Asymmetric All-Pay Contests with Heterogeneous Prizes

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Abstract

This paper studies complete-information, all-pay contests with asymmetric players competing for multiple heterogeneous prizes. In these contests, each player chooses a performance level or "score". The first prize is awarded to the player with the highest score, the second, less valuable prize to the player with the second highest score, etc. Players are asymmetric in that they incur different costs of score. The players are assumed to have ordered marginal costs, and the prize sequence is assumed to be either quadratic or geometric. I show that each such contest has a unique Nash equilibrium and exhibit an algorithm that constructs the equilibrium. I then apply the main result to study: (a) the issue of tracking students in schools, (b) the incentive effects of "superstars", and (c) the optimality of winner-take-all contests.

Keywords: all-pay, contest, asymmetric, heterogeneous

1   Introduction

The winner of the 2011 US Open tennis tournament was awarded a prize of $1.65M. The runner-up won $800K whereas those in joint third position—that is, the losing semi-finalists—won $400K each. This prize sequence was convex—the difference in the prizes for winner

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and the runner-up was greater than the difference in the prizes for the runner-up and the semi-finalists. In fact, at this tournament the prize for a particular rank was roughly twice the prize for the next rank.[1] In research and development competitions, the winner may win a major contract while other participants receive smaller contracts. Similar examples include the competition among students for grades; the competition among employees for different promotion opportunities, etc. The key characteristics common to these contests are: heterogeneous prizes awarded solely on the basis of relative performance; convex prize sequences; participants with possibly different abilities; and sunk costs of participants' investments.

This paper studies complete-information all-pay contests in which participants with differing abilities compete for heterogeneous prizes. The participants have different costs of performance, and their marginal costs are ordered (a stronger participant's marginal cost is no more than that of a weaker participant at any performance level). The prize sequence is either *quadratic* (the second-order difference in prizes is a positive constant) or *geometric* (the ratio of successive prizes is a constant). Each player chooses a costly performance level—or "score", and the player with the highest performance receives the highest prize, the player with the second-highest performance receives the second highest prize and so on (the prizes may be allocated randomly in the case of a tie). A player's payoff is his winnings, if any, minus his cost of performance. Costs are incurred regardless of whether he wins a prize or not.

My main result is that such contests have a *unique* equilibrium. Moreover, I provide an algorithm that computes the equilibrium. The uniqueness result relies essentially on the fact that no two participants have exactly the same costs—when two or more participants have the same cost, there may be multiple equilibria (see Example 1 below). Moreover, as the example illustrates, different equilibria may lead to different allocations and different total expected score/effort/performance. In many applications, the total expected score is the objective of the designer or planner and so when there are multiple equilibria, it is difficult to compare different designs. Our result demonstrates, however, that the uniqueness of equilibrium is a generic property and so, in cases in which multiplicity occurs, the result can be used to select an equilibrium as a limit of the sequence of unique equilibria of arbitrarily close contests.

The fact that the unique equilibrium can be explicitly constructed allows us to address some interesting questions concerning competitions where relative performance is the key. Here are three examples. First, consider the issue of tracking students in schools. The

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[1]Similarly, at the 2011 US Open golf tournament, the winner received \$1.44M, the runner-up \$865K, the four players tied for third-place received \$364K each—the average of the prizes for positions 3 to 6.

tracking system typically identifies the students' abilities and groups students with similar abilities together.[2] Consider a situation in which a school wants to allocate a group of students with different abilities into different classrooms in order to maximize the students' total effort. Should the school track the students, i.e., group students with similar abilities together, or, should the school not track the students, i.e., group students with different abilities together? We demonstrate, in an example below, that the answer depends on the returns to education for the lower-ranked students in each classroom. In particular, if the returns are not too small, tracking is *better* than not tracking, but if the returns are small, tracking is *worse* than not tracking.

Second, consider a situation in which the designer of the contest has some fixed total amount as prize money. Is a winner-take-all prize structure—in which the whole amount is won by the highest-ranking participant—optimal (in the sense of maximizing total performance) or should the total amount be split into two or more prizes? In an interesting paper, Moldovanu and Sela (2001) have shown that when the participants are ex ante symmetric and the costs are linear, then a winner-take-all prize structure is indeed optimal. We show below that this result does not hold in our model when participants are asymmetric. It should be noted that their model is one of incomplete information whereas the model in this paper is one of complete information.

Third, consider a situation in which the set of contestants consists of one "superstar" of very high ability (very low cost) and a group of players of moderate ability. Brown (2011) has exhibited what is known as the "Tiger Woods" effect—the presence of a superstar in the contest causes the other players to decrease their effort levels. We show below that Brown's (2011) theoretical result relies on the assumption that the other players are symmetric.[3] Suppose we have a situation in which there is a group of asymmetric players, say, with one player who is a "star" but not a superstar. What happens if a "superstar" with very high ability replaces the weakest player? It turns out that in this case, the entry of the superstar may actually increase the effort of existing players.

Literature   There is a substantial literature on all-pay contests and, closely related, all-pay auctions. Since a very nice survey of the whole field can be found in the book by Konrad (2009), in what follows, we discuss only the work that is directly related to this paper.

Complete-information all-pay auctions can be shown to be isomorphic to all-pay contests. Complete-information all-pay auctions with a single prize were analyzed by Baye, Kovenock,

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[2]Students are generally placed into academic, general, or vocational tracks (see Shaw, 2000).

[3]Brown's (2011) analysis is in the context of a Tullock game.

and de Vries (1996). The case of multiple prizes with symmetric players was considered by Barut and Kovenock (1998). Both of these papers provide conditions under which there is a unique equilibrium and also demonstrate the possibility of multiple—actually a continuum of—equilibria.

The various studies of all-pay contests with multiple prizes differ along two dimensions: the structure of the sequence of prizes, $v^1 \geq v^2 \geq ... \geq v^m$, and the players' cost functions $c_i(s)$. Clark and Riis (1998) study contests in which the prizes are homogeneous while players are asymmetric but with linear (constant marginal) costs. They show that under these conditions there is a unique equilibrium. Siegel (2010) shows uniqueness also assuming a constant prize sequence but allowing for very general, possibly nonlinear, cost functions. Bulow and Levin (2006) consider situations in which prizes are different, assuming that the prize sequence is *arithmetic*, that is, the difference in successive prizes is a constant. Costs are assumed to be linear but may differ across players. Again, uniqueness obtains. González-Díaz and Siegel (2010) extend the work of Bulow and Levin (2006) by allowing for some special kinds of nonlinear costs. None of these papers, however, consider the case of convex prize sequences, the distinguishing feature of this paper. Table 1 provides an "at-a-glance" comparison of the various models along the two dimensions.

	PRIZE SEQUENCE	COSTS
Clark and Riis (1998)	Homogeneous $v^k = v^{k+1}$	Different linear
Siegel (2010)	Homogeneous $v^k = v^{k+1}$	Arbitrary
Bulow and Levin (2006)	Arithmetic $v^k - v^{k+1} = \beta$	Different linear
González-Díaz and Siegel (2010)	Arithmetic $v^k - v^{k+1} = \beta$	Nonlinear $\gamma_i c(s)$
This paper	Quadratic $(v^k - v^{k+1}) - (v^{k+1} - v^{k+2}) = \beta$	Nonlinear
This paper	Geometric $v^k = \alpha v^{k+1}$	Nonlinear

Table 1. All-Pay Contest Models

Our main result relies on an algorithm to construct a Nash equilibrium. We show that the algorithm results in only one equilibrium and there are no other equilibria. The key element of this algorithm is that the upper support (the least upper bound of the support) of a weaker player's strategy is a best response to the strategies in a contest in which only players stronger than him participate. This feature allows us to start with a set of strong

players and determine the upper support of the next strongest player, and therefore determine his equilibrium payoff. Then, we can derive his strategy and move on to determine the upper support of another, still weaker player.

The equilibrium in this paper cannot be constructed by the existing methods. This is because the equilibrium has two differences from the equilibria in the literature. First, the highest scores chosen by different players could be different. In contrast, the highest scores are the same in the contests, called simple, studied by Siegel (2010). This difference makes it hard to obtain equilibrium payoffs in the way that Siegel (2009) does.[4] Since the algorithm by Siegel (2010) starts with equilibrium payoffs, it cannot be used in my setting. Second, there could be gaps in the support of a player's equilibrium strategy. There is no gap in the contests studied by Bulow and Levin (2006), so their algorithm cannot be used here either.

The rest of this paper is organized as follows. Section 1.1 gives a simple example illustrating how to construct a Nash equilibrium and why the equilibrium is unique. Section 2 introduces the general model. Section 3 discusses equilibrium properties and Section 4 exhibits an algorithm and shows that it constructs the unique equilibrium for linear costs. Section 5 then extends the analysis to nonlinear costs. Section 6 applies our results to study the issue of tracking students in schools, whether winner-take-all contests are optimal, and the effect of "superstars". Section 7 discusses extension to general convex prize sequences and Section 8 concludes.

1.1   An Example

Let us start with a simple example. Consider a situation with three players competing for two prizes worth \$4 and \$1, respectively. Each player chooses a "score" (or performance level) $s \geq 0$. The players incur constant marginal costs of performance, and the costs of score are $c_1(s) = 4s$ for player 1, $c_2(s) = 6s$ for player 2, and $c_3(s) = 7s$ for player 3. The player with the highest score receives the first prize of \$4, the one with second-highest receives the second prize of \$1 and the one with lowest score receives \$0. If two or more players choose the same score, then the prizes are allocated among them, perhaps randomly, in a way that the expected prize accruing to each is positive. A player's payoff is the value of his prize less the cost of his performance.

In what follows, we construct a mixed strategy Nash equilibrium of this contest (it is easy to see that there cannot be a pure strategy equilibrium). It is assumed that no player's

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[4]Because the prizes are heterogeneous, players' *reach* (Siegel, 2009) is not well defined in the context of this paper, so his characterization of equilibrium payoffs does not apply here.

strategy assigns positive probability to any score $s > 0$ (this is a general property of equilibria and will be established later).

- Let $\bar{s} = 4/7$ be the upper support of all the players' strategies. Then their equilibrium payoffs must be $u_1 = v^1 - c_1(\bar{s}) = 12/7$, $u_2 = v^1 - c_2(\bar{s}) = 4/7$ and $u_3 = v^1 - c_3(\bar{s}) = 0$. This is because by choosing $\bar{s}$ a player wins the first prize for sure.

- For each $s \leq \bar{s}$, consider the following three-equation system in three variables G_1, G_2 and G_3 :

$$
\begin{aligned}
G_2 G_3 v^1 + (G_2(1 - G_3) + (1 - G_2)G_3)v^2 - c_1(s) &= u_1 \\
G_1 G_3 v^1 + (G_1(1 - G_3) + (1 - G_1)G_3)v^2 - c_2(s) &= u_2 \\
G_1 G_2 v^1 + (G_1(1 - G_2) + (1 - G_1)G_2)v^2 - c_3(s) &= u_3
\end{aligned}
$$

The first equation says that the mixed strategies G_2 and G_3 (cumulative distribution functions) for players 2 and 3 keep player 1 indifferent among any score $s \leq \bar{s}$, that is, his payoff from choosing any s is the same as his equilibrium payoff u_1. The other two equations are analogous.

- For $s \leq \bar{s}$, let $\hat{G}_1(s), \hat{G}_2(s), \hat{G}_3(s)$ be the solution to the system of equations above. Figure 1 depicts the three functions. Two facts are worth noticing. First, $\hat{G}_1(s)$ reaches zero at $\underline{s}_1 = 0.05$. Second, $\hat{G}_3$ is not monotone so cannot be a legitimate mixed strategy.[5]

- Define G_3^* to be the smallest monotone function G that satisfies $G \geq \hat{G}_3$. As depicted in Figure 2, G_3^* is constant over the interval $[0.05, 0.34]$. This will be the equilibrium strategy for player 3 and if he uses G_3^*, then this means that player 3 never chooses a score in this interval, that is, there is a *gap* in the support of his mixed strategy. Thus only players 1 and 2 choose scores $s \in [0.05, 0.34]$. For $s \in [0.05, 0.34]$, let G_1^* and G_2^* be the solution to the system

$$
\begin{aligned}
G_2 G_3^* v^1 + [G_2(1 - G_3^*) + (1 - G_2)G_3^*]v^2 - c_1(s) &= u_1 \\
G_1 G_3^* v^1 + [G_1(1 - G_3^*) + (1 - G_1)G_3^*]v^2 - c_2(s) &= u_2
\end{aligned}
$$

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[5]Throughout, by monotone, we mean non-decreasing.

Notice that this is the same as the system above except that we have fixed player 3's strategy to be G_3^*.

- For scores $s < \underline{s}_1 = 0.05$, only players 2 and 3 are active. For $s \in [0, 0.05]$, let G_2^* and G_3^* be the solution to the system

$$
\begin{aligned}
G_3 v^2 - c_2(s) &= u_2 \\
G_2 v^2 - c_3(s) &= u_3
\end{aligned}
$$

- To complete the construction of the equilibrium strategies, for $s \in [0.34, 0.57]$, let $G_i^* = \hat{G}_i$.
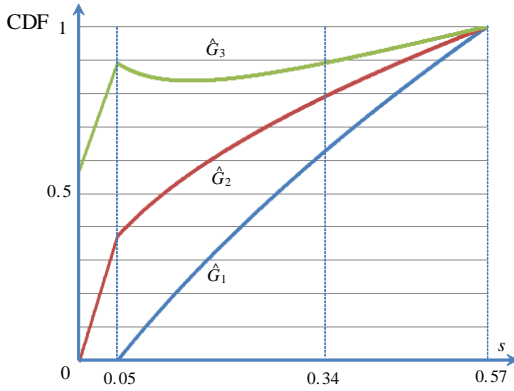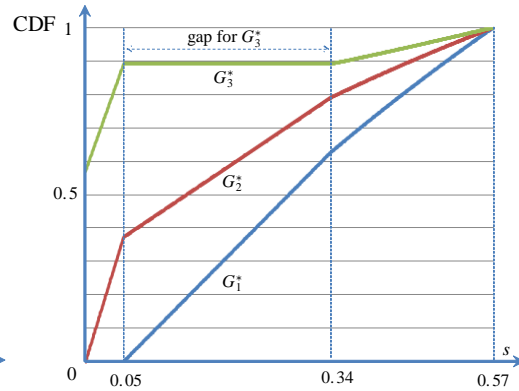

Figure 1

Figure 2

Figure 2 illustrates the equilibrium strategies (G_1^*, G_2^*, G_3^*). Why does player 3 not choose a score in the gap? For any score s in the gap, let us compare $G_i(s)$ and $G_i^*(s)$ for all the players. Since $G_3(s)$ is lower than $G_3^*(s)$, this means that the strategy G_3^* for player 3 is "less aggressive" than G_3. Therefore, if player 3 switched from G_3 to G_3^* while players 1 and 2 continued to play G_1 and G_2, this would cause their payoffs to increase. As a result, to maintain their payoffs u_1 and u_2, both players 1 and 2 would have to become more aggressive. That is, for any s in the gap, $G_1(s) < G_1^*(s)$ and $G_2(s) < G_2^*(s)$. As a result, player 3's payoff from playing any s in the gap would be less than u_3 after the change from (G_1, G_2) to (G_1^*, G_2^*), hence he would not deviate to any score in the gap.[6] Section 3 and 4 show that no other deviation is profitable.

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[6] If costs are linear and the prize sequence is arithmetic, then there cannot be any gaps (Bulow and Levin, 2006).

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Moreover, for any Nash equilibrium (G_1^*, G_2^*, G_3^*), if we start the algorithm with $\bar{s}_1^*$, the upper support of G_1^*, the algorithm constructs a unique Nash equilibrium according to Section 4. Therefore, there is no other equilibrium with the same maximum score. Are there other equilibria with different maximum scores? Suppose there is and the upper support is $\bar{s}_1^* + \varepsilon$. If we start the algorithm with $\bar{s}_1^* + \varepsilon$, the strategies we construct is G_i^* shifted by ε. If $\varepsilon > 0$, then the lower support of player 2's strategy is ε. Consequently, player 3 would not choose a score between 0 and ε and so player 2 would prefer to deviate to a score above 0, which is a contradiction. If $\varepsilon < 0$, then the strategies of both players 2 and 3 would have a mass point at 0 and this too is a contradiction.

2   Model

Consider a complete-information, all-pay contest with n players in $\mathcal{N} = \{1, 2, ..., n\}$. There are $m \leq n$ monetary prizes in amounts $v^1 > v^2 > ... > v^m > 0$ to be awarded. The ordered set of prizes $(v^k)_{k=1}^m$ is called a *prize sequence.*

Players choose their *scores* $s_i \geq 0$ simultaneously and independently. The player with the highest score wins the highest prize, v^1; the player with the second-highest score wins the second prize, v^2; and so on. In case of a tie, prizes are awarded in a way, perhaps randomly, that all tying players have a positive expected prize.[7]

The cost of score s for player i is $c_i(s)$, where $c_i(s)$ is differentiable and $c_i(0) = 0$.

Definition 1 *The players have* ordered marginal costs *if* $0 < c_1'(s) < ... < c_n'(s)$ *for all* $s \geq 0$.

Thus, players are *strictly* ordered according to "ability". In particular, player 1 is the strongest in the sense that his marginal cost is the lowest, player 2 is the second strongest, etc. If $i < j$, I will say that player i is *stronger* than player j and equivalently, or player j is *weaker* than player i. Note that both linear and nonlinear costs could have ordered marginal costs.

Player i's payoff is $v^k - c_i(s_i)$ if he chooses score s_i and wins the kth prize. All players are risk-neutral.

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[7]In many tournaments (for example, in golf), ties are resolved by a sharing of the prizes. As an example, if players i and i' tie with the second-highest score, then each receives $(v^2 + v^3)/2$. Our formulation allows this kind of sharing.

Definition 2 $\left(v^k\right)_{k=1}^m$ *is a* quadratic prize sequence *(QPS) if*

$$\left(v^k - v^{k+1}\right) - \left(v^{k+1} - v^{k+2}\right) = \beta$$

for $k = 1, ..., m - 3$ *where* $\beta > 0$ *is a constant.*

If we normalize the lowest prize $v^m = 1$, then $v^k = (m - k + 1)\left[(m - k)\beta + 2\right]/2$.

Definition 3 $\left(v^k\right)_{k=1}^m$ *is a* geometric prize sequence *(GPS) if* $m = n$ *(so that* $v^n > 0$*) and* $v^k = \alpha v^{k+1}$ *for* $k < n$, *where* $\alpha > 1$ *is a constant.*

In a GPS, the number of prizes must be the same as the number of players. If we normalize $v^n = 1$, then $v^k = \alpha^{n-k}$.

A profile of strategies constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players.[8] The main result of this paper is:

Theorem 1 *Every all-pay contest with a quadratic or a geometric prize sequence and ordered marginal costs has a unique Nash equilibrium.*

The following example shows that the conclusion of the theorem may fail if some players have the same cost function.

Example 1 *Suppose that there are four players* $(n = 4)$ *competing for two prizes* $(m = 2)$ *worth* $v^1 = 3$ *and* $v^2 = 1$. *The players' costs are:* $c_1' = 1/10$, $c_2' = 1$, $c_3' = c_4' = 6/5$.

There are at least two Nash equilibria. First, there is a "type-asymmetric" equilibrium—player 3 chooses positive scores while player 4, with the same costs as those of player 3, always chooses zero. The equilibrium strategies are:

$$
\begin{aligned}
G_1^* &= s/2 - 3/8, \quad s \in [3/4, 11/4] \\
G_2^* &= \begin{cases} 6s/5, & s \in [0, 3/4) \\ s/20 + 69/80, & s \in [3/4, 11/4] \end{cases} \\
G_3^* &= s + 1/4, \quad s \in [0, 3/4] \\
G_4^* &= 1
\end{aligned}
$$

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[8]The same definition is used by Siegel (2010).

Second, there is a "type-symmetric" equilibrium, in which the equilibrium strategies are:

$$
\begin{aligned}
G_1^{**} &= s/2 - 3/8, \quad s \in [3/4, 11/4] \\
G_2^{**} &= \begin{cases} (s+1/4)^{-1/2}\, 6s/5, & s \in [0, 3/4) \\ s/20 + 69/80, & s \in [3/4, 11/4] \end{cases} \\
G_3^{**} &= G_4^{**} = (s+1/4)^{1/2}, \quad s \in [0, 3/4]
\end{aligned}
$$

While the two equilibria are payoff equivalent, the allocations of prizes in the two are different—the probabilities with which players 2, 3 and 4 win the different prizes are not the same. More important, the total expected score (or performance levels) are also different. This is significant because in many applications, this may be the appropriate objective function of the planner.

If we consider a sequence of contests in which only player 4's costs are perturbed so that $c_4'^t < c_3'$ and $c_4'^t \uparrow c_4'$, then for each t, Theorem 1 implies that there is a unique equilibrium. The corresponding sequence of equilibria converges to the type-asymmetric equilibrium identified above.

3 Equilibrium Properties

In this section, I first study several properties of Nash equilibria of asymmetric contests with price sequences that satisfy either of the two conditions stated above—QPS or GPS. Equilibria typically involve mixed strategies.

We begin with the observation that a contest with QPS or GPS has at least one equilibrium. Siegel (2009) established the existence of an equilibrium when the prizes are homogeneous but his proof is readily adapted to include the kinds of prize sequences considered here. In the interests of space, I omit the minor details.[9]

The following properties of an all-pay contest are either well-known or easily derivable from known results in the literature[10]. In *any* equilibrium:

- No player chooses a score $s > 0$ with positive probability.

- Player $i > m + 1$ chooses score 0 with probability one.

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[9] If we replace "the probability of winning" by "probability of winning one prize", the proofs for Tie Lemma and Zero Lemma of Siegel (2009) are still true here. If we replace i's reach by the v^1/c_i, the proof of Corollary 1 of Siegel (2009) is also true here.

[10] See Bulow and Levin (2006) and Siegel (2009, 2010).

- At least two players choose each s between 0 and the highest score chosen by any player.

Since the first bullet above implies that there is no pure strategy equilibrium, a Nash equilibrium (henceforth, equilibrium) consists of a set of cumulative distribution functions $(G_i^*)_{i=1}^n$, where G_i^* represents i's mixed strategy. Let $(g_i^*)_{i=1}^n$ denote the corresponding densities, provided that they exist.

Let $\mathcal{P}(s)$ denote the set of players who, in equilibrium, choose scores both above and below s, that is,

$$\mathcal{P}(s) = \{i \mid G_i^*(s) \in (0,1)\}$$

Let $\mathcal{A}(s)$ denote the set of players that have positive densities around s, that is,

$$\mathcal{A}(s) = \{i \mid \text{there exist } s_l \to s \text{ such that for all } l, \ g_i^*(s_l) > 0\}$$

We refer to $\mathcal{A}(s)$ as the set of *active* players at s, and to $\mathcal{P}(s)$ as the set of *participating* players at s. Note that $\mathcal{A}(s) \subseteq \mathcal{P}(s)$ but if there is a gap containing s in the support of G_i^*, then i is in $\mathcal{P}(s)$, but not in $\mathcal{A}(s)$. The properties in the following lemmas are specific to the contests with QPS or GPS.

Lemma 1 (Stochastic Dominance) *For any player $i < n$, $G_i^*(s) \le G_{i+1}^*(s)$; if $i, i+1 \in \mathcal{P}(s)$, then $G_i^*(s) < G_{i+1}^*(s)$.*

The following lemma establishes that at any point s that is in the interiors of the equilibrium supports of two players, the densities associated with the equilibrium strategies can also be ordered. Of course, this implies that the supports of their mixed strategies must differ.

Lemma 2 (Ordered Densities) *If players $i, i+1 \in \mathcal{A}(s)$, and s is an interior point of the supports of both G_i^* and G_{i+1}^*, $g_i^*(s) > g_{i+1}^*(s)$.*

4 Linear Costs

Since the proof of the theorem is easier with linear costs, we first consider linear costs in this section, and extend the proof to nonlinear costs in Section 5. Since the costs are linear, their marginal costs are denoted as $c_1 < c_2 < ... < c_n$. We first introduce an algorithm that constructs a unique set of strategies. Second, I show that this set of strategies is actually the unique equilibrium.

4.1 Algorithm

The algorithm is schematically represented in Figure 3, and explained after that. The equilibrium construction in the example of Section 1.1 is a special case of this algorithm. For the general case, there is a complication if the upper supports of equilibrium strategies differ, and Step 1 of the algorithm deals with this complication.

Step 1.1 To start the algorithm, pick any $\bar{s} > 0$.

Step 1.2 Let $\bar{s}_2 = \bar{s}$. Suppose only players 1 and 2 compete for v^1 and v^2. Suppose that both choose scores only from $[0, \bar{s}_2]$. In that case, their payoffs must be $u_i = v^1 - c_i \bar{s}_2$. This is because by choosing $\bar{s}_2$, a player wins the first prize for sure. For $s \leq \bar{s}_2$, there exist unique $G_1(s) \leq 1$ and $G_2(s) \leq 1$ that solve the system[11]:

$$G_2 v^1 + (1 - G_2) v^2 - c_1 s = u_1$$
$$G_1 v^1 + (1 - G_1) v^2 - c_2 s = u_2$$

Extend the solution for $s < \bar{s}_2$ until $s = \underline{s}_1 > 0$ such that $G_1(\underline{s}_1) = 0$. Since $c_2 > c_1$, $G_2(\underline{s}_1) > 0$. The functions G_1, G_2 are now well-defined for $s \in [\underline{s}_1, \bar{s}_2]$. We call G_1, G_2 the *pseudo* strategies yielding u_1 and u_2.

Repeat the following step for $i = 3, ..., m + 1$.

Step 1.i Suppose players $1, 2, ..., i - 1$ use the strategies $G_1, G_2, ..., G_{i-1}$ determined in Step 1.$(i - 1)$. Let $\bar{s}_i$ be the infimum of all scores $s \in [\underline{s}_{i-2}, \bar{s}_{i-1}]$ that maximize player i's payoff against $G_1, G_2..., G_{i-1}$. If player i chooses scores from only $[0, \bar{s}_i]$, then his payoff must be u_i.

For $s \geq \bar{s}_i$, $G_1, G_2, ..., G_{i-1}$ remain the same because i does not choose above $\bar{s}_i$. However, for $s \leq \bar{s}_i$, the strategies of players $1, 2, ..., i - 1$ are different from those determined in Step 1.$(i-1)$. For $s \leq \bar{s}_i$, there exist unique $G_1(s), G_2(s), ..., G_i(s) \leq 1$ that solve the system of i equations: for $j \leq i$

$$W(\mathbf{G}_{-j}, \mathbf{v}) - c_j s = u_j \tag{1}$$

where $\mathbf{v} = (v^1, v^2, ..., v^i)$ and $W(\mathbf{G}_{-j}, \mathbf{v})$ represents the expected winnings of player j when the others use strategies $\mathbf{G}_{-j} = (G_1, ..., G_{j-1}, G_{j+1}, ..., G_i)$. Extend the solution

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[11] The verification of this and other claims can be found in Appendix C.

Step 1.2 $\boxed{\text{Pick any } \bar{s} > 0 \rightarrow u_1, u_2 \rightarrow G_1, G_2 \text{ yielding } u_1, u_2}$

replace i with $i+1$ $i = 3$

Step 1.i $\boxed{\text{Find } i\text{'s BR } \bar{s}_i \text{ against } G_1, \ldots, G_{i-1} \rightarrow u_i}$

$i \le m$

$\boxed{G_1, \ldots, G_i \text{ yielding } u_1, \ldots, u_i}$

Step 2 $i = m+1$

$\boxed{\text{Define } \hat{G}_1, \ldots, \hat{G}_{m+1} \text{ by shifting } G_1, \ldots, G_{m+1}}$

replace i with $i-1$

$\hat{G}_i$ is not monotone $\hat{G}_i$ is monotone

Step 3.i $\boxed{\text{Define } G_i^* \text{ and } \hat{G}_1, \ldots, \hat{G}_{i-1} \text{ after fixing } \hat{G}_i}$ $\boxed{\text{Define } G_j^* = \hat{G}_j \text{ for } j = 1, \ldots, i}$

$i > 3$ $i = 3$

$\boxed{\text{Define } G_i^* = \hat{G}_i \text{ for } i = 1, 2}$

Step 4 $\boxed{\begin{array}{l}\text{Define } G_i^* = 1 \text{ for } i = m+2, m+3, \ldots, n. \\ \text{Therefore, we have } G_i^* \text{ for } i = 1, \ldots, n\end{array}}$

Figure 3. Algorithm

for $s < \bar{s}_i$ until $s = \underline{s}_1 > 0$ such that $G_1\left(\underline{s}_1\right) = 0$. Since $c_{j+1} > c_j$, $G_{j+1}\left(\underline{s}_1\right) > G_j\left(\underline{s}_1\right) > 0$. Since $G_1\left(\underline{s}_1\right) = 0$, player 1 does not choose below $\underline{s}_1$.

For $s \le \underline{s}_1$, let $G_1\left(s\right) = 0$. For $s < \underline{s}_1$, substitute $G_1\left(s\right) = 0$ into (1) for $j > 1$. Again, find the unique solution $G_2\left(s\right), ..., G_i\left(s\right) \le 1$ for the resulting system. Extend this solution until $s = \underline{s}_2 > 0$ such that $G_2\left(\underline{s}_2\right) = 0$.

For $s \le \underline{s}_2$, let $G_1\left(s\right) = G_2\left(s\right) = 0$. Similarly, for $s < \underline{s}_2$, substitute $G_1\left(s\right)$ and $G_2\left(s\right)$ into (1) for $j > 2$. Again, find the unique solution $G_3\left(s\right), ..., G_i\left(s\right) \le 1$ for the resulting system. Extend this solution until $s = \underline{s}_3 > 0$ such that $G_3\left(\underline{s}_3\right) = 0$ and continue in this manner.

Consequently, we have constructed functions $G_1, ..., G_i$ for $s \in \left[\underline{s}_{i-1}, \bar{s}\right]$, where $G_{i-1}\left(\underline{s}_{i-1}\right) = 0$. We call $G_1, ..., G_i$ as the *pseudo strategies* yielding $u_1, ..., u_i$.

Step 2 Define $\hat{G}_i$ for $i = 1, ..., m+1$ as $\hat{G}_i\left(s\right) = G_i(s - \underline{s}_m)$. Note first that, $\hat{G}_i$ is continuous and lies in $[0, 1]$ for $i = 1, 2, ..., m+1$, and $\hat{G}_m\left(0\right) = 0$. Second, the payoffs associated with the pseudo strategies are $u_i^* = u_i + c_i\underline{s}_m$ for $i = 1, ..., m+1$ and $u_{m+1}^* = 0$. Third, $\hat{G}_i$ may be decreasing at some scores and so it may not be a legitimate mixed strategy.

We say that a continuous function $G\left(s\right)$ has a *dent* over $\left(s', s''\right)$ if i) $G\left(s'\right) = G\left(s''\right)$; ii) $G\left(s\right) \le G\left(s'\right)$ for $s \in \left(s', s''\right)$. Figure 4 illustrates a dent $\left(s', s''\right)$ for function G.

Next, we fix the non-monotonicity of the pseudo strategies $\hat{G}_1, ..., \hat{G}_{m+1}$ by replacing these with monotone functions in a way that yields the *same* payoffs. Repeat the following steps for $i = m+1, m, ..., 3$.
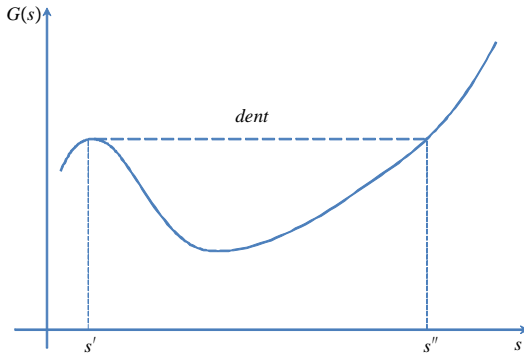

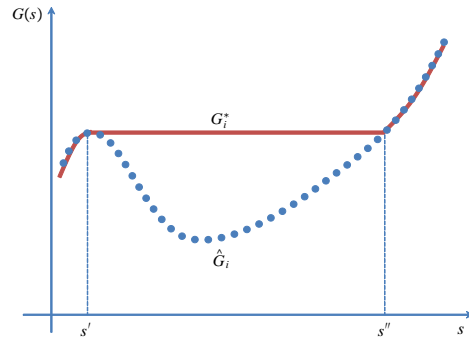Figure 4. Dent   Figure 5. Gap vs Dent

Step 3.i If $\hat{G}_i$ is monotone, let $G_j^* = \hat{G}_j$ for all $j \le i$ and move to Step 4. Otherwise, let G_i^* be the smallest monotone function that lies on or above $\hat{G}_i$.

14

Find all the dents of $\hat{G}_i$, and it can be verified that $\hat{G}_i$ has a finite number of dents. Pick any dent of $\hat{G}_i$, denote it as (s', s''). For any $s \in (s', s'')$, let $G_i^*(s) = \hat{G}_i(s')$ and substitute it into the system: for $j \in \mathcal{P}(s) \setminus \{i\}$,

$$W(\mathbf{G}_{-j}, \mathbf{v}) - c_j s = u_j^* \tag{2}$$

where W represents j's expected winnings in this contest, and $\mathbf{G} = (G_{i'})_{i' \in \mathcal{P}(s)}$, $\mathbf{v} = (v^k)_{k \in \mathcal{P}(s)}$ and $\mathcal{P}(s)$ is the participating players at s such that $\hat{G}_i(s) \in (0, 1)$. There exists a unique $G_j(s) \in [0, 1]$ for $j \in \mathcal{P}(s) \setminus \{i\}$ that solves the system above. Therefore, G_j is defined over all the dents of $\hat{G}_i$. For $j = 1, ..., i - 1$, *re-define* $\hat{G}_j(s) = G_j(s)$ over all the the dents of $\hat{G}_i$, and let $\hat{G}_j(s)$ remain the same if s is not contained in any dent of $\hat{G}_i$. We call $\hat{G}_1, ..., \hat{G}_{i-1}$ defined in this step as the pseudo strategies *after* fixing $\hat{G}_i$'s non-monotonicity. Note that $\hat{G}_1, ..., \hat{G}_{i-1}$ after fixing $\hat{G}_i$ are *different* from those after fixing $\hat{G}_{i+1}$.

Step 3.3 It can be verified that $\hat{G}_1, \hat{G}_2$ after fixing $\hat{G}_3$ are both monotone. Define $G_i^* = \hat{G}_i$ for $i \leq 2$ and move to Step 4.

Step 4 So far, we have defined G_i^* for $i = 1, ..., m$. Let $G_i^*(s) = 1$ for $i = m + 1, ..., n$ and for all s. Now all G_i^* for $i = 1, ..., n$ over $[0, \bar{s} - \underline{s}_m]$ have been defined. The algorithm ends.

4.2 Algorithm Properties

Next, let us introduce some properties of the algorithm.

Lemma 3 (Finiteness) *The algorithm ends in a finite number of steps.*

If the algorithm starts with a different value $\bar{s}' \neq \bar{s}$, it can be verified that the corresponding pseudo strategies constructed in Step 1 are the same functions with a horizontal shift. Therefore, after the shift in Step 2, $\hat{G}_1, ..., \hat{G}_{m+1}$ are the same as in the case starting with $\bar{s}$, which leads to the following lemma, which relies crucially on the assumption of *linear* costs.

Lemma 4 (Determinateness) *The algorithm uniquely determines* $(G_i^*)_{i \in \mathcal{N}}$, *and* $(G_i^*)_{i \in \mathcal{N}}$ *is independent of the initial value* $\bar{s}$.

The lemma above implies that $u_1^*, ..., u_{m+1}^*$ and the upper support of G_1^* are uniquely determined. If the algorithm starts with the upper support of G_1^*, there would be no shift in Step 2 and the pseudo strategies constructed in Step 1.i would yield the payoffs $u_1^*, ..., u_i^*$. Let $\underline{s}_j$ and $\bar{s}_j^*$ be the lower and upper supports of j's pseudo strategy. The following lemma implies that the algorithm finds the upper supports of the equilibrium strategies.

Lemma 5 (Upper Support) *The upper support of $i + 1$'s equilibrium strategy is the infimum of $i + 1$'s best responses in $\left[\underline{s}_{i-1}, \bar{s}_i^*\right]$ against the pseudo strategies yielding $u_1^*, ..., u_i^*$.*

Let us explain the idea used to prove this lemma. Consider a contest with only three players. For $s < \bar{s}_3^*$, let us compare equilibrium strategies G_1^*, G_2^* with the pseudo strategies $\hat{G}_1$ and $\hat{G}_2$ yielding u_1^* and u_2^*. Since 3 is absent at s, if the pseudo strategies are the same as G_1^*, G_2^*, 1 and 2 would have higher payoffs than u_1^* and u_2^*. Therefore, $\hat{G}_1$ and $\hat{G}_2$ have more competition than G_1^*, G_2^* do. As a result, player 3's payoff at s against G_1^*, G_2^* is higher than his payoff against $\hat{G}_1$ and $\hat{G}_2$. Therefore, 3's payoff at $s < \bar{s}_3^*$ is lower than u_3^* when he is facing $\hat{G}_1$ and $\hat{G}_2$. Notice that 3's payoff at $\bar{s}_3^*$ is u_3^* when he is facing $\hat{G}_1$ and $\hat{G}_2$, so $s < \bar{s}_3^*$ is never a best response against $\hat{G}_1$ and $\hat{G}_2$. Hence $\bar{s}_3^*$ is the infimum of 3's best responses in $[\underline{s}_1, \bar{s}_2^*]$ against $\hat{G}_1$ and $\hat{G}_2$.

The following lemma shows that the algorithm finds the gaps in the supports of the equilibrium strategies.

Lemma 6 (Gap vs. Dent) *The following two statements are equivalent:*

i) There is a gap (s_i', s_i'') in the support of i's equilibrium strategy.

ii) $\hat{G}_i$ has a dent over (s_i', s_i''), where $\hat{G}_i$ is player i's pseudo strategy after fixing $\hat{G}_{i+1}$'s non-monotonicity.

Let us briefly explain the idea used to prove this lemma. Consider a simple case when the lemma above is violated. In this case, equilibrium strategy G_i^* has a gap (s_i', s_i'') and $\hat{G}_i$ is higher than G_i^* at a score s in this gap, moreover, no other players have a gap containing (s_i', s_i''). Similar to the idea for Lemma 5, pseudo strategies $(\hat{G}_l)_{l \in \mathcal{A}(s)}$ give player i a higher payoff than equilibrium strategies $(G_l^*)_{\mathcal{A}(s) \setminus \{i\}}$ do, which is a contradiction because they should also give i the same payoff. Figure 5 illustrates that a dent of $\hat{G}_i$ coincides with a gap of G_i^*.

Lemma 7 (Nested Gaps) *Suppose i, j both choose above and below s and $i < j$ in an equilibrium. If the support of i's equilibrium strategy has a gap (s_i', s_i'') containing s, the support of j's equilibrium strategy also has a gap (s_j', s_j''), and $s_j' < s_i'$ and $s_j'' > s_i''$.*
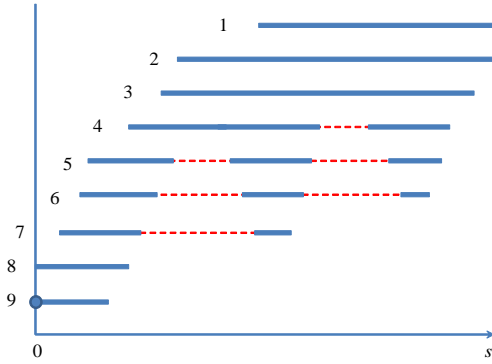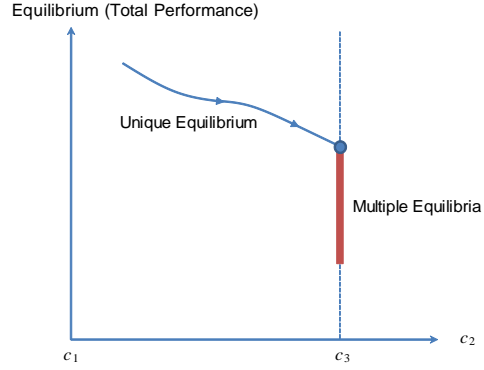
Figure 6. Nested Gaps

Figure 7. Equilibrium Selection

Figure 6 illustrates the supports of equilibrium strategies required by the lemma above.

Using Lemma 1 to 7, we can show that the algorithm constructs the unique Nash equilibrium for every all-pay contest with a quadratic or a geometric prize sequence and distinct linear costs. Therefore, Theorem 1 is established for linear costs.

5   Nonlinear Costs

Now let us consider the case with nonlinear costs. First, we can verify that all the results except Lemma 4 are also true for nonlinear costs. Therefore, given any equilibrium, if the algorithm starts with the upper support of player 1's equilibrium strategy $\bar{s}_1^*$, the algorithm constructs the equilibrium. Moreover, we have the following lemma.

Lemma 8 *Suppose the algorithm starts with $\bar{s}$. Then,*

i) $\underline{s}_m > 0$ and $u_i < u_i^*$ for all i if $\bar{s} > \bar{s}_1^*$, ii) $\underline{s}_m < 0$ and $u_i > u_i^*$ for all i if $\bar{s} < \bar{s}_1^*$,

where u_i is the payoff defined in Step 2 and $\underline{s}_m$ is the lower support of player m's pseudo strategy defined at the end of Step 2.

We can use this lemma to show Theorem 1. In particular, suppose there are two equilibria, and the corresponding maximum scores in these equilibria is $\bar{s}_1^*$ and $\bar{s}_1^{**}$. If $\bar{s}_1^* = \bar{s}_1^{**}$, Lemma 3 to 7 imply that the two equilibria must be the same. If $\bar{s}_1^* \neq \bar{s}_1^{**}$, Lemma 8 implies that the lowest score in the equilibrium is not 0, which cannot be true in an equilibrium. Therefore, we must have a unique equilibrium and Theorem 1 is proved.

Since Lemma 4 is no longer true for nonlinear costs, the algorithm in Section 4 may not construct the equilibrium. However, we can modify the algorithm to approximate the

equilibrium for nonlinear costs. Let us first explain the main idea before moving to the details of the modification. Given any number $\bar{s}$, we can determine whether $\bar{s}$ is above s_1^* or below it because of Lemma 8. Therefore, we can construct a sequence converging to s_1^* by repeating Step 1 and 2 and update $\bar{s}$ using Lemma 8. Similarly, we can also construct a sequence payoffs that converge to the equilibrium payoff u_i^*.

The algorithm for nonlinear costs is described below. There is some complication to approximate the upper supports of equilibrium strategies, and Step $1'.i$ deals with this complication.

Step $1'.1$ To start the algorithm, let $\bar{s}^l = 0$ and $\bar{s}^u = c_1^{-1}(v^1)$, and let $\bar{s}$ be the average of $\bar{s}^l$ and $\bar{s}^u$.

Step $1'.2$ The same as in Section 4. In particular, suppose only players 1 and 2 compete for v^1 and v^2. Suppose that both choose scores only from $[0, \bar{s}_2]$. In that case, their payoffs must be $u_i = v^1 - c_i \bar{s}_2$. This is because by choosing $\bar{s}_2$, a player wins the first prize for sure. For $s \le \bar{s}_2$, there exist unique $G_1(s) \le 1$ and $G_2(s) \le 1$ that solve the system[12]:

$$G_2 v^1 + (1 - G_2) v^2 - c_1 s = u_1$$
$$G_1 v^1 + (1 - G_1) v^2 - c_2 s = u_2$$

Extend the solution for $s < \bar{s}_2$ until $s = \underline{s}_1 > 0$ such that $G_1(\underline{s}_1) = 0$. Since $c_2 > c_1$, $G_2(\underline{s}_1) > 0$. The functions G_1, G_2 are now well-defined for $s \in [\underline{s}_1, \bar{s}_2]$. We call G_1, G_2 the *pseudo* strategies yielding u_1 and u_2.

Repeat the following step for $i = 3, ..., m + 1$.

Step $1'.i$ Suppose players $1, 2, ..., i - 1$ use the strategies $G_1, G_2, ..., G_{i-1}$ determined in Step $1.(i-1)$. Let u_i be the payoff of player i's best response in $[\underline{s}_{i-2}, \bar{s}_{i-1}]$ against strategies $G_1, G_2..., G_{i-1}$.

Step $1'.i.(i-1)$. If player i chooses the lower support $\underline{s}_i$ his strategy G_i, he should win prize v^i and his payoff is u_i. Therefore $\underline{s}_i$ solves $v^i - c_i(\underline{s}_i) = u_i$. For $s \ge \underline{s}_i$, there exist unique $G_{i-1}(s) \le 1$ and $G_i(s) \le 1$ that solve the system

$$G_i v^{i-1} + (1 - G_i) v^i - c_{i-1}(s) = u_{i-1} \tag{3}$$
$$G_{i-1} v^{i-1} + (1 - G_{i-1}) v^i - c_i(s) = u_i \tag{4}$$

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[12]The verification of this and other claims can be found in Appendix C.

Extend the solution for $s > \underline{s}_i$ until $s = \bar{s}_{i-1}$ such that $G_{i-1}(\bar{s}_{i-1}) = 1$. The functions G_i are now well defined for $s \in [\underline{s}_i, \bar{s}_{i-1}]$, and G_{i-1} is updated for the same interval. We call the newly defined G_{i-1}, G_i the *pseudo strategies* yielding u_{i-1} and u_i.

Repeat the following step for $j = i - 2, i - 3, ..., 1$.

Step $1'.i.j$. Suppose the pseudo strategies yielding $u_{j+1}, ..., u_i$ are $G_{j+1}, ..., G_i$, and suppose players $j + 1, j + 2, ..., i$ use these pseudo strategies. We can verify that there are scores in $[\underline{s}_i, \bar{s}_{j+1}]$ such that player j's payoff against $G_{j+1}, ..., G_i$ is 0. Let $\underline{s}_j$ be the infimum of all these scores. If $\underline{s}_j$ is the lower support of G_j, strategies $G_{j+1}, ..., G_1$ remain the same for $s < \underline{s}_j$ because player j does not choose any score below $\underline{s}_j$. However, for $s \geq \underline{s}_j$, the pseudo strategies of players $j + 1, ..., i$ are different from those determined in Step $1'.i.(j+1)$. For $s \geq \underline{s}_j$, there exists unique $G_j, G_{j+1}, ..., G_i \leq 1$ that solve the system of $i - j + 1$ equations: for $i' = j, j + 1, ..., i$,

$$W(\mathbf{G}_{-i'}, \mathbf{v}) - c_{i'}(s) = u_{i'} \tag{5}$$

where $\mathbf{v} = (v^j, v^{j+1}, ..., v^i)$ and $G_{-i'} = (G_j, ..., G_{i'-1}, G_{i'+1}, ..., G_i)$. Extend the solution for $s \geq \underline{s}_j$ until $s = \bar{s}_j$ such that $G_j(\bar{s}_j) = 1$.

Consequently, at the end of Step $1'.i.1$, we have constructed functions $G_1, ..., G_i$ for $s \in [\underline{s}_i, \bar{s}]$. We call $G_1, ..., G_i$ as the *pseudo strategies* yielding $u_1, .., u_i$. At the end of Step $1'.(m+1)$, we have constructed pseudo strategies $G_1, ..., G_{m+1}$ yielding $u_1, .., u_{m+1}$.

Step $3'$ Let $\hat{G}_i = G_i$ for $i = 1, ..., m + 1$, and the rest is the same as Step 3 in Section 4. Since the strategies constructed in this step may not be the equilibrium strategies, we use notation $\hat{G}_i^*$ to replace G_i^* in Step 3 and Step 4.

Step $4'$ The same as in Section 4. Suppose the outcome of this step is $\hat{G}_i^*$ for $i = 1, ..., n$, and let $\hat{G}_m^*(\underline{s}_m) = 0$. If $\underline{s}_m = 0$, the algorithm ends. If $\underline{s}_m > 0$, we update $\bar{s}^u$ with $\bar{s}$ and go back to Step $1'.1$ with the new $\bar{s}^u$. If $\underline{s}_m < 0$, we update $\bar{s}^l$ with $\bar{s}$ and go back to Step 1 with the new $\bar{s}^l$.

Note that this algorithm does not have a counter part of Step 2 in the algorithm for linear costs. We call Step $1'$ to $4'$ an iteration. This algorithm either stops with equilibrium strategies or produces a sequence of strategy profiles that converges to the equilibrium. The convergence rate is characterized in the proposition below.

Proposition 1 *Suppose T is the number of iterations in the algorithm for nonlinear costs. Then, $|u_i - u_i^*| = O(2^{-T})$ for each player i, and $\left|\hat{G}_i^*(s) - G_i^*(s)\right| = O(2^{-T})$ for each s and i, where $\hat{G}_i^*(s)$ is the output of the algorithm after T iterations.*

Corollary 1 *Consider a sequence of contests in which $c_i(s) - c_j(s)$ pointwise converges to zero for players $i, j < m+2$, then $u_i^* - u_j^*$ also converges to zero and $G_i^*(s) - G_j^*(s)$ pointwise converges to zero.*

As illustrated in Example 1, there could be multiple equilibria if some players have the same cost functions. This corollary allows us to select an equilibrium as a limit of the sequence of unique equilibria of nearby contests. Moreover, the selected equilibrium has $m + 1$ players who choose scores above 0, and, among these players, the players with the same cost use the same strategy. Figure 7 illustrates this selection if there are three players with linear costs.

6 Applications

6.1 Tracking in Schools

Student tracking systems in schools have been frequently questioned (see Lockwood and Cleveland, 1998). These systems typically identify the students' abilities and group students with similar abilities together. Assuming that a school's objective is to maximize the students' total effort/performance, should the school track the students, i.e., group students with similar abilities together, or, should the school not track the students, i.e., group students with different abilities together?

The following example demonstrates that the answer depends on the returns to education for the lower-ranked students in each classroom. In particular, if the returns are not too small, tracking is *better* than not tracking, but if the returns are small enough, tracking is *worse* than not tracking. In all of the examples that follow, I use Corollary 1 to select a unique equilibrium in cases where there are possibly many equilibria (the multiplicity arises because of ties in the costs of different players).

Example 2 *Consider a school with two classrooms with four seats in each classroom. Suppose there are four H-type students of high ability and four L-type students of low ability. The H-type students have a marginal cost of $c_H = 1$ and the L-type students have a marginal*

cost of $c_L = 2$. In each classroom, the four students choose effort levels (scores) to compete in an all-pay contest with two prizes: v^1 and $v^2 = 4 - v^1$, where $v^1 \in (8/3, 4)$.[13]

We compare two scenarios. In scenario one, students are tracked so that four H-type students are assigned to one classroom and four L-type students are assigned to the other classroom. In the classroom with H-type students, each student gets an equilibrium payoff of 0, which implies that the total expected cost equals the total value of prizes, $v^1 + v^2 = 4$. Since all the students have the same marginal cost in this classroom, the total expected effort is the total expected cost divided by the marginal cost: $4/c_H$. Similarly, the total expected effort of the classroom with L-type students is $4/c_L$. Therefore, the total expected effort of all the students is $\Pi_{Track} = 6$.

Now consider scenario two in which each classroom is mixed and contains two H-type students and two L-type students. It can be verified that only the H-type students choose positive effort levels in the equilibrium, and the equilibrium strategies are

$$G_H^* = s/\left(2v^1 - 4\right), \qquad G_L^* = 1.$$

The resulting total expected effort of all the students is $\Pi_{Mixed} = 4v^1 - 8$.

Thus, $\Pi_{Track} > \Pi_{Mixed}$ if $v^2 > 0.5$ and $\Pi_{Track} < \Pi_{Mixed}$ if $v^2 < 0.5$.

Why does the value of the lower prize matter? Compared to not tracking, tracking has an advantage of facilitating greater competition in each classroom by assigning students of similar abilities together. However, tracking also has a disadvantage. It does not use the highest prizes to motivate the best students. As a result, if the value of the lower prize is not too small, the advantage dominates, and tracking is better than not tracking. Now suppose the value of the lower prize is very small. If the students are tracked, only half of the prize money is used to motivate H-type students. However, if the students are not tracked, most of the prize money is used to motivate H-type students. Hence, the disadvantage of tracking may dominate its advantage, and tracking could be worse than not tracking.

6.2 Winner-Take-All?

Consider a situation in which the designer of a contest has some fixed amount of prize money, and he wants to choose the optimal prize structure to maximize the total expected score (performance). Is it optimal for the designer to adopt a winner-take-all prize structure,

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[13]The prizes represent the discounted future returns to education. Moreover, $v^1 \in (8/3, 4)$ ensures that the prize sequence is QPS.

in which the whole amount is won by the highest-ranking player, or, should the total amount be split into two or more prizes?

Moldovanu and Sela (2001) consider a contest with incomplete information, and they find that winner-take-all prize structure is optimal if the players are ex ante symmetric and the costs are linear. However, this result does not hold in our model if the players are asymmetric. If the players are ex ante symmetric, participation is not important because all the players choose positive effort in the symmetric equilibrium. However, when the players are asymmetric in this model, more prizes could encourage more participation and therefore introduce more competition. The following example demonstrates that the total expected score can actually be higher if the total amount is split into two prizes.[14]

Example 3 *Consider a contest with three players with costs $c_1 = 2, c_2 = c_3 = 3$. The total amount of prize money is 4.*

First, consider the contest with *one* prize of value 4. Player 3 always chooses 0, and players 2 and 3 compete for the prize. The equilibrium payoffs are $4/3$ for player 1, and 0 for the others. The equilibrium strategies are

$$
\begin{aligned}
G_1^* &= 3s/4 \text{ for } s \in [0, 4/3] \\
G_2^* &= s/2 + 1/3 \text{ for } s \in [0, 4/3] \\
G_3^* &= 1
\end{aligned}
$$

The total expected score is 1.11.

Second, consider a contest with *two* prizes: $v^1 = 3$ and $v^2 = 1$. The equilibrium payoffs (in the equilibrium selected as a limit of equilibria of contests for which $c_3 < c_2 = 3$) are 1 for player 1 and 0 for others. Players 2 and 3 use the same strategy in this equilibrium, and the equilibrium strategies are

$$
\begin{aligned}
G_1^{**} &= \frac{1}{2\sqrt{s+1}} \left(3\sqrt{2}s - 2\sqrt{s+1} + \sqrt{2}\right) \text{ for } s \in \left[\sqrt{13}/9 - 2/9, 1\right] \\
G_2^{**} &= G_3^{**} = \begin{cases} \sqrt{2}\sqrt{s+1} - 1 & \text{for } s \in \left[\sqrt{13}/9 - 2/9, 1\right] \\ 3s & \text{for } s \in [0, \sqrt{13}/9 - 2/9) \end{cases}
\end{aligned}
$$

Total expected score is 1.19.

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[14]There are some papers demonstrating similar results in different setups. For example, Szymanski and Valletti (2005) show a similar result in a three-player logit contest if the cost of the strongest player is close to 0; Cohen and Sela (2008) demonstrate in a three-player all-pay auction that a small asymmetry of players' valuations may lead to a similar result.

Therefore, the total expected score with two prizes 1.19 is *larger* than that with one prize, 1.11.

6.3 Effects of Superstars

Consider a situation in which the set of contestants consists of one "superstar" of very high ability (very low costs) and a group of players of moderate ability. Brown (2011) exhibits, in a Tullock game, what is known as the "Tiger Woods" effect—the presence of a superstar in the contest causes average players to decrease their effort levels. The effect of a superstar can also be studied in our model. In the two examples below, I show that Brown's theoretical result relies on the assumption that the other players are symmetric. In particular, Example 4 studies the situation in which the other players are symmetric, and exhibits the same phenomenon as in Brown (2011). However, Example 5 illustrates that, if the other players are asymmetric, the entry of a superstar may actually increase the expected scores (effort levels) of other players.

Why does the asymmetry of the other players matter? The presence of a superstar has two effects: first, it reduces the expected winnings of other players and therefore discourages competition; second, it increases the competition for the top prizes and motivates the other players with strong abilities. If the other players are symmetric, the second effect is small, so the presence of a superstar discourages competition. However, if some of the other players have similar abilities with the superstar, the second effect may dominate the first, so the presence of a superstar may lead to more competition.[15]

Example 4 *Consider a contest with three players and two prizes: $v^1 = 3, v^2 = 1$.*

First, suppose that the contest does not have a superstar. Let the set of players $\mathcal{N} = \{2, 3, 4\}$ with costs $c_2 = c_3 = c_4 = 1$. Since players are symmetric, and the marginal cost is 1, the total expected score of all the players equals their expected winnings minus their total payoff. The total expected winnings are just 4 and the total expected payoff in equilibrium is just 0. Thus the expected score of each player is 1.33.

Now suppose that we introduce a superstar with cost $c_1 = 0.1$ who displaces player 4 in the contest. Now $\mathcal{N} = \{1, 2, 3\}$. The equilibrium payoffs are $u_1^* = 2.7$, $u_2^* = u_3^* = 0$, and the

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[15]Cohen and Sela (2008) demonstrate, in a three-player contest, that a small asymmetry in players' valuations may lead to a similar result.

equilibrium strategies are

$$G_1^* = \frac{2}{\sqrt{0.4s + 14.8}} \left(s - 0.5\sqrt{0.4s + 14.8} + 1 \right) \text{ for } s \in (0.95, 3)$$

$$G_2^* = G_3^* = \begin{cases} 0.5\sqrt{0.4s + 14.8} - 1 & \text{for } s \in [0.95, 3] \\ s & \text{for } s \in [0, 3) \end{cases}$$

Therefore, the expected score of 2 or 3 is 0.55.

Therefore, player 1's presence *reduces* the expected score of 2 or 3 from 1.33 to 0.55.

Example 5 *Consider a contest with three players and two prizes* $v^3 = 3, v^2 = 1$.

As above, first suppose that the contest does not have a superstar and the set of players $\mathcal{N} = \{2, 3, 4\}$ with costs $c_2 = c_3 = 1$ and $c_4 = 2$. The equilibrium payoffs in this case are $u_2^{**} = u_3^{**} = 1$, $u_4^{**} = 0$, and the equilibrium strategies are

$$G_3^{**} = G_2^{**} = s/2 \text{ for } s \in [0, 2]$$
$$G_4^{**} = 1$$

Therefore, the expected score of 2 or 3 is 1.

Now suppose that a superstar with cost $c_1 = 0.8$ displaces the weakest player, player 4. The new equilibrium payoffs are $u_1^* = 0.6$, $u_2^* = u_3^* = 0$, and the equilibrium strategies are

$$G_1^* = \frac{2(s - 0.5\sqrt{3.2s + 6.4} + 1)}{\sqrt{3.2s + 6.4}} \text{ for } s \in [0.38, 3]$$

$$G_2^* = G_3^* = \begin{cases} 0.5\sqrt{3.2s + 6.4} - 1 & \text{for } s \in [0.38, 3] \\ s & \text{for } s \in [0, 0.38) \end{cases}$$

Therefore, the expected score of 2 or 3 is 1.07.

In this case, player 1's presence *increases* the expected score of 2 or 3 from 1 to 1.07.

Moreover, if we fix the prizes and c_2, c_3, c_4 as above and decrease c_1 from 1 to 0, the increase in 2 or 3's expected score caused by player 1's presence decreases, and eventually this increase becomes negative and player 1's presence decreases 2 or 3's expected score.

7   Extension to General Convex Sequences

Our results require the prize sequence to be either quadratic or geometric. Although, most common prize structures can be well-approximated by one or the other specification, one

would still like to extend the results of this paper to general *convex* prize sequences. The quadratic/geometric specification plays a key role in the proof—it guarantees that there is a unique solution to system of nonlinear equations resulting from the players' indifference conditions defining the equilibrium (Claim 9 in Appendix A). How to extend this result to general convex sequences remains an open problem at present.

8   Conclusion

In this paper, I studied a *complete* information model of all-pay contests with asymmetries among players and (two classes) of convex prize sequences. While it would be desirable to study a similar environment under *incomplete* information, the problems associated with multiple prizes and asymmetric players under incomplete information are well known from auction theory. For instance, even with symmetric players very little is known about discriminatory (pay-as-you-bid) auctions for the sale of multiple units. Similar difficulties arise when considering all-pay auctions with multiple prizes.[16] The complete information setting allows us to study environments that, as yet, cannot be studied under an incomplete information setting.

I hope to explore some extensions of the model. It would be interesting to investigate, more generally, what an optimal prize sequence looks like. Must it be convex? Does the uniqueness result hold for general convex (possibly non-quadratic and non-geometric) prize sequences? These and other questions will be explored in subsequent work.

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Appendices

A Equilibrium Properties

This appendix provides proofs for the results in Section 3. For the consideration of space, some technical details in linear algebra is omitted and put in supplementary notes of this paper.

Claim 1 *Player i's expected winning $W\left(\mathbf{G}_{-i}, \mathbf{v}\right)$ is symmetric in the variables of $\mathbf{G}_{-i}$; it is linear in $\mathbf{v}$; and it is strictly increasing in each variable of $\mathbf{G}_{-i}$, G_j, if $G_j \in (0, 1)$.*

Proof. It is easy to see that $W\left(\mathbf{G}_{-i}, \mathbf{v}\right)$ is symmetric in the variables in $\mathbf{G}_{-i}$, and that it is linear in $\mathbf{v}$.

To see that W is increasing in G_j, notice

$$W\left(\mathbf{G}_{-i}, \mathbf{v}\right) = G_j W\left(\mathbf{G}_{-i,j}, \mathbf{v}_{-k'}\right) + \left(1 - G_j\right) W\left(\mathbf{G}_{-i,j}, \mathbf{v}_{-k''}\right)$$

where k' is the lowest prize in $\mathbf{v}$ and k'' is the highest prize in $\mathbf{v}$

$$
\begin{aligned}
dW/dG_j &= W\left(\mathbf{G}_{-i,j}, \mathbf{v}_{-k'}\right) - W\left(\mathbf{G}_{-i,j}, \mathbf{v}_{-k''}\right) \\
&= W\left(\mathbf{G}_{-i,j}, \mathbf{v}_{-k'} - \mathbf{v}_{-k''}\right) \\
&= \sum_{l=k''}^{k'-1}(v^l - v^{l+1})P_i^l\left(s\right)
\end{aligned}
$$

where $P_i^l\left(s\right)$ is i's probability of winning the lth prize if i chooses s and the players in $\mathcal{N} \backslash \{i, j\}$ choose strategies in $\mathbf{G}_{-i,j}$. Since $P_i^l\left(s\right)$ is non-negative and $v^l - v^{l+1} > 0$, $dW/dG_j > 0$ where the strict inequality comes from the fact that $P_i^l\left(s\right)$ cannot be 0 for all l. ∎

Claim 2 (No Atom) *No score $s > 0$ is chosen with positive probability, and only player $i > m$ may choose 0 with positive probability.*

Proof. Suppose player i chooses $s > 0$ with positive probability. If no player chooses score immediately below s, then player i could benefit by moving the probability on s to a score slightly below s. If there is a sequence of scores $\{s_l\}$ that are chosen by some players and they converge to s, there exists player j who chooses infinite many scores in this sequence.

Then, player j's payoff at s is strictly more than at the scores in the sequence, contradiction. Hence, there is no score that is chosen with positive probability.

Suppose $i < m+1$ chooses 0 with positive probability. Consider two cases. First, suppose no other player chooses 0 with positive probability. Then, player i's payoff is 0 at 0, so $u_i^* = 0$. Since player i can guarantee himself a payoff no less than u_j^* by choosing slightly above the highest score chosen by player j for $j > i$, we have $u_i^* \geq u_j^*$. Therefore, $u_j^* = 0$ for $j > i$. However, player $i+1$ can get a positive payoff by choosing 0. Contradiction. Second, suppose there is another player besides i choosing 0 with positive probability, say player j. If $u_i^* = 0$, there is a contradiction as in the first case, so $u_i^* > 0$. Hence, player i and j have positive probability to win a prize at $s = 0$. Then, player j would deviate to a score slightly above 0 because the cost is almost the same but he does not have to split the prizes with player i. In sum, the two cases imply that player i does not choose 0 with positive probability. ∎

Claim 3 (Participation) *Players weaker than $m + 1$ choose score 0 with probability one.*

Proof. Zero Lemma by Siegel (2009) is also true here. If we replace the probability of winning (one of the homogeneous prizes) with the probability of winning at least one prize (one of the heterogenous prizes), his proofs also work in this context.

Zero Lemma implies that at least $n - m$ players have zero expected payoff. Recall that $u_i^* \geq u_j^*$ for $j > i$ as in the proof of Claim 2, $u_i^* = 0$ for $i \geq m + 1$. Suppose player $i > m + 1$ assigns positive probability on a set of positive scores. Suppose s is any score from that set, consider two cases. First, if player $m + 1$ does not choose above score s, player $m + 1$'s expected winnings at s are the same as player i's. Second, consider the case in which player $m + 1$ chooses above s. Player $m + 1$'s expected winning at s is $W(\mathbf{G}^*_{-(m+1)}(s), \mathbf{v})$, where $\mathbf{G}^* = (G_j^*)_{j \in \mathcal{P}(s)}$ and $\mathbf{v} = (v^k)_{k \in \mathcal{P}(s)}$. Similarly, i's expected winnings at s are

$$W(\mathbf{G}^*_{-(m+1)}(s), G_{m+1}^*(s), \mathbf{v}) < W(\mathbf{G}^*_{-(m+1)}(s), G_i^*(s), \mathbf{v}) = W(\mathbf{G}^*_{-(m+1)}(s), \mathbf{v})$$

where $G_i^*(s) = 1$ and the inequality comes from the monotonicity of W. Therefore, $m + 1$'s expected winnings at s are more than i's in the second case. In sum, $m + 1$'s expected winnings at s are no less than i's, and $m + 1$'s cost is lower at s than i's, so $m + 1$ gets a higher payoff than i does at s. Contradiction. ∎

Claim 4 *In an equilibrium, the highest score that stronger player chooses is no less than the highest score that a weaker player chooses. That is, $\bar{s}_{i+1}^* \leq \bar{s}_i^*$.*

Proof. Denote the upper support of player i's strategy as $\bar{s}_i^*$, and the upper support of $i+1$'s equilibrium strategy as $\bar{s}_{i+1}^*$. Suppose $\bar{s}_i^* < \bar{s}_{i+1}^*$. Similar to Claim 3, $i+1$'s expected winnings are the same as i's at $\bar{s}_{i+1}^*$, and $i+1$'s expected winnings are strictly higher than i's at $\bar{s}_i^*$. Namely, $P_i\left(\bar{s}_{i+1}^*\right) = P_{i+1}\left(\bar{s}_{i+1}^*\right)$ and $P_{i+1}\left(\bar{s}_i^*\right) > P_i\left(\bar{s}_i^*\right)$, where $P_j\left(s\right)$ denotes player j's expected winnings at s.

From the definition of payoffs, we have $u_i^* = P_i\left(\bar{s}_i^*\right) - c_i(\bar{s}_i^*)$ and $u_{i+1}^* = P_{i+1}\left(\bar{s}_{i+1}^*\right) - c_{i+1}(\bar{s}_{i+1}^*)$.

Player i's payoff at $\bar{s}_{i+1}^*$ should not be more than u_i^*, so we have

$$P_i\left(\bar{s}_{i+1}^*\right) - c_i(\bar{s}_{i+1}^*) \le P_i\left(\bar{s}_i^*\right) - c_i(\bar{s}_i^*)$$

Note that $P_i\left(\bar{s}_{i+1}^*\right) = P_{i+1}\left(\bar{s}_{i+1}^*\right)$, so the equation above implies

$$
\begin{aligned}
P_{i+1}\left(\bar{s}_{i+1}^*\right) - c_i(\bar{s}_{i+1}^*) &\le P_i\left(\bar{s}_i^*\right) - c_i(\bar{s}_i^*) \\
P_{i+1}\left(\bar{s}_{i+1}^*\right) - P_i\left(\bar{s}_i^*\right) &\le c_i(\bar{s}_{i+1}^*) - c_i(\bar{s}_i^*)
\end{aligned}
\tag{6}
$$

Player $i+1$'s payoff at $\bar{s}_i^*$ should not be more than u_{i+1}^*, so we have

$$P_{i+1}\left(\bar{s}_i^*\right) - c_{i+1}(\bar{s}_i^*) \le P_{i+1}\left(\bar{s}_{i+1}^*\right) - c_{i+1}(\bar{s}_{i+1}^*)$$

Note that $P_{i+1}\left(\bar{s}_i^*\right) > P_i\left(\bar{s}_i^*\right)$, so we have

$$
\begin{aligned}
P_i\left(\bar{s}_i^*\right) - c_{i+1}(\bar{s}_i^*) &< P_{i+1}\left(\bar{s}_i^*\right) - c_{i+1}(\bar{s}_i^*) \le P_{i+1}\left(\bar{s}_{i+1}^*\right) - c_{i+1}(\bar{s}_{i+1}^*) \\
P_{i+1}\left(\bar{s}_{i+1}^*\right) - P_i\left(\bar{s}_i^*\right) &> c_{i+1}(\bar{s}_{i+1}^*) - c_{i+1}(\bar{s}_i^*)
\end{aligned}
\tag{7}
$$

(6) and (7) contradict with each other. As a result, $\bar{s}_{i+1}^* \le \bar{s}_i^*$. ∎

Claim 5 *At any score lower than the maximum one, the difference between two players' equilibrium payoffs is no more than the difference in their costs. That is, $u_i^* + c_i(s) \le u_{i+1}^* + c_{i+1}(s)$ if $s \le \bar{s}_{i+1}^*$; and $u_i^* + c_i(s) < u_{i+1}^* + c_{i+1}(s)$ if $s < \bar{s}_{i+1}^*$.*

Proof. Since $\bar{s}_{i+1}^* \le \bar{s}_i^*$, i's expected winnings at $\bar{s}_{i+1}^*$ are no less than that of $i+1$:

$$u_{i+1}^* + c_{i+1}(\bar{s}_{i+1}^*) \le u_i^* + c_i(\bar{s}_{i+1}^*)$$

$$u_i^* + c_i(\bar{s}_{i+1}^*) - \left(u_{i+1}^* + c_{i+1}(\bar{s}_{i+1}^*)\right) \ge 0 \tag{8}$$

Therefore we have

$$
\begin{aligned}
& u_i^* + c_i(s) - (u_{i+1}^* + c_{i+1}(s)) \\
= & \ u_i^* - u_{i+1}^* - (c_{i+1}(s) - c_i(s)) \\
> & \ u_i^* - u_{i+1}^* - (c_{i+1}(\bar{s}_{i+1}^*) - c_i(\bar{s}_{i+1}^*)) \\
\geq & \ 0
\end{aligned}
$$

where the last inequality comes from (8). ∎

Let us introduce some notations before we move to the next claim.

For a given interval, let $\mathcal{A}$ be the set of active players and $\mathcal{P}$ be the participating players, $\mathcal{A} \subset \mathcal{P}$. Then for any s in this interval, equilibrium strategies $(G_i)_{i \in \mathcal{A}}$ solve

$$
W\left(\mathbf{G}_{-i}, \mathbf{v}\right) - c_i(s) = u_i \qquad \text{for } i \in \mathcal{A} \tag{9}
$$

where $\mathbf{G} = (G_i)_{i \in \mathcal{P}}$ and $\mathbf{v} = (v^k)_{k \in \mathcal{P}}$.[17]

Suppose $\mathcal{P} = \{1, 2, ..., j'\}$. If $\mathcal{P} \neq \{1, 2, ..., i'\}$, we can order the elements and rename them, and the argument below would be the same. If $\mathbf{v} \neq (v^k)_{k \in \mathcal{P}}$, we can also rename the prizes similarly, and the analysis below applies as well.

For any $i' \in \mathcal{A} \setminus \{i\}$, take derivatives of both sides of (9) with respect to $G_{i'}$, we have

$$
\sum_{l \in \mathcal{A} \setminus \{i, i'\}} \frac{dW\left(\mathbf{G}_{-i}, \mathbf{v}\right)}{dG_l} \frac{dG_l}{dG_{i'}} = -\frac{dW\left(\mathbf{G}_{-i}, \mathbf{v}\right)}{dG_{i'}} \quad \text{for } i \neq i' \tag{10}
$$

We can write (10) into matrix form

$$
\hat{\mathbf{W}}_{j-1} \boldsymbol{\delta} = -\mathbf{d} \tag{11}
$$

where $j = \#\mathcal{A}$ and $\hat{\mathbf{W}}_{j-1}$ is a $(j-1) \times (j-1)$ matrix, $\boldsymbol{\delta}$ and $\mathbf{d}$ are vectors of $j-1$ rows. The diagonal entries of $\hat{\mathbf{W}}$ are zero and the entry at (j_1, j_2) is $dW(\mathbf{G}_{-j_1}, \mathbf{v})/dG_{j_2}$; the element in row j_1 of $\boldsymbol{\delta}$ is $dG_{j_1}/dG_{i'}$ for $j_1 \neq i'$; the element in row j_1 of $\mathbf{d}$ is $dW\left(\mathbf{G}_{-j_1}, \mathbf{v}\right)/dG_{i'}$ for $j_1 \neq i'$. Define $\mathbf{W}_j = \begin{pmatrix} \hat{\mathbf{W}}_{j-1} & \mathbf{w} \\ \mathbf{w} & 0 \end{pmatrix}$ and w_{j_1, j_2} as the entry in row j_1 and column j_2 of $\mathbf{W}_j$.

---

[17]Since the indices of $\mathbf{G}$ and $\mathbf{v}$ in $W(\mathbf{G}_{-i}, \mathbf{v})$ are the same below, we sometime only mention the indices for $\mathbf{G}$.

Define first order difference as $\Delta_k^{(1)} = v^{k-1} - v^k$, for $k = 2, ..., j'$. The lth order difference is $\Delta_k^{(l)} = \Delta_{k-1}^{(l-1)} - \Delta_k^{(l-1)}$ for $l = 1, ..., j' - 1$, and $k = l + 1, ..., j'$.

Claim 6 *Suppose the prize sequence is either a QPS or GPS, $i, j \in \mathcal{P}(s)$, and $i < j$. If $j \in \mathcal{A}(s)$, $i \in \mathcal{A}(s)$.*

The claim is true at the highest score chosen by the players, $\bar{s}_2^*$. Suppose that this claim is true from $\tilde{s}$ to $\bar{s}_2^*$, we are going to show the two claims below. Then, we will find a contradiction at the supremum of scores that violate this claim, hence the claim is true for all s and so are the two claims below.

Claim 7 *Suppose Claim 6 is true for s in $(\tilde{s}, \bar{s}_2^*]$. For any s in $(\tilde{s}, \bar{s}_2^*]$ and any i such that i, $i + 1 \in \mathcal{N}$, $G_i^*(s) \leq G_{i+1}^*(s)$; if $i, i + 1 \in \mathcal{P}(s)$, $G_i^*(s) < G_{i+1}^*(s)$.*

Proof. Since the upper support of a weaker player is no less than that of a stronger player, $G_i^*(\bar{s}_2^*) \leq G_{i+1}^*(\bar{s}_2^*)$. We are going to consider the case if $s < \bar{s}_2^*$. If i is not in $\mathcal{P}(s)$, $G_i^*(s)$ is 0 or 1, the claim is true. Similarly, the claim is true if i is not in $\mathcal{P}(s)$. Therefore, it is sufficient to examine the case with $i, i + 1 \in \mathcal{P}(s)$.

Consider three possibilities. First, suppose both i and $i + 1$ are active at s, so $i, i + 1 \in \mathcal{A}(s)$. Then, $(G_l^*(s))_{l \in \mathcal{A}(s)}$ is the solution of (9) for $\mathcal{A}(s)$ and $\mathcal{P}(s)$. Let us compare the equation for i and $i+1$. Claim 5 implies that the $u_i^* + c_i(s) \geq u_{i+1}^* + c_{i+1}(s)$, so $W(\mathbf{G}_{-(i+1)}^*, \mathbf{v}) \leq W(\mathbf{G}_{-i}^*, \mathbf{v})$. Since $W(\mathbf{G}_{-i}, \mathbf{v})$ is increasing in $\mathbf{G}_{-i}$, we have $G_i^*(s) \leq G_{i+1}^*(s)$.

Second, suppose one of i and $i + 1$ are active at s, then $i, i + 1 \in \mathcal{P}(s)$, $i \in \mathcal{A}(s)$ and $i + 1 \notin \mathcal{A}(s)$. Then, there exists s_{i+1}'' such that $i + 1$ is active above it. Since i and $i + 1$ are active at s_{i+1}'', $G_i^*(s_{i+1}'') \leq G_{i+1}^*(s_{i+1}'')$. $i + 1$ is not active over (s, s_{i+1}''), so $G_{i+1}^*(s) = G_{i+1}^*(s_{i+1}'') \geq G_i^*(s_{i+1}'') > G_i^*(s)$.

Third, suppose neither i nor $i + 1$ is active at s, so $i, i + 1 \notin \mathcal{A}(s)$, but $i, i + 1 \in \mathcal{P}(s)$. Then, let i is active at s_i'' and $i + 1$ is active at s_{i+1}'' and $s_i'' \leq s_{i+1}''$. Therefore $G_{i+1}^*(s) = G_{i+1}^*(s_{i+1}'') \geq G_i^*(s_{i+1}'') = G_i^*(s)$, where the inequality comes from the first two cases.

Now we are going to prove the second part of the claim. If $i, i+1 \in \mathcal{P}(s)$, Claim 5 implies that $u_i^* + c_i(s) < u_{i+1}^* + c_{i+1}(s)$. Similar to the analysis above, we have $G_i^*(s) < G_{i+1}^*(s)$ if $i, i + 1 \in \mathcal{P}(s)$. ∎

Claim 8 (Ordered Densities) *Suppose the prize sequence is either a QPS or GPS, and Claim 6 is true for s in $(\tilde{s}, \bar{s}_2]$. For any s in $(\tilde{s}, \bar{s}_2]$ and any i such that $i, i + 1 \in \mathcal{A}(s)$, if s is an interior point of i and $i + 1$'s supports, $g_i^*(s) > g_{i+1}^*(s)$.*

Proof. Without loss of generality, suppose $\mathcal{A}(s) = \{1, 2, ..., a\}$.[18]

First, consider players 1 and 2. Suppose s is an interior point of the supports of G_1^* and G_2^*. Consider the equations in (9) for $i = 1$ and 2,

$$W(\mathbf{G}_{-1}^*, \mathbf{v}) = u_1^* + c_1(s)$$
$$W(\mathbf{G}_{-2}^*, \mathbf{v}) = u_2^* + c_2(s)$$

Take derivatives w.r.t. s for both sides of the equations, we have

$$\frac{dW(\mathbf{G}_{-1}^*, \mathbf{v})}{dG_2} g_2^* + \frac{dW(\mathbf{G}_{-1}^*, \mathbf{v})}{dG_3} g_3^* + ... + \frac{dW(\mathbf{G}_{-1}^*, \mathbf{v})}{dG_a} g_a^* = c_1' \tag{12}$$

$$\frac{dW(\mathbf{G}_{-2}^*, \mathbf{v})}{dG_1} g_1^* + \frac{dW(\mathbf{G}_{-2}^*, \mathbf{v})}{dG_3} g_3^* + ... + \frac{dW(\mathbf{G}_{-2}^*, \mathbf{v})}{dG_a} g_a^* = c_2' \tag{13}$$

Since $W(\mathbf{G}_{-1}, \mathbf{v})$ is linear in G_2 for $i = 2, ..., a$, $dW(\mathbf{G}_{-1}^*, \mathbf{v})/dG_2$ is independent of G_2^*, therefore

$$\frac{dW(\mathbf{G}_{-1}^*, \mathbf{v})}{dG_2} = \frac{dW(\mathbf{G}_{-2}^*, \mathbf{v})}{dG_1} \tag{14}$$

If the prize sequence satisfies QPS, we have[19]

$$dW(\mathbf{G}_{-1}, \mathbf{v})/dG_j = \beta \left(\sum_{j' \in \mathcal{P}(s)} G_{j'} - G_1 - G_j \right) + v^{\max \mathcal{P}(s)}.$$

Therefore, $dW(\mathbf{G}_{-1}, \mathbf{v})/dG_j$ is increasing in G_i for $i \neq 1, j$, and Lemma 1 implies

$$\frac{dW(\mathbf{G}_{-1}^*, \mathbf{v})}{dG_j} \geq \frac{dW(\mathbf{G}_{-2}^*, \mathbf{v})}{dG_j} \tag{15}$$

Similarly, $dW(\mathbf{G}_{-1}, \mathbf{v})/dG_j$ is increasing in G_i for $i \neq 1, j$ if the prize sequence satisfies GPS, therefore (15) is also true.

Let us compare (12) and (13). The terms except the first one are bigger in the LHS of (12), and the RHS is smaller in (12), therefore the first term on the LHS must be smaller in (12):

$$\frac{dW(\mathbf{G}_{-1}^*, \mathbf{v})}{dG_2} g_2^* < \frac{dW(\mathbf{G}_{-2}^*, \mathbf{v})}{dG_1} g_1^* \tag{16}$$

Then, (14) implies $g_2^* < g_1^*$.

---

[18] If $\mathcal{A}(s) \neq \{1, ..., a\}$, we can rank the players from the strongest to the weakest, and rename them to $1, 2, ..., a$. Then, the analysis would be the same.

[19] As in Claim 9 and 10 in the supplementary notes.

Similarly, $g_{i+1}^* < g_i^*$. ∎

Claim 9 (Local Solution) *Consider system (9) for $\mathcal{A}(s)$ and $\mathcal{P}(s)$ and $u_i = u_i^*$. This system has a unique local solution $(\hat{G}_i(s))_{i \in \mathcal{A}(s)}$, and $(\hat{G}_i(s))_{i \in \mathcal{A}(s)}$ is differentiable at s.*

Proof. Take derivatives w.r.t. to s of the system, we have

$$\mathbf{W}_{\#\mathcal{A}(s)}\mathbf{g} = \mathbf{c}'$$

where $\mathbf{g} = (g_i)_{i \in \mathcal{A}(s)}$ and $\mathbf{c}' = (c_i')_{i \in \mathcal{A}(s)}$. Since $G_i^*(s) \in [0,1]$ for $i \in \mathcal{A}(s)$, Claim 10 and 13 in the supplementary notes imply that matrix $W_{\#\mathcal{A}(s)}$ is invertible if the prize sequence is either quadratic or geometric. Therefore, we have an ordinary differential equation system

$$\mathbf{g} = \mathbf{W}_{\#\mathcal{A}(s)}^{-1}\mathbf{c}' \tag{17}$$

with initial condition

$$\mathbf{G}(s) = \mathbf{G}^*(s)$$

Theorem of 20.7 of Olver (2007) implies that there is a local solution $\hat{\mathbf{G}}$ to (17), and this solution extends to s' as long as $\hat{G}_l(s'') \geq 0$ for $s'' \in (s, s')$ and all $l \in \mathcal{A}(s)$. It is obvious that $\hat{\mathbf{G}}$ is differentiable at s and it solves (9) for $\mathcal{A}(s)$ and $\mathcal{P}(s)$ and $u_i = u_i^*$. ∎

Proof of Claim 6 (Nested Gaps) . Suppose s_i'' is the supremum of scores that violate this claim. Therefore, there is a player j weaker than i such that i is not active immediately below s_i'' but j is. Since s_i'' is not the lower support of G_i^* by definition, there are scores below s_i'' such that i is active or j is inactive, and let s_j'' be the supremum of these scores. There are two possible situations at s_j'': i is active or i is inactive.

Consider Case 1: i is active at s_j''. Since j is also active at s_j'' by definition, both i and j are active at s_j''. Since G_i^* and G_j^* satisfy $W(\mathbf{G}_{-i}^*(s_j''), \mathbf{v}) = u_i^* + c_i(s_j'')$ and $W(\mathbf{G}_{-j}^*(s_j''), \mathbf{v}) = u_j^* + c_j(s_j'')$, so $G_i^*(s_j'') < G_j^*(s_j'')$, hence $g_i^*(s_j'') > g_j^*(s_j'')$. Since $g_j^*(s_j''+) \geq 0$, $g_i^*(s_j''-) > 0$. Claim 9 shows that there is a differentiable local solution $(\hat{G}_{j'})_{j' \in \mathcal{A}(s_i)}$ at s_j'' to (9) for $\mathcal{A}(s_j'')$ and $\mathcal{P}(s_j'')$. Moreover, $\hat{g}_i(s_j'') > 0$ where $\hat{g}_i(s_j'')$ is the derivative of $\hat{G}_i(s_j'')$. Therefore, i would deviate to slightly above s_j'' according to Claim 11 in the supplementary notes. Contradiction. Figure 8 illustrates $\hat{G}_i$ and $\hat{G}_j$ in this case, where the horizontal lines demonstrate the supports.
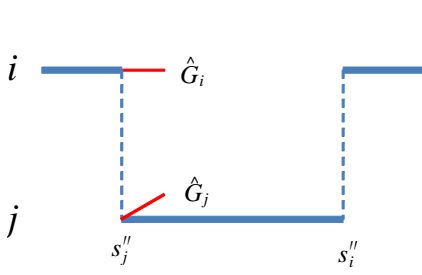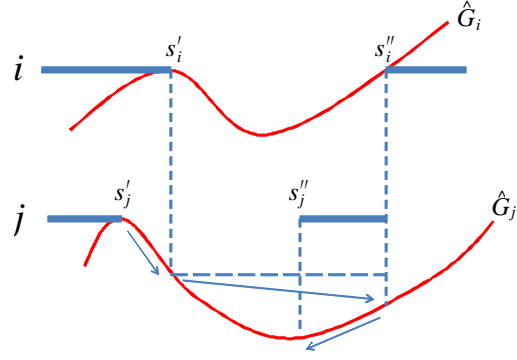
Figure 8. Case 1

Figure 9. Case 2

Consider Case 2: i is inactive at s_j''. Let s_i' be the supremum of scores below s_j'' such that i is active. If j is active at s_i', we would have a similar contradiction as in Case 1 above. Let s_j' be the supremum of scores below s_i' such that j is active, therefore $s_j' < s_i' < s_j'' < s_i''$, where (s_i', s_i'') is a gap for i. We proceed the analysis in two steps.

In the first step, for $s \in (s_i', s_i'')$, consider two equations

$$W(\mathbf{G}^*_{-i,j}, G_j, \mathbf{v}) - c_i(s) = u_i^* \tag{18}$$

$$W(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v}) - c_j(s) = u_j^* \tag{19}$$

where $\mathbf{G}^*_{-i,j} = (G_l^*)_{l \in \mathcal{N} \setminus \{i,j\}}$ and $\mathbf{v}$ is the prizes available at s. Case 1 implies that the players in $\mathcal{P}(s_i')$ who are stronger than i are active at s_i', therefore $(G_i^*(s_i'), G_j^*(s_i'))$ solves (18) and (19) for $s = s_i'$. According to Claim 15 in the supplementary notes, this is also the unique solution. Since s_i'' is the first violation, Claim 8 implies that players in $\mathcal{P}(s_i'')$ who are stronger than j are active at s_i'', so $(G_i^*(s_i''), G_j^*(s_i''))$ is the unique solution to (18) and (19) for $s = s_i''$ similarly.

Take derivatives of both sides of (18) and (19) w.r.t. s, we have

$$\sum_{j' \in \mathcal{N} \setminus \{i,j\}} \left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}} g_{j'}^* \right) + \frac{dW(\mathbf{G}^*_{-i,j}, G_j, \mathbf{v})}{dG_j} g_j = c_i'$$

$$\sum_{j' \in \mathcal{N} \setminus \{i,j\}} \left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}} g_{j'}^* \right) + \frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_i} g_i = c_j'$$

$$g_j = \left[c_i' - \sum_{j' \in \mathcal{N} \setminus \{i,j\}} \left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}} g_{j'}^* \right) \right] \Big/ \frac{dW(\mathbf{G}^*_{-i,j}, G_j, \mathbf{v})}{dG_j} \tag{20}$$

$$g_i = \left[c'_j - \sum_{j' \in \mathcal{N} \setminus \{i,j\}} \left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}} g^*_{j'} \right) \right] \bigg/ \frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_i} \tag{21}$$

With initial conditions $G_i(s'_i) = G^*_i(s'_i)$ and $G_j(s'_i) = G^*_j(s'_i)$, (20) and (21) have a unique local solution $(\hat{G}_i(s), \hat{G}_j(s))$ at s'_i as in Claim 9. Moreover, the solution can be extended to s''_i as long as $\hat{G}_i(s), \hat{G}_j(s)$ are finite. Notice that $\hat{G}_j(s)$ also satisfies (18) and W in (18) is increasing in G_j, so $\hat{G}_j(s)$ must be finite for s in (s'_i, s''_i). Similarly $\hat{G}_i(s)$ is also finite for s in (s'_i, s''_i). Therefore, $\hat{G}_i(s)$ and $\hat{G}_j(s)$ are well defined for s in (s'_i, s''_i). Since $\hat{G}_i(s)$ and $\hat{G}_j(s)$ are differentiable, denote $\hat{g}_i(s)$ and $\hat{g}_j(s)$ as their derivatives.

In the second step, consider the interval (s'_i, s''_i), (20) and (21) imply $\hat{g}_i \geq \hat{g}_j$ over this interval, therefore $\hat{G}_i(s''_i) - \hat{G}_j(s''_i) \geq \hat{G}_i(s'_i) - \hat{G}_j(s'_i)$. Note that $\hat{G}_i(s''_i) = \hat{G}_i(s'_i)$, so

$$\hat{G}_j(s''_i) \leq \hat{G}_j(s'_i) \tag{22}$$

Recall that (G^*_i, G^*_j) solves (18) and (19) for $s = s'_i$ and s''_i, so we have

$$\hat{G}_j(s''_i) = G^*_j(s''_i)$$
$$\hat{G}_j(s'_i) = G^*_j(s'_i)$$

Therefore,

$$G^*_j(s''_j) < G^*_j(s''_i) = \hat{G}_j(s''_i) \leq \hat{G}_j(s'_i) = G^*_j(s'_i) = G^*_j(s'_j) \tag{23}$$

where the first inequality comes from G^*_j increases over (s''_j, s''_i), the second inequality comes from (22). Figure 9 illustrates $\hat{G}_i$ and $\hat{G}_j$ in Case 2, and the arrows represent the steps in (23). However, mixed strategy G^*_j should be non-decreasing over (s'_j, s''_j), which contradicts (23). ∎

Claim 6 is true for all $s \in [0, \bar{s}^*_2]$, therefore, Claim 7 and 8 are also true for all $s \in [0, \bar{s}^*_2]$. Hence, we have Lemma 1 and 2.

B Linear Costs

This appendix provides proofs for the results in Section 4. Because there always exists an equilibrium, suppose $\bar{s}^*_1$ is the maximum score chosen by player 1 in an equilibrium. In this appendix, we first discuss the properties of the algorithm if it starts with $\bar{s} = \bar{s}^*_1$. Then, we discuss the general case if the algorithm starts with any value $\bar{s}$.

Claim 10 *Suppose the algorithm starts with $\bar{s}_1^*$, and $\bar{s}_j = \bar{s}_j^*$ for $j = 1, ..., i$ after Step 1.i. There exists a unique solution to (1) in $[0, \infty)^i$.*

Proof. The proof is similar to the first step in Case 2 in the proof to Claim 6.

Let $\mathcal{P} = \{i', i' + 1, ..., i\}$ be the set of players such that $G_j(\bar{s}_i) > 0$. For any $j < i'$, $G_j(\bar{s}_i) = 0$, so we define $G_j(s) = 0$ and substitute it into (1). As a result, (1) becomes

$$W(\mathbf{G}_{-j}, \mathbf{v}) - c_j(s) = u_j \text{ for } j \in \mathcal{P} \tag{24}$$

where $\mathbf{G} = (G_j)_{j \in \mathcal{P}}$ and $\mathbf{v} = (v^k)_{k \in \mathcal{P}}$. Then, for $s < \bar{s}_i$, $(G_j)_{j \in \mathcal{P}}$ is the solution to (24). Hence $(G_j)_{j \in \mathcal{P}}$ also satisfies the differential equation system

$$\mathbf{W}\mathbf{g} = \mathbf{c}' \tag{25}$$

where $\mathbf{g} = (g_j)_{j \in \mathcal{P}}$, $w_{j,j'}$ is the derivative of $W(\mathbf{G}_{-j}, \mathbf{v})$ w.r.t. $G_{j'}$ for $j' \in \mathcal{P} \backslash \{j\}$, and $\mathbf{c}' = (c_j')_{j \in \mathcal{P}}$. Since W is invertible according to Claim 10 and 13 in the supplementary notes, the differential equation can be rewritten as

$$\mathbf{g} = \mathbf{W}^{-1}\mathbf{c}' \tag{26}$$

By definition, we already know $G_j(\bar{s}_i)$ for $j = i', ..., i - 1$ and $G_i(\bar{s}_i) = 1$, which are the initial conditions of the different equation above. Theorem 20.7 of Olver (2007) implies that (26) has a local solution $\mathbf{G}$ around $\bar{s}_i$, and this solution can be extended to $s < \bar{s}_i$ as long as $G_j > 0$ for all $j \in \mathcal{P}$.[20]

By similar analysis of Lemma 1, we have $G_{i'}$ is the smallest in $\mathbf{G}$. It is easy to see in (24) that the solution cannot extend to $-\infty$. Therefore, there must exist a score $\underline{s}_{i'}$ such that $G_{i'}(\underline{s}_{i'}) = 0$. Moreover, $G_{i'}$ is strictly increasing. To see why, suppose otherwise and $g_{i'}(s_0) \leq 0$. By similar analysis of Lemma 2, we have $g_j(s_0) < g_{i'}(s_0) \leq 0$ for all $j \in \mathcal{P}$. Therefore, (25) is violated. Since $G_{i'}$ is strictly increasing, $\underline{s}_{i'}$ is the only score such that $G_{i'}(\underline{s}_{i'}) = 0$.

The uniqueness comes from Claim 15 in the supplementary notes. ∎

Claim 11 *Suppose the prize sequence is geometric. For any subset $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$, G_a decreases if G_j increases in (9), where j is the weakest player in $\mathcal{A}$ and a is the second weakest player in $\mathcal{A}$.*

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[20] See Olver (2007), pp. 1102-1103.

Proof. Suppose $\mathcal{P} = \{1, 2, ..., j\}$. The other cases can be proved similarly.

The solution to (11) is

$$\delta_i = -\frac{\det \tilde{\mathbf{W}}_i}{\det \hat{\mathbf{W}}_{j-1}} \text{ for } i = 1, ..., j-1 \tag{27}$$

where $\tilde{\mathbf{W}}_i$ is $\hat{\mathbf{W}}_{j-1}$ with the ith column replaced by $\mathbf{d}$. We want to show that δ_{j-1} is negative. Notice that we already know that $\det \hat{\mathbf{W}}_{j-1}$ has sign of $(-1)^j$ because of Claim 10 in the supplementary notes, and we can also verify that $\det \tilde{\mathbf{W}}_{j-1}$ has sign $(-1)^j$ (see Claim 16 in the supplementary notes). Therefore, δ_{j-1} is negative and the claim is proved. ∎

Claim 12 *Suppose the prize sequence is quadratic. For any subset $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$, G_a decreases if G_j increases in (9), where j is the weakest player in $\mathcal{A}$ and a is the second weakest player in $\mathcal{A}$.*

Proof. Suppose $\mathcal{A} = \mathcal{P} = \{1, 2, ..., j\}$. The other cases can be shown similarly.

Let $\boldsymbol{\delta}$ satisfies (11). The rest of the proof has two steps. First, we are going to show that $\delta_1 \geq \delta_2 \geq ... \geq \delta_{j-1}$, second, we are going to show that $\delta_{j-1} < 0$.

Step 1. As in the previous claim, the solution to (11) is also characterized by (27), but $\tilde{\mathbf{W}}_i$ and $\hat{\mathbf{W}}_{j-1}$ would be different that those in the previous claim because the prize sequence here is quadratic rather than geometric. Since $\det \hat{\mathbf{W}}_{j-1}$ has sign $(-1)^j$, it is sufficient to show that $\det \tilde{\mathbf{W}}_i - \det \tilde{\mathbf{W}}_{i-1}$ also has sign $(-1)^j$ for $i = 2, ..., j-1$. It can be verified that this is indeed true (see Claim 17 in the supplementary notes).

Step 2. Suppose $\delta_{j-1} \geq 0$, then $\delta_1 > \delta_2 > ... > \delta_{j-1} \geq 0$. Therefore $\mathbf{W}_{j-1}\boldsymbol{\delta} \gg 0$. Contradiction.

As a result of the two steps, we have $\delta_{j-1} < 0$.

If $\mathcal{A} = \mathcal{P} \neq \{1, 2, ..., j\}$, we can rename the players in $\mathcal{A}$ and the proof is the same. If $\mathcal{A} \subsetneq \mathcal{P}$, the corresponding $\hat{\mathbf{W}}_{j-1}$ has $(G_i)_{i \in \mathcal{P} \setminus \mathcal{A}}$ in each entry, then we can define h_i similarly as in Claim 11 in the supplementary notes. The rest of the proof is the same. ∎

Claim 13 *There exists a unique solution $(\hat{G}_i)_{i \in \mathcal{P}}$ in $[0, 1]^{\#\mathcal{P}}$ to (9) for $\mathcal{A} = \mathcal{P} = \mathcal{P}(s)$ and $s \in [\underline{s}^*_{p'}, \bar{s}^*_{p''}]$, where p' and p'' are the weakest and strongest players in $\mathcal{P}(s)$. Moreover, $\hat{G}_{p'}(s) \leq G^*_{p'}(s)$ for $s \in [\underline{s}^*_{p'}, \bar{s}^*_{p''}]$.*

Proof. If $g^*_{p'}(s) > 0$, Claim 6 implies that $g^*_i(s) > 0$ for any $i \in \mathcal{P}$. Therefore, $(G^*_i(s))_{i \in \mathcal{P}}$ is the solution to (9) and it is unique according to Claim 15 in the supplementary notes.

Suppose there is a gap (s', s'') in the support of the weakest player's strategy, $G_{p'}^*$. Claim 10 implies that it is sufficient to show $\hat{G}_i(s) \leq 1$ for $s \in (s', s'')$. Since $\hat{G}_i(s) \leq \hat{G}_{p'}(s)$ by similar analysis to Lemma 1, it is sufficient to show that $\hat{G}_{p'}(s) \leq G_{p'}^*(s)$ for $s \in (s', s'')$.

The rest of the proof has three steps.

Step 1. Suppose $\#\mathcal{A}(s) = \#\mathcal{P}(s) - 1$ and $\hat{G}_{p'}(s) > G_{p'}^*(s)$. If we decrease $\hat{G}_{p'}(s)$ to $G_{p'}^*(s)$, Claim 11 and 14 in the supplementary notes imply that $W(\mathbf{G}_{-p'}^*, \mathbf{v}) > u_{p'}^* + c_{p'}(s)$. Contradiction. Therefore, $\hat{G}_{p'}(s) \leq G_{p'}^*(s)$.

Suppose $\#\mathcal{A}(s) < \#\mathcal{P}(s) - 1$ and $\hat{G}_{p'}(s) > G_{p'}^*(s)$ for some s in (s', s''). Denote the lower bound of $p' - 1$'s gap as $s_d^{p'-1}$. Then, we have $\hat{G}_{p'}(s_d^{p'-1}) < G_{p'}^*(s')$, otherwise, $p' - 1$ would deviate to slightly above $s_d^{p'-1}$ as in Case 1 of Claim 6. Then, intermediate value theorem implies $\hat{G}_{p'}(s_0) = G_{p'}^*(s_0)$ for some $s_0 \in (s_d^{p'-1}, s)$. See Figure 9. We are going to find a contradiction in the next two steps.

Step 2. We claim that $\hat{G}_i(s_0) > G_i^*(s_d^i)$, where i is any player in $\mathcal{P}(s_0) \backslash \mathcal{A}(s_0)$ and s_d^i is the lower bound of i's gap.

Since $\hat{g}_{p'} \leq \hat{g}_{p'-1}$ by similar analysis in Claim 8, $\hat{G}_{p'-1}$ increases faster than $\hat{G}_{p'}$ does. Notice that $\hat{G}_{p'}(s_d^{p'-1}) < G_{p'}^*(s_0)$, so $\hat{G}_{p'-1}(s_d^{p'-1}) < \hat{G}_{p'-1}(s_0)$. Moreover, $G_{p'-1}^*(s_d^{p'-1}) = \hat{G}_{p'-1}(s_d^{p'-1})$, then, we have $G_{p'-1}^*(s_d^{p'-1}) < \hat{G}_{p'-1}(s_0)$. Similarly, $G_{p'-1}^*(s_d^{p'-1}) < \hat{G}_{p'-1}(s_0)$ implies $G_{p'-2}^*(s_d^{p'-2}) < \hat{G}_{p'-2}(s_0)$, and so on.
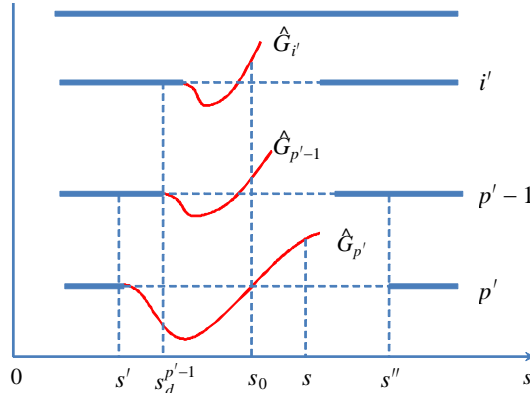

Figure 9

Step 3. We claim that i' would deviate to s_0, where i' is the strongest player in $\mathcal{P}(s_0) \backslash \mathcal{A}(s_0)$.

Decrease $\hat{G}_j(s_0)$ to $G_j^*(s_0)$ for $j = p', p' - 1$. Denote $(\tilde{G}_i)_{i \in \mathcal{P}(s) \backslash \{p', p'-1\}}$ as the solution to (9) for $\mathcal{A} = \mathcal{P}(s) \backslash \{p', p' - 1\}$ and $\mathcal{P} = \mathcal{P}(s)$. Claim 11 and 12 imply that $\tilde{G}_{p'-2}(s_0) > \hat{G}_{p'-2}(s_0)$. Similar to Step 1, $\tilde{G}_i(s_0) > G_i^*(s_d^i)$ for $i \in \mathcal{P}(s) \backslash \{p', p' - 1\}$. Repeat this process until $\tilde{G}_i(s_0) > G_i^*(s_d^i)$ for $i \in \mathcal{A}(s) \cup \{i'\}$ where i' is the strongest player in $\mathcal{P}(s) \backslash \mathcal{A}(s)$. This would contradict with Step 1. ■

38

Claim 14 *If $\bar{s}_j = \bar{s}_j^*$ for $j = 1, ..., i$, there exists a unique solution to (1) in $[0,1]^i$.*

Proof. Because of Claim 10, it is sufficient to show that $G_i \leq 1$ for $j \leq i$.

Claim 13 shows that this claim is true for $i = m + 1$, and denote the solution as $(G_i^1)_{1 \leq i \leq m+1}$. Now, let $i = m$ and denote the solution as $(G_i^2)_{1 \leq i \leq m}$. Claim 11 and 12 imply that $G_m^2(s) < G_m^1(s) \leq 1$. Therefore, the claim is also true for $i = m$. Similarly, we can always exclude the weakest remaining player and show that the claim is true for a smaller i, therefore the claim is true for any $i = 3, ..., m + 1$. ∎

Similarly, unique solution can also be proved for the other parts of Step 1.i.

Claim 15 *The upper support of $i + 1$'s equilibrium strategy is the infimum of $i + 1$'s best responses in $[\underline{s}_{i-1}, \bar{s}_i^*]$ against the pseudo strategies yielding $u_1^*, ..., u_i^*$.*

Proof. If we exclude the players weaker than $i + 1$, there are pseudo strategies G_i yielding the equilibrium payoffs for the remaining player according to Claim 14. Suppose **G** is the pseudo strategies yielding $u_1^*, ..., u_i^*$, Claim 11 and 14 in the supplementary notes imply $W(\mathbf{G}, \mathbf{v}) < u_{k+1}^* + c_{k+1}(s)$ for s between the lower support of G_{i-1} and $\bar{s}_{i+1}^*$.

Claim 11 and 12 imply that $G_i(s) \leq G_i^*(s)$, so the lower support of G_i is bigger than G_i^*. This is not a problem because $\bar{s}_{i+1}^*$ cannot be less than the lower support of G_i.

Therefore, $\bar{s}_{i+1}^*$ is the infimum of the best responses to the pseudo strategies $G_1, ..., G_i$ yielding $u_1^*, ..., u_i^*$. ∎

Lemma 6 implies $\bar{s}_2^* \leq \bar{s}_1^*$. Since there is no aggregate gap, $\bar{s}_2^* = \bar{s}_1^*$. If we let the algorithm start with $\bar{s} = \bar{s}_1^*$, G_1 and G_2 in Step 1.2 yield u_1^*, u_2^*. Therefore, Claim 15 implies that $\bar{s}_3 = \bar{s}_3^*$ in Step 1.3, then Claim 14 implies the existence of G_1, G_2, G_3 yielding u_1^*, u_2^*, u_3^*. Similarly, Claim 15 implies that $\bar{s}_4 = \bar{s}_4^*$ and Claim 14 implies the existence of $G_1, ..., G_4$ yielding $u_1^*, ..., u_4^*$, and so on. As a result, if the algorithm starts with $\bar{s} = \bar{s}_1^*$, we have $\bar{s}_i = \bar{s}_i^*$ for $i = 1, ..., m + 1$. Since G_i^* is increasing slightly below its upper support $\bar{s}_i^*$, Lemma 2 implies all participating players are active slight below $\bar{s}_i^*$. Therefore, $u_i = u_i^*$ for $i = 1, ..., m + 1$, which means that the payoffs defined in Step 1 are the equilibrium payoffs.

Similar to (14), there exists a unique solution to (2) in $[0,1]^i$ for each $i = m + 1, ..., 3$ in Step 2 of the algorithm.

Claim 16 *If the algorithm starts with $\bar{s}_1^*$, the following two statements are equivalent:*

i) There is a gap (s_i', s_i'') in the support of i's equilibrium strategy.

ii) $\hat{G}_i$ *has a dent over* (s'_i, s''_i) *, where* $\hat{G}_i$ *is player* i *'s pseudo strategy after fixing* $\hat{G}_{i+1}$ *'s non-monotonicity.*

Proof. First, consider the following statement: $\hat{G}_i$ is strictly increasing if and only if G_i^* does not has a gap in the equilibrium.

"$\Rightarrow$": Suppose $\hat{G}_i$ is strictly increasing and G_i^* has a gap (s', s'') in its support. One the one hand, $G_i^* = \hat{G}_i$ at s' and s'', i's payoff at the boundaries of the gap should be u_i^* for both $\hat{G}_i$ or G_i^*. On the other hand, recall that $\hat{G}_3$ is strictly increasing, but G_i^* is constant over (s', s''), therefore $\hat{G}_i(s') < G_i^*(s')$. Therefore, if we replace $\hat{G}_i$ with G_i^* at s', Claim 11 and 14 in the supplementary notes imply that i's payoff at s' is less than u_i^*. Contradiction.

"$\Leftarrow$": Suppose G_i^* has no gap and $\hat{G}_i$ has a dent (s', s''). By definition, G_i^* and $\hat{G}_i$ are different, so there are multiple solutions to (9) for $\mathcal{A} = \mathcal{P} = \{a'', a'' + 1, ..., i\}$. Contradiction to Claim 13.

Second, consider the following statement: $\hat{G}_i$ has a dent over (s', s'') if and only if G_i^* has gap (s', s'').

"$\Leftarrow$": Suppose G_i^* has gap (s', s''). It is easy to see that $G_i^* = \hat{G}_i$ at s' and s''. Claim 13 implies $\hat{G}_i$ has a dent over s' and s''.

"$\Rightarrow$": Suppose $\hat{G}_i$ has a dent over (s', s''), but G_i^* does not have a gap. There are two solutions to (2) for $j = 1, ..., i$. Contradiction. Suppose $\hat{G}_i$ has a dent over (s', s'') and G_i^* has a gap (s'_g, s''_g) but the gap is different from (s', s''). Let us discuss in four different cases.

Consider the first case: $s' < s'_g$. By definition, $\hat{G}_i$ is not increasing on (s', s'_g), but G_i^* is. Therefore, we have two different solutions for (9) for $\mathcal{A} = \mathcal{P} = \{a'', a'' + 1, ..., i\}$. Contradiction.

Consider the second case: $s'_g < s'$. Therefore, $\hat{G}_i(s') > G_i^*(s')$, which contradicts Claim 13.

Consider the third case: $s''_g > s''$. Since $s'_g = s'$, $G_i^*(s'_g) = \hat{G}_i(s')$. By definition, $G_i^*(s') = G_i^*(s'')$ and $\hat{G}_i(s'_g) = \hat{G}_i(s''_g)$. Because $s''_g > s''$, $\hat{G}_i(s''_g) > \hat{G}_i(s'') = G_i^*(s'') = G_i^*(s') = \hat{G}_i(s')$. Contradiction. (start at the same value but did not end at the same value.)

Consider the fourth case: $s''_g < s''$. We have a contradiction as in the third case. ∎

Claim 17 *If the algorithm starts with* $\bar{s}_1^*$ *, it ends in a finite number of steps.*

Proof. Consider equation system $W(\mathbf{G}_{-i}, \mathbf{v}) - c_i(s) = u_i$ for $i \in \mathcal{A}$. Suppose solution $\mathbf{G}$ exists in a neighborhood of s_0. Take derivatives of both hand w.r.t. s, we have

$$\mathbf{W}_j \mathbf{g} = \mathbf{c}'$$

40

where $j = \#\mathcal{A}$, $\mathbf{W}_j$ is the $j \times j$ matrix defined above, $\mathbf{g} = (G'_i)_{i \in \mathcal{A}}$ and $\mathbf{c} = (c'_i)_{i \in \mathcal{A}}$. We can verify (see Claim 10 and 13 in the supplementary notes) that $\det \mathbf{W}_j$ is not zero, hence we have an ordinary differential equation system

$$\mathbf{g} = \mathbf{W}_j^{-1} \mathbf{c}'$$

with the initial condition that $\mathbf{G}$'s value at s_0 is $\mathbf{G}(s_0)$. Since $\mathbf{W}_j^{-1} \mathbf{c}'$ is an analytic function in $\mathbf{G}$, Theorem 20.10 of Olver (2007) implies that the solution to this system is analytic in a small neighborhood of s_0. Then, $\mathbf{g} = \mathbf{W}_j^{-1} \mathbf{c}'$ is a composition of analytic functions, hence is also an analytic function in the neighborhood.

Recall that g_i is defined over an bounded interval in Step 1.i. Since g_i is analytic, Identity Theorem[21] implies that g_i either has a finite number of roots in its domain or $g_i = 0$. Either case implies that G_i has only a finite number of dents and the algorithm ends in finite steps.
∎

Lemma 7 (Nested Gaps): Suppose i, j *both choose above and below s and $i < j$ in the equilibrium. If the support of G_i^* has a gap (s'_i, s''_i) containing s, the support of G_j^* also has a gap (s'_j, s''_j) such that $s'_j < s'_i$ and $s''_j > s''_i$.*

Proof. Suppose i has a gap with lower bound s'_i, and j has a gap with lower bound s'_j. Claim 6 implies that $s'_i \geq s'_j$. Suppose $s'_i = s'_j$, therefore at $g_i^*(s'_i) > g_j^*(s'_i)$. Lemma 6 implies that $g_i^*(s'_i) = 0$, therefore $g_j^*(s'_i) < 0$ contradiction. Hence, $s'_i > s'_j$.

Suppose i has a gap with upper bound s''_i, and j has a gap with upper bound s''_j. Claim 6 implies that $s''_i \leq s''_j$. Suppose $s''_i = s''_j$. Since $s'_i > s'_j$,

$$G_i^*(s'_i) < \hat{G}_j(s'_i) < G_j^*(s'_j) \tag{28}$$

Since $\hat{g}_i(s) > \hat{g}_j(s)$ for $s \in (s'_i, s''_i)$, $\hat{G}_i$ increases faster than $\hat{G}_j$, so $\hat{G}_j(s''_i) < \hat{G}_i(s''_i) = G_i^*(s''_i) = G_i^*(s'_i) < G_j^*(s'_j) = G_j^*(s''_i)$, where the last inequality comes from (28). Contradiction. ∎

The above lemma is a stronger version of Claim 6. This lemma ensures that we only need to fix monotonicity of $\hat{G}_i$ in Step 2.i in the gaps of $\hat{G}_{i+1}$. Therefore, we only need to update $\hat{G}_i$ over the gaps of $\hat{G}_{i+1}$ in Step 3.$i + 1$.

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[21]See Chapman (2002), pp. 256.

Note that $G_m^*(0) = 0$ in the equilibrium. Consider $\hat{G}_1, ..., \hat{G}_{m+1}$ in Step 3. Only players m and $m+1$ are participating below $\underline{s}_{m-1}^*$, therefore, $\hat{G}_i(s) = G_i^*(s)$ for $i = m, m+1$ and $s < \underline{s}_{m-1}^*$. As a result, $\hat{G}_m(0) = 0$, so we do not update $\bar{s}$ in Step 3, and the algorithm ends after one iteration.

Claim 14 to 16 and Lemma 7 imply that the algorithm *constructs* the equilibrium strategies if it starts with $\bar{s}_1^*$, the highest score in the equilibrium. Now let us consider the case in which the algorithm starts with an arbitrary score $\bar{s}$. It is easy to see that Claim 14 to 16 are also true if the algorithm starts with arbitrary $\bar{s}$ instead of $\bar{s}_1^*$.

Lemma 4 (Determinateness): *If the costs are linear, the algorithm uniquely determines* $(G_i^*)_{i \in \mathcal{N}}$, *and* $(G_i^*)_{i \in \mathcal{N}}$ *is independent of the initial value* $\bar{s}$.

Proof. It is sufficient to show that G_i is a function of $\bar{s} - s$.

Substitute $u_i = v^1 - c_i(s)$ into $W(\mathbf{G}_{-i}, \mathbf{v})$ for $i \in \{1, 2\}$, we have

$$\begin{aligned} W(G_2, v^1, v^2) &= u_1 + c_1(s) = v^1 - (c_1(\bar{s}) - c_1(s)) \\ W(G_1, v^1, v^2) &= u_2 + c_2(s) = v^1 - (c_2(\bar{s}) - c_2(s)) \end{aligned}$$

therefore G_1 and G_2 in Step I are functions of $\bar{s} - s$.

Suppose $G_1, ..., G_{i-1}$ are the pseudo strategies yielding $u_1, ..., u_{i-1}$, and they are functions of $\bar{s} - s$. Since $\bar{s}_i$ is i's best response in $[\underline{s}_{i-2}, \bar{s}_{i-1}]$ to $G_1, ..., G_k$, $\bar{s}_1 - \bar{s}_i$ is constant and $u_i + c_i s$ is a function of $\bar{s} - s$. The right hand sides of system (1) are functions of $\bar{s} - s$. As a result, the pseudo strategies $G_1, ..., G_i$ are also functions of $\bar{s} - s$. Similarly, at the end of Step 1.$(m+1)$, pseudo strategies $G_1, ..., G_{m+1}$ are also functions of $\bar{s} - s$. Therefore, $(G_i^*)_{i \in \mathcal{N}}$ is independent of the initial value $\bar{s}$, and the algorithm uniquely determines $(G_i^*)_{i \in \mathcal{N}}$. ∎

Now we can prove the theorem for linear costs. The theorem is also true for nonlinear costs, and the proof is contained in Appendix C.

Theorem 1: *If the costs are linear, the algorithm constructs the unique Nash equilibrium for every all-pay contest with a quadratic or a geometric prize sequence and ordered marginal costs.*

Proof. Suppose there are two equilibria, and the corresponding maximum scores in these equilibria is $\bar{s}_1^*$ and $\bar{s}_1^{**}$. If $\bar{s}_1^* = \bar{s}_1^{**}$, Lemma 3 to 7 imply that the two equilibria must be the

same. If $\bar{s}_1^* \neq \bar{s}_1^{**}$, Lemma 4 would be violated. Therefore, we have a unique equilibrium and it is constructed by the algorithm. ∎

C   Nonlinear Costs

This appendix contains the proofs for Section 5.

Lemma 8: *Suppose the algorithm starts with $\bar{s}$. Then,*

i) $\underline{s}_m > 0$ and $u_i < u_i^*$ for all i if $\bar{s} > \bar{s}_1^*$, ii) $\underline{s}_m < 0$ and $u_i > u_i^*$ for all i if $\bar{s} < \bar{s}_1^*$,

where u_i is defined in Step 2 and $\underline{s}_m$ is the lower support of player m's pseudo strategy defined at the end of Step 2.

Proof. We are going to use induction. Suppose $\bar{s} > \bar{s}_1^*$.

First, we are going to show that $u_2 < u_2^*$ and $\underline{s}_2 > \underline{s}_2^*$. Since $\bar{s}_1 > \bar{s}_1^*$, $u_1 < u_1^*$ and $u_2 < u_2^*$. Player 2's payoff at $\underline{s}_2$ should be:

$$0 - c_2(\underline{s}_2) = u_2 < u_2^* = 0 - c_2(\underline{s}_2^*)$$

Therefore, $\underline{s}_2 > \underline{s}_2^*$.

Second, suppose $u_l < u_l^*$, we want to show that $\underline{s}_{l+1} > \underline{s}_{l+1}^*$ and $u_{l+1} < u_{l+1}^*$. In particular, since $u_l < u_l^*$, when we construct pseudo strategies for $1, ..., l+1$, player l's payoff at $\underline{s}_l$ should be:

$$v^{l+1} - c_l(\underline{s}_l) = u_l < u_l^* = v^{l+1} - c_l(\underline{s}_l^*)$$

Therefore, $\underline{s}_l > \underline{s}_l^*$. Then, we have $\underline{s}_{l+1} > \underline{s}_{l+1}^*$ because $\underline{s}_l = \underline{s}_{l+1}$ and $\underline{s}_l^* = \underline{s}_{l+1}^*$. Player $l+1$'s payoff at $\underline{s}_{l+1}$ should be:

$$u_{l+1} = v^{l+1} - c_{l+1}(\underline{s}_{l+1}) < v^{l+1} - c_{l+1}(\underline{s}_{l+1}^*) = u_{l+1}^*$$

Therefore, induction implies $\underline{s}_m > \underline{s}_m^*$. Moreover, $u_i < u_i^*$ for $i = 1, ..., m+1$.

Similarly, if $\bar{s}_1 < \bar{s}_1^*$, we have $u_i > u_i^*$ for all i and $\underline{s}_m < \underline{s}_m^*$. ∎

Claim 18 *In any equilibrium, if a player deviates to a score below the lower support of his strategy, his payoff is strictly less than his equilibrium payoff.*

Proof. Take any score s below the lower support $\bar{s}_i^*$ of player i's equilibrium strategy, let

the set of active players at score s be $\mathcal{P}(s)$. Then, consider the equation system

$$W(\mathbf{G}_{-j}, \mathbf{v}) - c_j(s) = u_j^*$$

for $j \in \mathcal{P}(s) \cup \{i\}$. Similar to the analysis in Appendix C, this system has a unique solution G_j for $j \in \mathcal{P}(s) \cup \{i\}$. Lemma 1 implies that i is stronger than the players in $\mathcal{P}(s)$, therefore, similar to Lemma 2, $G_i'(s)$ should be larger than $G_j'(s)$ for any $j \in \mathcal{P}(s)$. As a solution to the equation system above, at least one of of G_j for $j \in \mathcal{P}(s) \cup \{i\}$ should be increasing. Therefore, $G_i'(s) > 0$ for all s below $\bar{s}_i^*$ so $G_i(s) < 0$. Moreover, Claim 11 and 14 in the supplementary notes imply that the payoff of player i at s is strictly less than u_i^*. ∎

If the algorithm starts with $\bar{s}_1^*$, the claim above ensures that the algorithm for nonlinear costs finds the lower supports of the equilibrium strategies. Similar to the analysis for linear costs, the algorithm constructs the unique equilibrium if it starts with $\bar{s}_1^*$.

Proposition 1: *Suppose T is the number of iterations in the algorithm for nonlinear costs. Then, $|u_i - u_i^*| = O(2^{-T})$ for each i, and $|\hat{G}_i^*(s) - G_i^*(s)| = O(2^{-T})$ for each s and i, where $\hat{G}_i^*(s)$ is the output of the algorithm after T iterations.*

Proof. If $\bar{s} > \bar{s}_1^*$, Lemma 8 implies that $\bar{s}_1^*$ is in the interval $[\bar{s}, \bar{s}^u]$. Since $\bar{s}$ is also in the same interval, and the interval shrinks by half after each iteration, we have $|\bar{s} - \bar{s}_1^*| = O(2^{-T})$. Similarly, $|u_i - u_i^*| = O(2^{-T})$ for all i.

Now we are going to show that $|\hat{G}_i^*(s) - G_i^*(s)| = O\left(2^{-T}\right)$ for each i and s. Similar to (26), the solution to (3) and (4) is also a solution to an ordinary differential equation system

$$\begin{pmatrix} G_{i-1}' \\ G_i' \end{pmatrix} = \begin{pmatrix} 0 & v^{i-1} - v^i \\ v^{i-1} - v^i & 0 \end{pmatrix}^{-1} \begin{pmatrix} c_{i-1}' \\ c_i' \end{pmatrix}$$

with the following initial conditions

$$G_{i-1}(\underline{s}_i) = 0,$$
$$G_i(\underline{s}_i) v^{i-1} + (1 - G_i(\underline{s}_i)) v^i - c_{i-1}(\underline{s}_i) = u_{i-1}.$$

Since $|u_i - u_i^*| = O(2^{-T})$, we have $|\underline{s}_i - \underline{s}_i^*| = O(2^{-T})$ where $\underline{s}_i^*$ is the counter part of $\underline{s}_i$ if the algorithm starts with $\bar{s}_1^*$. Observe that the supports of pseudo strategies G_i and G_{i-1}

44

are bounded, we have $|G_i'(s) - G_i'^+(s)| = O(2^{-T})$ and $|G_i(s) - G_i^+(s)| = O(2^{-T})$, where $G_i^+(s)$ is the counter part of $G_i(s)$ if the algorithm starts with $\bar{s}_1^*$. Similarly, for all the pseudo strategies defined in Step $1'.i$ satisfies $|G_i(s) - G_i^+(s)| = O(2^{-T})$.

In Step $3'.i$, pseudo strategy $\hat{G}_i(s)$ is replaced by the smallest monotone function $\hat{G}_i^*(s)$ that lies on or above it. It can be verified that, after this step, we still have $|\hat{G}_i^*(s) - \hat{G}_i^+(s)| = O(2^{-T})$, where $\hat{G}_i^+(s)$ is the counter part of $\hat{G}_i^*(s)$ if the algorithm starts with $\bar{s}_1^*$. Similarly, we have $|\hat{G}_i^*(s) - \hat{G}_i^+(s)| = O(2^{-T})$ for each i and s at the end of Step $3'$. ∎

Corollary 1: *Consider a sequence of contests in which $c_i(s) - c_j(s)$ pointwise converges to zero for players $i, j < m + 2$, then $u_i^* - u_j^*$ also converges to zero and $G_i^*(s) - G_j^*(s)$ pointwise converges to zero.*

Proof. Suppose $c_i(s)$ pointwise converges to $c_{i+1}(s)$. Let us consider the equilibrium in the limit. Lemma 1 implies $\bar{s}_i^* \geq \bar{s}_{i+1}^*$. Suppose $\bar{s}_i^* > \bar{s}_{i+1}^*$, therefore i's expected winnings at $\bar{s}_{i+1}^*$ are more than $i+1$'s, therefore $i+1$ would deviate to $\bar{s}_i^*$ for a higher payoff. Therefore, $\bar{s}_i^* = \bar{s}_{i+1}^*$, and the payoffs of i and $i+1$ are also the same.

From the way we construct the strategies for i and $i+1$, their strategies $G_i^*(s)$ and $G_{i+1}^*(s)$ must also converge at any s in the common supports. ∎

Supplementary Notes for "Asymmetric All-Pay Contests with Heterogeneous Prizes"

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Abstract

These notes contain the results in linear algebra that are important to "Asymmetric All-Pay Contests with Heterogeneous Prizes". There are two main results, Claim 11 and 14, which are used to prove the unique solution of equation system (1), equation system (2) in "Asymmetric All-Pay Contests with Heterogeneous Prizes", and their counter parts in the N-player contests.

We follow the same notations as in "Asymmetric All-Pay Contests with Heterogeneous Prizes", which is referred as to the "main paper" hereafter.

Define $\mathbf{D}_j \equiv \mathbf{1}_j \mathbf{1}_j' - \mathbf{I}_j$, where $\mathbf{1}_j$ is a j-dimensional vector of ones. The diagonal entries of $\mathbf{D}_j$ are zeros, and all the other entries are 1. $\mathbf{B}_j$ is $\mathbf{D}_j$ with the entry at position $(1,1)$ replaced with 1.

Claim 1 $\det \mathbf{D}_j = (j-1)(-1)^{j-1}$.

Proof. We use induction in this proof. When $j = 3$, it is easy to see that

$$\det \mathbf{D}_j = (j-1)(-1)^{j-1} \tag{1}$$

$$\det \mathbf{B}_j = (-1)^{j-1} \tag{2}$$

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Suppose the two equations above are true for $j-1$. Expand $\det \mathbf{D}_j$ according the first column, we get a sum of $j-1$ terms of alternating signs. For the j_1th term, put its j_1th column to left and move columns 1 to $j_1 - 1$ one position to the right. Then, each term is $- \det \mathbf{B}_{j-1}$, and we have

$$\det \mathbf{D}_j = - (j-1) \det \mathbf{B}_{j-1} \tag{3}$$

Expand $\det \mathbf{B}_j$ according to the first column, we get a sum of j terms of alternating signs. For the $(j_1 + 1)$th term and $1 \leq j_1 \leq j - 1$, put its j_1th column to left and move columns 1 to $j_1 - 1$ one position to the right. Then, each of the last $j - 1$ term is $- \det \mathbf{B}_{j-1}$, so

$$\begin{aligned}
\det \mathbf{B}_j &= \det \mathbf{D}_{j-1} - (j-1) \det \mathbf{B}_{j-1} \\
&= \det \mathbf{D}_{j-1} - \det \mathbf{D}_j
\end{aligned} \tag{4}$$

where the second equality comes from (3). Therefore, (3) and (4) imply (1) and (2) are also true for j. ∎

If a non-zero entry of $\mathbf{D}_j$ is replace with 0, then we say that the resulting matrix has an *off-diagonal zero* at position (j_1, j_2).

Denote $\mathcal{M}_j$ as the set of all $j \times j$ matrices such that i) it has at most j off-diagonal zeros, ii) each column has at most one off-diagonal zero.

Claim 2 *If i) $\mathbf{A}_j \in \mathcal{M}_j$, ii) it has j off-diagonal zeros and iii) each row has an off-diagonal zero, $\det \mathbf{A}_j$ is 0 or has sign $(-1)^{j-1}$.*

Proof. Suppose the off-diagonal zero in row 1 is at column j_2, where $j_2 \neq 1$. Add all other rows to row 1 and divide it by $j - 2$, then we get a row of ones. It easy to see that column j_2 does not have an off-diagonal zero. Suppose the off-diagonal zero in row j_2 is at column j_3.

Deduct column j_2 from column j_3, then column j_3 becomes zeros except -1 in row j_3.

Expand the determinant according to column j_3, we get $- \det \mathbf{A}_{j-1}^1$ where $\mathbf{A}_{j-1}^1$ is a $(j-1) \times (j-1)$ matrix with ones in the first row[1]. Moreover, two column of $\mathbf{A}_{j-1}^1$ has no off-diagonal zeros and any other column has one off-diagonal zero. Suppose the two columns without off-diagonal zeros are column j_1' and j_2'.

Multiply row 1 by $j - 3$ and deduct all others rows from it, then the first row has two off-diagonal zeros at column j_1' and j_2'. Hence the resulting matrix is in $\mathcal{M}_{j-1}$, so its determinant is either 0 or of the sign $(-1)^{j-2}$. As a result, $- \det \mathbf{A}_{j-1}^1$ is 0 or has sign $(-1)^{j-1}$. ∎

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[1] The rest of the Appendix uses elementary operation of matrix, and we use supscripts to index the matrices in a sequence of such operations.

Claim 3 *If i)* $\mathbf{A}_j \in \mathcal{M}_j$, *ii) it has* j *off-diagonal zeros, iii) at least one row has no off-diagonal zero,* $\det \mathbf{A}_j$ *is 0 or has the sign* $(-1)^{j-1}$.

Proof. Denote the row without an off-diagonal zero as row j_1. Suppose row j_2 is a row with an off-diagonal zero. Add all the other rows to row j_2, then divide it by $j - 1$, then row j_2 only has ones.

Deduct row j_1 from row j_2, we get a row of zeros except 1 at column j_1.

Expand the determinant according to row j_2, we have $(-1)^{j_1+j_2} \det \mathbf{A}_{j-1}^1$ where $\mathbf{A}_{j-1}^1$ is the (j_2, j_1) minor matrix of $\mathbf{A}_j$.

It is easy to see that $j_1 \neq j_2$. Suppose $j_1 > j_2$. Move column j_2 of $\mathbf{A}_{j-1}$ to the left and shift the column 1 to $j_2 - 1$ to the right by one position. We have $(-1)^{j_1+j_2} (-1)^{j_2-1} \det \mathbf{A}_{j-1}^2$. Move row $j_1 - 1$ to the top and shift all the rows above row $j_1 - 1$ down by one position, we have $(-1)^{j_1+j_2} (-1)^{j_2-1} (-1)^{j_1-2} \det \mathbf{A}_{j-1}^3 = -\det \mathbf{A}_{j-1}^3$. The first row of $\mathbf{A}_{j-1}^3$ only has ones, and each column has at most one off-diagonal zero. If $j_1 < j_2$, we get the same result similarly.

Multiply row 1 of $\mathbf{A}_{j-1}^3$ by $j - 3$ and deduct all the other rows from it, the resulting matrix has one off-diagonal zero in each column. Therefore, this matrix is in $\mathcal{M}_{j-1}$, and has a determinant that is either 0 or of the sign $(-1)^{j-2}$. Since $\det \mathbf{A}_j$ has the opposite sign of the determinant of the resulting matrix, $\det \mathbf{A}_j$ is 0 or has the sign $(-1)^{j-1}$. ∎

Off-Diagonal Condition: Column j_1 has an off-diagonal zero if there is a column with an off-diagonal zero in row j_1.

Claim 4 *If i)* $\mathbf{A}_j \in \mathcal{M}_j$, *ii) it has less than* j *off-diagonal zeros, iii) it does not satisfy the off-diagonal condition,* $\det \mathbf{A}_j$ *is 0 or has the sign* $(-1)^{j-1}$.

Proof. Since $\mathbf{A}_j$ does not satisfy the off-diagonal condition, there is a column, j_2, such that i) column j_2 has an off-diagonal zero in row j_1, ii) column j_1 has no off-diagonal zero.

Deduct column j_2 from column j_1, and and then column j_1 becomes zeros except a 1 in row j_2.

The following analysis is similar to Claim 2. Expand the determinant according to column j_1, and then we have $(-1)^{j_1+j_2} \det \mathbf{A}_{j-1}^1$, where $\mathbf{A}_{j-1}^1$ is the (j_2, j_1) minor matrix of $\mathbf{A}_j$. Recall that $j_2 \neq j_1$, so first consider $j_2 < j_1$. Move row $j_1 - 1$ to the top and column $j_2 - 1$ to the right. The determinant becomes $-\det \mathbf{A}_{j-1}^2$. Note that the entry at $(1,1)$ in $\mathbf{A}_{j-1}^2$ is the entry at (j_1, j_2) in $\mathbf{A}_j$, which is 0 by assumption. Therefore, $\mathbf{A}_{j-1}^2 \in \mathcal{A}_{j-1}$, so $\det \mathbf{A}_j$ is either 0 or of the sign $(-1)^{j-1}$. If $j_1 > j_2$, we can get the same result similarly. ∎

3

Claim 5 *If i)* $\mathbf{A}_j \in \mathcal{M}_j$, *ii) it has less than j off-diagonal zeros, iii) it satisfies the off-diagonal condition,* $\det \mathbf{A}_j$ *is 0 or has the sign* $(-1)^{j-1}$.

Proof. First, we claim that column j_1 has an off-diagonal zero if row j_1 has an off-diagonal zero. To see why, suppose otherwise. Then, row j_1 has an off-diagonal zero at column j_2 and column j_1 does not. Column j has an off-diagonal zero in row j_1, and column j_1 has no off-diagonal zero, which contradicts the off-diagonal condition.

As a result, if column j_1 has no off-diagonal zero, row j_1 has no off-diagonal zero. Denote $\mathcal{J}$ as $\{1, 2, ..., j\}$ and $\mathcal{H}$ as the columns with an off-diagonal zero, then $\mathcal{J}\backslash\mathcal{H}$ is a set of rows without off-diagonal zeros.[2]

Pick any row with an off-diagonal zero and add all the other rows with off-diagonal zeros to it. The resulting row is either $\hat{j}$ or $\hat{j} - 1$, where $\hat{j}$ is the number of rows with off-diagonal zeros. Moreover, in this row, $\hat{j} - 1$ is at the columns in $\mathcal{H}$ and $\hat{j}$ is at the columns in $\mathcal{J} \backslash \mathcal{H}$.

Pick a row in $\mathcal{J} \backslash \mathcal{H}$ and add the rest to this row, the resulting row has entries equal $\bar{j}$ or $\bar{j} - 1$, where $\bar{j} = \#(\mathcal{J}\backslash\mathcal{H})$. Moreover, $\bar{j}$ is in the columns in $\mathcal{H}$, and $\bar{j} - 1$ is in the columns in $\mathcal{J} \backslash \mathcal{H}$.

Add the row with $\hat{j}$ and $\hat{j} - 1$ to the one with $\bar{j}$ and $\bar{j} - 1$, and divide it by $\bar{j} + \hat{j} - 1$. The resulting row has only ones.

This row of ones replaces a row with one off-diagonal zero. Deduct a row without an off-diagonal zero from this row of ones, then we get a row of zeros except one entry as 1. Similar to Claim 3 and 4, if we expand the determinant according to this row and move some rows and columns, $\det \mathbf{A}_j$ becomes $-\det \mathbf{A}_{j-1}$, where $\mathbf{A}_{j-1} \in \mathcal{M}_{j-1}$. Hence, $\det \mathbf{A}_j$ *is 0 or has the sign* $(-1)^{j-1}$. ∎

Claim 6 *If* $\mathbf{A}_j \in \mathcal{M}_j$, $\det \mathbf{A}_j = 0$ *or* $\det \mathbf{A}$ *has the sign* $(-1)^{j-1}$.

Proof. By induction.

It is easy to verify that the statement is true for $j' = 2$. Suppose the statement is true for $j' = j - 1$, Claim 2 to 5 show that it is also true for $j' = j$. Therefore the claim is true if integer j is bigger than 1. ∎

Claim 7 $\det \mathbf{H}_j$ *has sign* $(-1)^{j-1}$, *where* $\mathbf{H}_j$ *is a $j \times j$ matrix with zero diagonal entries and*
$$h_{j_1, j_2} = \sum_{l=1}^{j} h_l - h_{j_1} - h_{j_2}, \text{ where } h_l > 0 \text{ for any } l.$$

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[2] There might be more than one such set.

Proof. Column 1 of $\mathbf{H}_j$ is a sum of $j-1$ vectors, $\sum_{l=2}^{j} h_l \mathbf{1}_{-1,-l}$, where $\mathbf{1}_{-j_1,-j_2}$ is a column

vector with ones except two zeros in row j_1 and j_2. Therefore, $\det \mathbf{H}_j = \sum_{l=2}^{j} \det \mathbf{H}_j^1$, where

$\mathbf{H}_j^1$ is a $j \times j$ matrix with column 1 as $h_l \mathbf{1}_{-1,-l}$ and the other columns the same as in $\mathbf{H}_j$.
Note that column 1 in $\mathbf{H}_l^1$ only contains 0 or h_l.

For any $\mathbf{H}_l^1$, its second column is $\sum_{l=1,3,...,j} h_l \mathbf{1}_{-2,-l}$, so $\det \mathbf{H}_l^1$ also equals a sum of $j-1$

determinants of $j \times j$ matrices. Moreover, the first two columns of these matrices only have
one h_l.

Repeat this step for the other columns until $\det \mathbf{H}_j$ become a sum of determinants of
$j \times j$ matrices that have zero or the same h_l in each column. Moreover, these determinants
have other properties. First, if column j_2 of these matrices has only h_l, then it is $h_l \mathbf{1}_{-l,-j_2}$.
Second, column j_2 cannot have h_{j_2}.

Denote $\mathcal{J} = \{1, ..., j\}$ and

$$\mathfrak{K}_j = \left\{ (\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j) \mid \mathcal{J}_l \cap \mathcal{J}_{l'} = \varnothing, \ \cup_{l=1}^{j} \mathcal{J}_l = \mathcal{J}, \text{ and } \#\mathcal{J}_l < j \right\}$$

$(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j)$ is a j-set partition of $\mathcal{J}$ except that $\mathcal{J}_l$ can be empty. For any $(\mathcal{J}_1, ..., \mathcal{J}_j) \in$
$\mathfrak{K}_j$, replace the entry of (j_1, j_2) in $\mathbf{D}_j$ with 0 if $j_2 \in \mathcal{J}_{j_1}$, and denote the resulting matrix as
$\mathbf{A}_{\mathcal{J}_1,...,\mathcal{J}_j}$. $\det \mathbf{H}_j$ is a polynomial of order j, and each term has the same order. That is

$$\det \mathbf{H}_j = \sum_{(\gamma_1,...,\gamma_j)} \left(\eta_{\gamma_1,...,\gamma_j} \prod_{l=1}^{j} h_l^{\gamma_l} \right)$$

where the sum is over the set $\left\{ (\gamma_1, ..., \gamma_j) \mid \gamma_l < j, \ \sum_{l=1}^{j} \gamma_l = j \text{ and } \gamma_l \in \mathbb{Z}_+ \right\}$. Denote $\mathfrak{K}_{\gamma_1,...,\gamma_j} =$
$\{(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j) \in \mathfrak{K}_j \mid \#\mathcal{J}_l = \gamma_l\}$. Then, $\eta_{\gamma_1,...,\gamma_j} = \sum_{(\mathcal{J}_1,...,\mathcal{J}_j)} \det \mathbf{A}_{\mathcal{J}_1,...,\mathcal{J}_j}$, where the sum is
over the set $\mathfrak{K}_{\gamma_1,...,\gamma_j}$.

For each $(\mathcal{J}_1, ..., \mathcal{J}_j)$ in $\mathfrak{K}_{\gamma_1,...,\gamma_j}$, $\mathbf{A}_{\mathcal{J}_1,...,\mathcal{J}_j}$ is in $\mathcal{A}_j$, so $\eta_{\gamma_1,...,\gamma_j}$ is either 0 or has sign $(-1)^{j-1}$
by Claim 6. As a result, $\det \mathbf{H}_j$ either is 0 or has sign $(-1)^{j-1}$. Now we are going to show
$\det \mathbf{H}_j \neq 0$ by proving that one of the coefficients is not zero.

Consider the coefficient of $h_1^{j-2} h_2 h_3$ in $\det \mathbf{H}_j$, it is a sum of determinants and one of them
is associated with $(\mathcal{J}_1, ..., \mathcal{J}_j)$ such that $\mathcal{J}_2 = \{3\}, \mathcal{J}_3 = \{1\}$ and $\mathcal{J}_1 = \mathcal{J} \setminus (\mathcal{J}_2 \cup \mathcal{J}_3)$.
Such determinant has j off-diagonal zeros: one at column 3 to row 2, another at column

5

1 row 3 and all the others are in the first row. The resulting matrix has zeros in the first row except a 1 in column 3. Expand this determinant according to the first row, we get a $(j-1) \times (j-1)$ determinant. Switch the first two row and we get $-\det \mathbf{D}_{j-1}$ which is not zero. ∎

Claim 8 $\det \hat{\mathbf{H}}_j$ *has sign* $(-1)^{j-1}$, *where* $\hat{\mathbf{H}}_j$ *is a* $j \times j$ *matrix with zeros diagonal entries and* $\hat{h}_{j_1,j_2} = \sum_{l=1}^{j'} h_l - h_{j_1} - h_{j_2}$, *where* $h_l > 0$ *for any* l *and* $j' \geq j$.

Proof. Add $\sum_{l=j}^{j'} h_l$ to the entries off the diagonal of $\mathbf{H}_j$, we have $\hat{\mathbf{H}}_j$. Denote $h_l' = h_l +$

$\frac{1}{j-2} \sum_{l=j}^{j'} h_l > 0$, then $\hat{h}_{j_1,j_2} = \sum_{l=1}^{j} h_l' - h_{j_1}' - h_{j_2}'$ and the previous claim implies that $\det \hat{\mathbf{H}}_j$ *has sign* $(-1)^{j-1}$. ∎

Let us introduce some notations before we move to the next claim.

For a given interval, let $\mathcal{A}$ be the active players and $\mathcal{P}$ be the participating players, $\mathcal{A} \subset \mathcal{P}$. Then for any s in this interval, equilibrium strategies $(G_i)_{i \in \mathcal{A}}$ solve

$$W(\mathbf{G}_{-i}, \mathbf{v}) - c_i(s) = u_i \qquad \text{for } i \in \mathcal{A} \tag{5}$$

where $\mathbf{G} = (G_i)_{i \in \mathcal{P}}$ and $\mathbf{v} = (v^k)_{k \in \mathcal{P}}$.[3]

Suppose $\mathcal{P} = \{1, 2, ..., j'\}$. If $\mathcal{P} \neq \{1, 2, ..., i'\}$, we can order the elements and rename them, and the argument below would be the same. If $\mathbf{v} \neq (v^k)_{k \in \mathcal{P}}$, we can also rename the prizes similarly, and the analysis below applies as well.

For any $i' \in \mathcal{A} \backslash \{i\}$, take derivatives of both sides of (5) with respect to $G_{i'}$, we have

$$\sum_{l \in \mathcal{A} \backslash \{i,i'\}} \frac{dW(\mathbf{G}_{-i}, \mathbf{v})}{dG_l} \frac{dG_l}{dG_{i'}} = -\frac{dW(\mathbf{G}_{-i}, \mathbf{v})}{dG_{i'}} \quad \text{for } i \neq i' \tag{6}$$

$$\sum_{l \in \mathcal{A} \backslash \{i'\}} \frac{dW(\mathbf{G}_{-i'}, \mathbf{v})}{dG_l} \frac{dG_l}{dG_{i'}} = 0 \tag{7}$$

---

[3]Since the indices of $\mathbf{G}$ and $\mathbf{v}$ in $W(\mathbf{G}_{-i}, \mathbf{v})$ are the same below, we sometime only mention the indices for $\mathbf{G}$.

We can write (6) and (7) into matrix form

$$\hat{\mathbf{W}}_{j-1}\boldsymbol{\delta}= -\mathbf{d} \tag{8}$$

$$\mathbf{d}'\boldsymbol{\delta} > 0$$

where $j = \#\mathcal{A}$ and $\hat{\mathbf{W}}_{j-1}$ is a $(j-1)\times(j-1)$ matrix, $\boldsymbol{\delta}$ and $\mathbf{d}$ are vectors of $j-1$ rows. The diagonal entries of $\hat{\mathbf{W}}$ are zero and the entry at (j_1, j_2) is $dW\left(\mathbf{G}_{-j_1}, \mathbf{v}\right)/dG_{j_2}$; the element in row j_1 of $\boldsymbol{\delta}$ is $dG_{j_1}/dG_{i'}$ for $j_1 \neq i'$; the element in row j_1 of $\mathbf{d}$ is $dW\left(\mathbf{G}_{-j_1}, \mathbf{v}\right)/dG_{i'}$ for $j_1 \neq i'$. Define $\mathbf{W}_j = \begin{pmatrix} \hat{\mathbf{W}}_{j-1} & \mathbf{w} \\ \mathbf{w} & 0 \end{pmatrix}$ and w_{j_1, j_2} as the entry in row j_1 and column j_2 of $\mathbf{W}_j$.

Define first order difference as $\Delta_k^{(1)} = v^{k-1} - v^k$, for $k = 2, ..., j'$. The lth order difference is $\Delta_k^{(l)} = \Delta_{k-1}^{(l-1)} - \Delta_k^{(l-1)}$ for $l = 1, ..., j'-1$, and $k = l+1, ..., j'$.

Claim 9

$$w_{j_1, j_2} = \Delta_j^{(1)} + \sum_{l'=2}^{j'-1} \Delta_{j'}^{(l')}\left(\sum_{\{i_1,...,i_{l'-1}\}\subset \Gamma_{j_1, j_2}}\left(\prod_{l=1}^{l'-1}G_{i_l}\right)\right)$$

where $j_1 \neq j_2$ and $\Gamma_{j_1, j_2} = \{G_1, ..., G_{j'}\}/\{G_{j_1}, G_{j_2}\}$.

Proof. We are going to prove by induction. First, it is easy to verify the statement is true for $j' = 3$. Suppose the statement is true for $j' = j'_I - 1$, we are going to show that it is also true for $j' = j'_I$.

For the purpose of cleaner exhibition, the following proof only focuses on w_{12}.

We know that

$$
\begin{aligned}
w_{12} &= W\left(G_3, ..., G_{j'_I}, \Delta_2^{(1)}, ..., \Delta_{j'_I}^{(1)}\right)\\
&= G_{j'_I}W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)}, ..., \Delta_{j'_I-1}^{(1)}\right) + \left(1 - G_{j'_I}\right)W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right)\\
&= W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right)\\
&\quad + G_{j'_I}\left(W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)}, ..., \Delta_{j'_I-1}^{(1)}\right) - W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right)\right)\\
&= W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right)\\
&\quad + G_{j'_I}W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)} - \Delta_3^{(1)}, \Delta_3^{(1)} - \Delta_4^{(1)}, ..., \Delta_{j'_I-1}^{(1)} - \Delta_{j'_I}^{(1)}\right)\\
&= W\left(G_3, ..., G_{k'_I-1}, \Delta_3^{(1)}, ..., \Delta_{k'_I}^{(1)}\right) + G_{k'_I}W\left(G_3, ..., G_{k'_I-1}, \Delta_3^{(2)}, ..., \Delta_{k'_I}^{(2)}\right) \tag{9}
\end{aligned}
$$

Since the statement is true for $k' = j'_I - 1$, we have

$$W(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}) = \Delta_{j'_I}^{(1)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l')} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \quad (10)$$

$$W(G_3, ..., G_{j'_I-1}, \Delta_3^{(2)}, , ..., \Delta_{j'_I}^{(2)}) = \Delta_{j'_I}^{(2)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l'+1)} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \quad (11)$$

where $\Gamma'_{12} = \{G_1, ..., G_{j'_I-1}\}/\{G_1, G_2\}$ and

Substitute (10) and (11) into (9), then we have

$$
\begin{aligned}
w_{12} &= \Delta_{j'_I}^{(1)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l')} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \\
&\quad + G_{j'_I} \left(\Delta_{j'_I}^{(2)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l'+1)} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \right)
\end{aligned}
$$

therefore the coefficient of $\Delta_{j'_I}^{(j')}$ is

$$
\begin{aligned}
&\left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) + G_{j'_I} \left(\sum_{\{i_1,...,i_{l'-2}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-2} G_{i_l} \right) \right) \\
&= \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12} \cup \{G_{j'_I}\}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right)
\end{aligned}
$$

As a result, $w_{12} = \Delta_{j'}^{(1)} + \sum_{l'=2}^{j'-1} \Delta_{j'}^{(l')} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right)$.

Similarly, we can extend the analysis above to w_{j_1,j_2} for $j_1 \neq j_2$. Hence, the statement is also true for j'_I. ∎

Under the assumption of QPS, $\Delta_{j'}^{(l)} = 0$ for $l > 2$. Therefore, both $\mathbf{W}_j$ and $\hat{\mathbf{W}}_j$ are

simplified, and

$$
\begin{aligned}
w_{j_1,j_2} &= \left(\sum_{l=1}^{j'} G_l - G_{j_1} - G_{j_2}\right)\left(v^{j'-2} - 2v^{j'-1}\right) + \left(v^{j'-1} - v^{j'}\right) \\
&= \left(\sum_{l=1}^{j'} G_l - G_{j_1} - G_{j_2}\right)\Delta_{j'}^{(2)} + \Delta_{j'}^{(1)}
\end{aligned}
$$

if $j_1 \neq j_2$.

Claim 10 $\det \mathbf{W}_j$ *and* $\det \hat{\mathbf{W}}_j$ *have sign* $(-1)^{j-1}$ *if the prizes satisfy QPS and* $G_l > 0$ *for* $l \in \mathcal{P}$.[4]

Proof. First, suppose $\mathcal{A} = \mathcal{P}$, so $j = j'$.

$\det \mathbf{W}_j = \left(\Delta_{j'}^{(1)}\right)^j \det \mathbf{Z}_j$, where $z_{j_1,j_2} = \sum_{l=1}^{j} h_l - h_{j_1} - h_{j_2}$, $h_l = G_l\Delta_{j'}^{(2)}/\Delta_{j'}^{(1)} + \frac{1}{j-2}$.

Assume any $G_{i'}$ equals G_i for $i' \in \mathcal{P}\backslash\{i,j\}$, we have $dW_i/dG_j = (j-2)\Delta_{j'}^{(2)}G_i + \Delta_{j'}^{(1)} > 0$, where the inequality comes from Claim 1 in the main paper. As a result, $h_i = G_i\Delta_{j'}^{(2)}/\Delta_{j'}^{(1)} + \frac{1}{j-2} > 0$.

Claim 7 implies $\det \mathbf{Z}_j$ is of the sign $(-1)^{j-1}$, and so it is $\det \mathbf{W}_j$.

Second, suppose $\mathcal{A} \subsetneq \mathcal{P}$, then we have $j < j'$.

$$
w_{j_1,j_2} = \left[\Delta_{j'}^{(2)}\left(\sum_{l\in\mathcal{A}} G_l - G_{j_1} - G_{j_2}\right) + \Delta_{j'}^{(1)}\right] + \Delta_{j'}^{(2)}\sum_{l\in\mathcal{P}\backslash\mathcal{A}} G_l
$$

if $j_1 \neq j_2$ and $j_1, j_2 \in \mathcal{A}$.

$\det \mathbf{W}_j = (\Delta_{j'}^{(1)})^j \det \mathbf{Z}_j$, where $z_{j_1,j_2} = \sum_{l\in\mathcal{A}} h_l - h_{j_1} - h_{j_2}$, $h_l = G_l\Delta_{j'}^{(2)}/\Delta_{j'}^{(1)} + \frac{y}{j-2}$, $y = 1 + \left(\Delta_{j'}^{(2)}/\Delta_{j'}^{(1)}\right)\sum_{l\in\mathcal{P}\backslash\mathcal{A}} G_l$. $\det \mathbf{Z}_j$ has sign $(-1)^{j-1}$ according to Claim 7.

Let us consider $\det \hat{\mathbf{W}}_j$. Suppose $\mathcal{A}' = \mathcal{A}\backslash\{i'\}$ and consider equation (1) in the main paper with $\mathcal{A}'$ and $\mathcal{P}$. The corresponding $\mathbf{W}_{j-1}$ of the new system is just $\hat{\mathbf{W}}_{j-1}$ in the original system. Therefore, $\det \hat{\mathbf{W}}_{j-1}$ has sign $(-1)^{j-2}$ according to Claim 7. ∎

Claim 11 *Suppose the prizes satisfies QPS. For any* $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$ *and* $i \in \mathcal{A}$*, LHS of (5) for* i *decreases if* G_i *increases in other equations of (5).*

---

[4]This claim may fail if the prizes are not QPS or GPS. Consider a four-player contest with prizes $v_1 = 7, v_2 = 2, v_3 = 1$ and $v_4 = 0$. When G_1 and G_2 are close to 0, G_3 and G_4 are close to 1, $\det \mathbf{F}_4$ is close to 5.

Proof. Suppose i is the weakest player in $\mathcal{A}$. Claim 10 shows that $\det \mathbf{W}_j$ has sign $(-1)^{j-1}$ and $\det \hat{\mathbf{W}}_{j-1}$ has sign $(-1)^{j-2}$, then $\hat{\mathbf{W}}_{j-1}$ is invertible and

$$\mathbf{d}'\mathbf{g} = -\mathbf{d}'\hat{\mathbf{W}}_{j-1}^{-1}\mathbf{d} = \det \mathbf{W}_j / \det \hat{\mathbf{W}}_{j-1} < 0$$

Therefore, we have $\hat{\mathbf{W}}_{j-1}\mathbf{g} = -\mathbf{d}$ and $\mathbf{d}'\mathbf{g} > \mathbf{0}$, so the claim is true for $i = a'$.

Since players in $\mathcal{A}$ are symmetric in this problem, the claim is also true for other players in $\mathcal{A}$. ∎

Now consider geometric prize sequences.

Claim 12 $\det \mathbf{H}_j$ *has sign* $(-1)^{j-1}$, *where* $\mathbf{H}_j$ *is a* $j \times j$ *matrix with zeros diagonal entries and* $h_{j_1,j_2} = \left(\prod_{l=1}^{j'} h_l\right) / (h_{j_1} h_{j_2})$ *with* $h_l > 0$ *for any* l *and* $j' \geq j$.

Proof. Multiply row $j_1 > 1$ by h_{j_1}.

Divided column j_2 by $\left(\prod_{l=1}^{j'} h_l\right) / h_{j_2}$. Let us describe the resulting matrix. First, the entries in the first row are $1/h_1$ except a zero at the first column. Second, the diagonal entries are zero. Third, $h_{j_1,j_2} = 1$ for $j_1 > 1$ and $j_1 \neq j_2$.

Multiply the first row by h_1, we get $\det \mathbf{D}_j = (j-1)(-1)^{j-1}$ by Claim 1. ∎

Claim 13 $\det \mathbf{W}_j$ *and* $\det \hat{\mathbf{W}}_j$ *have sign* $(-1)^{j-1}$ *if the prizes satisfies GPS and* $G_l > 0$ *for* $l \in \mathcal{P}$.

Proof. We can verify that $\Delta_{j'}^{(l)} = (\alpha - 1)^l v^j$, so $w_{j_1,j_2} = \prod_{l \in \mathcal{P}\backslash\{j_1,j_2\}} ((\alpha - 1)G_l + 1)$. Denote $h_l = (a-1)G_l + 1$, and $h_l > 0$ since $\alpha > 1$ and $0 < G_l$. Therefore, Claim 12 implies $\det \mathbf{W}_j$ has sign of $(-1)^{j-1}$.

Similar to the case of QPS, $\det \hat{\mathbf{W}}_{j-1} = \det \mathbf{W}_{j-1}$ for $\mathcal{A}' = \mathcal{A}\backslash\{i'\}$ and $\mathcal{P}$, so $\det \hat{\mathbf{W}}_{j-1}$ has sign $(-1)^{j-2}$. ∎

Claim 14 *Suppose the prizes satisfies GPS, then for any subset* $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$ *and* $i \in \mathcal{A}$, *LHS of (5) for* i *decreases if* G_i *increases in other equations of (5).*

Proof. Given the previous claim, the proof is the same as Claim 11. ∎

Claim 15 *Equation system (5) has at most one solution in* $[0, 1]^{\#\mathcal{A}}$.

Proof. Suppose there are two sets of solutions, $(\tilde{G}_i)_{i \in \mathcal{A}}$ and $(\hat{G}_i)_{i \in \mathcal{A}}$. Since the solutions are different, suppose $\tilde{G}_i(s_0) > \hat{G}_i(s_0)$ without loss of generality. Therefore Claim 11 and 14 imply $W(\tilde{\mathbf{G}}_{-i}, \mathbf{v}) < W(\hat{\mathbf{G}}_{-i}, \mathbf{v})$, so they cannot both equal $u_i + c_i(s)$, which contradicts the definition of $\tilde{G}_i$ and $\hat{G}_i$. ∎

The following two claims provide the omitted proofs in Claim 10 and 11 in the main paper.

Denote $h_l \equiv (\alpha - 1)G_l + 1$ and a $j \times j$ matrix

$$\mathbf{H}_j \equiv \begin{pmatrix} \hat{\mathbf{W}}_{j-1} & \mathbf{d} \\ \mathbf{d}' & 0 \end{pmatrix}.$$

Switch the last two columns of $\mathbf{H}_j$, then drop the last column and last row, we have a $(j-1) \times (j-1)$ matrix, which is exactly $\tilde{\mathbf{W}}_{j-1}$ in Claim 10 in the main paper.

Claim 16 *If the prize sequence is geometric,* $\det \tilde{\mathbf{W}}_{j-1}$ *has sign of* $(-1)^j$.

Proof. Now we are going to use induction to show that $\det \tilde{\mathbf{W}}_{j-1}$ has sign of $(-1)^j$.

First, when $j = 3$, we have $\det \tilde{\mathbf{W}}_2 = \det \begin{pmatrix} 0 & h_2 \\ h_3 & h_1 \end{pmatrix} < 0$.

Suppose $\det \tilde{\mathbf{W}}_{j'-1}$ has sign of $(-1)^{j'}$. Consider $\det \tilde{\mathbf{W}}_{j'}$. First, divide all columns except the last one by $h_{j'+1}$, then times column $j'-1$ by $h_{j'-1}$ and deduct it from the last column. The last column has zeros except in row $j'-1$. Expand the determinant according to the last column, and we have (-1) times a $(j'-1)$-dimensional determinant. Take the transpose of the matrix, we get $\det \tilde{\mathbf{W}}_{j'-1}$ which has the sign of $(-1)^{j'}$. As a result, $\det \tilde{\mathbf{W}}_{j'}$ has sign of $(-1)^{j'-1}$. ∎

Claim 17 *If the prize sequence is quadratic, we have* $\det \tilde{\mathbf{W}}_i - \det \tilde{\mathbf{W}}_{i-1}$ *has sign of* $(-1)^j$ *for* $i = 2, ..., j-1$.

Proof. If the prize sequence is quadratic, the H_j has entry $h_{i,i'} = \sum_{l=1}^{j} h_l - h_i - h_{i'}$ for $i \neq i'$ and zero diagonal elements.

We are going to show that $\det \tilde{\mathbf{W}}_i - \det \tilde{\mathbf{W}}_{i-1}$ has sign of $(-1)^j$ after a series of elementary operations. If we compare $\tilde{\mathbf{W}}_i$ and $\tilde{\mathbf{W}}_{i-1}$, they are the same except the $(i-1)$th and ith columns. The $(i-1)$th and ith columns in $\tilde{\mathbf{W}}_i$ are $\mathbf{h}_{i-1}$ and $\mathbf{d}$, and the $(i-1)$th and ith

columns in $\tilde{\mathbf{W}}_{i-1}$ are $\mathbf{d}$ and $\mathbf{h}_i$. Therefore

$$\det \tilde{\mathbf{W}}_i - \det \tilde{\mathbf{W}}_{i-1}$$
$$= \det(\mathbf{h}_1, ..., \mathbf{h}_{i-1}, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}) - \det(\mathbf{h}_1, ..., \mathbf{d}, \mathbf{h}_i, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1})$$
$$= \det(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_{i-1}, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}) + \det(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_i, \mathbf{d}, \mathbf{h}_{i+1}..., \mathbf{h}_{j-1})$$
$$= \det(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_{i-1} + \mathbf{h}_i, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}) \qquad (12)$$

where the second equality comes from switching the $(i-1)$th and ith columns in $\det \tilde{\mathbf{W}}_{i-1}$.

Deduct ith row from $(i-1)$th row, the $(i-1)$th row in the resulting determinant is $h_i - h_{i-1}$ except 0 in the $(i-1)$th column. Divide the $(i-1)$th row by $h_i - h_{i-1}$. Since $h_i - h_{i-1} > 0$, the resulting determinant has the same sign.

Deduct column i from all other columns except the $(i-1)$th column. The $(i-1)$th row of the resulting determinant has zeros except 1 at column i. The other rows are the same as in (12).

Expand the determinant according to the $(i-1)$th row, the result is $-\det \mathbf{Y}_{j-2}$, where $\mathbf{Y}_{j-2}$ is a $(j-2)$-dimensional determinant. Let us describe $\mathbf{Y}_{j-2}$. The $(i-1)$th column of $\mathbf{Y}_{j-2}$ is $h_{i-1} + h_i$ excluding the $(i-1)$th row; Column $i' > i-1$ of $\mathbf{Y}_{j-2}$ has $h_j - h_{i'}$ except $-\sum_{l=1}^{j-1} h_l + h_{i'}$ in column i'; Column $i' < i-1$ of $\mathbf{Y}_{j-2}$ has $h_j - h_{i'+1}$ except $-\sum_{l=1}^{j-1} h_l + h_{i'+1}$ in column i'.

Add all other columns to column $(i-1)$ of $\mathbf{Y}_{j-2}$, the $(i-1)$th column has only $(j-2) h_{j-1}$. Since $h_{j-1} > 0$, we can normalize column $i-1$ to ones without change the sign of the determinant.

Multiply column $i-1$ with $h_j - h_{i'}$ and deduct it from column $i' \neq i-1$. Then the i'th column has only zeros except $-\sum_{l=1}^{j-1} h_l + h_{i'+1} - (h_j - h_{i'}) = h_{i'} - \sum_{l \in \mathcal{A}\backslash\{i'\}} h_l < 0$; row $i-1$ has only zeros except 1 at column $i-1$, therefore we can set column $i-1$ to zeros except in row $i-1$ and not affect the sign of the determinant.

The resulting determinant is a diagonal matrix, therefore the determinant equals

$$-\prod_{i' \in \mathcal{A}\backslash\{i-1, j\}} \left(h_{i'} + \sum_{l \in \mathcal{A}\backslash\{i'\}} h_l \right)$$

which is a product of $1 + (j-3)$ negative numbers, therefore $\det \tilde{\mathbf{W}}_i - \det \tilde{\mathbf{W}}_{i-1}$ has sign of $(-1)^j$. ∎