# Comparison of methods for evaluation of the convolution of two compound $\mathcal{R}_1$ distributions

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**Abstract**

In the present paper we compare four methods for evaluating the convolution of two compound $\mathcal{R}_1$ distributions by counting the numbers of elementary algebraic operations required. Two of the methods are applicable in general whereas the remaining two are restricted to the case when the two compound distributions have the same severity distribution. This case is discussed separately. We consider in particular the special case when this common severity distribution is concentrated in one, that is, evaluation of the convolution of two $\mathcal{R}_1$ distributions.

## 1 Introduction

1A. With the notation of Sundt (1992) the class $\mathcal{R}_1$ consists of all probability functions on the non-negative integers with a positive mass at zero, for which there exist constants $\alpha$ and $\beta$ such that

$$p(n) = \left(\alpha + \frac{\beta}{n}\right) p(n-1). \qquad (n = 1, 2, \ldots) \tag{1.1}$$

Sundt & Jewell (1981) showed that this class consists of the binomial ($\alpha < 0$), Poisson ($\alpha = 0$), and negative binomial ($\alpha > 0$) distributions. We denote the probability function given by (1.1) by $R_1[\alpha, \beta]$.

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1B. Let $h$ be a probability function on the positive integers. Panjer (1981) showed that the compound probability function

$$f = p \vee h = \sum_{n=0}^{\infty} p(n)\, h^{n*}$$

can be evaluated by the recursion

$$f(y) = \sum_{z=1}^{y} \left( \alpha + \beta \frac{z}{y} \right) h(z)\, f(y-z) \qquad (y = 1, 2, \ldots) \tag{1.2}$$

$$f(0) = p(0). \tag{1.3}$$

1C. For $i = 1, 2$ let $p_i$ be the probability function $R_1[\alpha_i, \beta_i]$, let $h_i$ be a probability function on the positive integers, and let $f_i = p_i \vee h_i$. We also introduce the convolution $g = f_1 * f_2$.

In the present paper we shall compare four methods for evaluation of $g(y)$ for $y = 0, 1, \ldots, x$. Methods A and B are applicable in general whereas Methods C and D are restricted to the case when $h_1 = h_2$.

Our measure of efficiency is the number of elementary algebraic operations required. Like in Kuon, Reich, & Reimers (1987) and Sundt & Dickson (1998) we distinguish between bar operations (addition and subtraction) and dot operations (multiplication and division). These measures can only give a rough idea of which method is most efficient. In Sundt & Dickson (1998) we discuss reasons for not drawing too strong conclusions from the numbers of algebraic operations.

1D. In Method A we first evaluate $f_1$ and $f_2$ by (1.2) with the starting value (1.3). Then we evaluate $g$ by

$$g(y) = \sum_{z=0}^{y} f_1(z)\, f_2(y-z). \qquad (y = 0, 1, 2, \ldots) \tag{1.4}$$

In Method B we first evaluate recursively the De Pril transforms of $f_1$ and $f_2$. Then we find the De Pril transform of $g$ as the sum of these two De Pril transforms, and from the De Pril transform of $g$ we can evaluate $g$ recursively.

In Sections 2 and 3 we consider the number of algebraic operations required for Methods A and B respectively, and in Section 4 we compare the two methods.

We do not make any simplifying assumptions for the severity distributions. In particular we do not consider simplifications obtained when $h_i(y) = 0$ for some values of $y$.

On the other hand, for each of the counting distributions we distinguish between the cases $(\alpha_i \neq 0;\ \beta_i \neq 0)$, $(\alpha_i > 0;\ \beta_i = 0)$, and $(\alpha_i = 0;\ \beta_i > 0)$ (when one of these two parameters is equal to zero, the other one must be positive as $\alpha_i + \beta_i \geq 0$ with equality only in the degenerate case with the distribution concentrated at zero, cf. Sundt & Jewell (1981)), and when considering the convolution we look at combinations of these cases for the two counting distributions, except the case $\alpha_1 = \alpha_2 = 0$. In that case $p_1$ and $p_2$ are Poisson probability functions with parameters $\beta_1$ and $\beta_2$ respectively, and e.g. from Theorem 10.1 in Sundt (1993) we obtain that $g = p \vee h$, where $p$ is the Poisson probability function $R_1[0, \beta_1 + \beta_2]$ and

$$h = \frac{\beta_1 h_1 + \beta_2 h_2}{\beta_1 + \beta_2}.$$

Thus, we can easily evaluate $g$ by (1.2) and (1.3), and it is clear that it is inefficient to apply Method A. On the other hand, applying (1.2) in this case is very close to Method B. We shall show that also in the remaining cases Method B is more efficient than Method A.

We do not consider possible simplifications that could be obtained when one of the parameters is equal to one.

1E. The case when $h_1 = h_2$, does not seem to imply any significant simplifications in Methods A and B. However, in this case we can also express $g$ as $q \vee h$ with $q = p_1 * p_2$. In Section 5 we compare two methods based on this compound representation of $g$ with Methods A and B. Method C is based on a recursion for $g$ whereas in Method D we apply a recursion for the De Pril transform of $g$. In our comparisons we do not count the numbers of algebraic operations for all the cases of the parameters like we did for Methods A and B in the general case, but concentrate on the case $(\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 = 0;\ \beta_2 > 0)$. In this case we find Method D less attractive than Methods B and C. However, our conclusion of the comparison between Methods B and C is less clear.

1F. In Section 6 we consider the degenerate case when $h_1$ and $h_2$ are concentrated in one, that is, evaluation of the convolution of two probability functions in $\mathcal{R}_1$. In this case it turns out that Method C is more efficient than the three other methods except possibly when $x$ is small.

1G. Finally, in Section 7 we give some concluding remarks.

1H. In general we apply the notation $b(x)$ and $d(x)$ with appropriate subscript to denote the number of respectively bar and dot operations required when applying some procedure for all $y \leq x$. In particular, for $i =$ A,B,C,D we apply subscript $i$ to the number of bar and dot operations respectively required to evaluate $g(y)$ for $y = 0, 1, 2, \dots, x$ by Method $i$. For comparison between the methods we introduce

$$b_{ij} = b_i - b_j; \qquad d_{ij} = d_i - d_j. \qquad (i, j = \text{A,B,C,D})$$

1I. We make the convention that $\sum_{i=u}^{v} = 0$ when $v < u$.

# 2 Method A

2A. Let us first consider evaluation of $f(y)$ by (1.2) for $y = 0, 1, \dots, x$. We assume that $p(0)$ is known. We denote by $b_f(x)$ and $d_f(x)$ the number of bar operations and dot operations respectively required to evaluate $f(y)$ for $y = 1, 2, \dots, x$.

2B. In the case ($\alpha \neq 0$; $\beta \neq 0$) we rewrite (1.2) as

$$f(y) = \frac{1}{s(y)} \sum_{z=1}^{y} k(z, y) f(y - z) \qquad (y = 1, 2, \dots) \tag{2.1}$$

with

$$k(z, y) = (\kappa z + y - z) h(z)$$
$$s(y) = \sigma y$$
$$\kappa = \frac{\beta}{\alpha} + 1; \qquad \sigma = \frac{1}{\alpha}.$$

We can evaluate $k$ recursively by

$$k(z, y) = k(z, y - 1) + h(z) \qquad (z = 1, 2, \dots, y - 1) \tag{2.2}$$

$$k(y, y) = t(y) h(y) \tag{2.3}$$

with

$$t(y) = \kappa y,$$

$t$ by

$$t(y) = t(y - 1) + \kappa \tag{2.4}$$
$$t(1) = \kappa,$$

and $s$ analogously.

To evaluate the constants $\kappa$ and $\sigma$ we need one bar operation and two dot operations.

Let us first consider the case $y = 1$. To evaluate $k(1, 1)$ by (2.3) we need one dot operation, and to evaluate $f(1)$ by (2.1) we need two dot operations, that is, totally we need three dot operations.

We now turn to the case $y > 1$. To update $t(y)$ by (2.4) we need one bar operation, and the same is required for $s(y)$. We need one dot operation to evaluate $k(y, y)$ by (2.3), and to evaluate $k(z, y)$ by (2.2) for $z = 1, 2, \ldots, y-1$ we need $y - 1$ bar operations. Finally we need $y - 1$ bar operations and $y + 1$ dot operations to evaluate $f(y)$ by (2.1). Thus, totally we need $2y$ bar operations and $y + 2$ dot operations.

From the above results we obtain

$$b_f(x) = 1 + \sum_{y=2}^{x} 2y = x^2 + x - 1$$

$$d_f(x) = 2 + 3 + \sum_{y=2}^{x} (y + 2) = \frac{x^2 + 5x + 4}{2}.$$

2C. In the case $(\alpha > 0;\ \beta = 0)$ (1.2) can be written as

$$f(y) = \alpha \sum_{z=1}^{y} h(z) f(y - z). \qquad (y = 1, 2, \ldots) \tag{2.5}$$

To evaluate $f(y)$ by (2.5) we need $y - 1$ bar operations and $y + 1$ dot operations. Thus,

$$b_f(x) = \sum_{y=1}^{x} (y - 1) = \frac{x(x-1)}{2}; \qquad d_f(x) = \sum_{y=1}^{x} (y + 1) = \frac{x(x+3)}{2}.$$

2D. In the case $(\alpha = 0;\ \beta > 0)$ we write (1.2) as

$$f(y) = \frac{1}{s(y)} \sum_{z=1}^{y} k(z) f(y - z) \qquad (y = 1, 2, \ldots) \tag{2.6}$$

with

$$k(y) = yh(y) \tag{2.7}$$

$$s(y) = \frac{y}{\beta}.$$

We can evaluate $s$ recursively by

$$s(y) = s(y - 1) + \sigma \tag{2.8}$$

$$s(1) = \sigma = \frac{1}{\beta}.$$

Let us first consider the case $y = 1$. To find $s(1) = \sigma$ we need one dot operation, and to find $f(1)$ by (2.6) we need two dot operations, that is, totally we need three dot operations.

| Case | $b_f(x)$ | $d_f(x)$ |
|---|---|---|
| $\alpha \neq 0;\ \beta \neq 0$ | $x^2 + x - 1$ | $\dfrac{x^2 + 5x + 4}{2}$ |
| $\alpha > 0;\ \beta = 0$ | $\dfrac{x(x-1)}{2}$ | $\dfrac{x(x+3)}{2}$ |
| $\alpha = 0;\ \beta > 0$ | $\dfrac{x^2 + x - 2}{2}$ | $\dfrac{x(x+5)}{2}$ |

*Table 2.1: Numbers of operations required for evaluation of $f$.*

We now turn to the case $y > 1$. To evaluate $s(y)$ by (2.8) we need one bar operation, and evaluation of $k(y)$ by (2.7) requires one dot operation. Finally, to evaluate $f(y)$ by (2.6) we need $y - 1$ bar operations and $y + 1$ dot operations. Thus, we need totally $y$ bar operations and $y + 2$ dot operations.

From the above results we obtain

$$b_f(x) = \sum_{y=2}^{x} y = \frac{x^2 + x - 2}{2}; \qquad d_f(x) = 3 + \sum_{y=2}^{x} (y + 2) = \frac{x(x+5)}{2}.$$

2E. The results of subsections 2B–D are summarised in Table 2.1.

2F. Like in Section 2 of Sundt & Dickson (1998) we conclude that for given $f_1(y)$ and $f_2(y)$ for $y = 0, 1, \ldots, x$, the numbers of operations required for evaluation of $g(y)$ for $y = 0, 1, \ldots, x$ by (1.4) are

$$b_*(x) = \frac{x(x+1)}{2}; \qquad d_*(x) = \frac{(x+1)(x+2)}{2} \tag{2.9}$$

2G. We now obtain

$$b_{\mathrm{A}}(x) = b_{f_1}(x) + b_{f_2}(x) + b_*(x); \quad d_{\mathrm{A}}(x) = d_{f_1}(x) + d_{f_2}(x) + d_*(x), \tag{2.10}$$

and insertion of the expressions for $b_{f_1}$, $b_{f_2}$, $d_{f_1}$, and $d_{f_2}$ from Table 2.1 and the expressions for $b_*$ and $d_*$ from (2.9) gives the expressions displayed in Table 2.2.

# 3 Method B

3A. Let $\mathcal{P}_0$ denote the class of probability functions on the non-negative integers. Sundt (1995) defined the De Pril transform $\varphi_k$ of $k \in P_0$ by

$$\varphi_k(y) = \frac{1}{k(0)} \left( yk(y) - \sum_{z=1}^{y-1} \varphi_k(z)\, k(y-z) \right). \qquad (y = 1, 2, \ldots) \tag{3.1}$$

| Case | $b_{\mathrm{A}}(x)$ | $d_{\mathrm{A}}(x)$ |
|---|---|---|
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 \neq 0;\ \beta_2 \neq 0$ | $\frac{5x^2+5x-4}{2}$ | $\frac{(3x+10)(x+1)}{2}$ |
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 > 0;\ \beta_2 = 0$ | $(x+1)(2x-1)$ | $\frac{(x+3)(3x+2)}{2}$ |
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 = 0;\ \beta_2 > 0$ | $2(x^2+x-1)$ | $\frac{3x^2+13x+6}{2}$ |
| $\alpha_1 > 0;\ \beta_1 = 0;\ \alpha_2 = 0;\ \beta_2 > 0$ | $\frac{(x+1)(3x-2)}{2}$ | $\frac{3x^2+11x+2}{2}$ |
| $\alpha_1 > 0;\ \beta_1 = 0;\ \alpha_2 > 0;\ \beta_2 = 0$ | $\frac{x(3x-1)}{2}$ | $\frac{3x^2+9x+2}{2}$ |

*Table 2.2: Numbers of operations required for Method A.*

By solving (3.1) for $k(y)$ we obtain

$$k(y) = \frac{1}{y}\sum_{z=1}^{y} \varphi_k(z)\, k(y-z). \qquad (y = 1, 2, \ldots) \tag{3.2}$$

As a probability function sums to one, the De Pril transform determines the probability function uniquely.

Sundt (1995) showed that

$$\varphi_{k_1 * k_2} = \varphi_{k_1} + \varphi_{k_2}, \tag{3.3}$$

and Sundt & Ekuma (1998) showed that

$$\varphi_f(y) = (\alpha+\beta)\, y h(y) + \alpha \sum_{z=1}^{y-1} h(z)\, \varphi_f(y-z) \qquad (y = 1, 2, \ldots) \tag{3.4}$$

in the situation of subsection 1B.

3B. Method B consists in first evaluating the De Pril transforms of $f_1$ and $f_2$ by (3.4). Then we find the De Pril transform of $g$ by (3.3), and finally we evaluate $g$ by (3.2) with initial value

$$g(0) = p_1(0)\, p_2(0). \tag{3.5}$$

3C. We first consider evaluation of $\varphi_f(y)$ for $y = 1, 2, \ldots, x$ by (3.4). Let $b_{\varphi_f}(x)$ and $d_{\varphi_f}(x)$ denote the number of bar and dot operations respectively required to evaluate $\varphi_f(y)$ for $y = 1, 2, \ldots, x$ by (3.4).

3D. In the case ($\alpha \neq 0$; $\beta \neq 0$) we rewrite (3.4) as

$$\varphi_f(y) = \sigma y h(y) + \alpha \sum_{z=1}^{y-1} h(z)\,\varphi_f(y-z) \qquad (y = 1, 2, \ldots)$$

with

$$\sigma = \alpha + \beta.$$

To find $\sigma$ we need one bar operation. To evaluate $\varphi_f(y)$ we need one dot operation for $y = 1$ and $y - 1$ bar operations and $y + 2$ dot operations for each $y = 2, 3, \ldots$. From these results we obtain

$$b_{\varphi_f}(x) = 1 + \sum_{y=2}^{x} (y-1) = 1 + \frac{x(x-1)}{2} = \frac{x^2 - x + 2}{2}$$

$$d_{\varphi_f}(x) = 1 + \sum_{y=2}^{x} (y+2) = \frac{x^2 + 5x - 4}{2}.$$

3E. In the case ($\alpha > 0$; $\beta = 0$) (3.4) becomes

$$\varphi_f(y) = \alpha \left( y h(y) + \sum_{z=1}^{y-1} h(z)\,\varphi_f(y-z) \right). \qquad (y = 1, 2, \ldots)$$

We easily see that evaluation of $\varphi_f(y)$ requires one dot operation for $y = 1$ and $y - 1$ bar operations and $y + 1$ dot operations for each $y = 2, 3, \ldots, x$. From this we obtain

$$b_{\varphi_f}(x) = \sum_{y=2}^{x} (y-1) = \frac{x(x-1)}{2}; \qquad d_{\varphi_f}(x) = 1 + \sum_{y=2}^{x} (y+1) = \frac{x^2 + 3x - 2}{2}.$$

3F. In the case ($\alpha = 0$; $\beta > 0$) (3.4) reduces to

$$\varphi_f(y) = \beta y h(y), \qquad (y = 1, 2, \ldots)$$

and we immediately see that to evaluate $\varphi_f(y)$ we need one dot operation when $y = 1$ and two when $y > 1$. This gives

$$b_{\varphi_f}(x) = 0; \qquad d_{\varphi_f}(x) = 2x - 1.$$

3G. The results of subsections 3D–F are summarised in Table 3.1.

3H. To evaluate $\varphi_g(y)$ by (3.3) for $y = 1, 2, \ldots, x$ we obviously need $x$ bar operations and no dot operations.

| Case | $b_{\varphi_f}(x)$ | $d_{\varphi_f}(x)$ |
|---|---|---|
| $\alpha \neq 0;\ \beta \neq 0$ | $\frac{x^2 - x + 2}{2}$ | $\frac{x^2 + 5x - 4}{2}$ |
| $\alpha > 0;\ \beta = 0$ | $\frac{x(x-1)}{2}$ | $\frac{x^2 + 3x - 2}{2}$ |
| $\alpha = 0;\ \beta > 0$ | $0$ | $2x - 1$ |

*Table 3.1: Numbers of operations required for evaluation of $\varphi_f$.*

3I. To evaluate $g(0)$ by (3.5) we need one dot operation, and evaluation of $g(1)$ by (3.2) requires one dot operation. For $y > 1$, we need $y - 1$ bar operations and $y + 1$ dot operations to evaluate $g(y)$ by (3.2).

From these results we obtain that the numbers of operations required to evaluate $g(y)$ for $y = 0, 1, \ldots, x$ by (3.5) and (3.2) are

$$b_{\varphi^{-1}}(x) = \sum_{y=2}^{x} (y - 1) = \frac{x(x-1)}{2} \tag{3.6}$$

$$d_{\varphi^{-1}}(x) = 1 + 1 + \sum_{y=2}^{x} (y + 1) = \frac{x(x+3)}{2}. \tag{3.7}$$

3J. We now obtain

$$b_{\mathrm{B}}(x) = b_{\varphi_{f_1}}(x) + b_{\varphi_{f_2}}(x) + x + b_{\varphi^{-1}}(x) \tag{3.8}$$

$$d_{\mathrm{B}}(x) = d_{\varphi_{f_1}}(x) + d_{\varphi_{f_2}}(x) + d_{\varphi^{-1}}(x), \tag{3.9}$$

and insertion of the expressions for $b_{\varphi_{f_1}}$, $b_{\varphi_{f_2}}$, $d_{\varphi_{f_1}}$, and $d_{\varphi_{f_2}}$ from Table 3.1 and the expressions for $b_{\varphi^{-1}}$ and $d_{\varphi^{-1}}$ from (3.6) and (3.7) gives the expressions displayed in Table 3.2.

# 4 Comparison

4A. From the expressions for $b_{\mathrm{A}}$ and $b_{\mathrm{B}}$ in Tables 2.2 and 3.2 we find the expressions for $b_{\mathrm{AB}}$ and $d_{\mathrm{AB}}$ displayed in Table 4.1. We see that in all the displayed cases Method A requires at least as many bar operations and dot operations as Method B. Furthermore, as indicated in subsection 1D, the same is the case for the remaining case ($\alpha_1 = 0;\ \beta_1 > 0; \alpha_2 = 0;\ \beta_2 > 0$). Thus, we conclude that Method B is always preferable.

| Case | $b_{\mathrm{B}}(x)$ | $d_{\mathrm{B}}(x)$ |
|---|---|---|
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 \neq 0;\ \beta_2 \neq 0$ | $\frac{3x^2 - x + 4}{2}$ | $\frac{3x^2 + 13x - 8}{2}$ |
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 > 0;\ \beta_2 = 0$ | $\frac{3x^2 - x + 2}{2}$ | $\frac{3x^2 + 11x - 6}{2}$ |
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 = 0;\ \beta_2 > 0$ | $x^2 + 1$ | $x^2 + 6x - 3$ |
| $\alpha_1 > 0;\ \beta_1 = 0;\ \alpha_2 = 0;\ \beta_2 > 0$ | $x^2$ | $x^2 + 5x - 2$ |
| $\alpha_1 > 0;\ \beta_1 = 0;\ \alpha_2 > 0;\ \beta_2 = 0$ | $\frac{x(3x - 1)}{2}$ | $\frac{3x^2 + 9x - 4}{2}$ |

*Table 3.2: Numbers of operations required for Method B.*

| Case | $b_{\mathrm{AB}}(x)$ | $d_{\mathrm{AB}}(x)$ |
|---|---|---|
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 \neq 0;\ \beta_2 \neq 0$ | $(x + 4)(x - 1)$ | 9 |
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 > 0;\ \beta_2 = 0$ | $\frac{(x + 4)(x - 1)}{2}$ | 6 |
| $\alpha_1 \neq 0;\ \beta_1 \neq 0;\ \alpha_2 = 0;\ \beta_2 > 0$ | $(x + 3)(x - 1)$ | $\frac{x^2 + x + 12}{2}$ |
| $\alpha_1 > 0;\ \beta_1 = 0;\ \alpha_2 = 0;\ \beta_2 > 0$ | $\frac{(x + 2)(x - 1)}{2}$ | $\frac{x^2 + x + 6}{2}$ |
| $\alpha_1 > 0;\ \beta_1 = 0;\ \alpha_2 > 0;\ \beta_2 = 0$ | 0 | 3 |

*Table 4.1: Comparison between Methods A and B.*

4B. We stress that in this section we have compared Methods A and B, but we have not considered whether there could be other more efficient ways to evaluate $g$.

As an example, let us look at the case when $p_1$ and $p_2$ are Bernoulli distributions, that is,

$$p_i(1) = 1 - p_i(0) = \pi_i.$$

Then

$$f_i(y) = \begin{cases} 1 - \pi_i & (y = 0) \\ \pi_i h_i(y), & (y = 1, 2, \ldots) \end{cases} \tag{4.1}$$

and the most efficient way to evaluate $g$ seems to be by (4.1) and (1.4).

# 5 The case $h_1 = h_2$

5A. In this section we shall consider the special case when

$$h_1 = h_2 = h. \tag{5.1}$$

In this case we can express $g$ in the form $q \vee h$ with $q = p_1 * p_2$.

Sundt (1992) defined $\mathcal{R}_2$ as the class of probability functions $p$ on the non-negative integers with a positive mass at zero, for which there exist constants $\gamma_1$,$\gamma_2$, $\delta_1$, and $\delta_2$ such that

$$p(n) = \left(\gamma_1 + \frac{\delta_1}{n}\right) p(n-1) + \left(\gamma_2 + \frac{\delta_2}{n}\right) p(n-2) \qquad (n = 1, 2, \ldots) \tag{5.2}$$

with $p(-1) = 0$, and denoted this probability function by $R_2[(\gamma_1, \delta_1), (\gamma_2, \delta_2)]$. From Corollary 4 in Sundt (1992) we obtain that the convolution $q$ of $R_1[\alpha_1, \beta_1]$ and $R_1[\alpha_2, \beta_2]$ can be expressed as $R_2[(\gamma_1, \delta_1), (\gamma_2, \delta_2)]$ with

$$\gamma_1 = \alpha_1 + \alpha_2; \quad \delta_1 = \beta_1 + \beta_2; \quad \gamma_2 = -\alpha_1\alpha_2; \quad \delta_2 = -(\alpha_1\beta_2 + \alpha_2\beta_1). \tag{5.3}$$

The recursion

$$g(y) = \sum_{z=1}^{y} g(y-z) \left( \left(\gamma_1 + \delta_1 \frac{z}{y}\right) h(z) + \left(\gamma_2 + \frac{\delta_2}{2}\frac{z}{y}\right) h^{2*}(z) \right) \quad (y = 1, 2, \ldots) \tag{5.4}$$

is obtained from Corollary 5 in Sundt (1992) and the recursion

$$\varphi_g(y) = y \left( (\gamma_1 + \delta_1) h(y) + \left(\gamma_2 + \frac{\delta_2}{2}\right) h^{2*}(y) \right) + \sum_{z=1}^{y-1} \varphi_g(y-z) \left(\gamma_1 h(z) + \gamma_2 h^{2*}(z)\right) \qquad (y = 1, 2, \ldots) \tag{5.5}$$

from formula (2.7) in Sundt & Ekuma (1998).

In addition to Methods A and B, the recursions (5.4) and (5.5) suggest the following two methods for evaluation of $g$ in the special case (5.1):

- Method C. Evaluate $g$ by (5.4) and (3.5).
- Method D. Evaluate first $\varphi_g$ by (5.5) and then $g$ by (3.2) and (3.5).

It does not seem that Methods A and B can be simplified significantly under (5.1). We therefore still conclude that Method B is always preferable to Method A. In the following we can therefore concentrate on comparing Methods B, C, and D and discard Method A.

5B. In the special case when $\alpha_1 = \alpha_2 = \alpha$, we obtain from Theorem 5 in Sundt (1992) that $q$ is $R_1[\alpha, \alpha + \beta_1 + \beta_2]$, and in this case the most efficient way to evaluate $p = q \vee h$ seems to be by the Panjer recursion (1.2).

5C. We see that for both Methods C and D we need the values of $h^{2*}(y)$ for $y = 0, 1, \ldots, x$. For this evaluation Sundt & Dickson (1998) discuss a procedure, under which the numbers of operations required are

$$b_2(x) = x\left(\frac{x}{4} + 1\right) - r(x); \qquad d_2(x) = \frac{x^2}{4} + x + 1 - r(x)$$

with

$$r(x) = \begin{cases} 0 & (x \text{ even}) \\ \frac{1}{4}. & (x \text{ odd}) \end{cases}$$

5D. To discuss the number of algebraic operations like we did for Methods A and B, could be rather messy. Whereas for Methods A and B the main analysis could be performed on compound $\mathcal{R}_1$ distributions where we had to consider special cases for two parameters, we see that for Methods C and D we would have to discuss four parameters all the time. Instead of considering all these cases, we shall therefore concentrate on the case that seems to be of most interest in practice, namely the compound Delaporte distribution, and leave the other cases to the interested reader.

The Delaporte distribution is the convolution of a negative binomial distribution and a Poisson distribution, that is, we have $\alpha_1 > 0$ and $\alpha_2 = 0$. From (5.3) we obtain

$$\gamma_1 = \alpha_1; \quad \delta_1 = \beta_1 + \beta_2; \quad \gamma_2 = 0; \quad \delta_2 = -\alpha_1\beta_2,$$

and insertion in (5.4) and (5.5) gives

$$g(y) = \sum_{z=1}^{y} g(y-z)\left(\left(\alpha_1 + (\beta_1 + \beta_2)\frac{z}{y}\right)h(z) - \frac{\alpha_1\beta_2}{2}\frac{z}{y}h^{2*}(z)\right) \quad (5.6)$$

$$(y = 1, 2, \ldots)$$

$$\varphi_g(y) = y\left((\alpha_1+\beta_1+\beta_2)\,h(y) - \frac{\alpha_1\beta_2}{2}h^{2*}(y)\right) + \alpha_1\sum_{z=1}^{y-1}\varphi_g(y-z)\,h(z)$$

$$(y = 1, 2, \ldots) \qquad (5.7)$$

For simplicity we restrict to the case $\beta_1 \neq 0$, that is, we do not consider the convolution of a Poisson distribution and a geometric distribution.

The Delaporte distribution was introduced by Delaporte (1959) for the number of claims in a motor insurance portfolio and has later been discussed by i.a. Willmot & Sundt (1989) and Schröter (1991).

5E. For application of Method C we rewrite (5.6) as

$$g(y) = \frac{1}{s(y)}\sum_{z=1}^{y} k(z,y)\,g(y-z) \qquad (y = 1, 2, \ldots)$$

with

$$k(z,y) = (\kappa_1 z + y - z)\,h(z) - \kappa_2 z h^{2*}(z)$$

$$s(y) = \sigma y$$

$$\kappa_1 = \frac{\beta_1+\beta_2}{\alpha_1} + 1; \qquad \kappa_2 = \frac{\beta_2}{2}; \qquad \sigma = \frac{1}{\alpha_1}.$$

We can evaluate $k$ recursively by

$$k(z,y) = k(z,y-1) + h(z) \qquad (z = 1, 2, \ldots, y-1)$$

$$k(y,y) = t_1(y)\,h(y) - t_2(y)\,h^{2*}(y)$$

with

$$t_1(y) = \kappa_1 y; \qquad t_2(y) = \kappa_2 y.$$

We see that the situation is very similar to the situation of subsection 2B, so we can utilise much of what we did there.

To evaluate $h^{2*}(y)$ for $y = 0, 1, \ldots, x$ we need $b_2(x)$ bar operations and $d_2(x)$ dot operations. To evaluate the constants $\kappa_1$, $\kappa_2$, and $\sigma$ we need two bar operations and three dot operations. To evaluate $g(y)$ for $y = 0$ we need one dot operation, for $y = 1$ one bar operation and four dot operations, and for each $y = 2, 3, \ldots, x$, $2(y+1)$ bar operations and $y + 3$ dot operations. Totally this gives

$$b_{\mathrm{C}}(x) = b_2(x) + 2 + 1 + \sum_{y=2}^{x} 2(y+1) = \frac{5x^2+16x-4}{4} - r(x)$$

$$d_{\mathrm{C}}(x) = d_2(x) + 3 + 1 + 4 + \sum_{y=2}^{x}(y+3) = \frac{3x^2+18x+20}{4} - r(x).$$

| $i$ | $b_i(x)$ | $d_i(x)$ |
|---|---|---|
| B | $x^2+1$ | $x^2+6x-3$ |
| C | $\frac{5x^2+16x-4}{4}-r(x)$ | $\frac{3x^2+18x+20}{4}-r(x)$ |
| D | $\frac{5x^2+4x+8}{4}-r(x)$ | $\frac{5x^2+24x+4}{4}-r(x)$ |

*Table 5.1: Numbers of operations for Methods B, C, and D.*

5F. For application of Method D we rewrite (5.7) as

$$\varphi_g(y) = y\left(\sigma_1 h(y) - \sigma_2 h^{2*}(y)\right) + \alpha_1 \sum_{z=1}^{y-1} \varphi_g(y-z)\, h(z) \tag{5.8}$$

with

$$\sigma_1 = \alpha_1 + \beta_1 + \beta_2; \qquad \sigma_2 = \frac{\alpha_1\beta_2}{2}.$$

Once more, we need $b_2(x)$ bar operations and $d_2(x)$ dot operations for $h^{2*}$. Evaluation of the constants $\sigma_1$ and $\sigma_2$ requires two bar operations and two dot operations. To evaluate $\varphi_g(y)$ by (5.8) for $y = 1$ we need one bar operation and two dot operations and for each $y = 2, 3, \ldots, x$, $y$ bar operations and $y+3$ dot operations. Finally we need $b_{\varphi^{-1}}(x)$ bar operations and $d_{\varphi^{-1}}(x)$ dot operations to obtain the values of $g$ from the values of $\varphi_g$. Totally this gives

$$b_{\mathrm{D}}(x) = b_2(x) + 2 + 1 + \sum_{y=2}^{x} y + b_{\varphi^{-1}}(x) = \frac{5x^2+4x+8}{4} - r(x)$$

$$d_{\mathrm{D}}(x) = d_2(x) + 2 + 2 + \sum_{y=2}^{x} (y+3) + d_{\varphi^{-1}}(x) = \frac{5x^2+24x+4}{4} - r(x).$$

5G. In Table 5.1 we display the numbers of operations for Methods C and D found in subsection 5 together with the numbers of operations for Method B from Table 3.2 in the case ($\alpha_1 \neq 0$; $\beta_1 \neq 0; \alpha_2 = 0$; $\beta_2 > 0$), and in Table 5.2 we compare these three methods.

We immediately see that Method B is preferable to Method D both with respect to bar and dot operations.

Method C requires more bar operations than Method D for all $x > 1$, but less dot operations for all $x > 2$. However, we have

$$b_{\mathrm{CD}}(x) + d_{\mathrm{CD}}(x) = \frac{-x^2+3x+2}{2},$$

| $ij$ | $b_{ij}(x)$ | $d_{ij}(x)$ |
|---|---|---|
| BC | $\frac{-x^2-16x+8}{4}+r(x)$ | $\frac{x^2+6x-32}{4}+r(x)$ |
| BD | $-\frac{(x+2)^2}{4}+r(x)$ | $\frac{-x^2-16}{4}+r(x)$ |
| CD | $3x-3$ | $\frac{-x^2-3x+8}{2}$ |

*Table 5.2: Comparison between Methods B, C, and D.*

which is less than zero for all $x > 3$, that is, except for small values of $x$, the total number of elementary algebraic operations required is higher for Method D than for Method C. Method D also requires more dot operations than Method C. As pointed out in Sundt & Dickson (1998), the reason for distinguishing between bar operations and dot operations is that dot operations would normally be more time-consuming than bar operations. As both the number of dot operations and the total number of operations are higher for Method D than for Method C, we therefore conclude that Method C is preferable to Method D.

It remains to compare Methods B and C. Method B requires less bar operations than Method C for all values of $x$, but more dot operations for all $x > 3$. As

$$b_{\mathrm{BC}}(x) + d_{\mathrm{BC}}(x) = 2r(x) - \frac{5x+12}{2}$$

is less than zero for all $x$, the total number of operations is always lower for Method B than for Method C, so unfortunately we cannot apply the reasoning of the comparison between Methods C and D in the present case. In the literature one often counts only dot operations and discards bar operations, cf. e.g. Bühlmann (1984). If we do that, then Method C is preferable. On the other hand, Method B seems to be much more convenient to program than Method C.

5H. Willmot & Sundt (1989) proposed a method for recursive evaluation of the compound Delaporte distribution, which is very similar to Method D. As the performance of Method D was so bad, we also discard their method.

# 6 The convolution of two $\mathcal{R}_1$ distributions

In this section we shall compare Methods A-D for convolutions of two probability functions in $\mathcal{R}_1$, that is, the degenerate case when both $h_1$ and $h_2$ are concentrated in one.

Let us first consider Method C. In the present case (5.4) reduces to

$$g(y) = \begin{cases} (\gamma_1 + \delta_1)\, g(0) & (y = 1) \\ \left(\gamma_1 + \dfrac{\delta_1}{y}\right) g(y-1) + \left(\gamma_2 + \dfrac{\delta_2}{y}\right) g(y-2) & (y = 2, 3, \ldots) \end{cases} \tag{6.1}$$

with $\gamma_1$, $\delta_1$, $\gamma_2$, and $\delta_2$ given by (5.3). The case requiring the highest number of operations is obviously the case ($\alpha_1 \neq 0$; $\beta_1 \neq 0; \alpha_2 \neq 0$; $\beta_2 \neq 0$). In that case we need one dot operation to evaluate $g(0)$ by (3.5) and three bar operations and three dot operations to evaluate $\gamma_1$, $\delta_1$, $\gamma_2$, and $\delta_2$ by (5.3). To evaluate $g(y)$ by (6.1) we need one bar operation and one dot operation for $y = 1$ and three bar operations and four dot operations for each $y = 2, 3, \ldots, x$. Totally this gives

$$b_{\mathrm{C}}(x) = 3 + 1 + 3(x-1) = 3x + 1$$

$$d_{\mathrm{C}}(x) = 1 + 3 + 1 + 4(x-1) = 4x + 1.$$

Let us now compare Methods A and C. From (2.10) and (2.9) we obtain

$$b_{\mathrm{A}}(x) > b_*(x) = \frac{x(x+1)}{2}; \quad d_{\mathrm{A}}(x) > d_*(x) = \frac{(x+1)(x+2)}{2}.$$

We see that the numbers of operations for Method A have lower bounds of quadratic order whereas the numbers of operations for Method C are of linear order. Thus, we see that for $x$ sufficiently large, Method C will require less operations than Method A. More precisely, we see that when $x \geq 6$ both $b_{\mathrm{C}}(x) > b_*(x)$ and $d_{\mathrm{C}}(x) > d_*(x)$, that is, at least for all $x \geq 6$, Method C is more efficient than Method A.

Analogously, for $i =$ B,D we have

$$b_i(x) > b_{\varphi^{-1}}(x); \qquad d_i(x) > d_{\varphi^{-1}}(x),$$

and insertion of (3.6) and (3.7) gives

$$b_i(x) > \frac{x(x-1)}{2}; \qquad d_i(x) > \frac{x(x+3)}{2}.$$

Once again we obtain lower bounds of quadratic order, and we conclude that Method C is the most efficient of all the four methods, except possibly for low values of $x$. Since for Method C we have considered the case that requires the highest number of operations, whereas the bounds for the other methods are the same for the other cases, this conclusion also holds for the other cases.

# 7 Concluding remarks

In this paper we have compared four methods for evaluation of the convolution of two compound $\mathcal{R}_1$ distributions. In all the cases we studied, we could discard Method A. Furthermore, in the cases where Method D was applicable, we could also discard that method. We are then left with Methods B and C. In the general situation studied in Sections 2-4 Method C was not applicable, and in that case we recommend Method B. On the other hand, for the convolution of two distributions in $\mathcal{R}_1$ Method C performed much better than Method B except possibly when $x$ is small.

Analogous studies could be performed in other situations, like for evaluation of the convolution of $m$ compound distributions where for $i = 1, 2, \ldots, m$ the $i$th counting distribution belongs to the class $\mathcal{R}_{k_i}$ as defined by Sundt (1992). However, the counting of algebraic operations could soon become messy.


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