On the moments of ruin and recovery times

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August 4, 1998

Abstract

In this paper we present a recursive method to compute moments of the time to ruin and the duration of the first period of negative surplus. This method uses a discrete time compound Poisson process used by Dickson *et al.* (1995). With this method we will be able to calculate approximations for the corresponding quantities in the classical model. Furthermore, for the classical compound Poisson model we consider some asymptotic formulae, as initial surplus tends to infinity, for the severity of ruin, which allow us to find explicit formulae for the moments of the time to recovery.

Keywords: Time to ruin; probability of ruin; duration of negative surplus; severity of ruin; discrete time model; recursive calculation.

*This work was carried out during the author's stay at the Centre for Actuarial Studies at The University of Melbourne for the 1st semester of 1997. Support form JNICT and ISEG is gratefully acknowledged. Ref. PRAXIS XXI/BPB/9979/96.

1. Introduction

In this paper we consider recursive algorithms for the numerical approximation of the moments of the random variables time to ruin and the duration of the first period of negative surplus for the classical surplus process. Our approach for producing the algorithms is to approximate the classical surplus process by a discrete process, discrete time and discrete claim amount distribution, and then to derive an algorithm for the appropriate quantity for the discrete model. The discrete model we are using is a discrete time compound Poisson process first introduced by Dickson and Waters (1991) for a similar approximation purpose, and later retrieved by Dickson *et al.* (1995). As far as the time to ruin is concerned, much has been said in the actuarial literature about the study of the ruin probabilities, whether finite time or ultimate. The study of some existing moments like the expected value or the variance of the time to ruin can give another insight, although somehow limited. For practical purposes it may be useful for instance to have a quick look at how long it takes for ruin to occur, before going into more details like computing ruin probabilities. The proposed method is easy to implement and its outcome is easily interpreted.

The duration of negative surplus has been previously discussed by Egídio dos Reis (1993) and Dickson and Egídio dos Reis (1996). The first reference deals exclusively with the moments of this random variable only concerning the classical model, giving closed formulae which depend on the severity related quantities. However in many situations we cannot have explicit results other than for an initial surplus equal to zero. The alternative is to approximate these quantities via the severity of ruin using available methods, like the one presented by Dickson *et al.* (1995), who use the same discrete time model. The approach we present shows how we can do it dealing in full with the discrete model. Not only does it give exact results as far as the discrete model is concerned but it also provides the starting values (when the initial surplus is zero) for the recursions concerning the time to ruin, by enhancing the relationship between time to ruin and time to recovery when the initial surplus is zero, for the discrete model.

For the classical model we also present explicit formulae for the asymptotic moment generating function of the severity of ruin, as the initial surplus goes to infinity, which allow us to compute, at least numerically, moments for the severity of ruin as well as the related random variable time to recovery.

In the next section we introduce the basic continuous time surplus model as well as the discrete model that approximates the basic model, including definitions and notation. In Section 3 we present recursions for the moments of time to ruin for positive values of the initial surplus, considering the discrete model. In Section 4 we consider, for the discrete model the recursions for the duration of a first period of negative surplus, showing the relationship between time to recovery and severity of ruin. In Section 5 we discuss the starting values, with initial surplus equal to zero, for the recursions in the two previous sections, based on the fact that the time to ruin and the duration of negative surplus have the same distribution. In Section 6 we show some asymptotic formulae for the severity of ruin and, consequently, for the time to ruin, considering the classic model. Finally, in the last section we consider a couple of examples showing, where possible, the kind of approximations (for the classical model) we will expect to obtain from the recursions formerly discussed.

2. Models and notation

Let $\{U(t)\}_{t\geq 0}$ be a classical continuous time surplus process, so that

$$U(t) = u + ct - S(t),$$

where u is the insurer's initial surplus, c is the insurer's rate of premium income per unit time, $S(t) = \sum_{i=1}^{N(t)} X_i$ is the aggregate claim amount up to time t, $N(t)$ is the number of claims in the same time interval having a Poisson(λt) distribution, and $\{X_i\}_{i=1}^{\infty}$is a sequence of i.i.d. random variables representing the individual claim amounts. We denote by $P(x)$ and $p(x)$ the distribution and density function of X_i, respectively, with $P(0) = 0$, so that all claim amounts are non-negative. We also assume that the mean of X_i, which we denote by p_1, is finite. We will also assume throughout the paper the existence of higher moments than p_1, and we denote the k-th moment about the origin of X_i by p_k. We will further assume in some parts of the paper the existence of the moment generating function of X_i, which we denote by $m(t)$, and we state that clearly. We define a positive parameter θ to be such that

$$c = (1 + \theta)\lambda p_1$$

so that θ is the insurer's premium loading factor.

Without loss of generality, we will make the two following assumptions: $c = p_1 = 1$. We will refer to this process as our *basic process.*

We want to produce a discrete approximation to this basic process and we will consider Dickson *et al.* (1995) approach considering a discrete time Poisson process

$$U_d(t) = u + t - S_d(t)$$

for $t = 1, 2, ...,$ with an initial reserve u $(u = 0, 1, 2, ...)$ and $U_d(0) = u$. Also:

- $S_d(t)$ is the aggregate claim amount up to time t, and we denote by $F(x, t)$ and $f(x, t)$ the distribution and density function of $S_d(t)$;
- individual claims are i.i.d. random variables on the non-negative integers with mean $\beta > 0$. Like in the classical model, we will require the existence of higher moments of individual claims, and we will denote the k-th moment about the origin by b_k $(b_1 = \beta)$;
- premium income per unit time is 1;
- expected number of claims per unit time is $\lambda_d = \lambda/\beta$.

For simplicity we state $f_j = f_d(j, 1)$ and $F_j = F_d(j, 1)$ for $j = 0, 1, 2, ...$. Notice that $S_d(t)$ is the sum of i.i.d. random variables, each with probability function $\{f_j\}_{j=0}^{\infty}$. If $X_{d,n}$ denotes the aggregate claim amount from time $n-1$ until time n, then $S_d(t) = \sum_{n=1}^{t} X_{d,n}$ and f_j is the probability of $X_{d,n}$ taking value j. Values for f_j, $j = 0, 1, 2, ...$, can be obtained using Panjer's (1981) recursion.

Time to ruin in the basic process is denoted by T and defined as

$$T = \begin{cases} \inf\{t : U(t) < 0\} \\ \infty \text{ if } U(t) \geq 0 \text{ for all } t > 0 \end{cases},$$

finite time ruin probability from some initial surplus $u \geq 0$ is defined as

$$\psi(u,t) = \Pr[T \leq t \mid U(0) = u] \ ,$$

and ultimate ruin probability $\psi(u) = \Pr[T < \infty \mid U(0) = u]$. Finite time survival probability is denoted by $\delta(u,t) = 1 - \psi(u,t)$ and ultimate survival probability by $\delta(u) = 1 - \psi(u) = \Pr[T = \infty | U(0) = u]$. It is well known that $\psi(0) = \lambda p_1/c = 1/(1+\theta)$.

For the discrete time model we will use two definitions of ruin, depending on whether or not a surplus of zero, other than at time zero, is regarded as ruin. Accordingly, we define time to ruin as

$$T_d = \begin{cases} \min\{n : U_d(n) < 0, \ n = 1,2,3,...\} \\ \infty \text{ if } U_d(n) \geq 0 \text{ for } n = 1,2,3,.. \end{cases}$$

$$T_d^* = \begin{cases} \min\{n : U_d(n) \leq 0, \ n = 1,2,3,...\} \\ \infty \text{ if } U_d(n) > 0 \text{ for } n = 1,2,3,.. \end{cases}$$

Discrete and finite time ruin probabilities, for a given non-negative integer u are

$$\psi_d(u,t) = \Pr[T_d \leq t \mid U_d(0) = u]$$

$$\psi_d^*(u,t) = \Pr[T_d^* \leq t \mid U_d(0) = u] \ .$$

Ultimate ruin probabilities are defined as $\psi_d(u) = \Pr[T_d < \infty \mid U_d(0) = u]$ and $\psi_d^*(u) = \Pr[T_d^* < \infty \mid U_d(0) = u]$, with $\delta_d(u) = 1 - \psi_d(u)$ and $\delta_d^*(u) = 1 - \psi_d^*(u)$, denoting the corresponding probabilities of ultimate survival. From Dickson and Waters (1991) we have $\psi_d^*(0) = f_0\psi_d(0) = \psi(0)$. We denote the probability functions of T_d and T_d^* as $\phi_d(u,t)$ and $\phi_d^*(u,t)$, respectively.

We also need to define the (defective) distributions of the severity of ruin, probability density and probability functions, for the basic process and discrete model, respectively. Accordingly, we have for $u \geq 0$ and $y > 0$

$$G(u,y) = \Pr[T < \infty \text{ and } U(T) > -y | U(0) = u]$$

and, for $u = 0,1,2,...$ and $y = 1,2,3,...$

$$G_d(u,y) = \Pr[T_d < \infty \text{ and } U(T_d) \geq -y | U_d(0) = u]$$

$$G_d^*(u,y) = \Pr[T_d^* < \infty \text{ and } U(T_d^*) > -y | U_d(0) = u]$$

We denote the density and probability functions, respectively by $g(u,y)$, $g_d(u,y)$, and $g_d^*(u,y)$. Respective associated random variables are denoted as Y, Y_d and Y_d^*.

We will let the surplus process continue if it falls below zero, i.e. if ruin occurs. Given that ruins occurs, the process is certain to recover to positive levels. Let's define this time as the recovery time or the duration of a negative surplus. Eventually, the process will drift to infinity and the number of occasions it falls below zero can be multiple. For more details see Dickson and Egídio dos Reis (1996).

Let's denote by $\tilde{T}$ and $\tilde{T}_d$ the duration of the first period of negative surplus once ruin has occurred, in the classical and discrete time models, respectively. These random variables depend on the initial surplus u. For the discrete time model, $\tilde{T}_d$ stands for the recovery time up to non-ruin level zero, according to the first definition of ruin. Let $\alpha_d(u,t)$ be the probability function of $\tilde{T}_d$. We consider this function to be defective, i.e. $\alpha_d(u,t)$ represents the probability that ruin occurs from initial surplus u and the surplus takes t periods to reach the level 0 for the first time after T_d. Hence, we have that $t = 1, 2, \ldots$. We denote by $\tilde{T}_d^*$ the recovery time associated to the second definition of ruin, and $\alpha_d^*(u,t)$ is its probability function. In Section 6 we use some conditional random variables, given that $T < \infty$. When this is the case we write the variables with a subscript c.

According to Dickson *et al.* (1995, Section 2), $\psi_d(u\beta, \beta t)$ for some positive β, is an approximation for $\psi(u,t)$. Furthermore, they explain that $\psi_d^*(u\beta, \beta t)$ is a better approximation than $\psi_d(u\beta, \beta t)$. As far as approximations to the basic process is concerned, we will consider the second definition of ruin in the discrete model for the different quantities we want to compute in this paper. We will compute approximate values for the conditional moments of time to ruin and time to recovery, given that $T < \infty$, from initial surplus u, denoted by $E[T^k|u]/\psi(u)$ and $E[\tilde{T}^k|u]/\psi(u)$, for $k = 1, 2, \ldots$. Respective approximations we consider will be $\beta^{-k}E[{T_d^*}^k|\beta u]/\psi_d^*(\beta u)$ and $\beta^{-k}E[{\tilde{T}_d^*}{}^k|\beta u]/\psi_d^*(\beta u)$.

3. On the time to ruin

For the time to ruin we can retrieve de Vylder and Goovaerts' (1988) formulae. Considering aggregate claims at the end of the first period in the discrete time model we have

$$\psi_d(u,t) = \sum_{j=0}^{u+1} f_j \psi_d(u+1-j, t-1) + (1 - F_{u+1}) \text{ for } t > 1 \tag{3.1}$$

with $\psi_d(u,1) = \phi_d(u,1) = 1 - F_{u+1}$. As far as the probability function is concerned we get its respective version being, for $t > 1$

$$\phi_d(u,t) = \sum_{j=0}^{u+1} f_j \phi_d(u+1-j, t-1) \tag{3.2}$$

We assume that the r-th moment of T_d, denoted as $E[T_d^r|u]$, exists for a given initial surplus $u \geq 0$ ($r = 1, 2, \ldots$). As far as the classical model is concerned, Delbaen [1990] shows that the r-th moment of the time to ruin depends on the existence of p_{r+1}. We can use formula (3.2) to find a recursion for $E[T_d^r|u]$. The computation of $E[T_d^r|u]$ can then be used for the approximation of the corresponding quantity in the classical model. Unfortunately, we cannot make many comparisons of approximate values with exact ones since, with one exception, we don't have explicit formulae for the classical model for positive values of the initial surplus. We can use Gerber's (1979) exact formulae when $P(x)$ is exponential. For $u=0$ we can use formulae derived by Egídio dos Reis (1993), since in the classical model the conditional distribution of time to recovery and time to ruin, given ruin occurs, have the same distribution. We will consider the calculation of the expected value and variance only.

Using (3.2) we have, for a given value of $u \geq 0$,

$$\begin{aligned} E[T_d^r|u] &= \sum_{i=1}^{\infty} i^r \phi_d(u,i) = \phi_d(u,1) + \sum_{i=2}^{\infty} i^r \sum_{j=0}^{u+1} f_j \phi_d(u+1-j,i-1) \\ &= \phi_d(u,1) + \sum_{j=0}^{u+1} f_j \sum_{i=2}^{\infty} i^r \phi_d(u+1-j,i-1) \\ &= \phi_d(u,1) + \sum_{j=0}^{u+1} f_j \sum_{i=1}^{\infty} (i+1)^r \phi_d(u+1-j,i). \end{aligned}$$

Substituting $(i+1)^r = \sum_{k=0}^{r} \binom{r}{k} i^k$ and interchanging summations we get

$$\begin{aligned} E[T_d^r|u] &= \phi_d(u,1) + \sum_{j=0}^{u+1} f_j \sum_{k=0}^{r} \binom{r}{k} \sum_{i=1}^{\infty} i^k \phi_d(u+1-j,i) \\ &= \phi_d(u,1) + \sum_{j=0}^{u+1} f_j \sum_{i=1}^{\infty} \phi_d(u+1-j,i) \\ &\quad + \sum_{j=0}^{u} f_j \sum_{k=1}^{r} \binom{r}{k} \sum_{i=1}^{\infty} i^k \phi_d(u+1-j,i) \\ &= \phi_d(u,1) + \sum_{j=0}^{u+1} f_j \psi_d(u+1-j) + \sum_{j=0}^{u+1} f_j \sum_{k=1}^{r} \binom{r}{k} E[T_d^k|u+1-j]. \end{aligned}$$

since T_d is a defective random variable, i.e. $\sum_{i=1}^{\infty} \phi_d(u,i) = \psi_d(u)$. Or,

$$E[T_d^r|u] = 1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_d(u+1-j) + \sum_{k=1}^{r} \binom{r}{k} \sum_{j=0}^{u+1} f_j E[T_d^k|u+1-j]. \tag{3.3}$$

For instance, if we want to compute the mean of T_d we get, solving for $E[T_d|u+1]$ from (3.3) with $r=1$, and $u = 0,1,2,\dots$, .

$$\begin{aligned} E[T_d|u+1] &= f_0^{-1} \Big(E[T_d|u] - \Big(1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_d(u+1-j) \\ &\quad + \sum_{j=1}^{u+1} f_j E[T_d|u+1-j] \Big) \Big) . \end{aligned} \tag{3.4}$$

If we want to compute the variance of the (defective) random variable T_d we will need to compute $E[T_d^2|u]$. From (3.3) we have

$$\begin{aligned} E[T_d^2|u] &= 1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_d(u+1-j) \\ &\quad + 2\sum_{j=0}^{u+1} f_j E[T_d|u+1-j] + \sum_{j=0}^{u+1} f_j E[T_d^2|u+1-j] \end{aligned} , \tag{3.5}$$

from which we get, solving for $E[T_d^2|u+1]$, for $u = 0,1,2,\dots$,

$$\begin{aligned} E[T_d^2|u+1] &= f_0^{-1} \Big(E[T_d^2|u] - \Big(1 - F_{u+1} + \sum_{j=0}^{u+1} f_j \psi_d(u+1-j) \\ &\quad + 2\sum_{j=0}^{u+1} f_j E[T_d|u+1-j] + \sum_{j=1}^{u+1} f_j E[T_d^2|u+1-j] \Big) \Big) . \end{aligned} \tag{3.6}$$

Taking a look at the recursion for $E[T_d|u+1]$ in (3.4) we see that we need all the ultimate ruin probabilities $\psi_d(j)$ for $j = 0,1,\dots,u+1$, as well as all the mean values $E[T_d|j]$ for $j = 0,1,\dots,u$. The recursion for the second moment, (3.6), will also need the same ruin probabilities as well as all the mean values up to $E[T_d|u+1]$ and all the previous second moment values, from 0 up to u. For the computation of the ruin probabilities we can use the recursions by Dickson *et al.* [1995, formulae (3.3) and 3.8)].

We need to find a formula for the starting value, when $u = 0$. We will come to this later. Let's first deal with the computation of the moments for the duration of the first period of negative surplus, or time to recovery to positive values after ruin has occurred. The reason for this approach is that we will pay attention to the relationship between T_d and $\tilde{T}_d$ when $u = 0$.

To compute the moments of T_d^*, we note that Dickson and Waters(1991) explain that for $u = 1, 2, ...$, $\phi_d^*(u,t) = \phi_d(u-1,t)$, hence $E[T_d^{*\,r}\,|u] = E[T_d^{\,r}\,|u-1]$.

4. On the time to recovery

We consider the discrete time model. We will let the surplus process continue if it falls below zero, i.e. if ruin occurs. After ruin has occurred, the process is certain to recover to positive levels at some point. Eventually, the process will drift to infinity and the number of occasions on which it falls below zero can be multiple. For more details see Dickson and Egídio dos Reis (1996) who consider this discrete model. We note that the recovery time $\tilde{T}_d$ stands for the time that the surplus after having fallen below 0 recovers gets to level 0 for the first time. Its probability function has been defined as $\alpha(u,t)$, for $t = 1, 2, \ldots$.

Let's define $\tilde{T}_d(x)$ as the time that the surplus process $U_d(t)$ starting from 0 takes to reach a fixed positive level x ($x = 1, 2, ...$) for the first time. Its probability function is given by

$$\frac{x}{t} f_d(t-x,t) \tag{4.1}$$

with $t \geq x$ [see Gerber (1979, p. 21)]. That is, we consider

$$U_d(t) = t - S_d(t),$$

and

$$\tilde{T}_d(x) = \min\{t : U_d(t) = x | U_d(0) = 0\} \text{ for } x = 1, 2, 3, ...$$

where $S_d(t)$ is defined as in Section 2. $S_d(t)$ is the aggregate claim amount up to time t, so its distribution is a discrete compound Poisson with Poisson parameter λ_d. We further assume that the moment generating function of individual claims in this discrete model exists and we denote it by $m_d(r)$. Hence, the moment generating function of $S_d(t)$ is

$$M_{S_d(t)}(r) = [M_{X_d}(r)]^t = \exp\{\lambda_d t(m_d(r) - 1)\} \ ,$$

since $M_{X_d}(r) = \exp\{\lambda_d(m_d(r) - 1)\}$. We can find the moment generating function of $\tilde{T}_d(x)$ by means of the martingale method used by Gerber (1990) for the corresponding compound Poisson continuous time model. In this case we have a discrete time martingale

$$\{\exp\{-f(s)U_d(t) + s\,t\}\}$$

where $f(s)$ is some function of $s \leq 0$ such that

$$s = f(s) - \lambda_d\,[m_d(f(s)) - 1]\,.$$

Hence, we get that

$$E[e^{s\tilde{T}(x)}] = e^{f(s)x}.$$

The expression above leads to, for instance using the cumulant generator function [see Gerber (1990)],

$$\begin{aligned} E[\tilde{T}(x)] &= x/(1-\lambda_d\beta) \\ V[\tilde{T}(x)] &= x\lambda_d b_2/(1-\lambda_d\beta)^3 = xV[X_i]/(1-\lambda_d\beta)^3 \,. \end{aligned} \tag{4.2}$$

We see that, like in the continuous model, the mean of $\tilde{T}(x)$ equals x divided by the expected profit per unit time (c is equal to 1). Getting back to our main problem with the random variable $\tilde{T}_d$, we have that if the deficit at the time of ruin is j, $j = 1, 2, ...$, then the probability that the surplus returns to 0 at time $T_d + t$ ($t \geq j$) is given by (4.1). Hence,

$$\alpha_d(u,t) = \sum_{j=1}^{t} g_d(u,j)\frac{j}{t} f_d(t-j,t) \text{ for } t = 1,2,3,... \tag{4.3}$$

If we want to compute the r-th moment of $\tilde{T}_d$, for a given u, we have that

$$\begin{aligned} E[\tilde{T_d}^r|u] &= \textstyle\sum_{t=1}^{\infty} t^r \alpha_d(u,t) = \sum_{t=1}^{\infty} t^r \sum_{j=1}^{t} g_d(u,j)\frac{j}{t} f_d(t-j,t) \\ &= \textstyle\sum_{j=1}^{\infty} g_d(u,j) \sum_{t=j}^{\infty} j\, t^{r-1} f_d(t-j,t) \end{aligned}$$

after interchanging the order of the summations. In another way,

$$E[\tilde{T_d}^r|u] = \sum_{j=1}^{\infty} g_d(u,j)E[\tilde{T}(j)^r] = E[E[\tilde{T}(Y_d)^r|Y_d]|u],$$

where Y_d denotes the severity of ruin (defective) with probability function $g_d(u,y)$. From here we get, for instance,

$$\begin{aligned} E[\tilde{T}_d|u] &= E[Y_d|u]/(1-\lambda_d\beta) \\ E[\tilde{T_d}^2|u] &= E[Y_d|u]\lambda_d b_2/(1-\lambda_d\beta)^3 + E[{Y_d}^2|u]/(1-\lambda_d\beta)^2 \end{aligned}$$

We can use the recursions in Dickson *et al.* (1995, Section 4) for the moments of Y_d. Like its continuous analogue, once we get the moments of the severity of ruin we will be able to compute the moments of the duration of a negative surplus.

We will approximate the first two moments of $\tilde{T}$ by using the moments

$$\begin{aligned} E[\tilde{T}_d^*|u] &= E[Y_d^*|u]/(1-\lambda_d\beta) \\ E[{\tilde{T}_d^*}{}^2|u] &= E[Y_d^*|u]\lambda_d b_2/(1-\lambda_d\beta)^3 + E[{Y_d^*}^2|u]/(1-\lambda_d\beta)^2 \end{aligned}$$

calculating, for $u = 1, 2, ...$,

$$\begin{aligned} E[Y_d^*|u] &= E[Y_d|u-1] - \psi_d(u-1) \\ E[{Y_d^*}^2|u] &= E[Y_d^2|u-1] - 2E[Y_d|u-1] + \psi_d(u-1) \end{aligned} .$$

5. Formulae for $u = 0$

For $u = 0$ we have from (4.3) that

$$\alpha_d(0,t) = \sum_{j=1}^{t} g_d(0,j)\frac{j}{t} f_d(t-j,t) \text{ for } t = 1,2,3,\ldots .$$

and Dickson and Egídio dos Reis (1996) have defined, using the second definition of ruin in Section 2 that for $t = 1, 2, \ldots$

$$\alpha_d^*(0,t) = \sum_{j=1}^{t} g_d^*(0,j)\frac{j}{t} f_d(t-j,t) \ ,$$

where $g_d^*(0,y) = 1 - F_y = f_0 g_d(0,y)$ [see Dickson *et al.* (1995)], giving for $t = 1, 2, \ldots$ $\alpha_d^*(0,t) = f_0\alpha_d(0,t)$. Also, Dickson and Egídio dos Reis (1996) have showed that $\phi_d^*(0,t+1) = \alpha_d^*(0,t)$. It's easy to show that $\phi_d^*(u,t+1) = f_0\phi_d(0,t)$, hence $\alpha_d(0,t) = \phi_d(0,t)$. This is the discrete counterpart of the relationship between time to ruin and the duration of a negative surplus in the classical model explained by Egídio dos Reis (1993).

Hence, we can establish the starting values for recursions in Section 3, having for the first two moments

$$\begin{aligned} E[T_d|0] &= E[Y_d|0]/(1-\lambda_d\beta) \\ E[{T_d}^2\,|0] &= E[Y_d|0]\lambda_d b_2/(1-\lambda_d\beta)^3 + E[{Y_d}^2\,|0]/(1-\lambda_d\beta)^2 \end{aligned}$$

and, from Dickson *et al.* (1995),

$$\begin{aligned} E[Y_d^*|0] &= \tfrac{1}{2}\left(E[S_d(1)^2] - E[S_d(1)]\right) \\ E[{Y_d^*}^2\,|0] &= \tfrac{1}{3}E[S_d(1)^3] - \tfrac{1}{2}E[S_d(1)^2] + \tfrac{1}{6}E[S_d(1)] \end{aligned}$$

with $E[{Y_d}^k\,|0] = {f_0}^{-1}\,E[{Y_d^*}^k\,|0]$ for $k = 1, 2, \ldots$.

6. Some further comments on the classical model

In this section we will assume that the moment generating function of $P(x)$ exists. As in the discrete model, the moments of the duration of a negative surplus rely on the existence of the corresponding moments of the severity of ruin, in the classical model. In many cases we cannot find explicit formulae for these moments when $u > 0$. For $u = 0$ it is easy to show that $E[Y^k|u = 0] = \lambda p_{k+1}/c(k+1)$, $k = 1, 2, \ldots$, whenever p_{k+1} exists. If $m(t)$ exists we can also express the moment generating function of Y as

$$M_Y(0,t) = \frac{\lambda}{ct}\,[m(t) - 1]$$

[see for instance Egídio dos Reis (1993)]. We can also easily find a closed asymptotic formula for the moment generating function of Y as $u \to \infty$.

¿From Gerber (1974) we have that the conditional density of the severity of ruin, given that ruin occurs, denoted as $\tilde{g}(\infty, y)$, is given by

$$\tilde{g}(\infty,y) = \frac{\lambda R}{c\delta(0)}\int_0^\infty e^{Rx}\,[1 - P(x+y)]\,dx \qquad (6.1)$$

where R is the adjustment coefficient satisfying the following equation

$$\frac{\lambda}{c}\int_0^{\infty} e^{Rx}\left[1-P(x)\right]dx = 1 \tag{6.2}$$

Equation (6.1) can be expressed as

$$\tilde{g}(\infty,y) = \frac{\lambda R}{c\delta(0)}e^{-Ry}\int_y^{\infty} e^{Rx}\left[1-P(x)\right]dx \tag{6.3}$$

Hence, the moment generating function corresponding to the above density, denoted as $M_{Y_c}(\infty,t)$, becomes

$$M_{Y_c}(\infty,t) = \frac{\lambda R}{c\delta(0)}\int_0^{\infty} e^{(t-R)y}\int_y^{\infty} e^{Rx}\left[1-P(x)\right]dx$$

giving for $t \neq R$

$$M_{Y_c}(\infty,t) = \frac{\lambda R}{c\delta(0)(t-R)}\left(\int_0^{\infty} e^{tx}\left[1-P(x)\right]dx - 1\right)$$

interchanging the order of the integration and introducing (6.2). Expressing in another way we have

$$M_{Y_c}(\infty,t) = \frac{R}{\delta(0)(t-R)}\left(M_Y(0,t)-1\right)\ , \tag{6.4}$$

since $g(0,x) = \lambda[1-P(x)]/c$. For $t = R$ we get Gerber's (1974) formula, i.e

$$M_{Y_c}(\infty,R) = \delta(0)^{-1}\left(\frac{\lambda}{c}m'(R)-1\right),$$

where $m'(R)$ denotes the derivative of $m(t)$ evaluated at R.

¿From expression (6.4) we get

$$\begin{aligned} E[Y_c|\infty] &= 1/R - E[Y|u=0]/\delta(0) \\ E[Y_c^2|\infty] &= 2/R^2 - 2E[Y|u=0]/\delta(0) - E[Y^2|u=0]/\delta(0) \end{aligned}$$

If we take the expression for $E[Y_c|\infty]$, we obtain Gerber's (1979, p. 128) upper bound for R, since $E[Y_c|\infty] > 0$. If we express (6.3) as

$$\tilde{g}(\infty,y) = \frac{R}{\delta(0)}e^{-Ry}\left(1-\int_0^{y}\frac{\lambda}{c}e^{Rx}\left[1-P(x)\right]dx\right)$$

we see that this density can be viewed as a combination of an exponential (with mean $1/R$) and some other density, with weights $1/\delta(0)$ and $-\psi(0)/\delta(0)$, respectively.

Hence, we can find the asymptotic moment generating function of the duration of a negative surplus, conditional on $T < \infty$,

$$M_{\tilde{T}_c}(\infty,s) = M_{Y_c}(\infty,f(s))$$

where $f(s)$ is some function of s such that $s = f(s)c - \lambda[m(f(s))-1]$ and $s, f(s) \leq 0$ [see Egídio dos Reis (1993)]. Using $M_Y(0,t)$ above we can express $M_{\tilde{T}_c}(\infty,s)$ as

$$M_{\tilde{T}_c}(\infty,s) = \frac{Rs}{c\delta(0)f(s)(R-f(s))}.$$

7. Examples

In this section we present an example for which we considered the calculation of the first two moments of both ruin and the recovery times.

As far as time to ruin is considered we need to compute the first two moments of the severity of ruin when $u = 0$. For the time to recovery we need the corresponding moments of the severity of ruin for the corresponding initial surplus. For these situations, as shown by Dickson *et al.*al. distribution.. Panjer and Lutek (1993) describe a method which provides the discretization of the individual claim amount distribution that preserves the original moments, which we have adopted. Dickson *et al.* (1995) refer to some problems with this discretization method. We have used the software Mathematica for the calculations in the discretization procedure.

We considered exponentially distributed claim amounts, i.e $P(x) = 1 - e^{-\alpha x}$, and we have set $\alpha = 1$, so that it has mean one. Also, we have set $c = 1$, so that $\lambda = 1/(1+\theta)$. We can find the required moments for both the time to ruin and the time to recovery. Considering the time to ruin, we have an explicit expression for the moment generating function of T_c, which is given by Gerber (1979):

$$E[e^{sT}|T < \infty] = \frac{c}{\lambda}(\alpha - f(s))e^{-(f(s)-R)u}$$

where $f(s)$ is some function of s such that

$$s = f(s)c - \lambda[m(f(s)) - 1]$$

for $R \leq f(s) < \alpha$ and $R = \alpha - \lambda/c$ is the adjustment coefficient. We note that $f(s)$ is uniquely defined for $s \leq 0$ from $R \leq f(s) < \alpha$, and that $f(0) = R$ since $\lambda + cR = \lambda m(R)$. A graph of the above function is shown by Egídio dos Reis (1993). If we take the cumulant generating function it is easy to show that

$$E[T_c|u] = \frac{1 + \lambda u/c}{\alpha c - \lambda} = E[T_c|0]\left(1 + \frac{\lambda}{c}u\right)$$

and

$$V[T_c|u] = \frac{\alpha c + \lambda + 2\alpha\lambda u}{(\alpha c - \lambda)^3} = V[T_c|0] + \frac{2\alpha\lambda}{(\alpha c - \lambda)^3}u$$

and we can compare the approximations for these moments given by using the appropriate discrete model described earlier. In all the computations below we used a $\beta = 100$.

In Table 7.1 we show values for $E[T_c|u]$ and $E[{T_c}^2|u]$ together with the respective approximating values $\beta^{-1}E[T_d^*|\beta u]/\psi_d^*(\beta u)$ and $\beta^{-2}E[{T_d^*}^2|\beta u]/\psi_d^*(\beta u)$ for different values of initial surplus u. We don't show values for the same quantities of $\tilde{T}_c$ as it is obvious that they depend solely on the approximations for the respective moments of the severity of ruin and Dickson *et al.* (1995) already show a similar example for the severity of ruin random variable. The key for the table is the following: (1) and (4) show the true values of $E[T_c|u]$ and $E[{T_c}^2|u]$, (2) and (5) show the approximations for these quantities, respectively; columns 3 and 6 of the table show the ratios (2)/(1) and (5)/(4), respectively.

u	(1)	(2)	(2)/(1)	(4)	(5)	(5)/(4)
0	11	10.99500	0.99955	2662	2661.88985	0.99996
1	21	21.00500	1.00024	5402	5402.20968	1.00004
2	31	31.00499	1.00016	8342	8342.30952	1.00004
3	41	41.00499	1.00012	11482	11482.40935	1.00004
4	51	51.00499	1.00010	14822	14822.50918	1.00003
5	61	61.00499	1.00008	18362	18362.60902	1.00003
6	71	71.00499	1.00007	22102	22102.70885	1.00003
7	81	81.00499	1.00006	26042	26042.80868	1.00003
8	91	91.00499	1.00005	30182	30182.90852	1.00003
9	101	101.00499	1.00005	34522	34523.00835	1.00003
10	111	111.00499	1.00004	39062	39063.10818	1.00003
15	161	161.00499	1.00003	64762	64763.60735	1.00002
20	211	211.00499	1.00002	95462	95464.10651	1.00002
30	311	311.00499	1.00002	171862	171865.10481	1.00002
40	411	411.00499	1.00001	268262	268266.10298	1.00002
50	511	511.00499	1.00001	384662	384667.10113	1.00001
100	1011	1011.00633	1.00001	1266662	1266673.63702	1.00001

Table 7.1: 1st and 2nd moments of T_c for expoential claim size

For other claim amount distributions we don't have explicit formulae for the moments considered above, hence we cannot compare approximations (except for $u = 0$) and we don't show any more figures. However, we did compute figures for other Gamma claim size distributions. The only remark we make about this is that, unlike the exponential case, neither the expected value nor the variance of T_c seem to be linear functions of the initial surplus. The same actually happens for the time to recovery as we know from Egídio dos Reis (1993).

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