, (Rogers, 2002), (Haugh & Kogan, 2004), requires us to find a martingale M_t typically taken to have zero initial value. The upper bound is then given by

$$\mathbb{E}(\max_j(Z_j - M_j)).$$

The inventors of the method showed that if M_j is chosen to be the product itself with the optimal exercise strategy and correct reinvestment strategy the upper bound is the true price. The value is made zero by shorting numeraire bonds to fund the portfolio.

The reinvestment strategy is that at time of exercise, one must "buy" the product with one less exercise date using the exercise value. Since the strategy is optimal, this will always leave a positive surplus which is used to buy units of the numeraire, whose deflated value is necessarily constant. The Andersen–Broadie method is to use this reinvestment rule but to use a sub-optimal exercise strategy, since the optimal one is not known.

The nature of the method is such that we may need to consider several exercises on one path. It is therefore more convenient to think in terms of exercise indicators. This is natural in that most methodologies for developing exercise strategies approximate the continuation value and say to exercise when it is less than the exercise value. We let i_j be 1 if the exercise strategy says exercise at time j and zero otherwise. We require i_j to be $\mathcal{F}_j$ measurable. The stopping time τ_j is given by the first k such that $k \geq j$ with $i_k = 1$.

So for the AB hedge, we purchase one unit of the derivative paying Z_j with the exercise strategy (τ_j) at time zero, and the initial value is made zero via selling the same value of numeraire bonds. If we reach a point of exercise, say at time k, we exercise and purchase one unit of the derivative with the strategy $(\tau_j)_{j\geq k+1}$ and put the balance of the money in the numeraire. As the deflated value of a self-financing portfolio, this yields a martingale. We will denote the deflated value of this strategy by P_j and the deflated continuation value of the derivative with this exercise strategy by D_j. The hedge is therefore worth

$$P_j = (1 - i_j)D_j + i_j Z_j + \sum_{l<j}(Z_l - D_l)i_l.$$

The fundamental subtlety of the AB method is that D_j's value is not easily accessible. It is therefore estimated via Monte Carlo simulation. We denote the error caused by the estimation by ϵ_j. This will have mean zero. Andersen and Broadie observed that these errors cause upwards bias as well as variance for the upper bound estimate but did not analyze the magnitude of the bias. This bias is the topic of this paper.

The errors, ϵ_j, feed into the upper bound in a slightly non-obvious way. If j is not a time of exercise then the only effect is to replace $Z_j - M_j$ with $Z_j - M_j - \epsilon_j$. However, when j is an exercise time, the value of M_j is not affected since the continuation value is not needed for computing $Z_j - M_j$. It does affect the reinvestment into D_j however, and so affects the estimate of M_j at all subsequent

times. Thus the error at time j is

$$\delta_j = (1 - i_j)\epsilon_j - \sum_{l<j} i_l \epsilon_l.$$

Note that for the first time j such that $i_j = 1$, that is $j = \tau_0$, we have $\delta_j = 0$, and $M_j = Z_j$ so the pathwise maximum always contains a point which is perfectly matched and has zero error. This ensures that the upper bound estimate implied by a given lower bound strategy is always at least as big as the lower bound estimate.

Since ϵ_j is a Monte Carlo error for n paths, it will generally have an error with standard deviation proportional to $1/\sqrt{n}$. One would also expect the bias caused to have at most this size. In practice, practitioners tend to observe a bias of size $1/n$. We will show that this is indeed generally the case.

Before proceeding, we construct a cautionary example where the bias is $1/\sqrt{n}$. Let S_t be a non-dividend paying stock as in the risk-neutral measure of the Black–Scholes model. Let $r = 0$. A contract pays $Z_2 = (S_T - K)_+$ at time $T = 2$. It can be early exercised at time 1 with pay-off Z_1 equal to the Black–Scholes value of a call option with 1 year to go. So the continuation value and exercise value at time 1 are identical. The optimal exercise strategy is to exercise at time 2 always.

We apply the AB method with the optimal exercise strategy. Suppose we have a noisy estimate of the continuation value at time 1 with error

$$\frac{s}{\sqrt{n}} W$$

with W a standard normal and $s > 0$. Our pathwise estimate of the upper bound is the price of the call option with expiry 2 plus the duality gap:

$$\mathbb{E}(\max(sn^{-0.5}W, 0)) = n^{-0.5} s \mathbb{E}(\max(W, 0)).$$

The hedge is exact so the unbiased duality gap estimate is zero. Our bias is of magnitude $n^{-0.5}$. In consequence, if we want it to decay like n^{-1}, we need additional hypotheses. This example relies very heavily on the fact that the set where the continuation value and the exercise value agree is of positive probability. This renders it very artificial. In practice, the gradient of the difference of the exercise value and the continuation value is non-zero and its zero set is a hypersurface of probability zero. In order to avoid pathological situations, in what follows we will therefore assume that the points inside the maximum along each path (before errors are considered) have a jointly continuous density. A second lesson of this example is that the primary part of the bias arises from paths in which the values inside the maximum are similar. It is these paths and points that require close scrutiny.

We have so far discussed only the case where the product pays a single sum on exercise. For many interest rate products, it is more natural to consider a stream of cash-flows occurring until exercise which corresponds to cancelling the contract. This case is discussed in detail in M. S. Joshi (2006).

In practical terms, the AB hedge and the product generate the same stream of cash-flows which cancel each other precisely up to exercise so the impact is minimal.

3. Gaussian errors and bias

In this section, we will derive bounds when the errors are Gaussian of magnitude h. The reader thinking of applications to Monte Carlo simulation should think of $h = n^{-1/2}$ with n the number of paths.

Lemma 1. *Suppose γ is a continuous and bounded probability density on $\mathbb{R}$. Let $\sigma(y)$ be a bounded measurable function of y and $h > 0$ then*

$$\int_{-\infty}^{0} \gamma(y) \int e^{-\frac{z^2}{2}} \max(y + \sigma(y)hz, 0)\, dz\, dy = \mathcal{O}(h^2),$$

for h small.

Proof. The integral is trivially non-negative so we need to show that it is bounded by some multiple of h^2. The inner expectation is increasing in $\sigma(y)$ which is bounded so it is enough to consider the case where σ is constant. Dropping the zero set, we can rewrite the integral as

$$I(h) = \int_{-\infty}^{0} \gamma(y) \int_{-\frac{y}{\sigma h}}^{\infty} e^{-\frac{z^2}{2}} (y + \sigma hz)\, dz\, dy \leq \sigma h \int_{-\infty}^{0} \gamma(y) \int_{-\frac{y}{\sigma h}}^{\infty} e^{-\frac{z^2}{2}} z\, dz\, dy.$$

The inner integral is easily evaluated and we obtain

$$I(h) \leq h\sigma \int_{-\infty}^{0} e^{-\frac{y^2}{2\sigma^2 h^2}} \gamma(y)\, dy.$$

We assumed that γ is bounded so

$$I(h) \leq Ch \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2 h^2}}\, dy.$$

Performing the change of variables, $y = h\bar{y}$, the integrand becomes independent of h and we have

$$I(h) \leq C' h^2,$$

as required. □

Note that whilst we have assumed that γ is bounded for convenience, it is probably not strictly necessary for this result. However, any reasonable continuous density that actually arises in practice will have this property so we do not analyze the general case.

Let $\phi(z)$ denote the $N-$dimensional standard Gaussian density. When applying our results to the Andersen–Broadie method, the standard deviations of the Gaussian errors will depend on the

other state variables of the global simulation. We therefore introduce an auxiliary variable $x \in \mathbb{R}^d$ for some d.

Theorem 1. *Suppose $m = (m_1, m_2, \ldots, m_N)$ has a joint continuous density γ which is bounded. Let $d\Psi_m(x)$ denote the joint distribution of the state variables given m. Suppose that a_{ij} for $i, j = 1, 2, \ldots, N$ are bounded, measurable functions of x, then*

$$\int \gamma(m) \int\int \left(\max_i \left(m_i + h \sum_j a_{ij}(x) z_j \right) - \max_i m_i \right) \phi(z)\, dz\, d\Psi_m(x)\, dm = \mathcal{O}(h^2).$$

Proof. For each m, there must be some value of r with $m_r \geq m_i$ for all i. We can therefore divide up the domain of integration over m accordingly and write the integral as a sum of N terms:

$$\sum_r \int_{m_r \geq m_i} \gamma(m) \int\int \left(\max_i \left(m_i + h \sum_j a_{ij}(x) z_j \right) - m_r \right) \phi(z)\, dz\, d\Psi_m(x)\, dm.$$

It is enough to show that each of these is $\mathcal{O}(h^2)$. The hypotheses are symmetric so it is sufficient to study the case $r = 1$. Let

$$y_i = m_i - m_1,$$

for $i > 1$ and $y_1 = m_1$. We can subtract m_1 through all terms in the maximum, and also subtract $\sum_j a_{1j}(x) z_j$. Thus we can rewrite the integral as a constant times

$$\int_{y_p \leq 0, \forall p > 1} \bar{\gamma}(y) \int\int \max \left(0, \max_{i>1} \left(y_i + h \sum_j b_{ij}(x) z_j \right) \right) \phi(z)\, dz\, d\Psi_{m(y)}(x)\, dy$$

for some bounded measurable functions b_{ij} and $\bar{\gamma}$ the coordinate-changed density.

We replace the inner maximum by a sum to obtain an upper bound on the integral:

$$\sum_{i=2}^{N-1} \int_{y_p \leq 0, \forall p > 1} \bar{\gamma}(y) \int\int \max \left(0, y_i + h \sum_j b_{ij}(y) z_j \right) \phi(z)\, dz\, d\Psi_{m(y)}(x)\, dy.$$

It is enough to bound for each i. The functions $b_{ij}(x)$ are bounded so we can replace them by the maximum of their modulus and only increase values. We then have a sum of independent Gaussian random variables whose sum is also a Gaussian with a fixed standard deviation, s. We can therefore bound the inner integral by a one-dimensional integral, and obtain an upper bound of

$$\int_{y_p \leq 0, \forall p > 1} \bar{\gamma}(y) \int\int \max\left(0, y_i + hsw\right) \phi(w) dw\, d\Psi_{m(y)}(x)\, dy,$$

where w is one-dimensional and s is a fixed real number. We drop the integral with respect to x since it no longer does anything and obtain

$$\int_{y_p \leq 0, \forall p > 1} \bar{\gamma}(y) \int \max\left(0, y_i + hsw\right) \phi(w) dw\, dy.$$

Since everything is non-negative and well-behaved, we can integrate out all dimensions except i, and this reduces us to the one-dimensional case. The result then follows from Lemma 1.

□

4. General Monte Carlo errors and bias

It is often said that the Central Limit theorem states that the mean of a large number of i.i.d. random variables is approximately Gaussian with variance proportional to $1/n$. In order to prove that the bias of the AB method is indeed of size $1/n$, we need to make this statement precise and combine it with our results for the Gaussian case in Section 3.

The main result we will use is a restatement of Berry–Esseen bounds for Lipschitz functions. We shall say that h is Lipschitz with Lipschitz constant L if for all x, y

$$|h(x) - h(y)| \leq L|x - y|.$$

Theorem 2. *Let X_j be i.i.d. random variables with first three moments bounded, and given by μ, σ^2, ρ. Denote a standard normal, $N(0,1)$, random variable by Z. Let*

$$Y_n = \frac{1}{n}\sum_{j=1}^{n} X_j, \quad \text{and let} \quad Z_n = \frac{(Y_n - \mu)\sqrt{n}}{\sigma}.$$

If h is a Lipschitz function with Lipschitz constant L, then

$$|\mathbb{E}(h(Z_n)) - \mathbb{E}(h(Z))| \leq \frac{CL\rho}{\sqrt{n}}$$

for a fixed constant C that does not depend on h.

This is essentially a restatement of results in L. Chen, Goldstein, and Shao (2011).

First, we apply to estimates of the simulation of a single random variable.

Lemma 2. *Let X_j, Y_n, C be as in Theorem 2. Let*

$$W_n = \mu + \frac{\sigma}{\sqrt{n}} Z,$$

with Z a standard Gaussian random variable. Let K be a real number. We then have

$$|\mathbb{E}\left(\max(Y_n, K)\right) - \mathbb{E}\left(\max(W_n, K)\right)| \leq \frac{C\sigma\rho}{n}.$$

Note that we could regard Y_n as the result of an $n-$path Monte Carlo simulation. This lemma says that the difference between the normal approximation and the true value is of order $1/n$.

Proof. Let

$$B_n = \sqrt{n}\left(\frac{Y_n - \mu}{\sigma}\right).$$

We can write

$$\max(x, K) = (x - K)_+ + K,$$

and so we can estimate in that form instead. We obtain

$$\begin{aligned}
&\mathbb{E}\left((Y_n - K)_+ - (W_n - K)_+\right)\\
=&\mathbb{E}\left((\mu + \frac{\sigma}{\sqrt{n}}B_n - K)_+ - (\mu + \frac{\sigma}{\sqrt{n}}Z - K)_+\right),\\
=&\frac{\sigma}{\sqrt{n}}\mathbb{E}\left((B_n + \frac{\sqrt{n}}{\sigma}(\mu - K))_+ - (Z + \frac{\sqrt{n}}{\sigma}(\mu - K))_+\right).
\end{aligned}$$

We now let

$$h_n(s) = \left(\frac{\sqrt{n}}{\sigma}(\mu - K) + s\right)_+.$$

The functions h_n are Lipschitz continuous in s with constant 1. We can therefore apply the Berry–Esseen bound to conclude

$$|\mathbb{E}\left((Y_n - K)_+ - (W_n - K)_+\right)| \leq \frac{\sigma}{\sqrt{n}}\frac{C\rho}{\sqrt{n}} = \frac{C\sigma\rho}{n}.$$

□

In the Andersen–Broadie method, the set-up is that we have sub-simulations at each exercise time on each path to estimate the continuation value. Suppose the true values along the path are μ_i. These are approximated when the continuation value is needed. However, that value may not actually be used at the point for which it is calculated but instead only affect points further along the path. We can write the path-wise maximum as

$$\max_i \left(\mu_i + \sum_j \delta_{ij}\epsilon_j\right)$$

where ϵ_j is the Monte Carlo error for the computation of the continuation value at time j, and δ_{ij} takes values in the set $\{-1, 0, 1\}$. This is actually more general than necessary for Andersen–Broadie but is more than sufficient.

If we are using independent random variables for the sub-simulations, the errors, ϵ_j, will be independent for a given path. We now show that replacing a given error ϵ_j by $\frac{\sigma_j}{\sqrt{n}}Z_j$ with Z_j independent standard Gaussians and $\sigma_j \in \mathbb{R}$ introduces a perturbation of size at most $1/n$.

Theorem 3. *Let $\mu_j \in \mathbb{R}, j = 1, 2, \ldots, N$. Let $X_{j,k}$ be i.i.d. random variables for each j with mean zero, variance σ_j^2 and third moments bounded by ρ. Let $\sigma = \max(\sigma_j)$. Let Z_j be a Gaussian random variable with the same mean and variance as $X_{j,k}$. Let*

$$\epsilon_{j,n} = \frac{1}{n}\sum_{k=1}^{n} X_{j,k},$$

and suppose $\delta_{i,j} \in \{-1,0,1\}$. *We then have that for some* C'

$$\mathbb{E}\left(\max_i \mu_i + \sum_j \delta_{i,j}\epsilon_{j,n}\right) \leq \frac{C'\sigma\rho}{n} + \mathbb{E}\left(\max_i \mu_i + \sum_j \delta_{i,j} n^{-1/2} Z_j\right).$$

Proof. We show that we can replace one value of $\epsilon_{j,n}$ with $n^{-1/2}Z_j$ at a cost of increasing the bound by $C_1\sigma\rho/n$ for some C_1. We start with $j = 1$. The expectation is just an iterated integral and the reader should think of the integral being done from the inside out. The expectation in the following paragraph is the inmost integral.

$$I_1 = \mathbb{E}\left(\max_i \mu_i + \sum_j \delta_{i,j}\epsilon_{j,n}\right) = \mathbb{E}\left(\max_i \mu_i + \frac{\sigma}{\sqrt{n}}\delta_{i,1}\frac{\sqrt{n}}{\sigma}\epsilon_{1,n} + \sum_{j>1} \delta_{i,j}\epsilon_{j,n}\right).$$

Let $\tilde{\mu}_i = \mu_i + \sum_{j>1} \delta_{i,j}\epsilon_{j,n}$. We then have

$$I_1 = \frac{\sigma}{\sqrt{n}}\mathbb{E}\left(\max_i \left(\frac{\sqrt{n}}{\sigma}\tilde{\mu}_i + \delta_{i,1}\frac{\sqrt{n}}{\sigma}\epsilon_{1,n}\right)\right).$$

We now take the function in the Berry–Esseen theorem to be

$$h(t) = \max_i \left(\frac{\sqrt{n}}{\sigma}\tilde{\mu}_i + \delta_{i,1}t\right).$$

Since $|\delta_{i,1}| \leq 1$, h is Lipschitz continuous with constant 1. Using Lemma 2, it follows that

$$I_1 \leq \frac{C_1\sigma\rho}{n} + \mathbb{E}\left(\max_i \mu_i + \frac{1}{\sqrt{n}}\delta_{i,1}Z_1 + \sum_{j>1} \delta_{i,j}\epsilon_{j,n}\right).$$

We now need to do $i > 1$. We repeat the same argument for each i. Adding the constant term $\frac{C_1\sigma\rho}{n}$ to the integrand simply increases the upper bound by the same amount which is fine. We need to rearrange the order of integration to make the value of i we are doing inmost. However, this not an issue since we have assumed finiteness of three moments for the underlying independent random variables and normals have all moments existing. □

We can now apply our results to the case of interest. For our application to the AB method, we take m_i to be the deflated value of $Z_i - P_i$ and require these to have jointly continuous density. Combining Theorems 1 and 3.

Theorem 4. *In the Andersen–Broadie method, assume that the joint density of the pay-offs minus the hedges along paths is continuous and bounded. Assume also that the variances and third moments of the pathwise estimates of deflated continuation values in sub-simulations are uniformly bounded. Given these assumptions, the upwards bias of the method is* $1/n$ *where* n *is the number of paths.*

Note that if the pay-offs are uniformly bounded then the uniform bounds on the moments are immediate. For example, a put option on a basket in the multi-dimensional Black–Scholes model will have maximum value equal to the strike, and so the theorem applies. One could also turn linearly growing pay-offs into bounded pay-offs by choosing an appropriate numeraire. For example, in the Black–Scholes model if we use the stock as numeraire, the deflated value of a call option is bounded.

5. Estimating the bias and removing it

As above, we know that the value for a given path can be written in the form

$$\max_i \left(\mu_i + \sum_j \delta_{ij} \epsilon_j \right).$$

The values of δ_{ij} are determined by the exercise strategy and known once the outer path has been generated. In practice, on a given path we cannot separate the values of μ_i and ϵ_j since they are estimated together via Monte Carlo simulation. However, if we make a Gaussian approximation, we can use the sample estimates of means and standard errors to estimate their distribution. Once this is done, we have a maximum of a vector of correlated jointly Gaussian variables. There exist formulas for such maxima and so we can estimate the bias. The general formula is developed in Eggleston (2013) for M random variables. He proceeds first by subtracting the first term to reduce the problem to that of computing the expected maximum of a constant and $M-1$ random variables. He then divides the problem into a sum of terms according to which random variable achieves the maximum. He is able to develop a formula which involves a sum of $M-1$ dimensional cumulative normals. Since high-dimensional cumulative normals are themselves time consuming to compute, we suggest instead estimating the part using an appropriate small number of points on the path which only requires the bivariate cumulative normal. We use the approximation of Genz (1992) to compute it with little overhead. We need to decide which points to use. We know that bias occurs when points are close together, so we take the two or three largest points on the path.

Thus we proceed as follows. We give the algorithm for using the M greatest points.

(1) Develop a path for the underlying. Find the exercise indicators i_j for the path.
(2) Compute the continuation values $\hat{C}_j$ and their standard errors s_j along the path.
(3) Compute the portfolio approximations along the path

$$\hat{P}_j = (1 - i_j)\hat{C}_j + i_j Z_j + \sum_{l<j} (Z_l - \hat{C}_l) i_l.$$

(4) Identify the M points which give the greatest values to $Z_j - \hat{P}_j$. Denote these $j_1, \ldots, j_M$ in increasing order.

(5) Let $\mu_r = Z_{j_r} - \hat{P}_{j_r}$.
(6) Consider $H = \max(0, \mu_2 - \mu_1, \ldots, \mu_M - \mu_1)$. We compute its expected bias, B, by regarding its elements as correlated Gaussians and taking the difference of its implied expectation and its actual pathwise value.
(7) Our pathwise estimate of the maximum is

$$\max(Z_j - \hat{P}_j) - B.$$

Of course, point 6 is not trivial and requires further elucidation. Subtracting μ_1 will not affect the bias since subtracting the same random variable from all elements of a maximum does not introduce bias. We are making the approximation that $\hat{C}_j$ has a Gaussian error with mean zero and standard deviation s_j. This error is ϵ_j. Provided we use different random numbers for each j, we can take ϵ_j to be a vector of independent Gaussians. Our approximate error for $\mu_r - \mu_1$ is then a linear combination of these Gaussians. In particular, the error is

$$(1 - i_{j_r})\epsilon_{j_r} - \epsilon_{j_1} - \sum_{j_1 < t < j_r} i_t \epsilon_t.$$

Curiously, ϵ_{j_1} always appears: if j_1 is an exercise point, it enters into the reinvestment and so affects μ_r; otherwise, it affects μ_1 directly.

Once we have written the errors as linear combinations of independent Gaussians with known standard deviations, it is straight-forward to compute their covariance matrix. With known covariance matrix and means, we can compute the expectation of the maximum using known formulas (Eggleston, 2013) and we are done.

Clearly, the value of M to use represents a compromise between the time to compute the $M-1$-dimensional cumulative normals and the time to compute sub-simulations. In this paper, we explore M equal to 2 and 3, in these cases, the overhead is negligible. Clearly, taking M less than N is underestimating the bias, since we are effectively taking the error of the $N - M$ lowest points to be zero. We can therefore still expect our partially bias-corrected estimate to be biased high.

6. Numerical results for analytic bias removal

We focus on the case of cancellable swaps studied in Beveridge, Joshi, and Tang (2013). We choose this as it is challenging enough to be interesting without being esoteric and it yields a tough example already in the literature. The three products have 12, 20 and 40 underlying rates. Call the number of rates R. The first one starts in 0.5 years. Coupon payments occur at $0.5 + 0.5j$ for $j = 1, 2, \ldots, R$. It is cancellable at time 1.5 and every 0.5 years thereafter. No rebate is paid on cancellation. The swap pays a fixed rate of 0.04 and receive floating.

The calibration is that we set the forward rate from $0.5j$ to $0.5(j+1)$ to be

$$0.008 + 0.002j + x \quad \text{for} \quad j = 0, 1, 2, \ldots, R.$$

Here $x = 0.01, 0.005, 0.00$ for the three cases, $12, 20$ and 40, respectively. The well-used "abcd" time-dependent volatility structure is used, with

$$\sigma_i(t) = \begin{cases} 0, & t > T_i; \\ (0.05 + 0.09(T_i - t)) \exp(-0.44(T_i - t)) + 0.2, & \text{otherwise}, \end{cases}$$

and the instantaneous correlation between the driving Brownian motions is assumed to be of the form

$$\rho_{i,j} = \exp(-\phi |t_i - t_j|),$$

with $\phi = 2 \times 0.0669$. Displacements for all forward rates are assumed to be equal, with

$$\alpha_j = 1.5\%,$$

for all values of j.

We use a five-factor model. To obtain the reduced pseudo-square root matrices, we proceed as follows.

- Compute the full-factor covariance matrix for the step. This involves integrating the product of the volatility functions and multiplying by the instantaneous correlation for each entry.
- Perform a principal components analysis to obtain eigenvalues and eigenvectors, λ_j and e_j with λ_j decreasing.
- Form a column matrix $B = (\sqrt{\lambda_j} e_j), j = 1, 2, \ldots, 5$.
- Scale the rows of B so that the variances of the log-rates is the same as before factor reduction.

To develop the lower bound exercise strategy, we use the multiple regression version of least-squares suggested in M. S. Joshi (2014). We briefly review the method here. We take quadratic polynomials in the first forward, the co-terminal swap rate and final discount bond as basis functions. We fix a constant $\theta < 1$. We fit the basis functions to continuation values as in the original least-squares method. We then discard the fraction $1 - \theta$ of points estimated to be farthest from the exercise boundary, and repeat the least-squares fit. We do this 5 times. We choose $\theta = 0.25$. For further details of least squares for cancellable products, see Carrière (1996), Longstaff and Schwartz (2001), Amin (2003), and M. S. Joshi (2011). The lower bound strategy in each case is developed using 524288 Sobol paths. A second pass with a further 524288 paths is used to develop the lower bound estimate. Note, however, that the AB method and the extensions here actually estimate the duality gap between the lower and upper bounds so the accuracy of the lower bound values is not of primary importance.

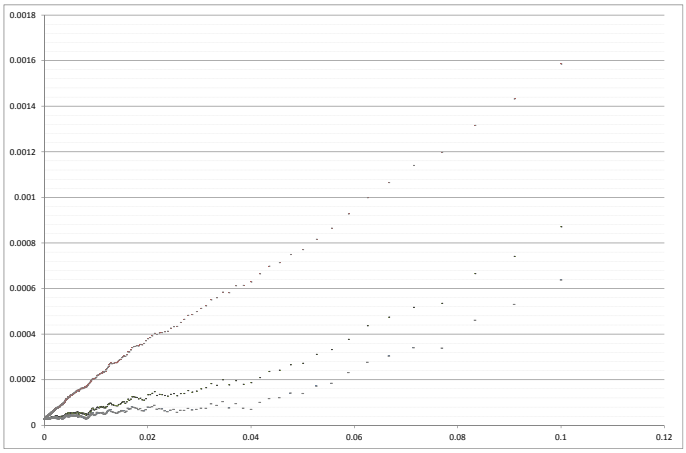

FIGURE 6.1. The duality gap as a function of the reciprocal of the number of paths for the 12 rate product. The upper line is no bias removal. The middle is two point and the bottom is three point.

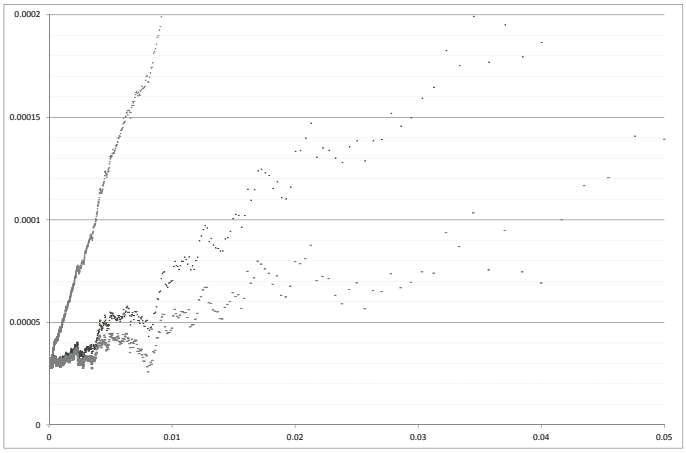

FIGURE 6.2. Figure 6.1 blown up near 0.

For the upper bounds, we use Mersenne twister random numbers. We use 2048 outer paths. The number of outer paths was chosen to be this large in order to reduce noise effects. We develop 8191 inner paths for each sub-simulation. However, we store partial results for each possible number of inner paths up to this to allow behaviour as a function of the number of paths to be examined.

For the 12 rate product, the lower bound estimate is 28.75 basis points. We present duality gap estimates in Table 1. Only 24 inner paths are needed with 3 point removal to obtain a duality gap of 1 basis point. This compares with 257 for no bias removal and 67 with 2 point removal. All 3 methods are heading for a bias gap of $2.76E-5$ which is about a quarter of a basis point. See also Figures 6.1 and 6.2. The case without bias removal clearly shows straight-line behaviour as a function of $1/n$ as expected.

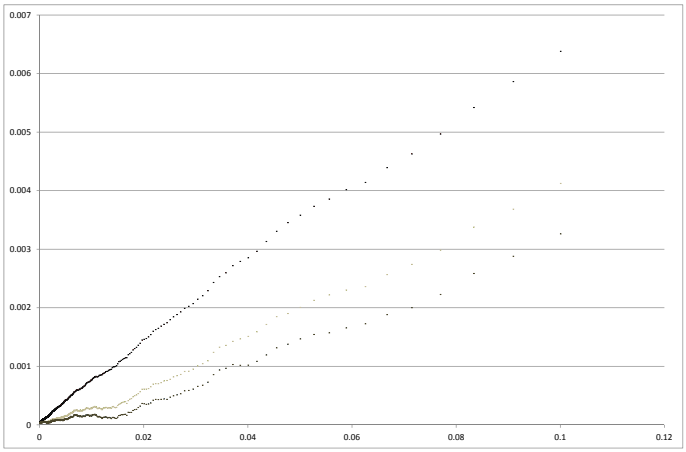

FIGURE 6.3. The duality gap as a function of the reciprocal of the number of paths for the 20 rate product. The upper line is no bias removal. The middle is two point and the bottom is three point.

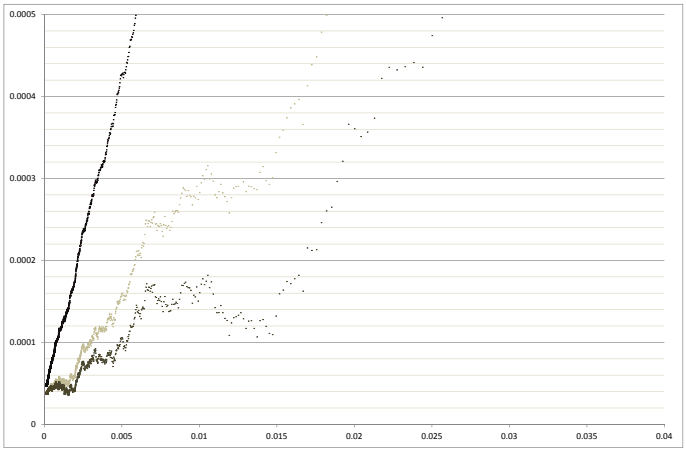

FIGURE 6.4. Figure 6.3 blown up near 0.

For the 20-rate product, the lower bound estimate is 184.7 basis points. We present duality gap estimates in Table 2. Only 190 inner paths are needed with 3 point removal to obtain a duality gap of 1 basis point. This compares with 1164 for no bias removal. All 3 methods are heading for a duality gap of $3.67E-5$ which is about a third of a basis point. See also Figures 6.3 and 6.4. The case without bias removal clearly shows straight-line behaviour as a function of $1/n$ as expected.

For the 40-rate product, the lower bound estimate is 1092.7 basis points. We present duality gap estimates in Table 3. Only 115 inner paths are needed with 3 point removal to obtain a duality gap of 5 basis points. This compares with 556 for no bias removal. To obtain two basis points, the numbers are 394 and 3154. All 3 methods suggest an unbiased duality gap of 1.5 basis points. See

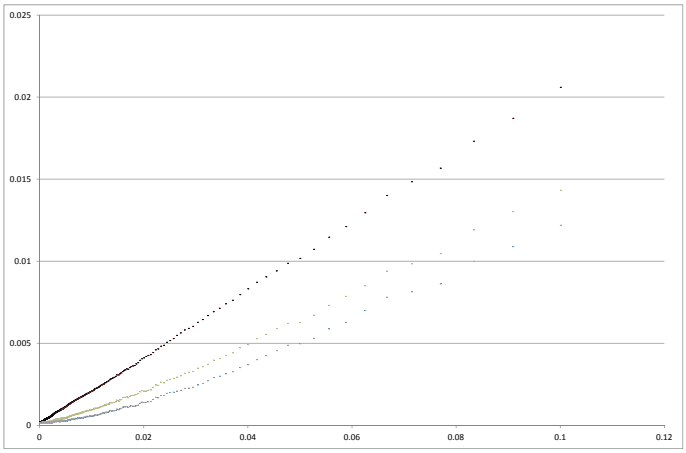

FIGURE 6.5. The duality gap as a function of the reciprocal of the number of paths for the 40 rate product. The upper line is no bias removal. The middle is two point and the bottom is three point.

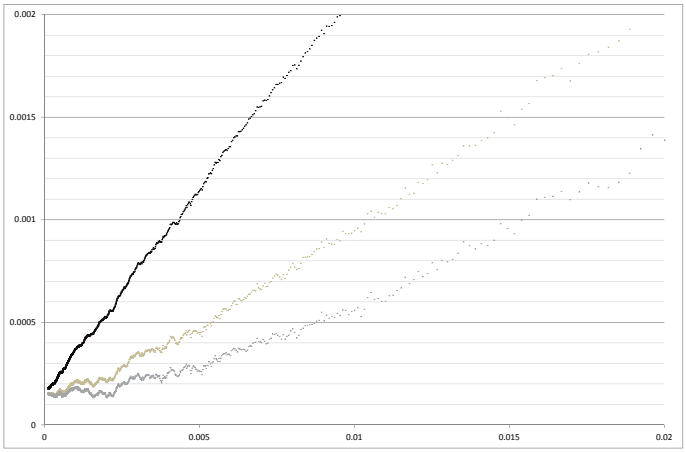

FIGURE 6.6. Figure 6.5 blown up near 0.

also Figures 6.5 and 6.6. Given the complexity of the product, this is a remarkably small duality gap for a not particularly sophisticated methodology for finding exercise strategies.

We do not present timings since the number of additional computations per outer path is very small. We need to find the three largest points, find some variances and compute two bivariate normals. This is a very small computational overhead compared with the cost of running a sub-simulation for every exercise date.

In all three cases, the number of paths to obtain a given bias level is greatly reduced by using the 3-point bias removal. Since the additional computational burden per path is miniscule, a similar proportional reduction in time needed will also be achieved. An interesting follow-up question would

paths	Mean 0	Se 0	Mean 2pt	Se 2pt	Mean 3pt	Se 3pt
10	0.001585	8.97E-05	0.00087	9.12E-05	0.000637	9.09E-05
24	0.000664	4.69E-05	0.000209	4.85E-05	0.00010	4.81E-05
67	0.000296	2.55E-05	0.000101	2.58E-05	6.45E-05	2.58E-05
100	0.00021	1.89E-05	7.04E-05	1.90E-05	4.72E-05	1.89E-05
257	0.000104	1.13E-05	4.27E-05	1.13E-05	3.54E-05	1.14E-05
512	6.81E-05	8.25E-06	3.59E-05	8.24E-06	3.33E-05	8.25E-06
1024	4.79E-05	6.11E-06	3.10E-05	6.05E-06	3.01E-05	6.06E-06
2048	4.07E-05	5.53E-06	3.21E-05	5.50E-06	3.19E-05	5.49E-06
4096	3.40E-05	4.93E-06	2.97E-05	4.94E-06	2.96E-05	4.94E-06
8191	3.01E-05	4.46E-06	2.77E-05	4.47E-06	2.76E-05	4.47E-06

TABLE 1. The duality gap for the 12-rate product with no bias removal, 2 point removal and 3 point removal. The first few levels of inner paths are selected to show how many inner paths are needed to obtain a given duality gap.

paths	Mean 0	Se 0	Mean 2pt	Se 2pt	Mean 3pt	Se 3pt
10	0.006381	0.000258	0.004124	0.000265	0.003263	0.000267
39	0.001845	9.78E-05	0.000819	9.98E-05	0.000496	0.0001
167	0.000503	4.03E-05	0.000207	4.06E-05	0.000137	4.08E-05
190	0.000436	3.53E-05	0.000158	3.58E-05	9.69E-05	3.59E-05
1164	0.000104	1.20E-05	4.81E-05	1.19E-05	4.31E-05	1.19E-05
2048	7.87E-05	9.74E-06	4.65E-05	9.70E-06	4.40E-05	9.73E-06
4096	5.82E-05	7.81E-06	4.22E-05	7.77E-06	4.14E-05	7.78E-06
8191	4.66E-05	7.64E-06	3.70E-05	7.64E-06	3.67E-05	7.63E-06

TABLE 2. The duality gap for the 20-rate product with no bias removal, 2 point removal and 3 point removal. The first few levels are selected to show how many inner paths are needed to obtain a given duality gap.

be to study how increasing the number of points used for bias removal increases the time taken versus the size of the bias reduction.

7. Monte Carlo bias removal

If we are working with 38 exercise dates and we want to fully estimate the bias using analytic formulas then we need to evaluate a $37-$dimensional multivariate cumulative normal function. This is a challenging problem in its own right. An alternative approach is to use another Monte Carlo

Inner paths	Mean 0	Se 0	Mean 2pt	Se 2 pt	Mean 3pt	Se 3pt
10	0.02059	0.00068	0.01432	0.00067	0.01219	0.00068
38	0.00548	0.00025	0.00293	0.00024	0.00208	0.00024
64	0.00318	0.00016	0.00157	0.00016	0.00102	0.00016
115	0.00188	0.00011	0.00085	0.00010	0.00049	0.00011
394	0.00066	0.00005	0.00029	0.00005	0.00020	0.00005
556	0.00050	0.00004	0.00022	0.00004	0.00016	0.00004
1024	0.00037	0.00003	0.00021	0.00003	0.00018	0.00003
2048	0.00026	0.00002	0.00017	0.00002	0.00016	0.00002
3154	0.00020	0.00002	0.00015	0.00002	0.00014	0.00002
4096	0.00019	0.00002	0.00015	0.00002	0.00014	0.00002
8191	0.00018	0.00002	0.00015	0.00002	0.00015	0.00002

TABLE 3. The duality gap for the 40-rate product with no bias removal, 2 point removal and 3 point removal. The first few levels of inner paths are selected to show how many inner paths are needed to obtain a given duality gap.

simulation. Since we are already running a large number, e.g. 37, sub-simulations on each path, adding one more need not have much incremental impact on time. In addition, if we make its driving random variables independent of those used for the rest of the path, its error will average across paths. We therefore get a slight increase in standard deviation but no additional bias coming from its estimation.

In concrete terms, we use Q paths to estimate the expectation over standard normals Z_j of the maximum, that is we compute

$$\mathbb{E}(\max(\mu_i + \sum \delta_{i,j} s_j Z_j)) - \max(\mu_i),$$

where $\mu_i, \delta_{i,j}, s_j$ are determined by the outer path. The computational complexity of this sub-simulation is $\mathcal{O}(QN^2)$. Compare this with the computational complexity of the other computations on each path. If the number of paths per sub-simulation is P, there are k steps remaining, n rates, and F factors then the complexity of each sub-simulation is of order $PknF$. We have one of these for each exercise date so we get a complexity of order

$$PnFN(N-1).$$

For any reasonably complex model, the computations per step will be higher than that of our new simulation, so provided QN^2 is small compared to $PnFN(N-1)$ this suggests that the additional

	0 time	0 mean	0 se	2 pt mean	2 pt se	3 pt mean	3 pt se	64 mean	64 se	8192 mean	8192 se
32	79.848	0.006456	0.00028	0.003562	0.00027	0.002578	0.000273	-7.66E-06	0.000286	-2.48E-05	0.000283
64	155.871	0.003431	0.00017	0.001797	0.000164	0.001257	0.000166	0.000235	0.000169	0.000226	0.000169
128	318.229	0.001716	0.000102	0.000776	0.0001	0.000459	0.000101	5.46E-05	0.000105	4.82E-05	0.000103
256	703.966	0.001011	7.09E-05	0.000487	6.97E-05	0.000362	7.01E-05	0.000244	7.10E-05	0.000239	7.02E-05
512	1690.3	0.000512	3.71E-05	0.000225	3.64E-05	0.000158	3.67E-05	0.000115	3.73E-05	0.000114	3.68E-05
1024	3551.26	0.000335	2.76E-05	0.000175	2.75E-05	0.000145	2.77E-05	0.000131	2.80E-05	0.000134	2.78E-05

TABLE 4. Mean and standard error of duality gap for 40 rate case using no bias reduction, 2 point removal, 3 point removal, and Monte Carlo removal with 64 and 8192 paths.

	0 99%	2pt 99%	3pt 99%	64 99%	8192 99%
32	0.007108	0.00419	0.003214	0.000658	0.000636
64	0.003826	0.002178	0.001643	0.000629	0.00062
128	0.001953	0.001009	0.000693	0.000298	0.000288
256	0.001176	0.00065	0.000524	0.000409	0.000403
512	0.000599	0.00031	0.000243	0.000202	0.000199
1024	0.0004	0.000239	0.000209	0.000197	0.000198

TABLE 5. 99th percentile upper bounds for duality gap for the 40 rate case using no bias reduction, 2 point removal, 3 point removal, and Monte Carlo removal with 64 and 8192 paths.

computation load will be light. Cancelling N, we need

$$Q/P < nF.$$

If $n = 40, F = 5$, this suggests that we could use 200 times as many paths for the bias estimation as for the sub-simulations before we achieve a substantial slowdown.

We focus on the 40-rate case since it is there that additional bias removal is most important. Note that when using a simulation to compute the bias removal, running a given number of paths does not subsume all lower numbers of paths unlike in our previous results. We use 2048 outer paths and vary the number of inner paths. We try $0, 64$ and 8192 paths for bias removal. We present numerical results in Tables 4 and 5. Since some of the means are negative and the use of extra simulations introduces more variance, it is perhaps more instructive to look at the 99th percentile upper bound (assuming normality) which we present in the second table. From the final two columns, using more than 64 paths for the bias removal appears unnecessary. In particular, the 99th percentile upper bound with 32 sub-simulation paths and 64 bias estimation paths is less than 7 basis points. Compare this with the no bias removal case of 512 sub-simulation paths which attains 6 basis points.

The crucial question is, of course, time. We present timings without the bias estimation in Table 4. The additional time for bias estimation for the 64-path case was 1.1 seconds and for the 8192-path case was 25 seconds. Thus the slowdown caused by using 64 paths is minimal in relative terms in all cases and even using 8192 paths increases time by less than 50%. The time taken for the additional estimation will be proportional to the number of outer paths so these results are robust across numbers of paths used. All times were computed using single-thread C++ code on a 3.30 GHz Xeon(R) CPU.

8. Conclusion

We have proven that the bias of the Andersen–Broadie method is indeed $1/n$ in realistic cases. We have also presented a methodology for greatly reducing the size of the bias. This methodology relies on a Gaussian approximation for the errors and is very easily and straight-forwardly computable. The additional time taken for the analytic approach is negligible. For the Monte Carlo simulation approach to bias removal, the additional time taken is small in comparison to the improvements attained. Overall, the number of paths and time taken for the Andersen–Broadie method to achieve good bounds is greatly reduced, rendering the method more useful in practical situations.

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