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by

David C M Dickson - University of Melbourne
and
Howard R Waters - Heriot-Watt University

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Centre for Actuarial Studies
Department of Economics
The University of Melbourne
Parkville, Victoria, 3052
Australia.

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Abstract

We study the effect of reinsurance on the probability of ultimate ruin in the classical surplus process and consider a retention level as optimal if it minimises the ruin probability. We show that optimal retention levels can be found when the reinsurer's premium loading depends on the retention level. We also show that when the aggregate claims process is approximated by a translated Gamma process very good approximations to both optimal retention levels and ruin probabilities can be obtained. Finally, we discuss the effect of reinsurance on the probability of ruin in finite time.

1. Introduction and notation

The purpose of this paper is to study the effect of reinsurance on the probability of ruin in the classical surplus process. Previous studies of the effect of reinsurance on the probability of ultimate ruin (for example Gerber (1979), Waters (1983), Centeno (1986) and Hesselager (1990)) have focussed on the effect of reinsurance on the adjustment coefficient. By finding a type of reinsurance arrangement or a retention level that maximises the value of the adjustment coefficient we can minimise the value of Lundberg's upper bound for the probability of ultimate ruin. The reason for considering the adjustment coefficient in the past was that it was relatively simple to calculate whereas the probability of ruin was not. However, the recent development of numerical algorithms for calculating, and approximating, the probability of ruin have made its calculation more feasible. (See, for example, Panjer (1986), De Vylder and Goovaerts (1988), Dickson and Waters (1991) and Dickson and Waters (1993).) We will use these algorithms to study the effect of different types and levels of reinsurance on the insurer's probability of ruin. Our aim is to find retention levels that minimise this probability given a type of reinsurance arrangement and a reinsurance premium loading.

In the classical surplus process, the insurer's surplus at time t is denoted U(t) and defined by

$$U(t) = u + ct - X(t)$$

where u is the initial surplus, c is the premium income per unit time, assumed to be received continuously, and X(t) denotes aggregate claims up to time t. The premium is calculated by the expected value principle with loading factor $\theta > 0$. The aggregate claims process is a compound Poisson process, and without loss of generality we can assume that the Poisson parameter is 1. Individual claim amounts have distribution function $P(x)$ and we assume (again without loss of generality) that this distribution has mean 1. With these assumptions, $c = 1 + \theta$. The ultimate ruin probability for this risk process is denoted $\psi(u)$ and defined by

$$\psi(u) = P(U(t) < 0 \text{ for some } t > 0)$$

Now suppose that the insurer effects reinsurance and that the amount paid by the insurer when the i-th claim, denoted X_i, occurs is $h(X_i)$ where $0 \le h(X_i) \le X_i$. We will assume throughout that reinsurance premiums are calculated with a loading factor ξ, where $\xi > \theta$. Then, assuming that reinsurance premiums are paid continuously, the insurer's surplus at time t, denoted $U(t;h)$, is

$$U(t;h) = u + (1 + \theta)t - (1 + \xi)(1 - E[h(X_i)]) - \sum_{i=1}^{N(t)} h(X_i)$$

where $N(t)$ denotes the number of claims up to time t. For this surplus process we denote the ultimate ruin probability by $\psi(u;h)$ and define it by

$$\psi(u;h) = P(U(t;h) < 0 \text{ for some } t > 0)$$

We will consider two forms of $h(X)$:

(i) $h(X) = \alpha X$, $0 < \alpha \le 1$, i.e. proportional reinsurance with retention level α, and

(ii) $h(X) = \min(X,M)$, i.e. excess of loss reinsurance with retention level M.

Our objective in each case is to find the retention level that minimises $\psi(u;h)$, and we will consider a retention level to be optimal if it minimises $\psi(u;h)$. We will also consider the question of whether finding the retention level that maximises the adjustment coefficient is a reasonable method of approximating this optimal retention level.

In sections 8 and 9 we study the effect of reinsurance on the probability of ruin in finite time. We demonstrate through a series of examples how the timescale affects the optimal retention level. We also show that considering discrete time ruin probabilities leads to different conclusions from a study of continuous time ruin probabilities. Steenackers and Goovaerts (1992) also consider the effect of reinsurance on the probability of ruin in finite and continuous time. However, their primary consideration is to determine an optimal form of reinsurance rather than an optimal retention level.

2. Reinsurance and the adjustment coefficient

Under the assumptions of the previous section, the insurer's adjustment coefficient is the unique positive number R satisfying

$$1 + (1 + \theta)R = M_X(R)$$

where $M_X(R)$ denotes the moment generating function of the individual claim amount distribution evaluated at R. When there is a reinsurance arrangement in force, $1 + \theta$ is replaced by premium income per unit time net of reinsurance, and X denotes the net individual claim amount. We will consider two individual claim amount distributions, exponential and Pareto. In our examples we will use the following combinations of θ and ξ.

θ	ξ
0.1	0.15
0.1	0.2
0.1	0.3
0.2	0.3
0.2	0.4

2.1 Proportional reinsurance

First, let $P(x) = 1 - \exp(-x)$. Then the insurer's adjustment coefficient (net of reinsurance) is given by

$$R(\alpha) = \frac{\theta - \xi(1 - \alpha)}{\alpha[1 + \theta - (1 + \xi)(1 - \alpha)]}$$

The values of α which maximise $R(\alpha)$ and the corresponding values of $R(\alpha)$ for our chosen combinations of θ and ξ are shown in Table 1.

Table 1

θ/ξ	α	$R(\alpha)$
0.1/0.15	0.644	0.1048
0.1/0.2	0.956	0.0911
0.1/0.3	1.000	0.0909
0.2/0.3	0.626	0.1965
0.2/0.4	0.923	0.1678

Consider next the case when $P(x) = 1 - (1 + x)^{-2}$, i.e. individual claims follow a Pareto(2,1) distribution. Then individual claims net of reinsurance follow a Pareto(2,α) distribution and so we cannot compute the adjustment coefficient in this case. Let us therefore take an alternative measure. For this distribution there is an asymptotic formula for the probability of ultimate ruin (see, for example, Panjer and Willmot (1992)), and so we could consider finding the value of α which minimises the asymptotic ultimate ruin probability. Since maximising $\exp\{-Ru\}$ is roughly equivalent to minimising $\psi(u;h)$ when u is large, we are adopting a similar criterion by minimising the ultimate ruin probability as $u \to \infty$.

Adapting formula (11.6.4) of Panjer and Willmot (1992) to allow for proportional reinsurance, we find that

$$\psi(u) \sim \alpha^2/(\alpha + u)(\theta - \xi + \alpha\xi) \tag{2.1}$$

It is easy to show that the value of α which minimises the right hand side of (2.1) as $u \to \infty$ is $min(2(1 - \theta/\xi),1)$. For our chosen combinations of θ and ξ, Table 2 shows the values of α which minimise the asymptotic ultimate ruin probability.

Table 2

θ/ξ	α
0.1/0.15	0.667
0.1/0.2	1
0.1/0.3	1
0.2/0.3	0.667
0.2/0.4	1

2.2 Excess of loss reinsurance

Consider first the case when $P(x) = 1 - \exp(-x)$, and let M denote the retention level under an excess of loss reinsurance arrangement. Then the equation defining the adjustment coefficient is

$$1 + \left(1 + \theta - (1 + \xi)\exp(-M)\right)R = (1 - R)^{-1}(1 - R\exp(-(1 - R)M))$$

which can be solved for R (= R(M)) by standard numerical techniques given values of θ, ξ and M. Table 3 shows the values of M which maximise R(M) together with the corresponding values of R(M) for our chosen combinations of θ and ξ.

Table 3

θ/ξ	M	R(M)
0.1/0.15	0.851	0.1642
0.1/0.2	1.533	0.1189
0.1/0.3	2.643	0.0993
0.2/0.3	0.832	0.3153
0.2/0.4	1.486	0.2265

Similarly, when $P(x) = 1 - (1 + x)^{-2}$ we can calculate R(M) from

$$1 + \left(1 + \theta - \frac{1 + \xi}{1 + M}\right)R = \int_0^M \exp(Rx)\frac{2}{(1 + x)^3}\,dx + \exp(RM)\left(\frac{1}{1 + M}\right)^2$$

This equation can also be solved using standard numerical techniques and Table 4 shows the values of M which maximise R(M) together with the corresponding values of R(M) for our chosen combinations of θ and ξ.

Table 4

θ/ξ	M	R(M)
0.1/0.15	1.111	0.1258
0.1/0.2	2.408	0.0757
0.1/0.3	5.326	0.0493
0.2/0.3	1.084	0.2420
0.2/0.4	2.325	0.1447

In Sections 4 and 5 we will compare these optimal retention levels with the optimal retention levels when the criterion for optimality will be that the ultimate ruin probability is minimised.

3. An algorithm to compute ultimate ruin probabilities

Dickson and Waters (1991) consider a discrete time compound Poisson risk model with the following characteristics:

- individual claim amounts are distributed on the non-negative integers with mean β, where β is greater than 1,
- the Poisson parameter for the expected number of claims per unit time is $1/[(1 + \theta)\beta]$,
- the premium income per unit time is 1.

For this model the surplus process, given initial surplus u which is assumed to be an integer, is denoted $\{Z(n)\}_{n\geq 0}$ and the ultimate ruin probability is given by $\psi_d(u) = P(\mathcal{T} < \infty)$ where

$$\begin{aligned}\mathcal{T} &= \min\{n: Z(n) \leq 0,\ n = 1,2,\ldots\} \\ &= \infty \text{ if } Z(n) > 0 \text{ for } n = 1,2,\ldots\end{aligned}$$

For reasons given by Dickson and Waters (1991, Section 1) we can regard $\psi_d(\beta u)$ as an approximation to $\psi(u)$.

Ultimate ruin probabilities under this model can be calculated recursively. Define g_k and $G(k)$ for $k = 0,1,2,\ldots$ to be the probabilities that aggregate claims per unit time are respectively equal and less than or equal to k. Then values of g_k can be calculated from Panjer's (1981) recursion formula. The algorithm to compute ultimate ruin probabilities is

$$\psi_d(0) = 1/(1 + \theta)$$

$$\psi_d(1) = g_0^{-1}\Big(\psi_d(0) - 1 + G(0)\Big)$$

$$\psi_d(u) = g_0^{-1}\Big(\psi_d(u - 1) - \sum_{j=1}^{u-1} g_j\psi_d(u - j) - 1 + G(u - 1)\Big) \quad \text{for } u \geq 2$$

(see Dickson and Waters (1991)). In the following sections we will explain how this algorithm can be modified to take account of reinsurance arrangements.

4. Proportional reinsurance

Let $\psi(u;\alpha)$ denote the probability of ultimate ruin when $h(X) = \alpha X$. Then

$$\psi(u;\alpha) = P\left(u + (1 + \theta - (1 - \alpha)(1 + \xi))t - \sum_{i=1}^{N(t)} \alpha X_i < 0 \quad \text{for some } t > 0\right)$$

$$= P\left(\frac{u}{\alpha} + (1 + \hat{\theta})t - \sum_{i=1}^{N(t)} X_i < 0 \quad \text{for some } t > 0\right)$$

where $\hat{\theta} = (\theta - \xi(1 - \alpha))/\alpha$, so that $\psi(u;\alpha) = \psi(u/\alpha)$ where $\psi(u/\alpha)$ is calculated using a loading of $\hat{\theta}$.

To apply the recursion algorithm we have discretised the distribution $P(x)$ on $0, 1/\alpha\beta, 2/\alpha\beta, \ldots$, where β is an integer, using the discretisation procedure of De Vylder and Goovaerts (1988). The calculated ruin probabilities give approximations to $\psi(j/\alpha\beta)$, for $j = 0,1,2,\ldots$ when the loading is $\hat{\theta}$. As we shall only consider integer values of u we find $\psi(u/\alpha)$ by setting $j = \beta u$ and this gives our approximation to $\psi(u;\alpha)$. The starting value for the algorithm becomes $\psi_d(0) = 1/(1 + \hat{\theta})$.

4.1 Exponential individual claims

In this section we assume that individual claims are exponentially distributed with mean 1. In this case there is no need to calculate approximate values of $\psi(u;\alpha)$ since claims (net of reinsurance) are exponentially distributed with mean α, and so we can calculate values of $\psi(u;\alpha)$ exactly from

$$\psi(u;\alpha) = \frac{\alpha}{c'} \exp(-R(\alpha)u)$$

where $c' = 1 + \theta - (1 + \xi)(1 - \alpha)$ and $R(\alpha) = (\theta - (1 - \alpha)\xi)/\alpha c'$.

Table 5 shows values of α which minimise $\psi(u;\alpha)$ for different values of u. We have calculated $\psi(u;\alpha)$ for values of α that are integer multiples of 0.001 and have then selected the value of α which minimises $\psi(u;\alpha)$ for a given value of u. If desired, the optimal retention level could be calculated to more than three decimal places. The final row of the table shows the values of α which maximise $R(\alpha)$.

Table 5

Loadings	0.1/0.15	0.1/0.2	0.1/0.3	0.2/0.3	0.2/0.4
u = 10	0.666	1.000	1.000	0.646	0.967
u = 20	0.655	0.980	1.000	0.636	0.944
u = 30	0.651	0.972	1.000	0.632	0.937
u = 40	0.649	0.968	1.000	0.631	0.933
u = 50	0.648	0.966	1.000	0.630	0.931
u = 60	0.648	0.964	1.000	0.629	0.930
u = 70	0.647	0.963	1.000	0.628	0.928
u = 80	0.647	0.962	1.000	0.628	0.928
u = 90	0.646	0.962	1.000	0.628	0.927
u = 100	0.646	0.961	1.000	0.628	0.927
R	0.644	0.956	1.000	0.626	0.923

When $\theta = 0.1$ and $\xi = 0.3$, the insurer's probability of ultimate ruin is always minimised when $\alpha = 1$. However, we can see that for the other combinations of θ and ξ that as u increases, the values of α that minimise the probability of ultimate ruin decrease, and appear to converge to the values of α that maximise the adjustment coefficient. However, reinsurance does not have a great effect on the probability of ultimate ruin. Table 6 shows mimimum values of $\psi(u;\alpha)$ together with values of $\psi(u)$. The tabulated figures show that the maximum reduction in the ultimate ruin probability is around 0.04. (In this table, the column headed 0.1/- gives $\psi(u)$ when $\theta = 0.1$; a similar interpretation applies to the column headed 0.2/-.)

Table 6

Loadings	0.1/0.15	0.1/0.2	0.1/-	0.2/0.3	0.2/0.4	0.2/-
u = 10	0.3267	0.3663	0.3663	0.1227	0.1571	0.1574
u = 20	0.1146	0.1475	0.1476	0.0172	0.0294	0.0297
u = 30	0.0402	0.0593	0.0595	0.0024	0.0055	0.0056
u = 40	0.0141	0.0239	0.0240	0.0003	0.0010	0.0011
u = 50	0.0049	0.0096	0.0097	0.0000	0.0002	0.0002

The figures in Table 5 show that choosing α to maximise $R(\alpha)$ is a reasonable alternative to choosing α to minimise $\psi(u;\alpha)$ at least for large values of u. Further, when u is large, say $u \geq 50$, $\exp\{-R(\alpha)u\}$ gives a good approximation to $\psi(u;\alpha)$ around the optimal value of α.

4.2 Pareto individual claims

Let $P(x) = 1 - (1 + x)^{-2}$. In this case we must compute approximate values of

$\psi(u;\alpha)$ from the algorithm described in Section 3. To find the optimal retention level we have set $\beta = 60$ to calculate values for $\psi(u;\alpha)$. This value of β ensured that for all values of α considered (i.e. those for which $\psi(u;\alpha) < 1$) the discretisation is on intervals of at most 1/20*th* of the mean individual claim amount. The optimal retention level has been taken to be the value of α which is an integer multiple of 0.001 and which minimises $\psi(u;\alpha)$. Finding the optimal value of α to a greater number of decimal places is of course possible. Table 7 shows values of α that minimise the computed value of the ultimate ruin probability for different values of u. The final row of the table shows the values of α that minimise the asymptotic ultimate ruin probabilities.

Table 7

Loadings	0.1/0.15	0.1/0.2	0.1/0.3	0.2/0.3	0.2/0.4
$u = 40$	0.762	1.000	1.000	0.739	1.000
$u = 80$	0.734	1.000	1.000	0.711	1.000
$u = 120$	0.720	1.000	1.000	0.698	1.000
$u = 160$	0.710	1.000	1.000	0.691	1.000
$u = 200$	0.703	1.000	1.000	0.686	1.000
$u \to \infty$	0.667	1.000	1.000	0.667	1.000

We can see that for three combinations of θ and ξ the optimal retention level in terms of minimising the ultimate ruin probability is 1. As in the case of exponential individual claims, proportional reinsurance causes little reduction in the ultimate ruin probability. Choosing α to minimise the asymptotic ultimate ruin probability does not appear to be a reasonable alternative to choosing α to minimise $\psi(u;\alpha)$, especially for smaller values of u.

4.3 General remarks

It is straightforward to show that $\psi(0;\alpha) > \psi(0)$ for any form of P(x) provided that $\alpha < 1$ since the net loading $\hat{\theta}$ is then strictly less than θ as $\theta < \xi$.
We can see from Table 5 that when $u = 10$ and the loading factors are 0.1/0.2 the optimal retention level is $\alpha = 1$. This is also true for smaller values of u (and when the loadings are different). This feature is also apparent when the individual claim distribution is Pareto(2,1).

5. Excess of loss reinsurance

Let $\psi(u;M)$ denote the probability of ultimate ruin when $h(X) = \min(X,M)$. In order to apply the algorithm described in Section 3 to calculate approximate values of $\psi(u;M)$ we discretised the individual claim amount distribution on 0,

$M/\beta, 2M/\beta, \ldots, M$ where β was always chosen to be an integer. If the value of u for which we wished to calculate $\psi(u;M)$ was not an integer multiple of M/β, then the value was calculated by linear interpolation as

$$\psi(u;M) = (k + 1 - \beta u/M)\psi(k;M) + (\beta u/M - k)\psi(k + 1;M)$$

where k is the integer such that $k \leq \beta u/M < k + 1$. Values of $\psi(u;M)$ were calculated for values of M that were integer multiples of 0.001. This allowed us to find to three decimal places the value of M that minimised $\psi(u;M)$.

For most of our calculations the value of β was 200. The calculated values were confirmed using a higher value of β (e.g. 300). If necessary, higher values of β were used until calculated optimal retention levels matched for differing values of β. For all combinations of θ and ξ, and for both distributions considered, the values of $\psi(u;M)$ around the optimal retention level were very close, often agreeing to as many as ten decimal places. Thus, the procedure of selecting the optimal value of M is very sensitive to the choice of β.

As in the previous section, we have discretised the individual claim amount distribution (net of reinsurance) using the method of De Vylder and Goovaerts (1988) as this method preserves the value of the insurer's expected individual claim payment.

5.1 Exponential individual claims

Again let $P(x) = 1 - \exp(-x)$. Table 8 shows values of M which minimise the computed values of $\psi(u;M)$ for selected values of u. The final row of this table shows the values of M which maximise the adjustment coefficient.

Table 8

Loadings	0.1/0.15	0.1/0.2	0.1/0.3	0.2/0.3	0.2/0.4
u = 10	0.865	1.583	2.821	0.845	1.529
u = 20	0.858	1.557	2.727	0.838	1.507
u = 30	0.856	1.549	2.698	0.836	1.500
u = 40	0.855	1.545	2.684	0.835	1.496
u = 50	0.854	1.543	2.676	0.833	1.494
R	0.851	1.533	2.643	0.832	1.486

The pattern in Table 8 is similar to that in Table 5. For each combination of θ and ξ, the value of M which minimises the probability of ultimate ruin decreases as u increases, and these values appear to converge to the value of M which maximises the value of the adjustment coefficient. One difference from Table 5 is that for each combination of θ and ξ that we have considered the ultimate ruin probability for these values of u is not minimised by retaining the entire risk.

The figures in Table 8 suggest that, at least for large values of u, finding the value of M which maximises the adjustment coefficient is a reasonable alternative to finding the value of M that minimises the ultimate ruin probability.

For this individual claim distribution we can assess the accuracy of the algorithm when u < M since it is possible to find an explicit expression for $\psi(u;M)$ in this case. For $0 < u < M$, it is easy to show that

$$k \frac{d}{du} \psi(u;M) = \psi(u;M) - \int_0^u p(u - x)\psi(x;M)dx - (1 - P(u)) \qquad (5.1)$$

where k denotes the insurer's premium income net of reinsurance per unit time and $k = 1 + \theta - (1 + \xi)\exp(-M)$. Equation (5.1) can be solved directly, but it is easier to note that the function F(u,x), representing the probability that ultimate ruin occurs and that the surplus immediately prior to ruin is less than x, satisfies the equation

$$(1 + \theta) \frac{d}{du} F(u,x) = F(u,x) - \int_0^u p(u - z)F(z,x)dz - (1 - P(u)) \qquad (5.2)$$

when u < x. Since $F(0,x) = (1 - \exp(-M))/(1 + \theta)$ and $\psi(0;M) = (1 - \exp(-M))/k$ we can use the solution for F(u,x) given by Dickson (1992, Section 4) to write down the solution for $\psi(u;M)$ for $0 \le u < M$ as

$$\psi(u;M) = \frac{1}{k}\left(1 + \frac{1}{k - 1} \exp(-M)\right)\exp(-(1 - 1/k)u) - \frac{1}{k - 1} \exp(-M)$$

Table 9 shows some exact and approximate values of $\psi(2;M)$ when $\theta = 0.1$ and $\xi = 0.15$. These figures show that the algorithm is producing good approximations and there is no reason to suspect that it will not produce good approximations when u > M.

Table 9

M	Exact value of $\psi(2;M)$	Calculated value of $\psi(2;M)$
2.25	0.73437	0.73437
2.50	0.74034	0.74034
2.75	0.74466	0.74467
3.00	0.74785	0.74785
3.25	0.75023	0.75023
3.50	0.75202	0.75203

Figure 1 shows calculated values of $\psi(u;M)$ (solid lines)and Lundberg's upper bound (dotted lines) for u = 10,20,30,40,50 when $\theta = 0.1$ and $\xi = 0.15$. We can see that at the optimal retention levels the probability of ultimate ruin is much smaller than $\psi(u)$. Further, the Lundberg bound is very close to the ruin probability at the optimal retention levels. Figure 2 illustrates the situation when $\theta = 0.1$ and $\xi = 0.3$. In this case the probability of ultimate ruin at the optimal retention level is at most 2% lower than $\psi(u)$.

5.2 Pareto individual claims

In this section we again consider $P(x) = 1 - (1 + x)^{-2}$. Table 10 shows values of M which minimise the calculated value of $\psi(u;M)$ for selected values of u. The final row of this table shows the values of M that maximise the adjustment coefficient.

Table 10

Loadings	0.1/0.15	0.1/0.2	0.1/0.3	0.2/0.3	0.2/0.4
u = 10	1.137	2.548	6.238	1.107	2.446
u = 20	1.124	2.475	5.716	1.095	2.383
u = 30	1.120	2.452	5.575	1.092	2.363
u = 40	1.118	2.441	5.509	1.090	2.354
u = 50	1.116	2.434	5.471	1.088	2.348
R	1.111	2.408	5.326	1.084	2.325

The pattern in this table is identical to that in Table 8. Again we can see that for large values of u at least, choosing M to maximise the adjustment coefficient is a reasonable alternative to choosing M to minimise the ultimate ruin probability.

Figure 3 shows calculated values of $\psi(u;M)$ and Lundberg's upper bound for $u = 10,20,30,40,50$ when $\theta = 0.1$ and $\xi = 0.15$, and Figure 4 illustrates the situation when $\theta = 0.2$ and $\xi = 0.4$. We can see from these figures that excess of loss reinsurance causes a significant reduction in the ultimate ruin probability. Once again, the Lundberg bound is close to the ruin probability at the optimal retention levels. However as M increases, the bound drifts away from $\psi(u;M)$ since $R(M) \to 0$ as $M \to \infty$.

5.3 General remarks

As in the case of proportional reinsurance, it is easy to show that when $u = 0$ the insurer should not effect reinsurance if the insurer's aim is to minimise the probability of ultimate ruin. We can show numerically that this is also true for small values of u.

6. Translated gamma processes

Dickson and Waters (1993) define a translated Gamma process $\{S_{TG}(t)\}_{t>0}$ by

$$S_{TG}(t) = S_G(t) + kt$$

for all $t > 0$ where $\{S_G(t)\}_{t>0}$ is a Gamma(α,β) process. (We are adopting the same notation as Dickson and Waters (1993) so that for a fixed value of t, $S_G(t)$ has a Gamma distribution with parameters αt and β, and $E[S(t)] = \alpha t/\beta$.) Dickson and Waters (1993) show that finite time ruin probabilities for a compound Poisson process can be reasonably approximated by ruin probabilities for a translated Gamma process for large values of u. In this section we will show that this approximation method can also be successfully applied to the problem of finding optimal retention levels. In the following we will approximate the compound Poisson process for net retained aggregate claims by a translated Gamma process and will find values of the retention level that maximise the adjustment coefficient and minimise the ultimate ruin probability for the approximating translated Gamma process.

The adjustment coefficient for a translated Gamma process with parameters α,β and k is the unique positive number R which satisfies

$$\exp\{R(c - k)\} = (1 - R/\beta)^{-\alpha}$$

where c ($> \alpha/\beta + k$) is the premium income per unit time. Given the parameter values, this equation is easily solved using standard numerical techniques. Following Dickson and Waters (1993) the parameter values will be chosen such that the mean, variance and coefficient of skewness of $S(t)$ and $S_{TG}(t)$ are the same for all values of t. The ultimate ruin probability $\psi(u)$ for a compound Poisson process where the premium loading factor is θ is approximated by $\psi_{SG}(\beta u)$, the probability of ultimate ruin for a standardised Gamma process (i.e. a Gamma(1,1) process) when the loading factor is $\hat{\theta} = (1 + k\beta/\alpha)\theta$. In the case when the insurer has effected reinsurance, we simply use the moments net of reinsurance of the process $\{S(t)\}_{t>0}$ to find the parameters of the translated Gamma process and calculate the loading factor, $\hat{\theta}$, using the net of reinsurance loading in place of θ.

Ultimate ruin probabilities for a standardised Gamma process have been calculated in a slightly different way to that described by Dufresne et al (1991). They show that ultimate ruin probabilities for a standardised Gamma process when the premium loading factor is θ can be calculated from

$$\psi_{SG}(u) = \sum_{n=0}^{\infty} \frac{\theta^n}{(1 + \theta)^{n+1}} G^{n*}(u)$$

where

$$G(x) = 1 - \exp\{-x\} + x \int_x^{\infty} (\exp\{-y\}/y)\,dy$$

We have calculated values of $\psi_{SG}(u)$ by discretising $G(x)$ on $0,h,2h,\ldots$ using crude rounding and then applying the method of Panjer (1981). To evaluate the integral expression in $G(x)$ we applied approximate formulae given by

Abramowitz and Stegun (1964). This approach, with $h = 0.005$, leads to values of $\psi_{SG}(u)$ which match those in Table 2 of Dufresne et al (1991) for almost all values of $u \geq 10$. In the cases where there was not an exact match to four decimal places, the difference in the computed values was one in the fourth decimal place.

6.1 Exponential individual claims

Consider first the case when individual claims are exponentially distributed with mean 1. For both proportional and excess of loss reinsurance the first three moments of the individual claim amount distribution net of reinsurance exist and so we can apply the approximation in both cases.

6.1.1 Proportional reinsurance

Table 11 shows values of α that maximise $R(\alpha)$ for the approximating translated Gamma process and the corresponding maximum values of $R(\alpha)$. The figures in parenthesis show the values for the compound Poisson process (as in Table 1).

Table 11

θ/ξ	α	$R(\alpha)$
0.1/0.15	0.644 (0.644)	0.1047 (0.1048)
0.1/0.2	0.956 (0.956)	0.0910 (0.0911)
0.1/0.3	1.000 (1.000)	0.0908 (0.0909)
0.2/0.3	0.624 (0.626)	0.1961 (0.1965)
0.2/0.4	0.920 (0.923)	0.1673 (0.1678)

The values of α that minimise $R(\alpha)$ for the translated Gamma process are very close to those for the compound Poisson process, as are the values of $R(\alpha)$. This is not particularly surprising since an approximate identity for $R(\alpha)$ for each process can be written in terms of the first three moments of the aggregate claims distribution.

Table 12 shows values of α that minimise ultimate ruin probabilities for the translated Gamma process and the corresponding ruin probabilities for two combinations of θ and ξ. The figures in parenthesis show the exact values.

Table 12

u	$\theta = 0.1/\xi = 0.15$		$\theta = 0.2/\xi = 0.4$	
	α	$\psi(u;\alpha)$	α	$\psi(u;\alpha)$
20	0.655 (0.655)	0.1146 (0.1146)	0.943 (0.944)	0.0295 (0.0294)
40	0.649 (0.649)	0.0141 (0.0141)	0.931 (0.933)	0.0010 (0.0010)
60	0.647 (0.648)	0.0017 (0.0017)	0.927 (0.930)	3.67×10^{-5} (3.57×10^{-5})
80	0.646 (0.647)	0.0002 (0.0002)	0.926 (0.928)	1.29×10^{-6} (1.25×10^{-6})
100	0.646 (0.646)	2.6×10^{-5} (2.6×10^{-5})	0.924 (0.927)	4.56×10^{-8} (4.34×10^{-8})

The values of α that minimise the ultimate ruin probabilities for the translated Gamma process are remarkably close to the exact values and the computed ruin probabilities give an excellent approximation to those for the compound Poisson process. The same pattern of results applies for other combinations of θ and ξ.

6.1.2 Excess of loss reinsurance

Table 13 shows values of M that maximise R(M) for the approximating translated Gamma process and the corresponding maximum values of R(M). The figures in parenthesis show the calculated values for the compound Poisson process (as in Table 3).

Table 13

θ/ξ	M	R(M)
0.1/0.15	0.851 (0.851)	0.1641 (0.1642)
0.1/0.2	1.532 (1.533)	0.1188 (0.1189)
0.1/0.3	2.639 (2.643)	0.0991 (0.0993)
0.2/0.3	0.830 (0.832)	0.3145 (0.3153)
0.2/0.4	1.480 (1.486)	0.2257 (0.2265)

This table displays the same characteristics as Table 11. Table 14 shows values of M that minimise ultimate ruin probabilities for the approximating translated Gamma process, together with the minimum ruin probabilities. The figures in parenthesis show the calculated values for the compound Poisson process.

Table 14

u	$\theta = 0.1/\xi = 0.15$		$\theta = 0.2/\xi = 0.4$	
	M	$\psi(u;M)$	M	$\psi(u;M)$
10	0.865 (0.865)	0.1853 (0.1854)	1.529 (1.529)	0.0094 (0.0094)
20	0.858 (0.858)	0.0359 (0.0359)	1.504 (1.507)	0.0099 (0.0098)
30	0.855 (0.856)	0.0070 (0.0070)	1.496 (1.500)	0.0010 (0.0010)
40	0.854 (0.855)	0.0014 (0.0013)	1.492 (1.496)	0.0001 (0.0001)
50	0.853 (0.854)	0.0003 (0.0003)	1.489 (1.494)	0.0000 (0.0000)

We can see that the values of M that minimise the ultimate ruin probability when the aggregate claims process is a translated Gamma process are again remarkably close to those that minimise the ultimate ruin probability when the aggregate claims process is a compound Poisson process, as are the values of the ultimate ruin probabilities at these values of M.

6.2 Pareto(2,1) claims

Since the second and third moments of the Pareto(2,1) distribution do not exist we cannot apply this approximation method when the reinsurance arrangement is proportional.

However, we can apply the approximation when the reinsurance arrangement is excess of loss. Table 15 shows values of M that maximise R(M) for the approximating translated Gamma process and the corresponding maximum values of R(M). The figures in parenthesis show the calculated values for the compound Poisson process (as in Table 4).

Table 15

θ/ξ	M	R(M)
0.1/0.15	1.111 (1.111)	0.1257 (0.1258)
0.1/0.2	2.406 (2.408)	0.0757 (0.0757)
0.1/0.3	5.318 (5.326)	0.0492 (0.0493)
0.2/0.3	1.081 (1.084)	0.2415 (0.2420)
0.2/0.4	2.317 (2.325)	0.1444 (0.1447)

The pattern of results in Table 15 is similar to that in Table 13. The values of M that maximise R(M) for the translated Gamma process are marginally smaller than the values for the compound Poisson process, and the same is true for the values of R(M).

Table 16 shows values of M that minimise ultimate ruin probabilities for the approximating translated Gamma process together with minimum ruin probabilities. The figures in parenthesis show the calculated values for the compound Poisson process. The pattern in this table is the same as that in Table 14.

Table 16

u	$\theta = 0.1/\xi = 0.15$		$\theta = 0.1/\xi = 0.2$	
	M	$\psi(u;M)$	M	$\psi(u;M)$
10	1.137 (1.137)	0.2727 (0.2729)	2.553 (2.548)	0.4457 (0.4462)
20	1.124 (1.124)	0.0776 (0.0776)	2.476 (2.475)	0.2093 (0.2094)
30	1.119 (1.120)	0.0221 (0.0221)	2.452 (2.452)	0.0982 (0.0982)
40	1.117 (1.118)	0.0063 (0.0063)	2.440 (2.441)	0.0461 (0.0461)
50	1.116 (1.116)	0.0018 (0.0018)	2.433 (2.434)	0.0216 (0.0216)

7. Varying the reinsurer's loading

In this section we illustrate how the optimal retention level can be found when the reinsurer's loading varies with the retention level. We will consider the case of excess of loss reinsurance and Pareto(2,1) individual claims.

To apply the algorithm to calculate $\psi(u;M)$ (given the individual claim amount distribution) the input parameters are simply the initial surplus, the retention level, the insurer's loading and the reinsurer's loading. Thus it is straightforward to calculate ruin probabilities when $\xi = \xi(M)$ by simply changing the value of ξ when we change the value of M.

Figure 5 shows calculated values of $\psi(50;M)$ and $\exp(-50R(M))$ when $\theta = 0.1$ and

$$\begin{aligned} \xi(M) &= 0.15 && \text{for } 0.5 \le M < 3 \\ &= 0.2 && \text{for } 3 \le M < 5 \\ &= 0.3 && \text{for } 5 \le M < \infty \end{aligned}$$

This figure is consistent with the figures shown by Centeno (1991) who considers the effect of reinsurance on the adjustment coefficient. In this case we see that the optimal retention level for each value of u is the same as it is when $\xi(M) = 0.15$ for $0.5 \le M < \infty$.

Figure 6 shows calculated values of $\psi(u;M)$ and $\exp(-R(M)u)$ for $u = 10,20,30,40,50$ when $\theta = 0.1$ and

$$\begin{aligned} \xi(M) &= (1 + M/5)/6 && \text{for } 1 \le M < 25 \\ &= 1 && \text{for } M \ge 25 \end{aligned}$$

i.e. a loading that increases linearly from 20% at $M = 1$ to 100% at $M = 25$ and remains at 100% for $M > 25$. The features of this figure are similar to those in Figures 1 to 4.

8. Finite and continuous time ruin and reinsurance

So far in this paper we have considered the effect of reinsurance on the probability of ruin in *infinite* (and continuous) time. In this section we consider, using numerical examples, the effect of reinsurance on the probability of ruin in finite and continuous time - finite and discrete time ruin will be considered in the next section.

Recall from Section 1 that $U(t;h)$ denotes the insurer's net surplus at time t given a reinsurance arrangement, either proportional or excess of loss, defined by the function $h(x)$. Now define $\psi(u,T;h)$, the probability of ruin in continuous time before time T, given reinsurance defined by $h(x)$, as follows:

$$\psi(u,T;h) = P(U(t;h) < 0 \text{ for some } t,\ 0 < t \leq T)$$

In what follows we will replace h by α to indicate proportional reinsurance and by M to indicate excess of loss. (We may also replace h by a numerical value for α or for M, in which case it will be clear what form of reinsurance we are considering.) In all our examples, as in previous sections, the unit of time is such that we expect one claim per unit time, so that the Poisson parameter is 1.

Example 1: Individual claim amounts have an exponential distribution with mean 1; $\theta = 0.2$; $\xi = 0.3$; $u = 30$; proportional reinsurance.

Table 17 shows values of $\psi(30,T;\alpha)$ for $T = 100$, 500, 1000 and α in steps of 0.05 from 0.00 to 1 calculated in two different ways. The three columns headed $\psi(30,T;\alpha)$ have been calculated using the algorithm of De Vylder and Goovaerts (1988) (as rescaled by Dickson and Waters (1991, Section 2)). The individual claim amount distribution has been discretised in steps of 1/20 (before reinsurance) and the control parameter, ϵ, has been set at 3×10^{-9}. (See De Vylder and Goovaerts (1988) for details of these last two points.) The value of ϵ is such that the maximum error is $2.16\times T\times10^{-7}$ for $T = 100$, 500, 1000. The columns headed $\psi^*(30,T;\alpha)$ have been calculated by approximating the compound Poisson process for net retained aggregate claims by a translated Gamma process and then using the methods of Dickson and Waters (1993) to calculate the finite time ruin probability for this approximating process.

The points to note about Table 17 are:

a) Ruin probabilities for the translated Gamma process are very good approximations to ruin probabilities for the compound Poisson process for all values of α. (This is not surprising given the results of Dickson and Waters (1993).)

Table 17

α	T = 100		T = 500		T = 1000	
	$\psi(30,T;\alpha)$	$\psi^*(30,T;\alpha)$	$\psi(30,T;\alpha)$	$\psi^*(30,T;\alpha)$	$\psi(30,T;\alpha)$	$\psi^*(30,T;\alpha)$
0.00	0.0000	0.0000	1.0000	1.0000	1.0000	1.0000
0.05	0.0000	0.0000	1.0000	1.0000	1.0000	1.0000
0.10	0.0000	0.0000	0.9460	0.9481	1.0000	1.0000
0.15	0.0000	0.0000	0.3138	0.3133	1.0000	1.0000
0.20	0.0000	0.0000	0.0714	0.0709	0.8904	0.8910
0.25	0.0000	0.0000	0.0217	0.0215	0.3871	0.3868
0.30	0.0000	0.0000	0.0091	0.0090	0.1085	0.1081
0.35	0.0000	0.0000	0.0049	0.0049	0.0316	0.0314
0.40	0.0000	0.0000	0.0032	0.0032	0.0116	0.0115
0.45	0.0000	0.0000	0.0025	0.0025	0.0056	0.0056
0.50	0.0000	0.0000	0.0021	0.0021	0.0035	0.0035
0.55	0.0000	0.0000	0.0020	0.0020	0.0027	0.0027
0.60	0.0001	0.0001	0.0020	0.0021	0.0024	0.0024
0.65	0.0002	0.0002	0.0022	0.0022	0.0024	0.0024
0.70	0.0003	0.0003	0.0024	0.0024	0.0026	0.0026
0.75	0.0005	0.0005	0.0027	0.0027	0.0028	0.0029
0.80	0.0007	0.0008	0.0031	0.0031	0.0032	0.0032
0.85	0.0011	0.0011	0.0036	0.0036	0.0037	0.0037
0.90	0.0015	0.0015	0.0042	0.0042	0.0042	0.0043
0.95	0.0020	0.0020	0.0048	0.0049	0.0049	0.0049
1.00	0.0026	0.0026	0.0056	0.0056	0.0056	0.0057

b) For small values of α, several values of $\psi(30,T;\alpha)$ are shown as 0.0000 or 1.0000. It is easy to see how these values arise. Consider the case where α is 0. In this case the surplus process is deterministic: there are no (net) claims for the insurer to pay and the insurer's net premium income is $(\theta - \xi)$, which is negative. So for this particular example with $\alpha = 0$, the insurer will be ruined with certainty at time 30/0.1 = 300, but cannot be ruined earlier than this.

c) For the higher values of α, the value of $\psi(30,T;\alpha)$ does not increase very much as T increases from 500 to 1000. This indicates that $\psi(30,500;\alpha)$ is very close to $\psi(30;\alpha)$, the probability of ruin in infinite time.

The observation that $\psi^*(u,T;\alpha)$ gives a good approximation to $\psi(u,T;\alpha)$ (at least in this example) is interesting because it is far easier to calculate $\psi^*(u,T;\alpha)$ than it is to calculate $\psi(u,T;\alpha)$. We will make use of this observation in the next example.

Example 2: Individual claim amounts have an exponential distribution with mean 1; θ = 0.1; ξ = 0.2; excess of loss reinsurance.

Table 18 shows, for u = 10, 30 and 50 and for various values of T, the value of the excess of loss retention limit, M (restricted to be an integer multiple of 0.01), which minimises $\psi(u,T;M)$ and the corresponding minimum value of this probability. These probabilities have been calculated using a translated Gamma process approximation to the retained aggregate claims process.

Table 18

T	$\psi(10,T;M)$	M	$\psi(30,T;M)$	M	$\psi(50,T;M)$	M
200	0.1880	0.92				
300	0.2335	1.16				
400	0.2549	1.29	0.0078	0.76		
500	0.2666	1.37	0.0132	1.00		
600	0.2737	1.43	0.0170	1.15	0.0003	0.66
700	0.2781	1.47	0.0197	1.24	0.0007	0.90
800	0.2811	1.49	0.0216	1.31	0.0011	1.05
900	0.2832	1.51	0.0229	1.37	0.0014	1.16
1000	0.2846	1.53	0.0239	1.41	0.0016	1.24
1500	0.2875	1.57	0.0261	1.50	0.0022	1.43
2000	0.2882	1.58	0.0266	1.53	0.0024	1.50
2500	0.2884	1.58	0.0268	1.54	0.0024	1.53
3000	0.2885	1.58	0.0268	1.55	0.0025	1.54
3500	0.2885	1.58	0.0268	1.55	0.0025	1.54
∞	0.2885	1.58	0.0268	1.55	0.0025	1.54

The missing entries in Table 18 all correspond to $\psi(u,T;M)$ equal to 0 for M equal to 0. (See comment b) on Table 17.) The values of the optimal retention level for T equal to ∞ have been calculated using the methods of Section 6.

The most interesting feature of Table 18 is that it shows the convergence of the finite time optimal M's to the infinite time optimal M's as T increases - the smaller the value of u, the faster this convergence takes place, at least in this example.

A translated Gamma process approximation has the advantage that the calculation of the optimal finite time retention level is much faster than it would be using the original process together with the algorithm of De Vylder and Goovaerts (1988). However, the calculation of the optimal infinite time retention level using the methods of Sections 4 and 5 is even faster. Given this, we can ask whether it is reasonable to assume an optimal infinite time retention level is "approximately optimal" in finite time. Table 18 provides some answers to this question. For example, for u = 10, 30 or 50 the optimal infinite time retention level is also the optimal finite time retention level (to two decimal places) for T greater than 3000, but for u = 50 and T = 900 the optimal finite time retention level, 1.16, is not very close to the optimal infinite time level, 1.54. Another measure of the "accuracy" of this "approximation" is to calculate, for given values of u and T, the ratio:

$$\psi(u,T;\hat{M}_T)/\psi(u,T;\hat{M}_\infty)$$

where $\hat{M}_T$ and $\hat{M}_\infty$ are the optimal finite time and infinite time retention levels, respectively. The reasoning here is that, even though $\hat{M}_T$ may not be close to $\hat{M}_\infty$, if $\psi(u,T;\hat{M}_\infty)$ is close to $\psi(u,T;\hat{M}_T)$ then there is little to be gained from the extra effort required to calculate $\hat{M}_T$ rather than $\hat{M}_\infty$. We will refer to this ratio as the "efficiency" of the optimal infinite time retention level. Table 19 shows the efficiency for u = 10, 30 and 50 in this particular example.

The figures in Table 19 show, as could be predicted from Table 18, that the efficiency of the optimal infinite time retention level increases as the finite time horizon, T, increases. Two further points to note about Table 19 are:

a) For a given value of T, the efficiency (in this example) is a decreasing function of u.

b) The figures in Table 19 are more encouraging than those in Table 18 in the sense that, for example, for u = 50 and T = 900 there is a considerable

Table 19

T	Efficiency u = 10	Efficiency u = 30	Efficiency u = 50
200	0.87		
300	0.95		
400	0.98	0.57	
500	0.99	0.78	
600	1.00	0.88	0.33
700	1.00	0.93	0.58
800	1.00	0.96	0.79
900	1.00	0.97	0.88
1000	1.00	0.98	0.89
1500	1.00	1.00	0.96
2000	1.00	1.00	1.00
2500	1.00	1.00	1.00
3000	1.00	1.00	1.00
3500	1.00	1.00	1.00

difference between $\hat{M}_T$ and $\hat{M}_\infty$, 1.16 and 1.54, respectively, but much less difference between $\psi(50,700;1.16)$ and $\psi(50,700;1.54)$, 0.0014 and 0.0016, respectively.

9. Finite and discrete time ruin and reinsurance

In this section we consider briefly the effect of reinsurance on the probability of ruin in finite and discrete time, once again using examples. We define $\psi(u,T,d;h)$ as follows:

$$\psi(u,T,d;h) = P(U(t;h) < 0 \text{ for some } t,\ t = d,\ 2d,\ 3d,\ \ldots, T - d,\ T)$$

where T is assumed to be an integer multiple of d. Once the (net of reinsurance) individual claim amount distribution has been discretised on, say, $1/\beta$, $2/\beta$, ..., where β is some suitably large number: in Example 3 $\beta = 100$ and in Example 4 $\beta = 20$. We have used the algorithm of De Vylder and Goovaerts (1988) to calculate $\psi(u,T,d;h)$ in these two examples, with some adjustments in Example 3 as described below.

Example 3: Individual claim amounts have a Pareto (2,1) distribution; $\theta = 0.2$; $\xi = 0.4$; excess of loss reinsurance.

This example requires the algorithm of De Vylder and Goovaerts (1988) to be adjusted because the insurer's premium income net of reinsurance over the time period of length d is not in general an integer so that the surplus process need not be at integer levels at times d,2d,3d,... . Given that the aggregate claims distribution has been discretised on the integers and that we regard a surplus of zero as ruin (except at t = 0), note that

$$\psi(u,d,d;h) = \psi(\{u\},d,d;h)$$

where $\{u\}$ denotes the least integer greater than or equal to u.

Consider an integer initial surplus of u and a non-integer premium income of P in a time interval of length d. We define f_k and $F(k)$ to be the probabilities that aggregate claims in a time interval of length d are equal to and less than or equal to k, respectively.

The basic algorithm of De Vylder and Goovaerts (1988) now becomes

$$\psi(u,d,d;h) = 1 - F(u + [P])$$

where [P] is the greatest integer less than or equal to P, and, for T = 2d,3d,... ,

$$\psi(u,T,d;h) = \psi(u,d,d;h) + \sum_{j=0}^{u+[P]} f_j \psi(u+\{P\}-j,T-d,d;h)$$

For our calculations the relevant summations have been truncated using the same control parameter as in Section 8.

Table 20 shows for T = 100 to 1000 in steps of 100 and for u = 10, 30 and 50,

$\hat{M}$ (the value of the retention level which minimises $\psi(u,T,100;M)$, taken over values of M which are integer multiples of 0.01), the minimum value of $\psi(u,T,100;M)$ and the efficiency of the infinite (and *continuous*) time optimal retention level. (The optimal infinite and continuous time retention level, from Table 10, is 2.45 for u = 10, 2.36 for u = 30 and 2.35 for u = 50.)

Table 20

	u = 10			u = 30			u = 50		
T	$\hat{M}$	ψ	Eff'y	$\hat{M}$	ψ	Eff'y	$\hat{M}$	ψ	Eff'y
100	2.09	0.0413	0.99						
200	2.79	0.0627	0.99						
300	3.00	0.0714	0.97	1.68	0.0020	0.83			
400	3.09	0.0754	0.96	1.98	0.0031	0.95	1.34	4.3×10^{-5}	0.53
500	3.24	0.0774	0.95	2.24	0.0038	0.99	1.75	9.7×10^{-5}	0.78
600	3.24	0.0784	0.94	2.32	0.0042	1.00	1.91	1.5×10^{-4}	0.90
700	3.30	0.0790	0.94	2.39	0.0045	1.00	2.09	1.9×10^{-4}	0.96
800	3.34	0.0793	0.94	2.45	0.0047	1.00	2.24	2.1×10^{-4}	0.98
900	3.34	0.0794	0.94	2.48	0.0048	0.99	2.24	2.3×10^{-4}	0.99
1000	3.34	0.0795	0.93	2.48	0.0049	0.99	2.24	2.5×10^{-4}	1.00

The figures for u = 50 in Table 20 are unremarkable: as the time horizon, T, increases the optimal finite and discrete time retention level increases to about the optimal infinite and continuous time retention level and the efficiency of the latter converges to 1.00. The figures for u = 10 are a little more remarkable: as T increases, the optimal finite and discrete time retention level appears to be converging, but *not* to the optimal infinite and continuous time retention level and the efficiency of the latter is falling away from 1.00. These features will be even more apparent in our final example in this section.

Example 4: Individual claim amounts have an exponential distribution with mean 1; θ = 0.2; ξ = 0.3; proportional reinsurance.

Notice that this is the same as Example 1 in Section 8 except that we are now working in discrete time rather than continuous time. Table 21 corresponds to Table 20 for Example 3.

The significant message suggested by Table 20 and confirmed by Table 21 is that optimality in finite and discrete time can be a very different matter from optimality in infinite and continuous time, and even from finite and continuous time. Consider u = 30 and T = 500 in this example. The optimal retention level in finite and discrete time, from Table 21, is 0.63, which is the same as for infinite and continuous time, but different from the optimal finite and continuous time retention level, which, from Table 17, is in the interval (0.55, 0.6). As we saw in Example 3, as the time horizon, T, increases, the optimal finite and discrete time retention level actually *moves away from* the optimal infinite and continuous time retention level. However, these differences pale into insignificance when we consider the figures in Table 21 for u = 10! In this case the optimal finite and discrete time retention level is 1, i.e. no reinsurance, for all values of T in the table (the optimal infinite and continuous time retention level is 0.65 for u = 10) and the efficiency of the optimal infinite and continuous time retention level is *decreasing* as T increases.

Table 21

	u = 10			u = 30			u = 50		
T	$\hat{\alpha}$	ψ	Eff'y	$\hat{\alpha}$	ψ	Eff'y	$\hat{\alpha}$	ψ	Eff'y
100	1.00	0.0220	1.00						
200	1.00	0.0267	0.83						
300	1.00	0.0278	0.77	0.28	2.9×10^{-4}	0.60			
400	1.00	0.0281	0.74	0.55	6.1×10^{-4}	0.97			
500	1.00	0.0282	0.73	0.63	7.3×10^{-4}	1.00	0.27	4.5×10^{-6}	0.49
600	1.00	0.0282	0.72	0.66	7.8×10^{-4}	0.99	0.54	1.1×10^{-5}	0.89
700	1.00	0.0283	0.72	0.67	8.0×10^{-4}	0.98	0.60	1.3×10^{-5}	0.98
800	1.00	0.0283	0.72	0.68	8.1×10^{-4}	0.97	0.63	1.5×10^{-5}	1.00
900	1.00	0.0283	0.71	0.68	8.2×10^{-4}	0.97	0.64	1.6×10^{-5}	1.00
1000	1.00	0.0283	0.71	0.68	8.2×10^{-4}	0.96	0.65	1.6×10^{-5}	0.99

In both examples in this section the interval for the discrete time "checking" of the surplus process, i.e. the parameter d, has been chosen to be 100. In other words, we are checking for ruin following intervals of time in which we expect 100 claims. Presumably, the smaller the value of d, the closer "discrete time optimality" becomes to "continuous time optimality".

10. Conclusions

We have shown that (approximate) ultimate ruin probabilities can be calculated when the aggregate claims process is a compound Poisson process and when there is a reinsurance arrangement in force.

The value of the retention level (α or M) that minimises $\psi(u;h)$ is close to the value that maximises the adjustment coefficient, particularly when the initial surplus is large.

The values of the retention level which maximise the adjustment coefficient and minimise the ultimate ruin probability for a translated Gamma process are very close to the corresponding values for the compound Poisson process, as are the calculated values of the adjustment coefficient and $\psi(u;h)$.

Calculation of optimal retention levels, particularly for excess of loss, can be very sensitive to the discretisation of the individual claim amount distribution since the probability of ruin, as a function of the retention level, can be very flat around the optimal level. For this reason a high value of the discretisation parameter β may be required to obtain reliable results (see, for example, Section 5). It is interesting to note that the translated Gamma process approximation gives very good results even in these circumstances.

It seems clear from both Example 3 and Example 4 in Section 9 that it is difficult to infer anything about optimal retention levels in discrete time from information about optimality in continuous time.

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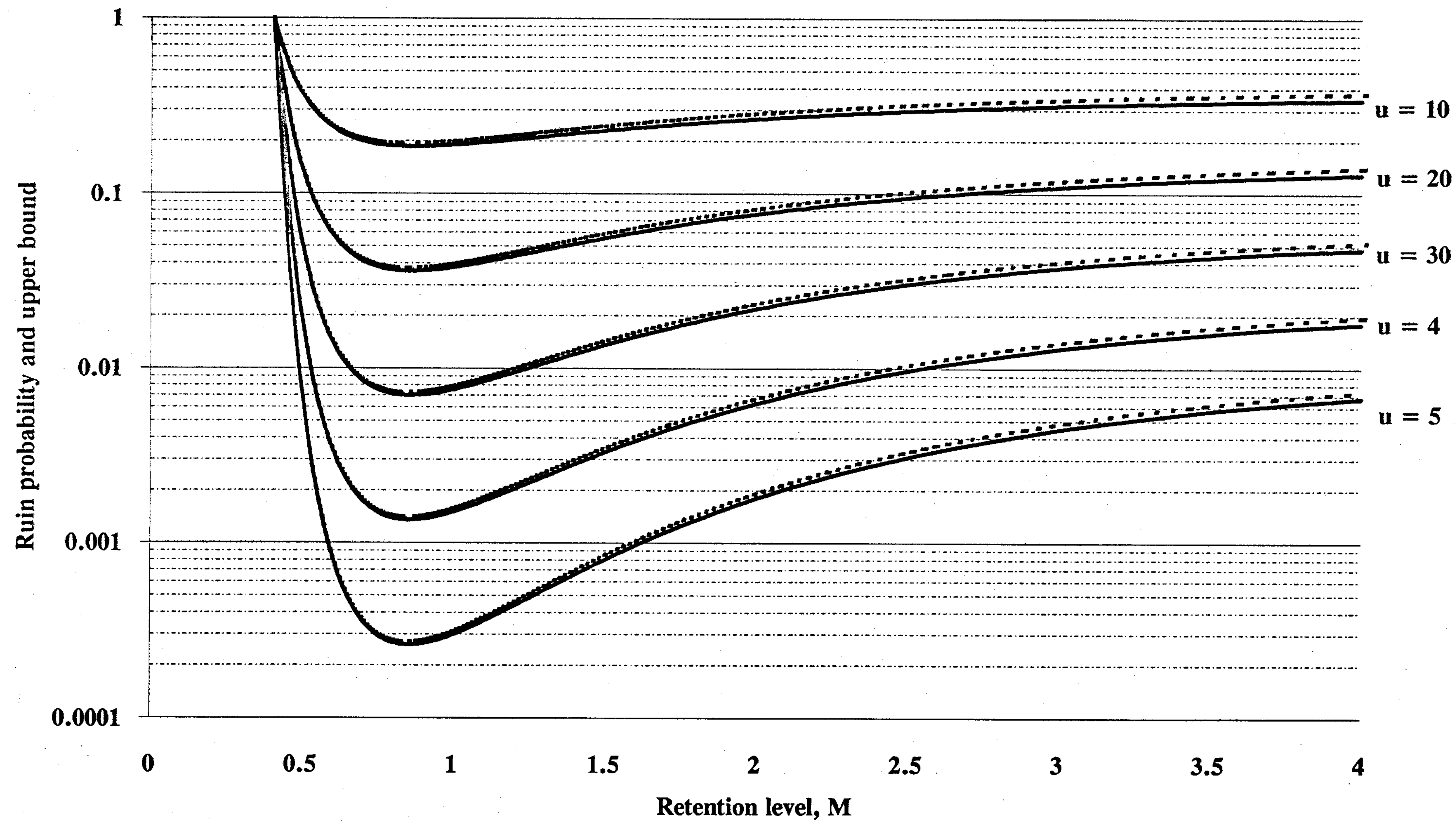

Figure 1: $\psi(u;M)$ and $\exp\{-R(M)u\}$ when individual claims are exponentially distributed, $\theta = 0.1$ and $\xi = 0.15$

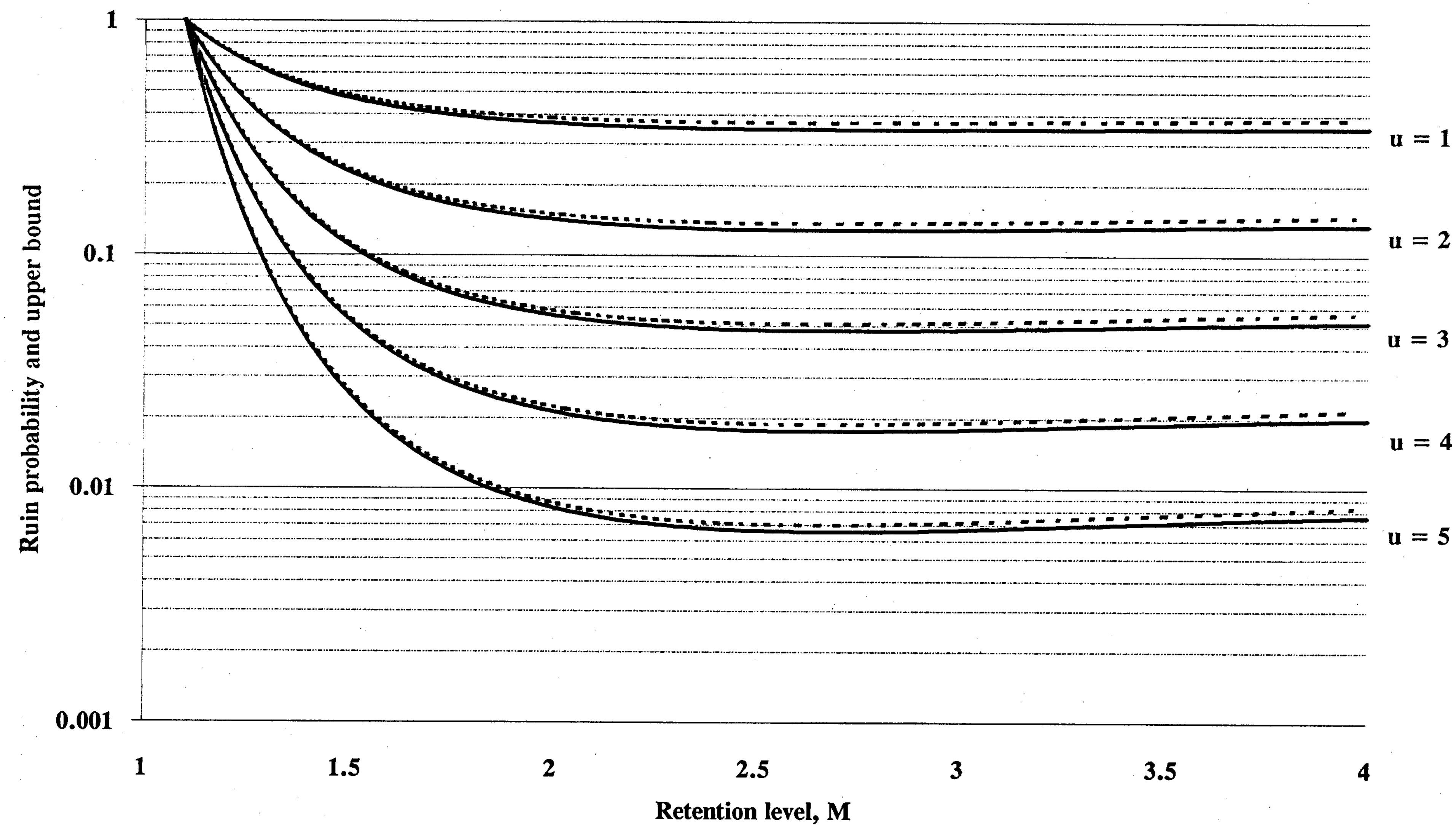

Figure 2: $\psi(u;M)$ **and** $\exp\{-R(M)u\}$ **when individual claims are exponentially distributed,** $\theta = 0.1$ **and** $\xi = 0.3$

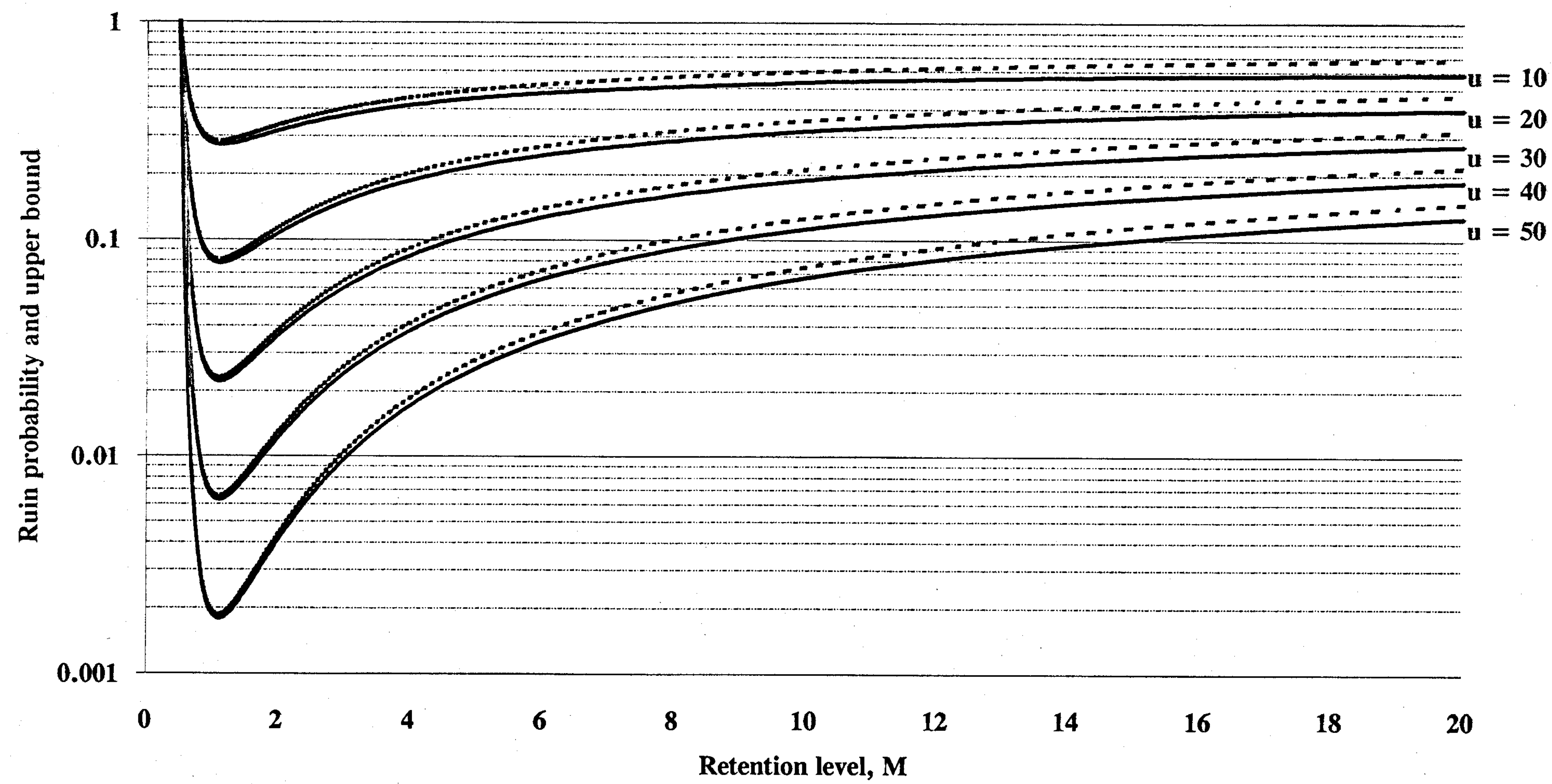

Figure 3: $\psi(u;M)$ and $\exp\{-R(M)u\}$ when individual claims are distributed as Pareto(2,1), $\theta = 0.1$ and $\xi = 0.15$

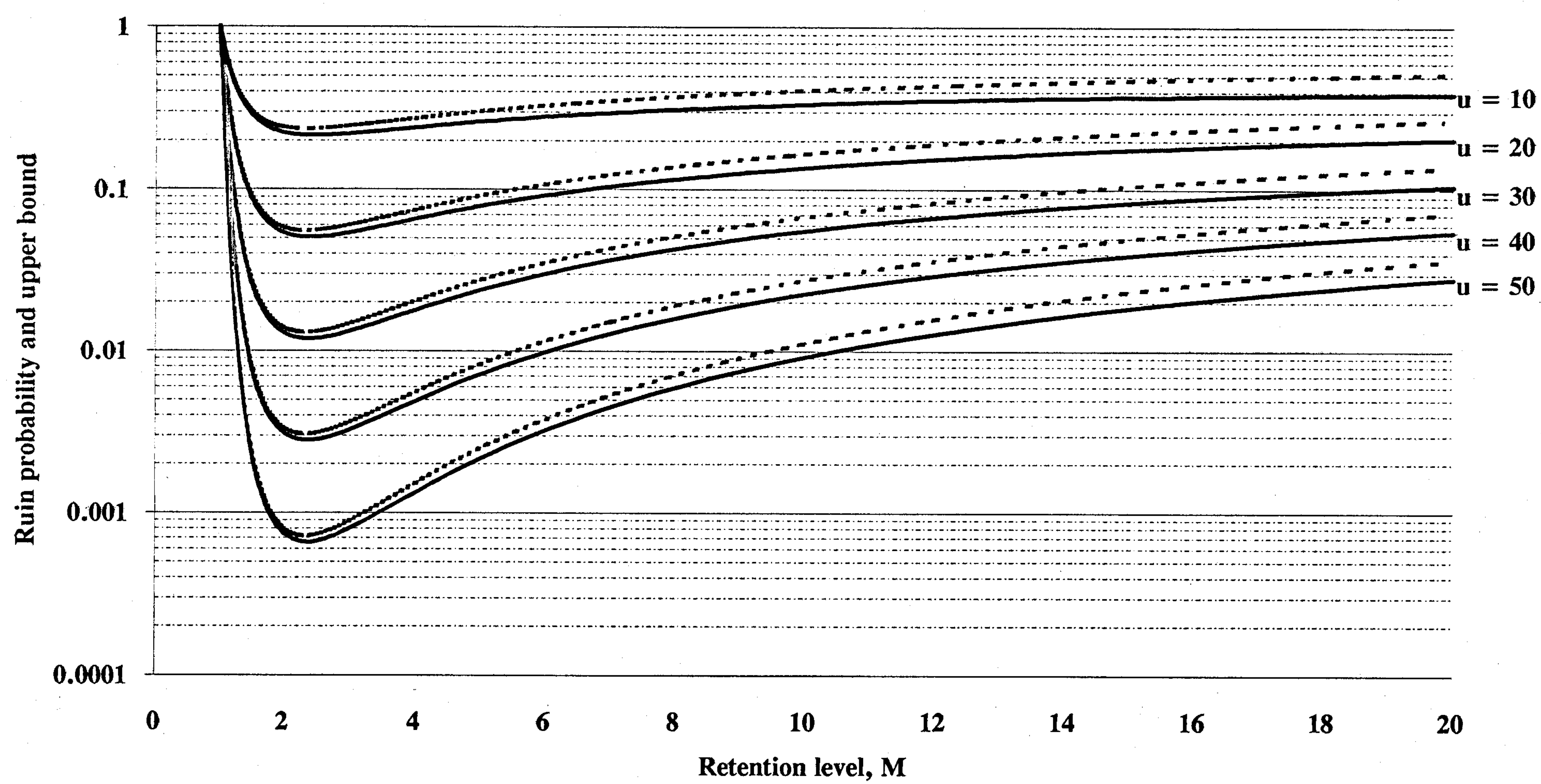

Figure 4: $\psi(u;M)$ and $\exp\{-R(M)u\}$ when individual claims are distributed as Pareto(2,1), $\theta = 0.2$ and $\xi = 0.4$

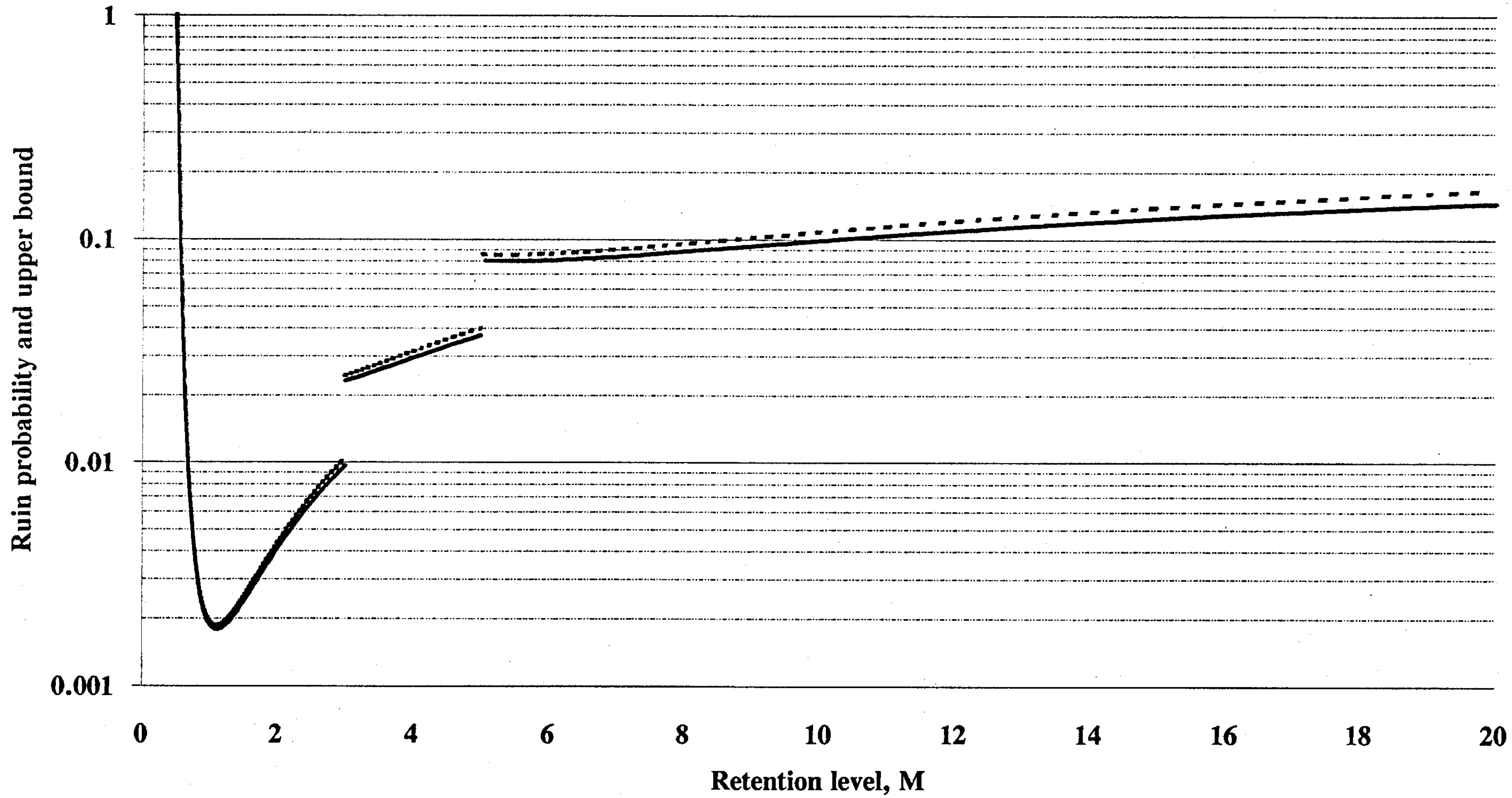

Figure 5: $\psi(50;M)$ **and** $\exp\{-50R(M)\}$ **when individual claims are distributed as Pareto(2,1),** $\theta = 0.1$ **and** $\xi = 0.15, 0.2$ **or 0.3**

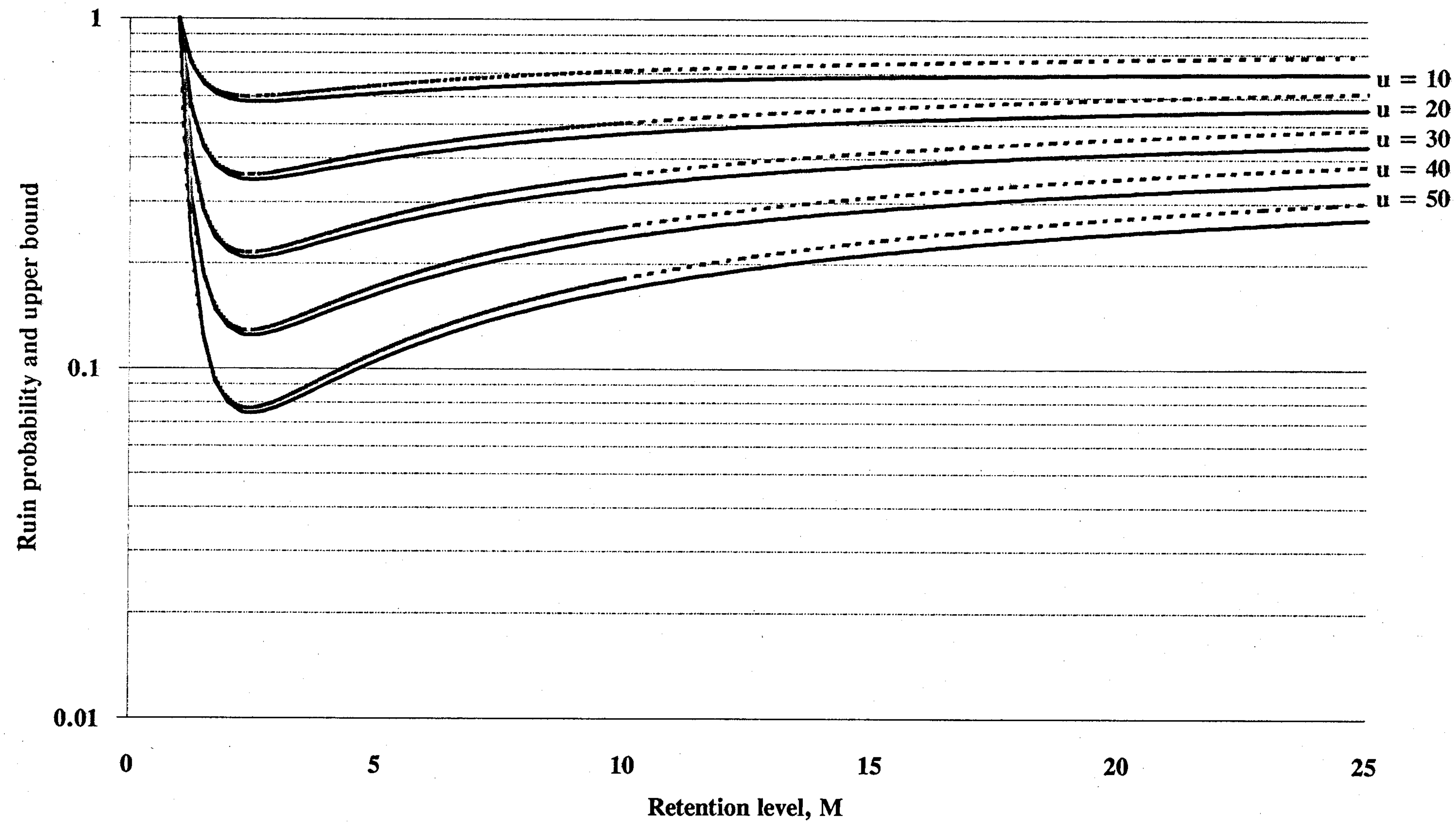

Figure 6: $\psi(u;M)$ and $\exp\{-R(M)u\}$ when individual claims are distributed as Pareto(2,1), $\theta = 0.1$ and $\xi(M) = (1 + M/5)/6$

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