Using the GB2 Income Distribution: A Review*

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February 8, 2018

JEL Classifications: I32, O15, C13

* The authors acknowledge support from ARC Grant DP140100673.

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Abstract

To use the GB2 distribution for the analysis of income and other positively-skewed distributions, knowledge of estimation methods and the ability to compute quantities of interest from the estimated parameters are required. We review estimation methodology that has appeared in the literature, and summarise expressions for inequality, poverty, and pro-poor growth that can be used to compute these measures from GB2 parameter estimates. An application to data from China and Indonesia is provided.

1. Introduction

Specification and estimation of parametric income distributions has a long history in economics. Much of the literature on alternative distributions can be accessed through the book by Kleiber and Kotz (2003), and the papers in Chotikapanich (2008). A series of papers by McDonald and his co-authors (McDonald 1984, McDonald and Xu 1995, Bordley et al 1997, McDonald and Ransom 2008, McDonald et al 2011) carry details of many of the distributions and the relationships between them. Our focus in this paper is on the generalised beta distribution of the second kind (GB2). It is a four-parameter distribution defined over the support $(0,\infty)$, and obtained by transforming a standard beta random variable defined on $(0,1)$. As described by McDonald and Xu (1995), it nests many popular three-parameter specifications of income distributions including the generalised gamma, beta2, Singh-Maddala and Dagum distributions. Two-parameter special cases of these distributions include the lognormal, gamma, Weibull, Lomax and Fisk distributions.[1] Parker (1999) describes a model of firm optimising behaviour that leads to a GB2 distribution for earnings. Applications have appeared in Butler and McDonald (1986), Cummins et al (1990), Feng et al (2006), Jenkins (2009), Graf and Nedyalkova (2014), and Jones et al (2014). In an extensive study examining global inequality, Chotikapanich et al (2012) estimate special case beta2 distributions for 91 countries in 1993 and 2000.

Estimation of a good-fitting parametric income distribution such as the GB2 facilitates further analysis. Once important quantities such as mean income, the Gini coefficient, the Lorenz curve, and the headcount ratio have been expressed in terms of the parameters of the distribution, they can be readily estimated from those parameters. If interest centres on a region which comprises a collection of countries or areas, a GB2 distribution can be estimated for each country/area; inequality, poverty and pro-poor growth for the region can be analysed by computing estimates of indicators expressed in terms of the parameters of a regional distribution which will be a population-weighted mixture of the GB2 distributions. If only grouped data are available, then estimating a distribution such as the GB2 provides a means for accommodating within-group variation, an important consideration for assessing inequality and poverty.

The purpose of this paper is to collect results on measures for inequality, poverty, and pro-poor growth, expressed as functions of the parameters of the GB2 distribution and its mixtures, and to summarise various methods of estimation that have appeared in the literature

[1] McDonald and Xu (1995) and McDonald and Ransom (2008) also consider a five-parameter generalised beta distribution which nests the GB2 and a GB1 distribution.

for estimating GB2 parameters from single observations or from grouped data. Expressions for the inequality, poverty, and pro-poor growth measures are given in Section 2. Section 3 contains a description of the various estimation techniques. The results from an application to 4 years of data for China and Indonesia are presented in Section 4. Some concluding remarks are offered in Section 5.

2. Inequality and Poverty Measures from the GB2 Distribution

Throughout we assume that income Y, for a given country or area, can be represented by a GB2 distribution whose probability density function (pdf) is given by

$$f\left(y \mid a,b,p,q\right)=\frac{ay^{ap-1}}{b^{ap}\mathrm{B}(p,q)\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q}} \qquad y>0 \tag{1}$$

where $a>0,\ b>0,\ p>0$ and $q>0$ are its parameters and $\mathrm{B}(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt$ is the beta function. The cumulative distribution function (cdf) corresponding to (1) is given by

$$F(y \mid a,b,p,q) = \frac{1}{\mathrm{B}(p,q)}\int_0^w t^{p-1}(1-t)^{q-1}dt = B\left(w \mid p,q\right) \tag{2}$$

where $w=\left(y/b\right)^{a}\Big/\left[1+\left(y/b\right)^{a}\right]$. The function $B(w \mid p,q)$ is the cdf for the normalized beta distribution, defined on the (0,1) interval, with parameters p and $q,$ and evaluated at w. It is a convenient representation because both it, and its inverse, are commonly included as readily-computed functions in statistical software. Properties of the GB2 distribution and its special cases have been considered extensively by McDonald (1984) and Kleiber and Kotz (2003). Three-parameter special cases which have been popular in the literature are the Singh-Maddala distribution where $p=1,$ the Dagum distribution where $q=1,$ and the beta2 distribution where $a=1.$ Extension to a 5-parameter GB distribution has been considered by McDonald and Xu (1996) and McDonald and Ransom (2008). Some further properties of the GB2 distribution are described by Graf and Nedyalkova (2014). In this Section we summarize main results from the GB2 distribution that are relevant for computing measures of inequality, poverty and pro-poor growth.

We envisage a scenario where GB2 distributions have been estimated for a number of countries, or for specific areas within a country such as urban and rural, and the objective is to evaluate inequality and poverty measures using the estimated parameters of the GB2 distributions. As well as evaluation of the measures from single GB2 distributions, we are interested in evaluating them for mixtures that arise when urban and rural GB2 distributions are combined to obtain a distribution for a country, or when country GB2 distributions are combined to obtain the distribution for a region. In most instances, we can express measures in terms of quantities such as beta and gamma functions that are readily computed by available software. Measures whose exact computation proves to be difficult can usually be written in terms of expectations which can be estimated by averaging values of the function over simulated draws from one or more of the GB2 distributions. Key quantities that are used for calculation of many measures, and for estimation of GB2 distributions, are the GB2 moments and moment distribution functions. We begin by giving expressions for them, as well as indicating how the GB2 Lorenz curve can be obtained. We then consider measures for inequality, poverty and pro-poor growth.

The k-th moment of the GB2 exists for $-ap < k < aq$ and is given by

$$\begin{aligned}\mu^{(k)} = E\left(Y^k\right) &= \frac{b^k \mathrm{B}\left(p+k/a, q-k/a\right)}{\mathrm{B}\left(p,q\right)} \\ &= \frac{b^k \Gamma\left(p+k/a\right)\Gamma\left(q-k/a\right)}{\Gamma(p)\Gamma(q)}\end{aligned} \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. The k-th moment distribution function for the GB2 is given by

$$\begin{aligned}F_k\left(y \mid a,b,p,q\right) &= \frac{1}{\mu^{(k)}}\int_0^y t^k f\left(t\right)dt \\ &= F\left(y \mid a,b,p+k/a, q-k/a\right)\end{aligned} \tag{4}$$

This result, that the GB2's moment distribution functions can be written in terms of its cdf evaluated at different parameter values, is particularly useful for deriving the Lorenz curve and for setting up and computing GMM estimates from grouped data. The Lorenz curve, relating the cumulative proportion of income η to the cumulative proportion of population u is given by

$$\begin{aligned}\eta(u) &= F_1\left[F^{-1}\left(u \mid a,b,p,q\right) \mid a,b,p,q\right] \\ &= F\left[F^{-1}\left(u \mid a,b,p,q\right) \mid a,b,p+1/a, q-1/a\right] \\ &= B\left[B^{-1}\left(u \mid p,q\right) \mid p+1/a, q-1/a\right] \qquad 0<u<1\end{aligned} \tag{5}$$

2.1 Inequality Measures

The most widely used inequality measure is the Gini coefficient. McDonald (1984) and McDonald and Ransom (2008) use hypergeometric functions to express the Gini coefficient in terms of the GB2 parameters. An algorithm for computing these functions has been proposed by Graf (2009). It has been our experience that it is easier computationally to compute the Gini coefficient via numerical integration than to numerically evaluate the hypergeometric functions. Another alternative is to estimate the Gini coefficient by simulating from the GB2 distribution. Specifically, noting that the Gini coefficient is given by

$$\begin{aligned} G &= -1+\frac{2}{\mu}\int_0^\infty yF\left(y\,|\,\phi\right)f\left(y\,|\,\phi\right)dy \\ &= -1+\frac{2}{\mu}E\left[yF\left(y\,|\,\phi\right)\right] \end{aligned} \tag{6}$$

where $\mu=\mu^{(1)}=E\left(y\right)=b\left[\Gamma\left(p+1/a\right)\Gamma\left(q-1/a\right)\right]\big/\left[\Gamma(p)\Gamma(q)\right]$ and $\phi'=\left(a,b,p,q\right)$, we can draw observations $\left(y_1,y_2,\ldots,y_M\right)$ from $f\left(y\,|\,\phi\right)$ and estimate G from

$$\hat{G}=-1+\frac{2}{\mu}\frac{1}{M}\sum_{m=1}^{M}y_mF\left(y_m\,|\,\phi\right) \tag{7}$$

The number of draws M can be made as large as necessary to achieve the derived level of accuracy. To draw observations from $f\left(y\,|\,\phi\right)$, we first draw observations $\left(w_1,w_2,\ldots,w_M\right)$ from a standard beta(p,q) distribution, defined on the $(0,1)$ interval, and then compute $y_m=b\left[w_m/\left(1-w_m\right)\right]^{1/a}$. If interest centres on one of the special case distributions where $p=1$, $q=1$ or $a=1$, then closed form expressions in terms of gamma or beta functions are available for the Gini coefficient. For example, if $a=1$, giving the beta2 distribution, then $G=2\mathrm{B}\left(2p,2q-1\right)\big/p\mathrm{B}^2\left(p,q\right)$.

Suppose now we have estimated GB2 income distributions for a number of different areas, such as countries within a region or urban and rural areas within a country, and we are interested in estimating the Gini coefficient for the combined area. The combined income distribution can be written as a population weighted mixture of the individual GB2 distributions. That is,

$$f\left(y\,|\,\Phi\right)=\sum_{j=1}^{J}\lambda_j f\left(y\,|\,\phi_j\right) \tag{8}$$

where $\Phi=\left(\phi_1,\phi_2,\ldots,\phi_J\right)$, λ_j is the proportion of the combined population in area j, and $\phi_j'=\left(a_j,b_j,p_j,q_j\right)$ is the vector of parameters of the distribution for area *j*. As noted by Chotikapanich et al (2007), in this case the Gini coefficient for a combination of J areas can be estimated from

$$G=-1+\frac{2}{\mu_C}\sum_{j=1}^{J}\sum_{\ell=1}^{J}\lambda_j\lambda_\ell\tau_{j\ell} \tag{9}$$

where

$$\tau_{j\ell}=\frac{1}{M}\sum_{m=1}^{M}y_{j,m}F\left(y_{j,m}\,|\,\phi_\ell\right) \tag{10}$$

$\mu_C=\sum_{j=1}^{J}\lambda_j\mu_j$ is the mean of the combined areas, μ_j is the mean for area j, and $y_{j,m}$ is the m-th draw from pdf $f\left(y\,|\,\phi_j\right)$. For the empirical work in this paper we estimated separate distributions for rural and urban areas in China and Indonesia and then combined them.

Next we consider the generalized entropy (GE) class of inequality measures whose expressions in terms of the parameters of the GB2 distribution have been provided by Jenkins (2009). The GE index is given by

$$I(\alpha)=\frac{1}{\alpha\left(\alpha-1\right)}\left[\frac{\mu^{(\alpha)}}{\mu^{\alpha}}-1\right] \qquad \text{for } \alpha\neq 0,1 \tag{11}$$

where, for the GB2 distribution, $\mu^{(\alpha)}=\int_0^\infty y^\alpha f(y\,|\,\phi)dy$ is given in (3), and $\mu^\alpha=\left[\mu^{(1)}\right]^\alpha$. Two popular special cases are obtained by taking limits as $\alpha\to 0$ and $\alpha\to 1$. The case where $\alpha\to 0$ is known as the mean logarithmic deviation or Theil(0) (Theil, 1967, p.127). Its general expression, and the result for the GB2 distribution, are[2]

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[2] See McDonald and Ransom (2008) or Jenkins (2009) for derivations. Equation (4) in Jenkins (2009) should read $I(1)=\nu_1/\mu-\log\mu$. Sarabia et al (2017) give details of the Theil indices for a wide range of distributions including the GB2.

$$
\begin{aligned}
I(0) &= \int_0^\infty \log\left(\frac{\mu}{y}\right) f\left(y \mid \phi\right) dy \\
&= \log(\mu) - E\left[\log(y)\right] \qquad (12) \\
&= \ln\left(\mu/b\right) - \psi(p)/a + \psi(q)/a
\end{aligned}
$$

The index obtained as $\alpha \to 1$ is known as Theil(1) (Theil, 1997, p.96). Its general expression, and result for the GB2 distribution, are

$$
\begin{aligned}
I(1) &= \int_0^\infty \frac{y}{\mu} \log\left(\frac{y}{\mu}\right) f(y \mid \phi) dy \\
&= \left[E\left(y \log(y)\right)\right]/\mu - \log\mu \qquad (13) \\
&= \left[\psi\left(p + 1/a\right) - \psi\left(q - 1/a\right)\right]/a + \log\left(b/\mu\right)
\end{aligned}
$$

where $\psi(c) = d \log\Gamma(c)/dc$ is the digamma function. The digamma function is computable by most software. In the event that it is not, draws $\left(y_1, y_2, \ldots, y_M\right)$ from $f\left(y \mid \phi\right)$ can be used to estimate $E\left[\log(y)\right]$ and $E\left[y \log(y)\right]$ with $\sum_{m=1}^{M} \log\left(y_m\right)/M$ and $\sum_{m=1}^{M} y_m \log\left(y_m\right)/M$, respectively.

The GE index for a mixture of income distributions and its decomposition into within and between group inequality has been considered by Sarabia et al (2017). For $\alpha \neq 0,1,$ it is given by

$$
\begin{aligned}
I_C(\alpha) &= \frac{1}{\alpha\left(\alpha-1\right)}\left[\int_0^\infty \left(\frac{y}{\mu_C}\right)^\alpha \sum_{j=1}^{J} \lambda_j f\left(y \mid \phi_j\right) dy - 1\right] \\
&= \frac{1}{\alpha\left(\alpha-1\right)}\left[\frac{1}{\mu_C^\alpha} \sum_{j=1}^{J} \lambda_j E_j\left(y^\alpha\right) - 1\right] \qquad (14) \\
&= \frac{1}{\alpha\left(\alpha-1\right)}\left[\frac{\sum_{j=1}^{J} \lambda_j \mu_j^{(\alpha)}}{\left(\sum_{j=1}^{J} \lambda_j \mu_j\right)^\alpha} - 1\right]
\end{aligned}
$$

where $\mu_j^{(\alpha)} = E_j\left(y^\alpha\right)$ is the α-moment with respect to $f\left(y \mid \phi_j\right)$, the distribution of the j-th component. For the case where $\alpha = 0,$ we have

$$I_C(0)=\int_0^\infty \log\left(\frac{\mu_C}{y}\right)\sum_{j=1}^J \lambda_j f\left(y\,|\,\phi_j\right)dy$$
$$=\log\mu_C-\sum_{j=1}^J\lambda_j E_j\left(\log y\right) \tag{15}$$

where, for the GB2 distribution, $E_j\left(\log y\right)=\left[\psi\left(p_j\right)-\psi\left(q_j\right)\right]\big/a_j+\log\left(b_j\right)$. For the case where $\alpha=1$,

$$I_C(1)=\int_0^\infty \frac{y}{\mu_C}\log\left(\frac{y}{\mu_C}\right)\sum_{j=1}^J \lambda_j f\left(y\,|\,\phi_j\right)dy$$
$$=\frac{1}{\mu_C}\sum_{j=1}^J\lambda_j E_j\left(y\log y\right)-\log\mu_C \tag{16}$$

with $E_j\left(y\log y\right)=\left(\mu_j/a_j\right)\left[\psi\left(p_j+1/a_j\right)-\psi\left(q_j-1/a_j\right)\right]+\mu_j\log b_j$.

An attractive feature of the GE index from a mixture is that it decomposes into a GE measure of inequality within the components of the mixture and a GE measure of inequality between components. To establish this decomposition, we write the index for the *j*-th area as

$$I_j(\alpha)=\frac{1}{\alpha\left(\alpha-1\right)}\left[\frac{\mu_j^{(\alpha)}}{\mu_j^{\alpha}}-1\right]$$

and note that

$$\frac{\mu_j^{(\alpha)}}{\mu_j^{\alpha}}=\alpha(\alpha-1)I_j(\alpha)+1$$

Substituting this expression into (14) yields

$$I_C(\alpha)=\frac{1}{\alpha\left(\alpha-1\right)}\left\{\sum_{j=1}^J\lambda_j\left(\frac{\mu_j}{\mu_C}\right)^{\alpha}\left[\alpha\left(\alpha-1\right)I_j(\alpha)+1\right]-1\right\}$$
$$=\sum_{j=1}^J\lambda_j\left(\frac{\mu_j}{\mu_C}\right)^{\alpha}I_j(\alpha^{\alpha})+\frac{1}{\alpha\left(\alpha-1\right)}\left\{\sum_{j=1}^J\lambda_j\left(\frac{\mu_j}{\mu_C}\right)-1\right\} \tag{17}$$
$$=I_C^{with}(\alpha)+I_C^{betw}(\alpha)$$

where $I_C^{with}(\alpha)$ is a weighted average of the inequalities for each area with weights given by $\lambda_j\left(\mu_j/\mu_C\right)^{\alpha}$, and $I_C^{betw}(\alpha)$ is a discrete version of the GE index for the J areas, measuring between inequality. Note that, unless $\alpha = 0$ or 1, the weights do not sum to 1. When $\alpha = 0$, the weights are the population shares λ_j; when $\alpha = 1$, the weights are the income shares $\lambda_j\mu_j \Big/ \sum_{j=1}^{J}\lambda_j\mu_j$. The components for these two cases are

$$
\begin{aligned}
I_C^{with}(0) &= \sum_{j=1}^{J}\lambda_j I_j(0) \\
&= \sum_{j=1}^{J}\lambda_j \log\left(\mu_j\right) - \sum_{j=1}^{J}\lambda_j E_j\left(\log y\right)
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
I_C^{betw}(0) &= \sum_{j=1}^{J}\lambda_j \log\left(\frac{\mu_C}{\mu_j}\right) \\
&= \log\mu_C - \sum_{j=1}^{J}\lambda_j \log\mu_j
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
I_C^{with}(1) &= \sum_{j=1}^{J}\lambda_j \frac{\mu_j}{\mu_C} I_j(1) \\
&= \frac{1}{\mu_C}\sum_{j=1}^{J}\lambda_j E_j\left(y\log y\right) - \sum_{j=1}^{J}\lambda_j \frac{\mu_j}{\mu_C}\log\left(\mu_j\right)
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
I_C^{betw}(1) &= \sum_{j=1}^{J}\lambda_j \frac{\mu_j}{\mu_C}\log\left(\frac{\mu_j}{\mu_C}\right) \\
&= \sum_{j=1}^{J}\lambda_j \frac{\mu_j}{\mu_C}\log\left(\mu_j\right) - \log\mu_C
\end{aligned}
\tag{21}
$$

Inequality is often also expressed in terms of the ratio of the income share of the richest to the income share of the poorest in the population. Graf and Nedyalkova consider the quintile share ratio (QSR) which is the ratio of the income share of the richest 20% relative to the income share of the poorest 20%. For the GB2 distribution, it is given by

$$
QSR = \frac{1 - B\left[B^{-1}\left(0.8 \mid p,q\right) \mid p + 1/a, q - 1/a\right]}{B\left[B^{-1}\left(0.2 \mid p,q\right) \mid p + 1/a, q - 1/a\right]}
$$

Noting that,

$$F_1\left(y\,|\,\Phi\right)=\frac{1}{\mu_C}\int_0^y t\sum_{j=1}^{J}\lambda_j f\left(t\,|\,\phi_j\right)dt$$

$$=\frac{1}{\mu_C}\sum_{j=1}^{J}\lambda_j\mu_j F_1\left(y\,|\,\phi_j\right)$$

the QSR for a mixture of GB2 distributions can be computed from

$$QSR_C=\frac{1-\sum_{j=1}^{J}\lambda_j\mu_j B\left(w_{j,0.8}\,|\,p_j+1/a_j\,,q_j-1/a_j\right)}{\sum_{j=1}^{J}\lambda_j\mu_j B\left(w_{j,0.2}\,|\,p_j+1/a_j\,,q_j-1/a_j\right)}$$

where $w_{j,0.8}=\left(y_{0.8}/b_j\right)^{a_j}\Big/\left[1+\left(y_{0.8}/b_j\right)^{a_j}\right]$ and $w_{j,0.2}=\left(y_{0.2}/b_j\right)^{a_j}\Big/\left[1+\left(y_{0.2}/b_j\right)^{a_j}\right]$, with $y_{0.2}$ and $y_{0.8}$ being the 20^{th} and 80^{th} percentiles from the mixture distribution. To obtain $y_{0.2}$ and $y_{0.8}$, the mixture distribution function needs to be inverted to obtain its corresponding quantile function, something that is not possible in closed form. As alternatives one can (1) attempt to solve the required equation numerically, or (2) generate a large number of observations from each component, combine and sort these components, choosing the 20^{th} and 80^{th} empirical percentiles as estimates.

2.2 Poverty Measures

Expressions for several poverty measures in terms of the parameters of the GB2 distribution have been provided by Chotikapanich et al (2013). The first is the headcount ratio which is simply the proportion of the population with income less than or equal to a poverty line z

$$H(z)=F\left(z\,|\,\phi\right)=B\left(v\,|\,p,q\right) \tag{22}$$

where $v=\left(z/b\right)^a\Big/\left[1+\left(z/b\right)^a\right]$. Setting the poverty line at 0.6 times the median gives what Graf and Nedyalkova (2014) term the at-risk-poverty rate ($ARPR$). It can be calculated from (22) after setting the poverty line at

$$z=0.6b\left(\frac{B^{-1}\left(0.5\,|\,p,q\right)}{1-B^{-1}\left(0.5\,|\,p,q\right)}\right)^{1/a} \tag{23}$$

A second poverty measure used extensively in the literature is the $FGT(\alpha)$ class of measures (Foster et al 1984) given by

$$FGT(\alpha) = \int_0^z \left(\frac{z-y}{z}\right)^{\alpha} f\left(y \mid \phi\right) dy \qquad \text{for}\,\alpha \geq 1 \tag{24}$$

For integer values of α, this expression can be written in terms of incomplete moments of the GB2 distribution as well as in terms of the income gap ratio, defined as the average amount of money that must be given to each of the poor to bring them up to the poverty line, expressed relative to the poverty line. Working in this direction, we define the k-th incomplete moment for the GB2 distribution, relative to poverty line *z*, as

$$\begin{aligned}\mu_z^{(k)} &= E\left(y^k \mid y < z\right) = \frac{1}{F\left(z \mid \phi\right)} \int_0^z y^k f\left(y \mid \phi\right) dy \\ &= \frac{\mu^{(k)} B\left(v \mid p + k/a, q - k/a\right)}{B\left(v \mid p, q\right)}\end{aligned}$$

Defining the income gap ratio as $g(z) = \left(z - \mu_z\right)\big/z$ where $\mu_z = \mu_z^{(1)}$ is mean income of the poor, we can write

$$\begin{aligned}FGT(1) &= B\left(v \mid p,q\right) - \left(\mu/z\right) B\left(v \mid p + 1/a, q - 1/a\right) \\ &= H(z) g(z)\end{aligned} \tag{25}$$

and

$$\begin{aligned}FGT(2) = B\left(v \mid p,q\right) &- \left(2\mu/z\right) B\left(v \mid p + 1/a, q - 1/a\right) \\ &+ \left(\mu^{(2)}/z^2\right) B\left(v \mid p + 2/a, q - 2/a\right) \\ = H\left(z\right) &\left[\left[g(z)\right]^2 + \left[1 - g(z)\right]^2 \frac{\sigma_z^2}{\mu_z^2}\right]\end{aligned} \tag{26}$$

where $\sigma_z^2 = \mu_z^{(2)} - \mu_z^2$ is the variance of the income of the poor. For non-integer values of α we can simulate values $y_1, y_2, \ldots y_M$ from the GB2 distribution and use the estimator

$$FGT\left(\alpha\right) = \frac{1}{M} \sum_{m=1}^{M} \left(\frac{z - y_m}{z}\right)^{\alpha} I\left(y_m \leq z\right) \tag{27}$$

where $I(\cdot)$ is an indicator function equal to 1 if its argument is true and zero otherwise.

As an alternative to the income gap ratio $g(z) = \left(z - \mu_z\right)\big/z$, Graf and Nedyalkova (2014) use a concept known as relative median poverty gap ($RMPG$). It is defined as the relative gap between a poverty line which is 0.6 times the median income of the population, and the median income of the poor. Specifically, with z defined as in (23),

$$RMPG = \frac{z - m_{poor}}{z}$$

where the median of the poor is defined as

$$m_{poor} = b\left(\frac{B^{-1}\left(A/2 \mid p,q\right)}{1 - B^{-1}\left(A/2 \mid p,q\right)} \right)^{1/a}$$

with A being the at-risk-poverty rate (the headcount ratio using the poverty line in (23)).

Considering the income shortfall in log format leads to the Watts index (Watts, 1968), defined as

$$\begin{aligned} W &= \int_0^\infty \left(\ln z - \ln y\right) f\left(y \mid \phi\right) dy \\ &= \ln\left(\frac{z}{b}\right) B\left(v \mid p,q\right) - \\ &\qquad \frac{1}{a}\left\{ D_p B\left(v \mid p,q\right) - D_q B\left(v \mid p,q\right) + B\left(v \mid p,q\right)\left[\psi\left(p\right) - \psi\left(q\right)\right] \right\} \end{aligned} \tag{28}$$

where $D_p B\left(v \mid p,q\right)$ and $D_q B\left(v \mid p,q\right)$ are the derivatives of the beta cdf $B\left(v \mid p,q\right)$ with respect to p and $q,$ respectively. These derivatives are available in some software (e.g., EViews), otherwise (28) can be estimated via simulation.

The last poverty measure that we describe is the Sen index (Sen, 1976) where the poverty gap is weighted by a person's rank in the ordering of the poor. This index is given by

$$\begin{aligned} S &= 2\int_0^z \left(\frac{z - y}{z}\right)\left(\frac{H(z) - F(y \mid \phi)}{H(z)}\right) f\left(y \mid \phi\right) dy \\ &= H(z)\left(g(z) + \left(1 - g(z)\right)G(z)\right) \end{aligned} \tag{29}$$

where $G(z)$ is the Gini coefficient for the poor given by

$$G(z) = -1 + \frac{2}{\mu_z H^2(z)} \int_0^z yF\left(y \mid \phi\right) f\left(y \mid \phi\right) dy \tag{30}$$

The last line in (29) shows how the index can be written in terms of the headcount ratio, the aggregate income gap ratio and the inequality of the poor measured using $G(z)$. Expressing S in terms of the parameters of the GB2 distribution is more difficult than it was for the other indices. In (29) we can use $H(z) = B\left(v \mid p,q\right)$ and $g(z) = 1 - \mu_z / z$, but evaluation of $G(z)$ is more troublesome. If we follow the simulation approach and draw M observations y_m, $m = 1,2,\ldots,M$ from $f\left(y \mid \phi\right)$, it can be estimated using

$$G(z) = -1 + \frac{2}{\mu_z H^2(z)} \frac{1}{M} \sum_{m=1}^{M} \left[y_m B\left(w_m \mid p,q\right) I\left(y_m \le z\right) \right] \tag{31}$$

where $w_m = \left(y_m/b\right)^a \Big/ \left[1 + \left(y_m/b\right)^a\right]$. More progress can be made analytically for the beta 2 special case where $a = 1$; for this case it can be shown that

$$G(z) = -1 + \frac{\mu}{\mu_z H^2(z)} \left[G \times B\left(v \mid 2p+1, 2q-2\right) + B^2\left(v \mid p+1, q-1\right) \right] \tag{32}$$

where $v = \left(z/b\right) \Big/ \left[1 + \left(z/b\right)\right]$ and $G = 2\mathrm{B}\left(2p, 2q-1\right) \big/ p\,\mathrm{B}^2(p,q)$ is the Gini coefficient for the whole population.

For aggregating poverty over a number of areas each of which has a GB2 distribution, the headcount ratio, FGT, and Watts indexes are simply population-weighted averages of the indexes for each area. That is, using obvious notation,

$$H_C(z) = \sum_{j=1}^{J} \lambda_j F\left(z \mid \phi_j\right) = \sum_{j=1}^{J} \lambda_j B\left(v_j \mid p_j, q_j\right) \tag{33}$$

$$FGT_C(\alpha) = \sum_{j=1}^{M} \lambda_j FGT_j(\alpha) \tag{34}$$

$$W_C = \sum_{j=1}^{M} \lambda_j W_j \tag{35}$$

This result does not hold for the at-risk-poverty rate and the relative median poverty gap where the poverty line is endogenous, nor does it hold for the Sen index which contains the cdf. For $ARPR$ and $RMPG$ the median of the mixture is required, and $RMPG$ also needs the median of the poor from the mixture distribution. These values can be estimated by simulating observations from the component distributions and ordering them as was suggested for the QSR. For the Sen index for the mixture, we have

$$S_C = 2\left[FGT_C(1) - \sum_{j=1}^{J}\sum_{\ell=1}^{J}\lambda_j\lambda_\ell \int_0^z \left(\frac{z-y}{z}\right) F\left(y\,|\,\phi_\ell\right) f\left(y\,|\,\phi_j\right) dy \right]$$
$$= 2\left[FGT_C(1) - \sum_{j=1}^{J}\sum_{\ell=1}^{J}\lambda_j\lambda_\ell\gamma_{j\ell} \right] \tag{36}$$

The term $\gamma_{j\ell} = \int_0^z \left[(z-y)/z\right] F\left(y\,|\,\phi_\ell\right) f\left(y\,|\,\phi_j\right) dy$ can be estimated from

$$\hat{\gamma}_{j\ell} = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{z-y_m}{z}\right) F\left(y_{j,m}\,|\,\phi_\ell\right) I\left(y_{j,m} \le z\right) \tag{37}$$

where the $y_{j,m}$ are draws from $f(y\,|\,\phi_j)$.

2.3 Measures of Pro-Poor Growth

In addition to examining changes in poverty incidence over time using measures such as the headcount ratio or refinements of it that take into account the severity of the poverty, it is useful to examine whether growth has favoured the poor relative to others placed at more favourable points in the income distribution. Following Duclos and Verdier-Chouchane (2010), we consider 3 such pro-poor measures, namely, measures attributable to Ravallion and Chen (2003), Kakwani and Pernia (2000), and a "poverty equivalent growth rate" (*PEGR*) suggested by Kakwani et al. (2003).

The first step towards the Ravallion-Chen measure is the construction of a "growth incidence curve" (*GIC*) which describes the growth-rate of income at each percentile *u* of the distribution. Specifically, if $F_A(y)$ is the income distribution function at time *A*, and $F_B(y)$ is the distribution function for the new income distribution at a later point *B*, then

$$GIC(u) = \frac{F_B^{-1}(u) - F_A^{-1}(u)}{F_A^{-1}(u)} \tag{38}$$

For computing values of $GIC(u)$ from the GB2 distribution, note that

$$F^{-1}(u\,|\,\phi) = b\left(\frac{B^{-1}(u\,|\,p,q)}{1 - B^{-1}(u\,|\,p,q)}\right)^{1/a} \tag{39}$$

where $B^{-1}(u\,|\,p,q)$ is the quantile function of the standardised beta distribution evaluated at *u*. When we have a regional distribution or a country distribution which is a mixture of rural and urban GB2 distributions, it is no longer straightforward to compute the quantile function.

In this case we require $F^{-1}(u|\Phi)$ which is the inverse function of $F(y|\Phi)=\sum_{j=1}^{J}\lambda_j F(y|\phi_j)$. One needs to either solve the resulting nonlinear equation numerically or estimate $F^{-1}(u|\Phi)$ using an empirical distribution function obtained by generating observations from the relevant GB2 distributions in the mixture. We followed the latter approach in our applications.

The *GIC* can be used in a number of ways. If $GIC(u)>0$ for all *u*, then the distribution at time *B* first-order stochastically dominates the distribution at time *A*. If $GIC(u)>0$ for all *u* up to the initial headcount ratio H_A, then growth has been *absolutely* pro-poor. If $GIC(u)>(\mu_B-\mu_A)/\mu_A$ for all *u* up to the initial headcount ratio H_A, that is, the growth rate of income of the poor is greater than the growth rate of mean income (μ), then growth has been *relatively* pro-poor.

For a single measure of pro-poor growth Ravallion and Chen suggest using the average growth rate of the income of the poor. It can be expressed as

$$RC=\frac{1}{H_A}\int_0^{H_A} GIC(u)du \tag{40}$$

For a GB2 distribution (not a mixture), this integral can be evaluated numerically. Alternatively, we can generate observations from a GB2 distribution or a mixture and compute

$$\widehat{RC}=\frac{1}{N_1}\sum_{i=1}^{N_1} GIC(i/N) \tag{41}$$

where *N* is the total number of observations generated, and $N_1=H_A N$.

The Kakwani-Pernia measure compares the change in a poverty index such as the change in the headcount ratio, H_A-H_B, with the change that would have occurred with the same growth rate, but with distribution neutrality, $H_A-H_{\tilde{B}}$. Here, $\tilde{B}$ denotes an income distribution that would be obtained if all incomes changed in the same proportion as the change in mean income that occurred when moving from distribution *A* to distribution *B*. To obtain $\tilde{B}$ in the context of single GB2 distributions, we can simply change the scale parameter *b* and leave the parameters a, p and q unchanged. The Lorenz curve and inequality measures obtained from a GB2 distribution depend on a, p and q, but do not depend on b. Thus, we have

$$a_{\tilde{B}} = a_A \qquad p_{\tilde{B}} = p_A \qquad q_{\tilde{B}} = q_A \qquad b_{\tilde{B}} = \left(\frac{\mu_B}{\mu_A}\right) b_A$$

Finding $\tilde{B}$ for a mixture of GB2 distributions – a situation that occurs when we combine rural and urban distributions to find a country distribution – is less straightforward. In this case the scale parameters in all components of the mixture change and the other parameters are left unchanged. For example, using the superscripts r and u to denote rural and urban, respectively, and $\left(\lambda_A^r, \lambda_A^u\right)$ and $\left(\lambda_B^r, \lambda_B^u\right)$ to denote the respective population proportions at times A and B, we first compute the combined means at times A and B as

$$\mu_A = \lambda_A^r \mu_A^r + \lambda_A^u \mu_A^u \qquad \mu_B = \lambda_B^r \mu_B^r + \lambda_B^u \mu_B^u$$

Then, we obtain the distribution function for $\tilde{B}$ as follows

$$a_{\tilde{B}}^j = a_A^j \qquad p_{\tilde{B}}^j = p_A^j \qquad q_{\tilde{B}}^j = q_A^j \qquad b_{\tilde{B}}^j = \left(\frac{\mu_B}{\mu_A}\right) b_A^j \qquad j = u, r$$

$$F\left(y \mid \phi_{\tilde{B}}^r, \phi_{\tilde{B}}^u\right) = \lambda_A^r F\left(y \mid \phi_{\tilde{B}}^r\right) + \lambda_A^u F\left(y \mid \phi_{\tilde{B}}^u\right) \qquad (42)$$

Thus, to obtain $\tilde{B}$ we assume that all incomes in the rural and urban sectors increase in the same proportion as their respective mean incomes, and the distributions of income and the population proportions in each of the sectors remain the same.

The Kakwani-Pernia measure is

$$KP = \frac{H_A - H_B}{H_A - H_{\tilde{B}}} \qquad (43)$$

Assuming the growth in mean income has been positive, a value $KP > 0$ implies the change in the distribution has been absolutely pro-poor, and a value $KP > 1$ implies the change in distribution has been relatively pro-poor.

The third measure of pro-poor growth is the poverty-equivalent growth rate (*PEGR*) suggested by Kakwani et al (2003). In the context of our description of the Kakwani-Pernia measure, it is the growth rate used to construct distribution $\tilde{B}$ such that $H_B = H_{\tilde{B}}$. In other words, it is the growth rate necessary to achieve the observed change in the headcount ratio when distribution neutrality is maintained. In terms of the GB2 distribution, it is the value g^* that solves the following equation

$$H_B = B(u \mid p_B, q_B) = B\left(u^* \mid p_A, q_A\right) \quad (44)$$

where $u = (z/b_B)^{a_B}\left[1+(z/b_B)^{a_B}\right]$ and

$$u^* = \frac{\left[z/\left(g^*+1\right)b_A\right]^{a_A}}{1+\left[z/\left(g^*+1\right)b_A\right]^{a_A}} \quad (45)$$

Thus, to find g^* we have $u^* = B^{-1}\left(H_B \mid p_A, q_A\right)$ and

$$g^* = \frac{z}{b_A}\left(\frac{1-u^*}{u^*}\right)^{1/a_A} - 1 \quad (46)$$

As was the case with previous calculations, for a mixture of GB2 distributions this procedure is less straightforward. As an alternative, to find an approximate g^* for a combined rural-urban distribution, we computed separate growth rates g_r^* and g_u^* for the two sectors and found a weighted average of them using weights from period B.

$$g^* = \lambda_B^r g_r^* + \lambda_B^u g_u^*$$

If $g^* < g = \left(\mu_B/\mu_A - 1\right)$, then, under distribution neutrality, the growth rate required to achieve the same outcome for the headcount ratio is less than realized growth rate, implying that the change in the distribution has not favoured the poor. Conversely, when $g^* > g$, a higher growth rate is required under distributional neutrality to equate the two headcount ratios. In this case the distributional effect must have favoured the poor.

3. Estimation

All the required quantities – the means of the distributions, the density and distribution functions, the Gini coefficients, the poverty measures, and the pro-poor growth measures – depend on the unknown parameters ϕ_j of the GB2 distributions. Potential methods of estimation of these parameters depend on whether the available data are in the form of single observations or are grouped, and, if they are grouped, whether information on group means, as well as the number of observations in each group, is available.

3.1 Estimation with Single Observations

For single observations, say a sample of observations $(y_1, y_2, \ldots, y_T)$, maximum likelihood estimation can be used with the log-likelihood given by

$$L(\phi) = \sum_{t=1}^{T} \log f(y_t \mid \phi) \tag{47}$$

For samples where sampling weights are available, a pseudo log-likelihood can be maximized to provide consistent parameter estimates, and their precision can be assessed with a sandwich covariance matrix estimator. Details of this estimation procedure are described by Graf and Nedyalkova (2014). With income equivalized over all household members, and sampling weights w_i attached to each household, their pseudo log-likelihood is given by

$$L(\phi) = \sum_{i=1}^{h} w_i n_i \log f(y_i \mid \phi)$$

where h is the number of households and n_i is the number of persons in household i.

A further estimation method that minimizes a weighted sum of squared distance between sample quantities for ($ARPR$, $RMPG$, QSR, Gini), and these quantities expressed in terms of GB2 parameters, has been suggested by Graf and Nedyalkova (2014). This method has some similarities to the grouped data methods of estimation we describe in the next subsection, where a weighted squared distance between empirical and theoretical quantiles and group means is minimized. One difference is that, for using quantiles and group means, an optimal weight matrix can be derived. Deriving an optimal weight matrix for the Graf-Nedyalkova proposal would appear to be a more difficult problem.

3.2 Estimation with Grouped Data

Suppose now that the observations $(y_1, y_2, \ldots, y_T,)$ have been grouped into N income classes $(x_0, x_1), (x_1, x_2), \cdots, (x_{N-1}, x_N)$ with $x_0 = 0$ and $x_N = \infty$. Let c_i be the proportion of observations in the i-th group, let $\bar{y}_i$ be mean income for the i-th group, and let $\bar{y}$ be overall mean income. In some instances, where income share data for each group $(s_1, s_2, \ldots s_N)$ are available, the group means may need to be calculated from $\bar{y}_i = s_i \bar{y} / c_i$. Choice of an estimation method depends on how much of the information just described is available. If the c_i and x_i are available, but the $\bar{y}_i$ are not, then the multinomial likelihood is a natural choice. In this case the log-likelihood is given by

$$L(\phi) \propto \sum_{i=1}^{N} c_i \log\left[F(x_i \mid \phi) - F(x_{i-1} \mid \phi)\right] \tag{48}$$

Another possibility is the minimum chi-squared estimator described in McDonald and Ransom (2008).

For the scenario where one also has data for the group means $\bar{y}_i$, and when the group bounds x_i may or may not be available, estimators based on moment conditions have been suggested by Chotikapanich et al (2007), Hajargasht et al (2012) and Griffiths and Hajargasht (2015). To describe the objective functions that are minimized to obtain these estimators, we need the moments of each group up to order 2, expressed in terms of ϕ and $\mathbf{x}' = (x_1, x_2, \ldots, x_{N-1})$. Working in this direction, we define

$$k_i = F(x_i \mid \phi) - F(x_{i-1} \mid \phi) \tag{49}$$

$$\mu_i = \mu\left[F_1(x_i \mid \phi) - F_1(x_{i-1} \mid \phi)\right] \tag{50}$$

$$\mu_i^{(2)} = \mu^{(2)}\left[F_2(x_i \mid \phi) - F_2(x_{i-1} \mid \phi)\right] \tag{51}$$

where $F_1(x_i \mid \phi)$ and $F_2(x_i \mid \phi)$ are the moment distribution functions defined in equation (4). Further, we define $v_i = k_i \mu_i^{(2)} - \mu_i^2$. Then, Hajargasht et al (2012) show that the GMM estimator that uses moments for c_i and $\tilde{y}_i = c_i \bar{y}$, and the optimal weight matrix, can be written as

$$GMM_1(\mathbf{x}, \phi) = \sum_{i=1}^{N} w_{1i}(c_i - k_i)^2 + \sum_{i=1}^{N} w_{2i}(\tilde{y}_i - \mu_i)^2 - 2\sum_{i=1}^{N} w_{3i}(c_i - k_i)(\tilde{y}_i - \mu_i) \tag{52}$$

where $w_{1i} = \mu_i^{(2)}/v_i$, $w_{2i} = k_i/v_i$ and $w_{3i} = \mu_i/v_i$. $GMM_1(\mathbf{x}, \phi)$ can be minimized with respect to both $\mathbf{x}$ and ϕ, or, if observations on $\mathbf{x}$ are available, with respect to ϕ only. Because the weights depend on $(\mathbf{x}, \phi)$, a variety of estimators can be used, depending on whether $GMM_1(\mathbf{x}, \phi)$ is minimized directly or a two-step or iterative procedure is employed. In a two-step procedure, initial estimates with weights that are not dependent on the parameters are obtained, and then estimates that minimize $GMM_1(\mathbf{x}, \phi)$, with weights computed from the initial estimates, are computed. Iterating this process leads to an iterative estimator.

An estimator that uses weights that do not depend on $(\mathbf{x},\phi)$, and which is useful for obtaining starting values for a two-step or iterative estimator from (52), is that proposed by Chotikapanich et al (2007). In contrast to (52), they considered moment conditions for c_i and $\bar{y}_i$ instead of c_i and $\tilde{y}_i = c_i\bar{y}_i$. Although they focused on the special case beta 2 distribution, their results also hold for the more general GB2 distribution. The function that they minimized is

$$GMM_2(\mathbf{x},\phi) = \sum_{i=1}^{N}\left(\frac{c_i - k_i}{c_i}\right)^2 + \sum_{i=1}^{N}\left(\frac{\bar{y}_i - \mu_i/k_i}{\bar{y}_i}\right)^2 \tag{53}$$

The weights used for this estimator $\left(c_i^{-2} \text{ and } \bar{y}_i^{-2}\right)$ are not optimal, but they have the intuitive appeal of minimizing the sum of squares of percentage errors. Also, computation of the second moment $\mu_i^{(2)}$ is not required.

A third GMM estimator is that described by Griffiths and Hajargasht (2015). Like (53), this estimator considers the moment conditions for c_i and $\bar{y}_i$, but uses the optimal weight matrix.[3] it is given by

$$GMM_3(\mathbf{x},\phi) = k_i^{-1}\sum_{i=1}^{N}(c_i - k_i)^2 + k_i^3 v_i^{-1}\sum_{i=1}^{N}(\bar{y}_i - \mu_i/k_i)^2 \tag{54}$$

Relative to the other optimal weight formulation in (52), this objective function avoids the term with the cross product of the moment conditions.

4. Applications

A major source of data for cross country study of income distributions, inequality and poverty is from the World Bank PovcalNet website. We used data on China and Indonesia, two Asian countries with relatively large populations. The years considered were 1992, 1999, 2006 and 2009 for China and 1993, 1999, 2005 and 2010 for Indonesia[4]. The data available are in grouped form comprising population shares and corresponding expenditure shares for a number of classes, together with mean monthly expenditure that has been reported from surveys, and then converted to purchasing-power-parity (PPP) using the World Bank's 2005

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[3] It may be better to describe the estimators that minimize $GMM_2(\mathbf{x},\phi)$ and $GMM_3(\mathbf{x},\phi)$ as minimum distance estimators rather than GMM estimators because the "moment condition" for $\bar{y}_i$ is $\text{plim}\,\bar{y}_i = \mu_i/k_i$ not $E(\bar{y}_i) = \mu_i/k_i$. The asymptotic distribution is the same, however. See, for example, Greene (2012, Ch.13).

[4] The version of the data that was used was downloaded on 15 October 2013 at http://research.worldbank.org/PovcalNet/index.html

PPP exchange rates for the consumption aggregate for national accounts. Also available are the data on population size. Throughout the paper we use the generic term *income* distributions although our example distributions are for expenditure. For both countries separate data were available for rural and urban populations and so distributions were estimated for each of these components. The distributions estimated were beta-2 distributions, a special case of the GB2 distribution where $a = 1$. They were estimated by minimizing the objective function $GMM_2(\mathbf{x}, \boldsymbol{\phi})$ given in (53).

Parameter estimates for each of the distributions are presented in Table 1, along with corresponding estimates for mean income and the populations for each region. The density functions for China and Indonesia, obtained as mixtures of the urban and rural densities, are plotted in Figures 1 and 2, respectively. In both cases there is an improvement over time in the sense that the distribution shifts to the right, and mean income increases, with the most dramatic improvement being from 1999 to 2006 (China) and 1999 to 2005 (Indonesia).

Inequality measures for the rural and urban areas and their combined distributions are presented in Table 2. We computed the Gini coefficient, QSR, $I(0)$ and $I(1)$. The within and between urban and rural components of $I(1)$ were also calculated. Tables 3 and 4 contain poverty measures and pro-poor growth measures, respectively. For poverty measures, the headcount, $FGT(1)$, $FGT(2)$ and Sen indices were computed using a poverty line of $38 per month, equivalent to $1.25 per day. Pro-poor growth measures, RC, KP and $PEGR$ were computed for the combined distributions; the GIC's for each time interval are depictured in Figures 3 to 8. From the tables and figures, we can make the following observations about China.

1. Inequality has increased over time in both rural and urban areas and in the country as a whole, with inequality being greater in the urban area although, in 2006, there was little difference between rural and urban inequality. From 2006-2009 there was only a small increase in inequality in urban China but a dramatic increase in rural inequality.
2. Inequality is much greater in the combined distribution than in its components, reflecting the large discrepancy in mean incomes between the rural and urban areas. Within inequality remains greater than between inequality, however.
3. The increasing inequality has been accompanied by large increases in mean income and large decreases in poverty. Over the period 1992-2009, the headcount ratio in

rural China declined from 62% to 20% and in urban China the decline was from 13% to 0.26%. All poverty measures show a consistent decline in poverty.

4. The $GIC's$ show that growth has favoured the rich more than the poor, although, from 1992-1999, the growth rate for the very poor was greater than the growth rate in mean incomes. For all time intervals, all of the scalar measures of pro-poor growth suggest growth has favoured the poor in an absolute, but not relative, sense $(0 < RC < g,\ 0 < KP < 1,\ PEGR < g)$.

Examining the results for Indonesia, we find:

1. In rural and urban Indonesia inequality declined slightly from 1993 to 1999, and then increased from 1999 to 2005. From 2005 to 2010 there was an increase in rural inequality but a decline in urban inequality. The combined distributions, that also depend on relative population changes, show increasing inequality over the first two intervals (1993-1999, 1999-2005), followed by a decline to 2010. The changes are far less dramatic than those in China and, comparing the two combined distributions, inequality is less in Indonesia than in China.
2. As expected, poverty incidence has declined over time, with particularly large falls in all measures in the interval 1999-2005. Making a comparison with China we find rural poverty is less than that in China but urban poverty is greater.
3. Growth in mean incomes was relatively large from 1999-2005 and only moderate in the time intervals before and after this period. The GIC for 1995-2005 shows that the large growth period tended to favour the rich relative to the poor. However, the growth from 1993-1999 favoured both the very poor and the very rich. From 2005-2010 growth was relatively more favourable to those on middle incomes; growth for the very poor and very rich was less than the average. As was the case with China, all scalar pro-poor growth measures, in all time intervals, suggest absolute but not relative pro-poor growth.

5. Concluding Remarks

Studying income distributions can provide valuable information about important aspects of a society's welfare such as the degree of inequality, the incidence of poverty, and whether there have been improvements in welfare over time. The GB2 is a popular and versatile distribution well suited to this purpose. We have reviewed some of the common indexes for measuring inequality, poverty and pro-poor growth, and described how values for these indexes can be computed from estimates of the parameters of the GB2 distribution. Optimal techniques for estimating the parameters using either single observations or grouped data are also reviewed. It is our hope that the bringing together of all these results into a single source will facilitate and promote use of the GB2 distribution.

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Table 1: Parameter Estimates, Mean Income and Population

	b	*p*	*q*	μ	Population (Millions)
China rural					
2009	3.4248	43.7389	2.6469	90.9578	753.73
2006	8.5819	21.2419	3.5612	71.1760	759.74
1999	8.8685	14.8871	3.7174	48.5855	815.90
1992	89.2711	3.9700	10.6417	36.7578	827.26
China urban					
2009	38.1788	14.8509	3.6246	216.1510	570.93
2006	52.5519	9.7164	4.1535	161.9202	544.76
1999	47.4128	9.2256	5.3686	100.1262	437.80
1992	38.4873	15.2744	8.5495	67.7526	351.18
Indonesia rural					
2010	2.9201	79.9000	4.0978	75.3164	111.0602
2005	0.06267	3372.53	4.3708	62.6988	116.7526
1999	0.2721	744.228	5.9189	41.1696	124.8501
1993	0.03580	4934.28	5.4397	39.7836	128.5010
Indonesia urban					
2010	12.0068	16.7127	3.2248	90.1939	128.8107
2005	0.02069	7457.82	2.7374	88.7973	113.1659
1999	0.01342	9867.12	3.3531	56.2740	85.7807
1993	0.03897	2996.36	3.2874	51.0483	65.0246

Table 2: Inequality Measures

	Gini	QSR	$I(0)$	$I(1)$	$I_C^{with}(1)$	$I_C^{betw}(1)$
China rural						
2009	0.4155	7.6117	0.2853	0.3447		
2006	0.3553	5.7866	0.2064	0.2311		
1999	0.3551	5.8409	0.2068	0.2284		
1992	0.3248	5.5429	0.1822	0.1734		
China urban						
2009	0.3598	5.9767	0.2126	0.2357		
2006	0.3504	5.8172	0.2026	0.2174		
1999	0.3171	4.9574	0.1653	0.1720		
1992	0.2462	3.4365	0.0979	0.1001		
China combined						
2009	0.4597	8.1131	0.3467	0.3653	0.2746	0.0907
2006	0.4160	6.4345	0.2884	0.3053	0.2226	0.0827
1999	0.3920	4.6738	0.2549	0.2634	0.1988	0.0646
1992	0.3329	3.0804	0.1990	0.1855	0.1412	0.0443
Indonesia rural						
2010	0.3138	4.6651	0.1591	0.1762		
2005	0.2963	4.2665	0.1412	0.1558		
1999	0.2488	3.3974	0.0989	0.1058		
1993	0.2605	3.5919	0.1085	0.1169		
Indonesia urban						
2010	0.3805	6.5739	0.2384	0.2709		
2005	0.3987	6.9425	0.2611	0.3147		
1999	0.3489	5.4833	0.1977	0.2273		
1993	0.3535	5.6035	0.2031	0.2344		
Indonesia combined						
2010	0.3449	5.7737	0.2057	0.2353	0.2313	0.0040
2005	0.3656	5.6309	0.2153	0.2627	0.2477	0.0150
1999	0.3091	4.0029	0.1511	0.1767	0.1646	0.0121
1993	0.2984	4.0024	0.1474	0.1704	0.1632	0.0073

Table 3: Poverty Measures

	HC	*FGT*(1)	*FGT*(2)	SEN
China rural				
2009	0.1984	0.0469	0.0162	0.0649
2006	0.2520	0.0616	0.0218	0.0850
1999	0.4987	0.1641	0.0724	0.2192
1992	0.6227	0.2448	0.1271	0.3245
China urban				
2009	0.0026	0.0003	0.0000	0.0005
2006	0.0158	0.0026	0.0007	0.0038
1999	0.0722	0.0142	0.0044	0.0201
1992	0.1334	0.0237	0.9966	0.0335
China combined				
2009	0.1140	0.0269	0.0092	0.0518
2006	0.1533	0.0370	0.0130	0.0704
1999	0.3498	0.1117	0.0487	0.1979
1992	0.4769	0.1789	0.0912	0.3019
Indonesia rural				
2010	0.1598	0.0301	0.0086	0.0422
2005	0.2479	0.0506	0.0152	0.0701
1999	0.5450	0.1437	0.0514	0.1928
1993	0.5840	0.1663	0.0631	0.2214
Indonesia urban				
2010	0.1636	0.0372	0.0126	0.0518
2005	0.1857	0.0408	0.0131	0.0565
1999	0.4000	0.1098	0.0415	0.1485
1993	0.4742	0.1434	0.0583	0.1917
Indonesia combined				
2010	0.1619	0.0339	0.0107	0.0646
2005	0.2173	0.0458	0.0142	0.0854
1999	0.4859	0.1299	0.0474	0.2183
1993	0.5471	0.1586	0.0615	0.2594

Table 4 Pro-poor Growth Measures

	Growth Rate	Growth rate for the poor (RC)	KP	PEGR
China				
2006 - 2009	0.3286	0.1134	0.4702	0.2409
1999 - 2006	0.6381	0.5361	0.8858	0.4592
1992 - 1999	0.4477	0.3149	0.6328	0.1898
Indonesia				
2005 - 2010	0.1027	0.0849	0.9681	0.0925
1999 - 2005	0.5965	0.3963	0.7638	0.3852
1993 - 1999	0.0861	0.0792	0.8915	0.0724

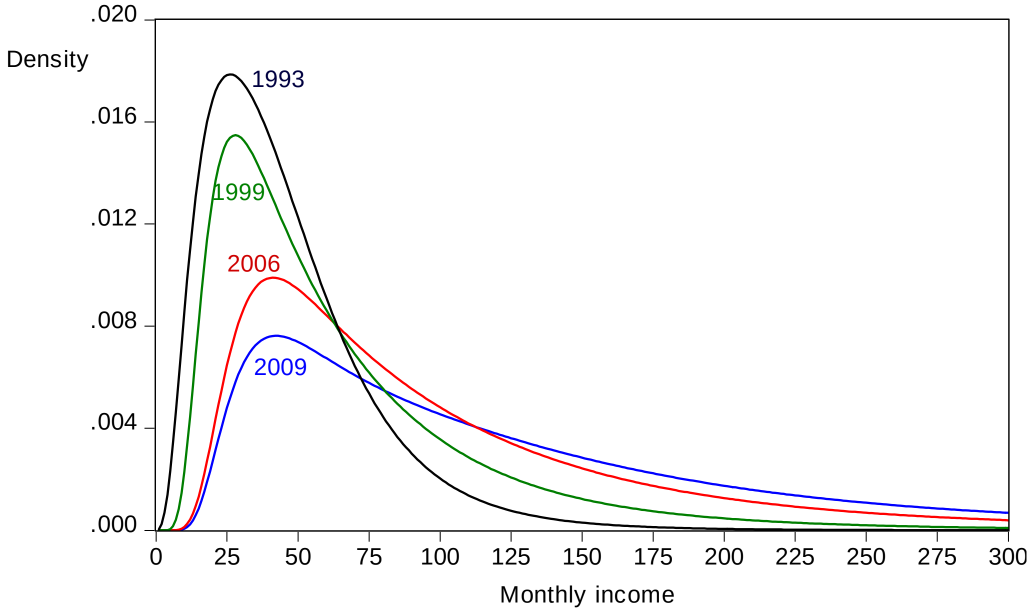

Figure 1 Income distributions for China

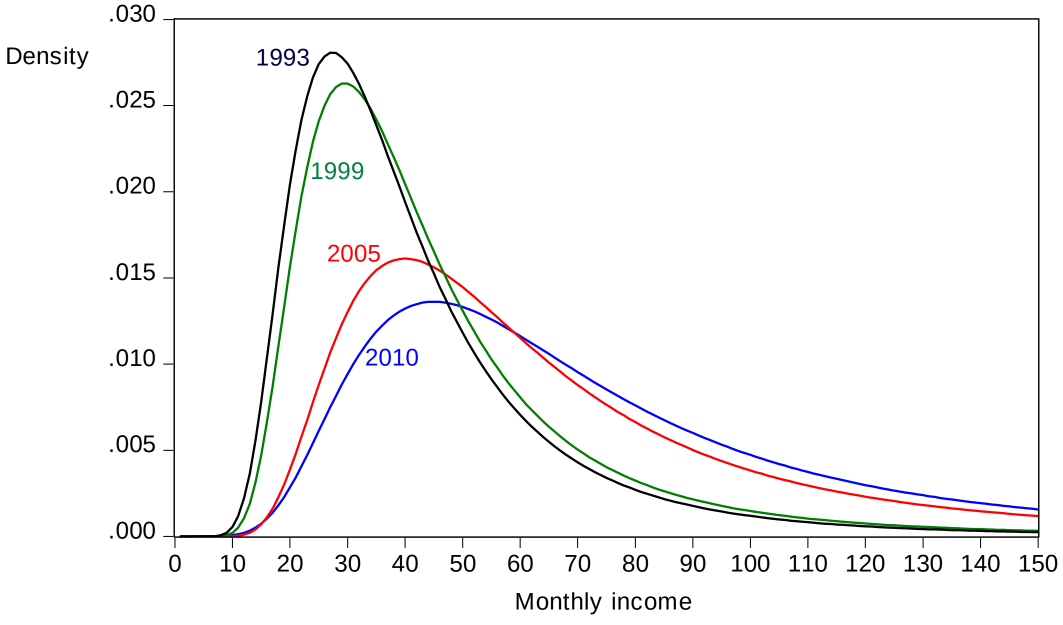

Figure 2 Income distributions for Indonesia

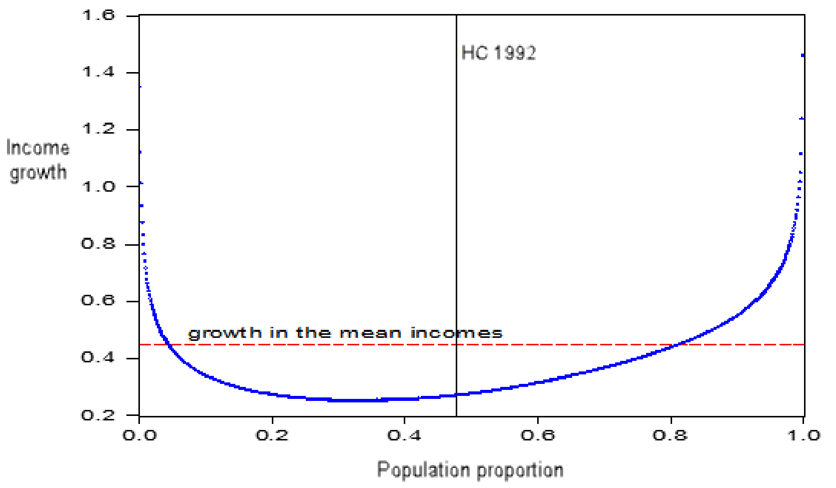

Figure 3 Growth Incidence Curve, China 1992-1999

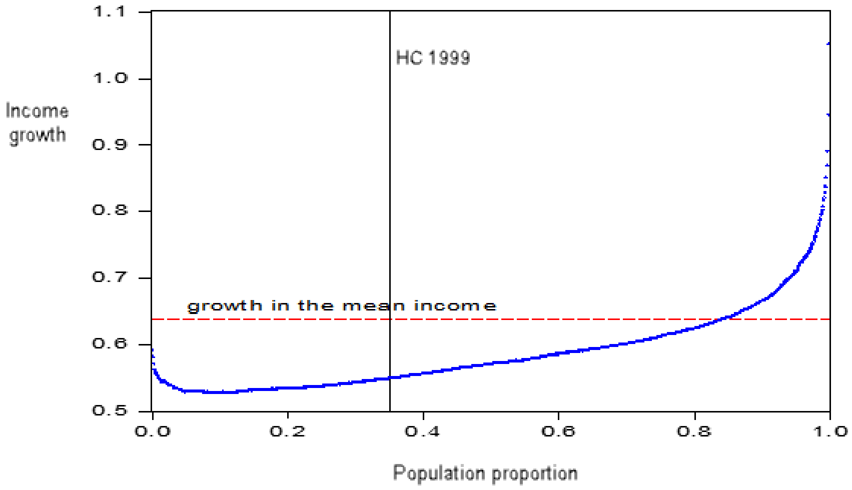

Figure 4: Growth Incidence Curve, China 1999-2006

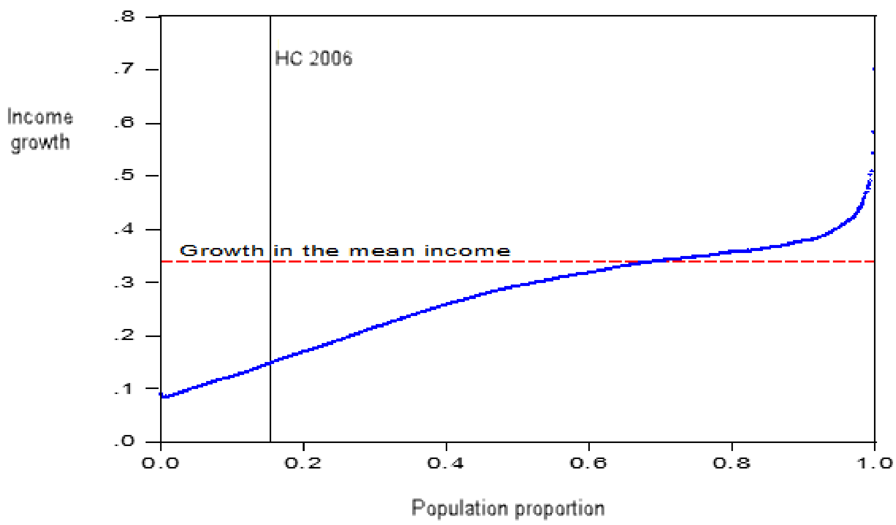

Figure 5: Growth Incidence Curve, China 2006-2009

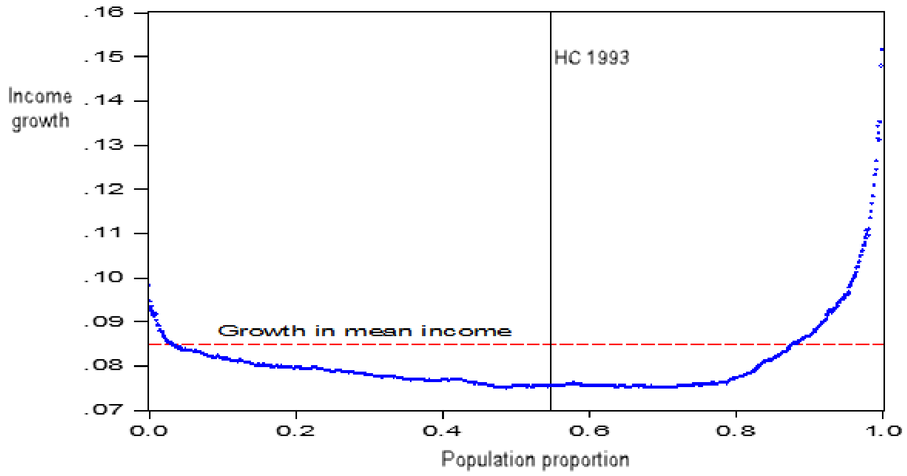

Figure 6 Growth Incidence Curve, Indonesia 1993-1999

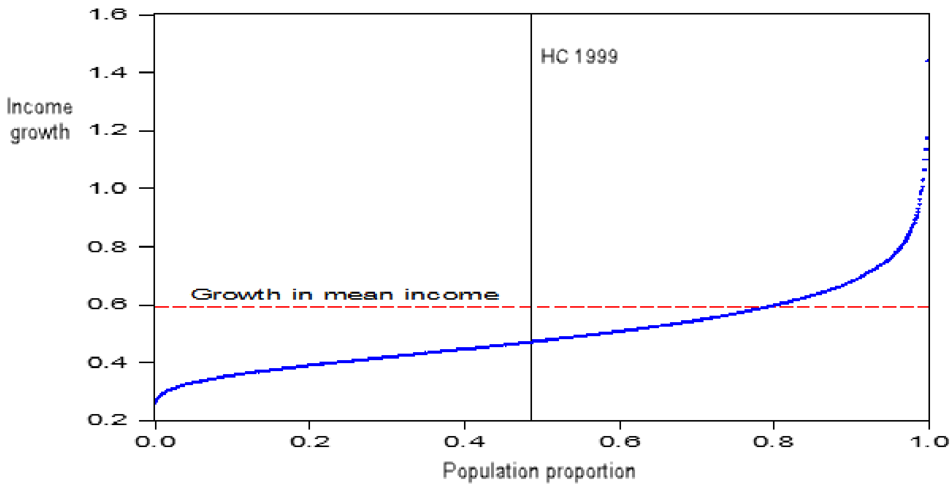

Figure 7 Growth Incidence Curve, Indonesia 1999-2005

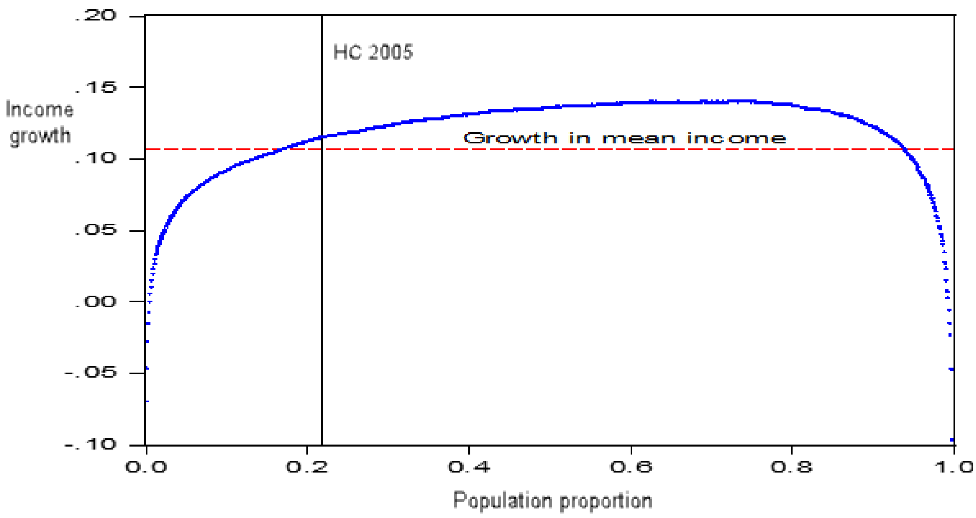

Figure 8 Growth Incidence Curve, Indonesia 2005-2010