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The Statistical Distribution of Incurred Losses and Its Evolution Over Time

II: Parametric Models

by

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Summary

The distribution of the total incurred losses of an accident year (or underwriting year) is considered. Before commencement of the accident year, there is a prior on this quantity. The distribution may evolve over time according to Bayesian revision which takes account of the accumulation of data with time.

The distribution in question can be made subject to various assumptions and restrictions. The different forms of these are explored in a sequence of papers that includes the present one.

A previous paper examined the situation in which no restrictions were imposed. The resulting models were referred to as **non-parametric**.

The present paper considers the case in which the posterior-to-data estimates of the subject distribution are restricted to a specific parametric family. These models are referred to as **parametric**. The subject distribution evolves with the evolution of its parameters under Bayesian revision. The credibility approximations to this revision are worked out in general (Section 4) and for the special case of normally distributed data (Section 5).

The results are illustrated by application to a very simply example (Section 6). They are illustrated further by application to a more extensive example involving real data (Section 7). The examples use the same respective data sets as in the previous paper.

All models to this point represent the loss experience of each development year in terms of a separate set of parameters applicable to just that year. Section 9 analyses the case in which all of these parameters are expressed as functions of a reduced parameter set. Again, a numerical example is given.

Keywords: distribution of incurred losses, credibility theory.

1. Introduction

This paper is written at the request of, and is partly funded by, the **Casualty Actuarial Society's** Committee on Theory of Risk. It is the first of a trio of papers whose purpose is to answer the following question, posed by the Committee:

> *Assume you know the aggregate loss distribution at policy inception and you have expected patterns of claims reporting, losses emerging and losses paid and other pertinent information, how do you modify the distribution as the policy matures and more information becomes available? Actuaries have historically dealt with the problem of modifying the expectation conditional on emerged information. This expands the problem to continuously modifying the whole distribution from inception until it decays to a point. One might expect that there are at least two separate states that are important. There is the exposure state. It is during this period that claims can attach to the policy. Once this period is over no new claims can attach. The second state is the discovery or development state. In this state claims that already attached to the policy can become known and their value can begin developing. These two states may have to be treated separately.*

In general terms, this brief requires the extension of conventional point estimation of incurred losses to their companion distributions. Specifically, the evolution of this distribution over time is required as the relevant period of origin matures.

Expressed in this way, the problem takes on a natural Bayesian form. For any particular year of origin (the generic name for an accident year, underwriting year, etc), one begins with a **prior distribution** of incurred losses which applies in advance of data collection. As the period of origin develops, loss data accumulate, and may be used for progressive Bayesian revision of the prior.

When the period of origin is fully mature, the amount of incurred losses is known with certainty. The Bayesian revision of the prior is then a single point distribution. The present paper addresses the question of how the Bayesian revision of the prior evolves over time from the prior itself to the final degenerate distribution.

This evolution can take two distinct forms. On the one hand, one may impose no restrictions on the posterior distributions arising from the Bayesian revisions. These posterior distributions will depend on the empirical distributions of certain observations. Such models are **non-parametric**.

Alternatively, the posterior distributions may be assumed to come from some defined family. For example, it may be assumed that the posterior-to-data distribution of incurred losses, as assessed at a particular point of development of the period of origin, is log normal. Any estimation questions must relate to the parameters which define the distribution within the chosen family.

These are **parametric models**. They are, in certain respects, more flexible than non-parametric, but lead to quite different estimation procedures.

The first paper (Taylor, 1999) dealt with non-parametric models only. This deals with certain parametric models. Within the parametric class one may identify two sub-classes according to whether or not the parameters which describe the distributions involved are fixed quantities, or themselves evolve over time. These are the cases of **static** and **dynamic** parameters respectively.

The present paper addresses the case of static parametric models. A future paper, the final one in the trio, will deal with dynamic parametric models.

Familiarity with the earlier paper will be assumed here. In particular, the Bayesian and credibility background introduced and described there will be assumed.

As far as possible, the notation used here will be common with the earlier paper.

2. Motivational example

Consider the same motivational example as in the earlier paper. The data were set out in the Table 2.2 of that paper, which is reproduced as Table 2.1 here.

Table 2.1 Payments per Claim Incurred

Accident Year	PPCI ($) in Development Year				
	0	**1**	**2**	**3**	**4**
1994	1,069	4,249	1,818	426	215
1995	1,033	3,896	2,128	496	
1996	1,138	3,722	1,863		
1997	1,126	3,960			
1998	915				
Prior mean	1,000	4,000	2,000	500	200

Let cell *(i,j)* represent development year *j* of accident year *i*, and let *X(i,j)* denote the Payments per Claim Incurred (PPCI) in respect of that cell.

Assume that, for each fixed *j*, the *X(i,j)* are an iid sample from a normal distribution with

$$E\,X(i,j)=\theta(j) \tag{2.1}$$

$$V\,X(i,j)=\tau^2(j), \tag{2.2}$$

with $\theta(j)$ and $\tau^2(j)$ independent of *i*.

Suppose that the *X(i,j)* (over all *i,j*) form a mutually stochastically independent set (for given $\theta(j)$). Suppose also that $\tau^2(j)$ is fixed and known, and that $\theta(j)$ is a sampling from some hyperdistribution.

This framework is very similar to that in the earlier paper. Indeed, (2.1) and (2.2) are identical to (2.2) and (2.3) of that paper. The essential difference is the imposition here of normality on *X(i,j)* whereas the distribution of *X(i,j)* was left free in the earlier paper.

As in the earlier paper, this example focuses attention on the incurred losses per claim in respect of accident year 1996:

$$\sum_{j=0}^{2} x(1996,j)+\sum_{j=3}^{4} X(1996,j), \tag{2.3}$$

with *x(i,j)* denoting the **realisation** of the random variable *X(i,j)*.

Consider the distribution of the $X(1996,3)$, say, conditional on the data in Table 2.1, ie $\{x(i,j), i \geq 1994, i+j \leq 1998\}$. It is known that this distribution is normal with variance τ_3^2. What must be estimated is $\theta(3)$ conditional on $\{x(i,j), i \geq 1994, i+j \leq 1998\}$.

This fixes an estimate of the distribution of $X(1996,3)$ conditional on the data. A similar conditional estimate of the distribution of $X(1996,4)$ may be obtained. These two distributions generate the distribution of the quantity (2.3).

The remainder of the paper will be concerned with the application of credibility theory to the estimation of the distribution of quantities like

$$\sum_{j=0}^{k} x(i,j) + \sum_{j=k+1}^{4} X(i,j), \qquad (2.4)$$

conditional on data, as they evolve from k = -1 to k = 4, under the convention that

$$\sum_{j=0}^{-1} (\text{anything}) = 0. \qquad (2.5)$$

3. Bayesian framework

The example of Section 2 is generalised as follows.

Let $X(i,j)$ denote some stochastic variable that is indexed by accident year i and development year j, $i \geq 0, 0 \leq j \leq J$ for fixed $J > 0$.

Let $k = i + j$ = experience year. As in the earlier paper, k labels diagonals in the rectangular array with rows and columns labelled by accident years and development years respectively.

Let

$$\begin{aligned} X'(k) &= \{X(i,j): i \geq 0,\ 0 \leq j \leq J,\ 0 \leq i+j \leq k\} \\ &= \text{data up to and including experience year } k \end{aligned} \tag{3.1}$$

where the prime indicates that the symbol to which it is affixed is being labelled by experience year.

Suppose that $X(i,j)$ has d.f. $G(\bullet | \theta(j))$ characterised by a real vector parameter $\theta(j) \in \mathcal{R}^p$, dependent on j. Suppose that the $X(i,j) | \theta(j)$ are all stochastically independent, and iid for fixed j. It will be convenient to adopt the abbreviated notation:

$$G_j^{(\theta)}(x) = G(x | \theta(j)), \tag{3.2}$$

the upper θ indicating the conditioning.

Now suppose that the $\theta(j)$ are unobservable parameters, representing iid samplings from a d.f. $F(\bullet)$.

Write

$$G_j(x) = \int G_j^{(\theta)}(x)\, dF(\theta), \tag{3.3}$$

which represents the expectation of $G^{(\theta)}(x)$ in the absence of any data.

Once data $X'(k)$ have accumulated, one may calculate the Bayesian revision of $\theta(j)$:

$$\theta^{(k)}(j) = E[\theta(j) | X'(k)]. \tag{3.4}$$

One will also be interested in the Bayesian revision of $G_j(\bullet)$. Ideally, this would be $E_{\tilde{\theta}} G_j^{\tilde{\theta}}(\bullet)$ where $\tilde{\theta} = \tilde{\theta}^{(k)}(j)$, the posterior random variable corresponding to $\theta(j)$. However, this would require distributional assumptions in respect of the latent parameters $\theta(j)$.

An alternative is to replace the strict Bayesian revision of $G_j(\bullet)$ by the quantity

$$G(\bullet|\theta^{(k)}(j)). \tag{3.5}$$

One may further approximate the true Bayesian revision of $G_j(\bullet)$ by replacing $\theta^{(k)}(j)$ in (3.5) by its credibility approximation, ie linearised Bayesian estimator.

Denote the quantity (3.5) with $\theta^{(k)}(j)$ approximated in this way by $G_j^{(k)}(\bullet)$, and adopt the convention that

$$G_j^{(j-1)}(\bullet) = G_j(\bullet). \tag{3.6}$$

Subsequent sections will be concerned with credibility approximations to (3.4) and their application to (3.5).

4. Credibility theory

4.1 Multi-dimensional credibility

Section 4.1 of the earlier paper recited elementary credibility theory. In the present paper, the subject of estimation $\theta(j)$ is in general a multi-dimensional quantity, and hence a multi-dimensional version of credibility theory is required. The theory summarised below derives ultimately from Jewell (1974).

Consider a real m-vector random variable Y_{ij}, relating to cell (i,j), with d.f. characterised by $\theta(j)$, and suppose that

$$E\Big[\underset{m\times 1}{Y_{ij}} \mid \theta(j)\Big] = \underset{m\times p}{A_{ij}}\,\theta(j), \tag{4.1}$$

where A_{ij} is a given design matrix, and vector and matrix dimensions are optionally written below their associated symbols.

Suppose that the $Y_{ij} \mid \theta(j)$ form a stochastically mutually independent set.

As assumed in Section 3, $\theta(0), \theta(1)$, etc are realisations of stochastically independent latent parameters.

Since attention will be confined initially to a fixed (but arbitrary) value of j, it will be convenient to suppress this subscript temporarily. On this understanding, (4.1) is re-written in the form:

$$E[Y_i \mid \theta] = A_i\theta. \tag{4.1a}$$

Write

$$\mu_i(\theta) = E[Y_i \mid \theta], \tag{4.2}$$

and abbreviate $\mu_i(\theta)$ to μ_i when this involves no ambiguity.

Unconditional operators such as $E[\bullet]$ and $V[\bullet]$ will be understood to have integrated θ out, eg

$$\begin{aligned} E[Y_i] &= E_\theta E[Y_i \mid \theta] \\ &= A_i E_\theta[\theta] \\ &= A_i\beta, \end{aligned} \tag{4.3}$$

where β denotes $E[\theta]$.

Let

$$\underset{p \times p}{\Gamma} = V[\theta] \tag{4.4}$$

$$\underset{m \times m}{V_i} = E_\theta V[Y_i \mid \theta]. \tag{4.5}$$

Let Y_{ir} denote the r-th component of Y_i, and μ_{ir} the corresponding component of μ_i. Also, write A_{ir}^T to denote the r-th row of A_i.

Suppose that Y_i exists for $i = 1, 2, \ldots, n$, and write

$$\underset{1 \times nm}{Y^T} = \left(Y_1^T, \ldots, Y_n^T\right) \tag{4.6}$$

$$\underset{p \times nm}{A^T} = \left(A_1^T, \ldots, A_n^T\right), \tag{4.7}$$

where the upper T denotes matrix transposition.

Consider estimators $\hat{Y}_{ir}$ of μ_{ir} that are linear in Y:

$$\underset{1 \times 1}{\hat{Y}_{ir}} = \underset{1 \times 1}{\alpha_0} + \underset{1 \times nm}{Y^T} \underset{nm \times 1}{\alpha}, \tag{4.8}$$

where α_0 and α are to be determined.

Define

$$\Phi_{ir} = E\left[\hat{Y}_{ir} - \mu_{ir}(\theta)\right]^2, \tag{4.9}$$

which is a measure of error in the linearised estimator of μ_{ir}.

By (4.1a), (4.2) and (4.8), it is possible to write (4.9) in the form:

$$\Phi_{ir} = E\left[\alpha_0 + Y^T \alpha - A_{ir}^T \theta\right]^2. \tag{4.10}$$

Credibility estimators of μ_{ir} are those estimators of form (4.8) which use optimal α_0 and α, ie α_0 and α are chosen to minimise Φ_{ir} in (4.10).

4.2 Inhomogeneous credibility

The required optimisation of (4.10) is carried out in Appendix A (which largely follows the technique laid down by Hachemeister (1975)), yielding the following result.

$$\hat{Y}_i = A_i\left(1-Z^T\right)\left[\beta+\sum_{h=1}^{n}\left(1-Z_h^T\right)^{-1}Z_h^T\hat{\theta}_h\right], \tag{4.11}$$

where $\hat{Y}_i$ is the m-vector with $\hat{Y}_{ir}$ as its r-th component, and

$$\hat{\theta}_h = \left(A_h^T V_h^{-1} A_h\right)^{-1} A_h^T V_h^{-1} Y_h \tag{4.12}$$

$$Z_h = M_h\left(1+M_h\right)^{-1} = \left(1+M_h\right)^{-1} M_h \tag{4.13}$$

$$M_h = A_h^T V_h^{-1} A_h \Gamma \tag{4.14}$$

$$Z = M\left(1+M\right)^{-1} = \left(1+M\right)^{-1} M \tag{4.15}$$

$$M = \sum_{h=1}^{n} M_h. \tag{4.16}$$

Note that $\hat{Y}_i$ involves a "weighted average" of the prior mean β of θ and the data based estimates $\hat{\theta}_h$. These latter estimates take the form of weighted least squares regression estimates.

The weighted average form of (4.11) can be made clearer if it is re-written as follows:

$$\hat{Y}_i = A_i\left[\left(1-Z^T\right)\beta+Z^T\bar{\theta}\right], \tag{4.11a}$$

where

$$\bar{\theta} = \left(Z^T\right)^{-1}\left(1-Z^T\right)\sum_{h=1}^{n}\left(1-Z_h^T\right)^{-1}Z_h^T\hat{\theta}_h$$

$$= \left(M^T\right)^{-1}\sum_{h=1}^{n} M_h^T\hat{\theta}_h \qquad \text{[by (4.13) and (4.15)]}$$

$$= \left[\sum_{h=1}^{n} M_h^T\right]^{-1}\sum_{h=1}^{n} M_h^T\hat{\theta}_h. \tag{4.17}$$

This shows $\bar{\theta}$ also to be a "weighted average" of the regression estimates $\hat{\theta}_h$. The matrix Z is a **credibility matrix** representing the weight given to the data based estimate $\bar{\theta}$ in $\hat{Y}_i$.

Estimators (4.11) and (4.11a) are called **inhomogeneous credibility** estimators because the first member inside the square bracket is a constant (ie independent of the data) and renders (4.11) and (4.11a) inhomogeneous in the data vector Y.

Note that a credibility estimator of the parameter vector θ is obtained by setting

$$A_i = \underset{p \times p}{1}$$

in (4.11a) (though not in (4.12) or (4.14)).

4.3 Homogenous credibility

The inhomogeneous credibility estimators (4.11) and (4.11a) require knowledge of the prior mean β. This is consistent with the situation outlined in Section 1. However, it is interesting to consider the alternative case, in which this quantity is unknown. It can be accommodated by setting $\alpha_0 = 0$ in (4.8), in which case this estimator becomes homogeneous in the data vector Y.

The estimator (4.8) which results when it is restricted to be unbiased as an estimator of Y_{ir} is referred to as a **homogeneous credibility** estimator.

In parallel with Section 4.2, this is obtained by minimising (4.10), but now with $\alpha_0 = 0$ and subject to the unbiasedness constraint

$$E\left[\hat{Y}_{ir}\right] = E_\theta\left[Y_{ir} \mid \theta\right]. \tag{4.18}$$

The necessary calculations are made in Appendix B, where it is found that

$$\hat{Y}_i = A_i \bar{\theta}. \tag{4.19}$$

Thus, in the absence of a prior mean for θ, this parameter is estimated by just $\bar{\theta}$. Note that this is not quite a classical estimator because, by (4.17), it is a weighted average of the $\hat{\theta}_h$ with weights M_h that depend on Γ, the prior variance of θ.

4.4 Diagonal case

A case worthy of special consideration in Sections 4.2 and 4.3 is that in which $m = p$ and A_i, V_i and Γ are diagonal:

$$A_i = \operatorname{diag}\left(a_{i1}, \ldots, a_{ip}\right) \tag{4.20}$$

$$V_i = \operatorname{diag}\left(v_{i1}, \ldots, v_{ip}\right) \tag{4.21}$$

$$\Gamma = \operatorname{diag}\left(\gamma_1, \ldots, \gamma_p\right). \tag{4.22}$$

Then (4.11) – (4.16) reduce to the following inhomogeneous result:

$$\hat{Y}_{ir} = a_{ir}\left[\left(1 - z_r\right)\beta_r + z_r \bar{\theta}_r\right], r = 1, 2, \ldots, p \tag{4.23}$$

with

$$z_r \quad = \gamma_r \sum_{h=1}^{n} a_{hr}^2 \, v_{hr}^{-1} / \left[1 + \gamma_r \sum_{h=1}^{n} a_{hr}^2 \, v_{hr}^{-1} \right] \tag{4.24}$$

$$\bar{\theta}_r \quad = \sum_{h=1}^{n} Y_{hr} \, a_{hr} \, v_{hr}^{-1} / \sum_{h=1}^{n} a_{hr}^2 \, v_{hr}^{-1} \tag{4.25}$$

and β_r denoting the r-th component of β.

Note that the multi-dimensional case is here reduced to p applications of 1-dimensional credibility.

It is also interesting to observe that dependency on Γ has vanished from some of these results. Thus, $\bar{\theta}_r$, and hence the homogeneous estimator (4.19), is now quite independent of the prior for θ. Indeed, (4.25) may be recognised as a classical estimator. Each quantity $Y_{hr} \, a_{hr}^{-1}$ is an estimator of θ_r, and (4.25) simply forms an average of these weighted by their reciprocal variances.

5. Normal cell distributions

5.1 Parameter estimation

Consider the case in which the $X(i,j)$ of Section 3 are conditionally normally distributed:

$$X(i,j) \mid \theta(j) \sim N[\theta_1(j), \theta_2(j)]. \tag{5.1}$$

Now suppose that the "observation vector" Y_{ij} of Section 4 is:

$$Y_{ij} = \begin{bmatrix} X(i,j) \\ \left[X(i,j) - \bar{X}(j)\right]^2 n_j / (n_j - 1) \end{bmatrix} \tag{5.2}$$

where

$$\bar{X}(j) = \sum_{i=1}^{n_j} X(i,j) / n_j, \tag{5.3}$$

and n_j is the number of data points $X(i,j)$ for given j.

In this case,

$$E\left[Y_{ij} \mid \theta(j)\right] = \theta(j), \tag{5.4}$$

which is (4.1) with

$$A_i = \underset{2\times 2}{1}. \tag{5.5}$$

It may be checked that

$$\text{cov}\left[X(i,j), \left[X(i,j) - \bar{X}(j)\right]^2\right] = 0, \tag{5.6}$$

taking into account the normality, and hence zero skewness, of the $X(i,j)$.

Because of (5.6), V_i defined in (4.5) is diagonal. Moreover, it is evident from (5.1) that the stochastic properties of $X(i,j)$ are independent of i. In particular,

$$v_{ir} = v_r, \text{ independent of } i. \tag{5.7}$$

If Γ is also diagonal, all conditions hold for the diagonal case of Section 4.4. This means that results (4.23) – (4.25) are applicable.

With (5.5) and (5.7) taken into account, these are as follows:

$$\hat{Y}_{ir} = (1-z_r)\beta_r + z_r\overline{\theta}_r, \qquad r=1,2, \tag{5.8}$$

with

$$z_r = n\gamma_r/(v_r + n\gamma_r) \tag{5.9}$$

$$\overline{\theta}_r = \sum_{h=1}^{n} Y_{hr}/n. \tag{5.10}$$

Recall that, in this notation from Section 4, the argument j is suppressed and so n_j is written as just n.

Note also that the special case (5.2) implies constraints on the parameters β, Γ and V. For example,

$$\begin{aligned}
\beta_2 &= E\,Y_{i2} \\
&= E\sum_{h=1}^{n_j} \left[X(h,j) - \overline{X}(j)\right]^2 / (n_j - 1) \\
&= E_{\theta(j)}E\left[\sum_{h=1}^{n_j} \left[X(h,j) - \overline{X}(j)\right]^2 / (n_j - 1) \mid \theta(j)\right] \\
&= E_{\theta(j)}V\left[X(h,j) \mid \theta(j)\right] \\
&= v_1,
\end{aligned} \tag{5.11}$$

by (4.5), (4.21) and (5.7). The second last step in arriving at (5.11) used the fact that the inner expectation is equal to a conditional sample variance.

Now write out the results (5.8) – (5.10) explicitly for the cases r = 1,2. For r = 2, taking (5.11) into account,

$$\hat{Y}_{i2} = (1-z_2)v_1 + z_2\overline{\theta}_2 \tag{5.12}$$

with

$$z_2 = n_j\gamma_2/(v_2 + n_j\gamma_2) \tag{5.13}$$

$$\overline{\theta}_2 = \sum_{h=1}^{n_j} \left[X(h,j) - \overline{X}(j)\right]^2 / (n_j - 1). \tag{5.14}$$

In (5.12), $\hat{Y}_{i2}$ is an estimator of β_2, and so of v_1 (because of (5.11)). It may therefore be inserted in place of v_1 in (5.9) for the case $r = 1$. In this case (5.8) – (5.10) yield

$$\hat{Y}_{i1} = (1 - z_1)\beta_1 + z_1\overline{\theta}_1 \tag{5.15}$$

with

$$z_1 = n_j\gamma_1 / \left(\hat{Y}_{i2} + n_j\gamma_1\right) \tag{5.16}$$

$$\overline{\theta}_1 = \sum_{h=1}^{n_j} X(h,j)/n_j. \tag{5.17}$$

Here, (5.15) is a credibility estimator of $\theta_1(j)$, the mean of *X(i,j)*, in terms of $\beta_1 = E[\theta_1(j)]$, the prior mean, and $\overline{\theta}_1$, the mean of observations *X(h,j)*. This is in fact the usual 1-dimensional credibility estimator derived by Bühlmann (1967) and with $\hat{Y}_{i2}$, the estimator of v_1, ultimately $(z_2 = 1)$ equal to the standard estimator according to De Vylder (1981).

Thus, in the present case, credibility estimation of a distribution consists of the usual credibility estimation of its mean, supplemented by one other equation, providing credibility estimation of its variance.

5.2 Forecasts

Section 3 introduced the notation $G_j^{(k)}(\bullet)$ for an approximated Bayesian revision of the d.f. of *X(i,j)* based on data $X'(k)$. These Bayesian revisions apply to **past** cells, $i + j \le k$.

Now consider forecasting *X(i,j)* for **future** cells $i + j > k$ on the basis of data $X'(k)$. Let $X^{(k)*}(i,j)$ denote such a forecast, and let $G^{(k)*}(i,j)$ denote the associated predictive d.f., ie the estimated d.f. of *X(i,j)* with mean $X^{(k)*}(i,j)$.

By assumption, $G^{(k)*}(i,j)$ is normal, and so will be characterised by its first two moments. Now

$$X(i,j) - X^{(k)*}(i,j) = [X(i,j) - \theta_1(j)] - [X^{(k)*}(i,j) - \theta_1(j)]. \tag{5.18}$$

By (5.1) and the fact that $X^{(k)*}(i,j)$ is an unbiased estimator of $\theta_1(j)$,

$$E[X(i,j) - X^{(k)*}(i,j)] = 0. \tag{5.19}$$

The forecast $X^{(k)*}(i,j)$ can be taken as $\hat{Y}_{i1}$ for the relevant value of *j*, given by (5.15). Then

$$X(i,j) - X^{(k)*}(i,j) = [X(i,j) - \theta_1(j)] - (1-z_1)[\beta_1 - \theta_1(j)] - z_1[\bar{\theta}_1 - \theta_1(j)]. \quad (5.20)$$

Hence,

$$V[X(i,j) - X^{(k)*}(i,j)] = E_{\theta(j)}V[X(i,j) \mid \theta_1(j)]$$
$$+ (1-z_1)^2 V[\theta_1(j)] + z_1^2 E_{\theta(j)}V[\bar{\theta}_1 \mid \theta_1(j)], \quad (5.21)$$

where use has been made of the fact that the three summands on the right side of (5.20) are stochastically independent. In particular, the independence of the first and last of these summands derives from the fact that the first relates entirely to the future, while the last, involving $\bar{\theta}_1$, relates entirely to past observations.

By (5.10), (4.5) and (4.21),

$$E_{\theta(j)}V[X(i,j) \mid \theta_1(j)] = v_1$$

$$E_{\theta(j)}V[\bar{\theta}_1 \mid \theta_1(j)] = v_1 / n_j.$$

Substitution of the results, and (4.4) and (4.22), in (5.21) yields

$$V[X(i,j) - X^{(k)*}(i,j)] = (1-z_1)^2\gamma_1 + v_1(1 + z_1^2 / n_j). \quad (5.22)$$

This quantity may be estimated by

$$(1-z_1)^2 \gamma_1 + \hat{Y}_{i2}(1 + z_1^2 / n_j). \quad (5.23)$$

Note that (5.16) may be rewritten in the form:

$$z_1 \hat{Y}_{i2} / n_j = (1-z_1)\gamma_1, \quad (5.24)$$

so that (5.22) has the alternative form:

$$V[X(i,j) - X^{(k)*}(i,j)] = (1-z_1)^2 \gamma_1 + v_1 + z_1(1-z_1)\gamma_1$$
$$= v_1 + (1-z_1)\gamma_1. \quad (5.25)$$

6. Application to motivational example

The results of Section 5 are illustrated by application to the data set out in Table 2.1. The last row of that table gave values of β_1 for the various j. These are incorporated in Table 6.1 which gives other parameters needed for application of Section 5.

Table 6.1 Credibility parameters

	$j=0$	1	2	3	4
β_1	1,000	4,000	2,000	500	200
$\beta_2 (= v_1)$	$(400)^2$	$(1{,}020)^2$	$(500)^2$	$(200)^2$	$(100)^2$
γ_1	$(100)^2$	$(200)^2$	$(200)^2$	$(150)^2$	$(100)^2$
γ_2 / v_2	0.5	0.5	0.5	0.5	0.5

It is evident from (5.13) that the ratio γ_2 / v_2, rather than individual values of γ_2 and v_2, is sufficient to determine z_2. Just this ratio is given in the table as it may be easier to form a prior view of it rather than its components.

Table 6.2 gives the summary statistics $\bar{\theta}_r$ which serve as input to the credibility results.

Table 6.2 Summary statistics

	$j=0$	1	2	3	4
At $k=0$:					
$\bar{\theta}_1$	1,069				
$\bar{\theta}_2$	400				
At $k=1$:					
$\bar{\theta}_1$	1,051	4,249			
$\bar{\theta}_2$	25	1,000			
At $k=2$:					
$\bar{\theta}_1$	1,080	4,072	1,818		
$\bar{\theta}_2$	53	268	500		
At $k=3$:					
$\bar{\theta}_1$	1,091	3,956	1,973	426	
$\bar{\theta}_2$	49	219	168	150	
At $k=4$:					
$\bar{\theta}_1$	1,056	3,957	1,936	461	215
$\bar{\theta}_2$	90	219	169	49	100

Note that when $j = k$, $\bar{\theta}_2$ is undefined in (5.14) since $n_j = 1$. Hence, $\hat{Y}_{i2}$ is taken to be v_1.

Tables 6.3 to 6.5 display:

- credibility factors
- credibility estimates
- credibility forecasts

respectively. Note that z_1 in Table 6.3 draws on Table 6.4 (see (5.16)). Table 6.4 applies estimates (5.12) and (5.15), ie "mean" is an estimate of $E[\theta_1(j) \mid X'(k)]$, while "s.d." relates to a Bayesian revision of the "within cell" variance v_1, also based on data $X'(k)$.

The estimates in Table 6.4 relate to past cells, as taking into account data up to and including the nominated experience year *k*. On the other hand, Table 6.5 gives forecast parameters at each value of *k*, deriving each **root mean square error of prediction** (RMSEP) from (5.23).

Table 6.3 Credibility factors

	$j = 0$	1	2	3	4
At $k = 0$:					
z_1	0.059				
z_2	0.333				
At $k = 1$:					
z_1	0.199	0.074			
z_2	0.5	0.5			
At $k = 2$:					
z_1	0.313	0.213	0.324		
z_2	0.6	0.6	0.6		
At $k = 3$:					
z_1	0.421	0.305	0.610	0.8	
z_2	0.667	0.667	0.667	0.667	
At $k = 4$:					
z_1	0.493	0.385	0.686	0.932	0.833
z_2	0.714	0.714	0.714	0.714	0.714

Note that the z_2 does not vary with *j* because of the assumption in Table 6.1 that γ_2 / v_2 does not vary.

Table 6.4 Credibility estimates of parameters

	$j = 0$	1	2	3	4
At $k = 0$:					
mean	1,004				
s.d.	400				
At $k = 1$:					
mean	1,010	4,018			
s.d.	283	1,000			
At $k = 2$:					
mean	1,025	4,015	1,941		
s.d.	256	666	500		
At $k = 3$:					
mean	1,038	3,986	1,984	441	
s.d.	234	604	320	150	
At $k = 4$:					
mean	1,028	3,983	1,956	463	212
s.d.	227	566	303	90	100

Table 6.5 Credibility forecasts of parameters

	$j = 0$	1	2	3	4
At $k = 0$:					
mean	1,004				
RMSEP	413				
At $k = 1$:					
mean	1,010	4,018			
RMSEP	303	1,021			
At $k = 2$:					
mean	1,025	4,015	1,941		
RMSEP	279	700	547		
At $k = 3$:					
mean	1,038	3,986	1,984	441	
RMSEP	260	643	389	220	
At $k = 4$:					
mean	1,028	3,983	1,956	463	212
RMSEP	253	608	374	179	146

As an example of forecasting outstanding losses, consider accident year 1997 at the end of $j = 1$, ie $k = 4$. The outstanding losses are

$$\sum_{j=2}^{4} X(1997, j),$$

and Table 6.5 gives the mean and variance of this quantity as:

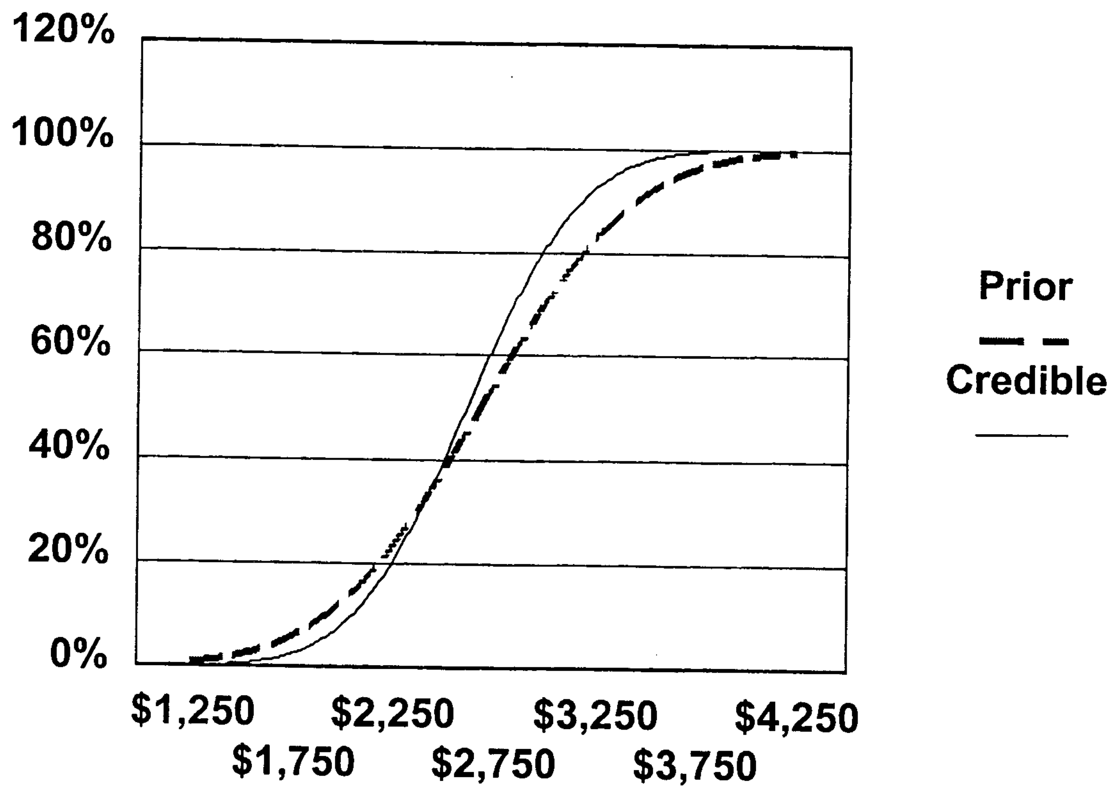
mean $= 1{,}956 + 463 + 212 \qquad = 2{,}632$
s.d. $= (374)^2 + (179)^2 + (146)^2 \qquad = (440)^2.$

These compare with prior estimates (Table 6.1):

mean $= 2{,}000 + 500 + 200 \qquad = 2{,}700$
s.d. $= (200)^2 + (150)^2 + (100)^2 + (500)^2 + (150)^2 + (100)^2 = (596)^2.$

Figure 6.1 provides a comparative plot of the two normal distributions, the prior and the credible distribution.

Fig 6.1
Development years 2 to 4



7. A more realistic example

The present section will illustrate the results of Section 5 by reference to the same real data set as used in Section 9 of the earlier paper. The data appeared there in Table 9.1 in the form of incremental paid losses.

Table 9.2 of that paper converted them to logged age-to-age factors, and for convenience these are reported here as Table 7.2. The underlying incurred losses, adjusted to constant dollar values for inflation, appear in Table 7.1.

Table 7.1 Incurred Losses

Period of origin	Incurred losses to end of development year n=																	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000
1978	9,268	18,263	20,182	22,383	22,782	26,348	26,172	26,184	25,455	25,740	25,711	25,452	25,460	25,422	25,386	25,520	25,646	25,469
1979	9,848	16,123	17,099	18,544	20,534	21,554	23,219	22,381	21,584	21,408	20,857	21,163	20,482	19,971	19,958	19,947	19,991	
1980	13,990	22,484	24,950	33,255	33,295	34,308	34,022	34,023	33,842	33,933	33,570	31,881	32,203	32,345	32,250	32,168		
1981	16,550	28,056	39,995	42,459	42,797	42,755	42,435	42,302	42,095	41,606	40,440	40,432	40,326	40,337	40,096			
1982	11,100	31,620	40,852	38,831	39,516	39,870	40,358	40,355	40,116	39,888	39,898	40,147	39,827	40,200				
1983	15,677	33,074	35,592	35,721	38,652	39,418	39,223	39,696	37,769	37,894	37,369	37,345	37,075					
1984	20,375	33,555	41,756	45,125	47,284	51,710	52,147	51,187	51,950	50,967	51,461	51,382						
1985	9,800	24,663	36,061	37,927	40,042	40,562	40,362	40,884	40,597	41,304	42,378							
1986	11,380	26,843	34,931	37,805	41,277	44,901	45,867	45,404	45,347	44,383								
1987	10,226	20,511	26,882	32,326	35,257	40,557	43,753	44,609	44,196									
1988	8,170	18,567	26,472	33,002	36,321	37,047	39,675	40,398										
1989	10,433	19,484	32,103	38,936	45,851	45,133	45,501											
1990	9,661	23,808	32,966	42,907	46,930	49,300												
1991	14,275	25,551	33,754	38,674	41,132													
1992	13,245	29,206	36,987	44,075														
1993	14,711	27,082	34,230															
1994	12,476	23,126																
1995	9,715																	

Table 7.2 **Logged incurred loss age to age factors**

Period of origin	Logged age to age factor from development year n to n+1 development year n=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1978	0.678	0.100	0.104	0.018	0.145	-0.007	0.000	-0.028	0.011	-0.001	-0.010	0.000	-0.001	-0.001	0.005	0.005	-0.007
1979	0.493	0.059	0.081	0.102	0.048	0.074	-0.037	-0.036	-0.008	-0.026	0.015	-0.033	-0.025	-0.001	-0.001	0.002	
1980	0.474	0.104	0.287	0.001	0.030	-0.008	0.000	-0.005	0.003	-0.011	-0.052	0.010	0.004	-0.003	-0.003		
1981	0.528	0.355	0.060	0.008	-0.001	-0.008	-0.003	-0.005	-0.012	-0.028	-0.000	-0.003	0.000	-0.006			
1982	1.047	0.256	-0.051	0.017	0.009	0.012	-0.000	-0.006	-0.006	0.000	0.006	-0.008	0.009				
1983	0.747	0.073	0.004	0.079	0.020	-0.005	0.012	-0.050	0.003	-0.014	-0.001	-0.007					
1984	0.499	0.219	0.078	0.047	0.089	0.008	-0.019	0.015	-0.019	0.010	-0.002						
1985	0.923	0.380	0.050	0.054	0.013	-0.005	0.013	-0.007	0.017	0.026							
1986	0.858	0.263	0.079	0.088	0.084	0.021	-0.010	-0.001	-0.022								
1987	0.696	0.270	0.184	0.087	0.140	0.076	0.019	-0.009									
1988	0.821	0.355	0.220	0.096	0.020	0.069	0.018										
1989	0.625	0.499	0.193	0.163	-0.016	0.008											
1990	0.902	0.325	0.264	0.090	0.049												
1991	0.582	0.278	0.136	0.062													
1992	0.791	0.236	0.175														
1993	0.610	0.234															
1994	0.617																
Average	0.699	0.250	0.124	0.065	0.049	0.020	-0.001	-0.013	-0.004	-0.006	-0.006	-0.007	-0.003	-0.003	0.001	0.004	-0.007
Standard deviation	0.169	0.121	0.095	0.045	0.052	0.033	0.017	0.019	0.013	0.018	0.021	0.014	0.013	0.002	0.004	0.002	

The quantities appearing in Table 7.2 provide the "observations" for the example. The prior estimates of parameters are as set out in Table 7.3.

Table 7.3 Prior parameter estimates

j	β_1	$(\beta_2)^{\frac{1}{2}}$
0	0.60	0.152
1	0.20	0.122
2	0.10	0.097
3	0.05	0.078
4	0.03	0.062
5	0.02	0.050
6 and later	0	0.19×0.8^j

These parameter values are consistent with their counterparts in Table 9.3 of the earlier paper.

Other parameters are:

$$\nu_1 = \beta_2$$
$$\nu_2 = (0.25 + 0.05j)^2 \nu_1$$
$$\gamma_1 / \nu_1 = 0.5$$
$$\gamma_2 / \nu_2 = 0.2.$$

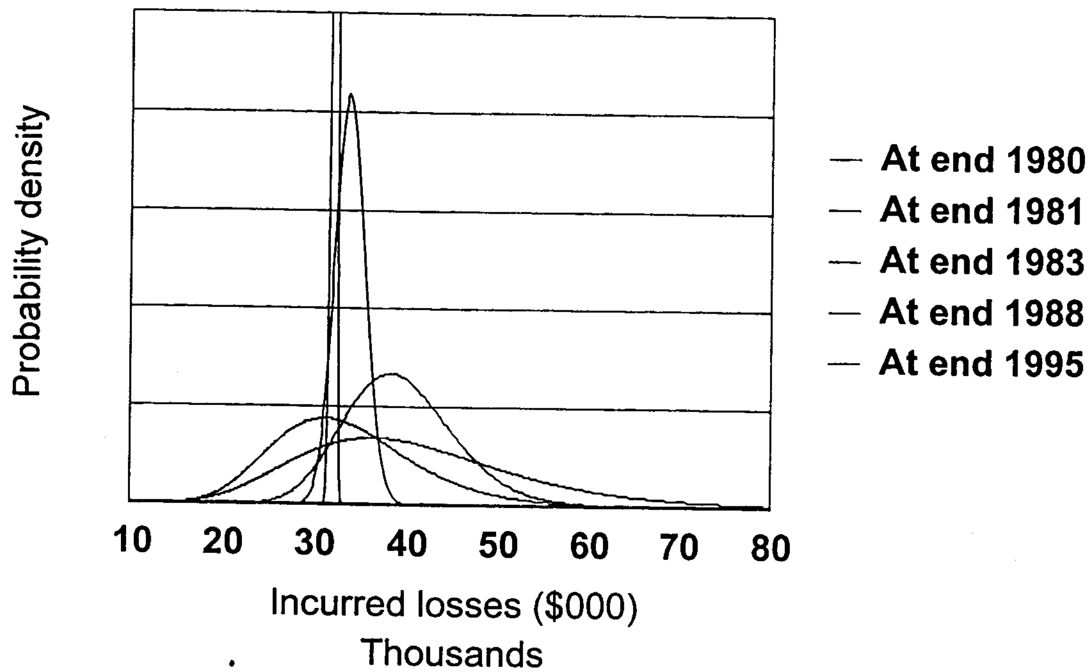
The results of Section 5 are applied to this example. The detailed calculations appear in Appendix C.

Tables C.11 and C.12 in particular give forecast parameters of the normal distributions associated with the ultimate incurred losses of each accident year, as it develops from its start to the completion of experience in 1995.

Figure 7.1 illustrates excerpts from these tables, showing a selection from the developing distribution of incurred losses associated with accident year 1980. The distributions plotted represent forecasts at the ends of 1980, 1981, 1983, 1988 and 1995 respectively.

The different distributions can be identified by their increasing concentration (and therefore increasing peak height) with increasing development. Thus, the distribution at the end of 1995, with only two years of development remaining, is highly concentrated.

Fig 7.1
Accident year 1980



8. Models spanning different development years

8.1 Theory

The models dealt with in Sections 5 to 7 were described by (5.1) which assumed that each development year was characterised by two parameters. Those parameters were specific to their own development year and no other.

As a result, the model applied in Section 7 assumed 32 parameters. It will often be possible to describe an accident year more parsimoniously than this, by assuming the parameters to be represented by parametric functions of (say) development year.

For example, suppose that the vector $\theta(j)=\left[\theta_1(j),...,\theta_p(j)\right]^T$ appearing in (4.1) can be expressed in the form:

$$\theta_r(j)=\sum_{q=1}^{Q_r}\lambda_q^{(r)}u_q^{(r)}(j), \quad r=1,2,...,p, \tag{8.1}$$

where $u_1^{(r)}(\bullet),...,u_{Q_r}^{(r)}(\bullet)$ are pre-defined functions and the $\lambda_q^{(r)}$ are constants to be estimated.

Now (8.1) may be written in matrix form:

$$\underset{J\times 1}{\theta_r}=\underset{J\times Q_r}{U^{(r)}}\ \underset{Q_r\times 1}{\lambda^{(r)}}, \quad r=1,2,...,p, \tag{8.2}$$

where

$$\theta_r \quad =\left[\theta_r(0),...,\theta_r(J-1)\right]^T$$
$$\lambda^{(r)} \quad =\left[\lambda_1^{(r)},...,\lambda_{Q_r}^{(r)}\right]^T$$

and $U^{(r)}$ is the matrix with (j,q)-element equal to $u_q^{(r)}(j)$.

Equations (8.2) may be stacked to yield

$$\underset{pJ\times 1}{\theta} \quad = \underset{pJ\times Q}{U}\ \underset{Q\times 1}{\lambda}, \tag{8.3}$$

with

$$Q \quad =\sum_{r=1}^{p}Q_r \tag{8.4}$$

$$\theta \quad =\left[\theta_1^T,...,\theta_p^T\right]^T \tag{8.5}$$

$$\lambda \quad =\left[\lambda^{(1)T},...,\lambda^{(p)T}\right]^T \tag{8.6}$$

and U is the matrix with diagonal block form:

$$U = \text{diag}\left[U^{(1)},...,U^{(p)}\right]. \tag{8.7}$$

Equation (8.3) reduces the number of parameters to be estimated from pJ to Q.

Now consider a real m-vector random variable Y_{ij}, relating to cell (i,j), with d.f. characterised by $\theta(j)$, and suppose that (4.1) holds.

Since $\theta(j)$ is just a sub-vector of θ, (4.1) may be expressed in the form:

$$E\left[\underset{m\times 1}{Y_{ij}} \mid \theta\right] = \underset{m\times pJ}{B_{ij}} \underset{pJ\times 1}{\theta}, \tag{8.8}$$

for a suitable matrix B_{ij}.

Substitution of (8.3) in (8.8) yields

$$\begin{aligned} E\left[Y_{ij} \mid \theta\right] &= B_{ij}\, U\, \lambda \\ &= \underset{m\times Q}{C_{ij}} \underset{Q\times 1}{\lambda} \end{aligned} \tag{8.9}$$

with

$$C_{ij} = B_{ij}\, U. \tag{8.10}$$

It is now possible to apply the credibility theory of Section 4 to obtain credibility estimates of

$$\mu_{ij}(\theta) = E\left[Y_{ij} \mid \theta\right]. \tag{8.11}$$

As in Section 4, assume that the $Y_{ij} \mid \theta$ form a stochastically mutually independent set. Also assume that λ is a single realisation of some latent variable.

Equations (8.9) and (8.11) correspond to (4.1a) and (4.2) respectively. The remainder of Sections 4.1 to 4.3 go through for the present model if certain changes are made.

In Section 4, each parameter $\theta(j)$ applied to a specific development year j. Hence, parameter estimation was carried out separately for each development year; the whole of the reasoning from (4.1a) to the end of Section 4 applies to a fixed (but arbitrary) development year. The development year argument is suppressed in the notation.

Parameter estimates, such as $\hat{\theta}_h$ (see (4.12)), relating to particular accident years h, were based on data Y_h (actually Y_{hj} with the j suppressed).

In the present case, this separate treatment of development years is not possible. The model (8.9) causes all development years to be dependent on the same set of parameters λ. This means that the parallel to (4.12) in the present case must depend on **all data in respect of accident year *h*.**

It will be necessary to change the earlier notation so that the argument j appears explicitly. Let Y_{ijr} denote the r-th component of Y_{ij} (denoted just Y_{ir} in Section 4). Then the totality of data in respect of accident year i is

$$Y_i^T = \left(Y_{i01}, Y_{i11}, \ldots, Y_{iJ_i1}, Y_{i02}, \ldots, Y_{iJ_i2}, \ldots, Y_{i0m}, \ldots, Y_{iJ_im}\right), \tag{8.12}$$

for some integer J_i.

By (8.9), Y_i may be expressed in the form:

$$\underset{m(J_i+1)\times 1}{E[Y_i \mid \theta]} = \underset{m(J_i+1)\times Q}{C_i} \; \underset{Q\times 1}{\lambda}, \tag{8.13}$$

with

$$C_i = \left(C_{i01}^T, C_{i11}^T, \ldots, C_{iJ_im}^T\right)^T, \tag{8.14}$$

and C_{ijr} denoting the r-th row of C_{ij}.

With these definitions, Sections 4.1 to 4.3 go through for the present model if the following replacements are made:

$$\theta \leftarrow \lambda \tag{8.15}$$

$$A_i \leftarrow C_i. \tag{8.16}$$

In the present context,

$$\underset{Q\times Q}{\Gamma} = V[\lambda] \tag{8.17}$$

$$\underset{m(J_i+1)\times m(J_i+1)}{V_i} = E_\lambda V[Y_i \mid \lambda] \tag{8.18}$$

$$C^T = \left(C_1^T, \ldots, C_n^T\right). \tag{8.19}$$

The inhomogeneous credibility estimator $\hat{Y}_i$ of $\mu_i(\theta)$ is given by (4.11) – (4.17) subject to the replacements (8.15) and (8.16). Similarly, the homogeneous credibility estimator is given by (4.19) with the same replacement.

8.2 Diagonal case

8.2.1 Credibility estimates

Consider now the diagonal case of Section 4.4. This cannot be extended directly to the present context because C_i (the counterpart of A_i in Section 4.4) will usually not be diagonal.

Suppose, however, that $m = p$ and B_{ij} in (8.8) takes **block diagonal** form:

$$B_{ij} = \text{diag}\left(B_{ij}^{(1)}, B_{ij}^{(2)}, \ldots, B_{ij}^{(p)}\right) \tag{8.20}$$

with $B_{ij}^{(r)}$, $r = 1, \ldots, p$, of dimension 1 x J.

By (8.7), U is also block diagonal with $U^{(r)}$ of dimension $J \text{ x } Q_r$ (see (8.2)). Then (8.10) gives

$$C_{ij} = \text{diag}\left(C_{ij}^{(1)}, \ldots, C_{ij}^{(p)}\right) \tag{8.21}$$

with $C_{ij}^{(r)}$ of dimension 1 x Q_r.

By (8.14), C_i also takes block diagonal form:

$$\underset{p(J_i+1)\text{x}Q}{C_i} = \text{diag}\left(C_i^{(1)}, \ldots, C_i^{(p)}\right), \tag{8.22}$$

with

$$\underset{(J_i+1)\text{x}Q_r}{C_i^{(r)}} = \left(C_{i0}^{(r)T}, C_{i1}^{(r)T}, \ldots, C_{iJ_i}^{(r)T}\right)^T. \tag{8.23}$$

Now suppose, in addition, that V_i and Γ have block diagonal form:

$$V_i = \text{diag}\left(V_i^{(1)}, \ldots, V_i^{(p)}\right) \tag{8.24}$$

$$\Gamma = \text{diag}\left(\Gamma^{(1)}, \ldots, \Gamma^{(p)}\right) \tag{8.25}$$

with $V_i^{(r)}$ having dimension $(J_i + 1) \text{x} (J_i + 1)$ and $\Gamma^{(r)}$ of dimension $Q_r \text{ x } Q_r$.

Let

$$\underset{Q_r\text{x}1}{\beta^{(r)}} = E\left[\lambda^{(r)}\right], \quad r = 1, \ldots, p. \tag{8.26}$$

Now apply the results of Sections 4.1 to 4.3 with replacements (8.15) and (8.16). The matrices M_h and Z_h in (4.14) and (4.13) respectively all take block diagonal form, eg

$$M_h \quad = \text{diag}\left(M_h^{(1)},...,M_h^{(p)}\right) \tag{8.27}$$

with

$$\underset{Q_r \times Q_r}{M_h^{(r)}} = \underset{Q_r \times (J_h+1)}{C_h^{(r)T}} \underset{(J_h+1)\times(J_h+1)}{\left[V_h^{(r)}\right]^{-1}} \underset{(J_h+1)\times Q_r}{C_h^{(r)}} \underset{Q_r \times Q_r}{\Gamma^{(r)}} . \tag{8.28}$$

This causes the counterpart of the main result (4.11) to decouple into p separate equations just as in (4.23), specifically:

$$\hat{Y}_i^{(r)} = C_i^{(r)}\left[\left[1-Z^{(r)T}\right]\beta^{(r)} + Z^{(r)T}\bar{\lambda}^{(r)}\right], \tag{8.29}$$

where $\hat{Y}_i^{(r)}$ is the credibility estimate of the r-th component of $E\left[Y_i \mid \theta\right]$, and

$$\bar{\lambda}^{(r)} \quad = \left[\sum_{h=1}^{n} M_h^{(r)T}\right]^{-1} \sum_{h=1}^{n} M_h^{(r)T}\hat{\lambda}_h^{(r)} \tag{8.30}$$

$$\hat{\lambda}_h^{(r)} \quad = \left[C_h^{(r)T}\left[V_h^{(r)}\right]^{-1}C_h^{(r)}\right]^{-1} C_h^{(r)T}\left[V_h^{(r)}\right]^{-1} Y_h^{(r)}, \tag{8.31}$$

with $Y_h^{(r)}$ denoting the vector $\left[Y_{h0}^{(r)},...,Y_{hJ_h}^{(r)}\right]^T$.

8.2.2 Forecasts

As in Section 5, let the forecast future values of *X(i)*, based on data up to period *k*, be denoted by $X^{(k)*}(i)$.

Consider the case in which $Y_{ij1} = X(i,j)$. If $\hat{Y}_i^{(1)}$ from (8.29) is adopted as the forecast, then the prediction error is

$$\begin{aligned} Y_i^{(1)} - \hat{Y}_i^{(1)} &= X(i) - C_i^{(1)}\left[\left(1-Z^{(1)T}\right)\beta^{(1)} + Z^{(1)T}\bar{\lambda}^{(1)}\right] \\ &= \left[X(i) - C_i^{(1)}\lambda^{(1)}\right] - C_i^{(1)}\left[1-Z^{(1)T}\right]\left[\beta^{(1)} - \lambda^{(1)}\right] \\ &\quad - C_i^{(1)}Z^{(1)T}\left[\bar{\lambda}^{(1)} - \lambda^{(1)}\right], \end{aligned} \tag{8.32}$$

and the MSEP is

$$V\left[Y_i^{(1)}-\hat{Y}_i^{(1)}\right]=V\left[X(i)-C_i^{(1)}\lambda^{(1)}\right]+C_i^{(1)}\left[1-Z^{(1)T}\right]V\left[\lambda^{(1)}\right]\left[1-Z^{(1)}\right]C_i^{(1)T}$$
$$+C_i^{(1)}Z^{(1)T}V\left[\bar{\lambda}^{(1)}\right]Z^{(1)}C_i^{(1)T}$$
$$=V_i^{(1)}+C_i^{(1)}\left[1-Z^{(1)T}\right]\Gamma^{(1)}\left[1-Z^{(1)}\right]C_i^{(1)T}$$
$$+C_i^{(1)}Z^{(1)T}V\left[\bar{\lambda}^{(1)}\right]Z^{(1)}C_i^{(1)T}. \quad (8.33)$$

With some routine algebra, it may be shown that

$$V\left[\bar{\lambda}^{(1)}\right]=\Gamma^{(1)}\left[M^{(1)}\right]^{-1}=\Gamma^{(1)}\left[1-Z^{(1)}\right]\left[Z^{(1)}\right]^{-1}. \quad (8.34)$$

Substitution of (8.34) in (8.33) yields

$$\text{MSEP}\left[\hat{Y}_i^{(1)}\right]=V_i^{(1)}+C_i^{(1)}\Gamma^{(1)}\left[1-Z^{(1)}\right]C_i^{(1)T}, \quad (8.35)$$

which is parallel with (5.25).

8.3 Example

Consider again the example dealt with in Section 7. Continue to assume that

$$X(i,j)\,|\,\theta(j)\sim N\left[\theta_1(j),\theta_2(j)\right], \quad (8.36)$$

where the $X(i,j)$ denote the logged age-to-age factors represented in Table 7.2. Also continue to let

$$Y_{ij}=\begin{bmatrix} X(i,j) \\ \left[X(i,j)-\bar{X}(j)\right]^2 n_j/(n_j-1) \end{bmatrix}, \quad (8.37)$$

with $\bar{X}(j)$ defined by (5.3).

Now assume the following parametric form for $\theta(j), j = 0, 1, 2, \ldots, 16$:

$$\theta_1(j) \;=\; \lambda_1^{(1)}(j+1)^{-2}+\lambda_2^{(1)}(j+1)^{-3} \quad (8.38)$$

$$\theta_2(j) \;=\; \lambda_1^{(2)}\,\text{x}(0.8)^{2j}. \quad (8.39)$$

Comparison of (8.38) and (8.39) with (8.1) yields

$$\mu_1^{(1)}(j)=(j+1)^{-2} \quad (8.40)$$
$$\mu_2^{(1)}(j)=(j+1)^{-3} \quad (8.41)$$
$$\mu_1^{(2)}(j)=(0.8)^{2j}. \quad (8.42)$$

Thus application of (8.3) – (8.7) to the present example gives

$$\underset{34\times1}{\theta} = \underset{34\times3}{U}\ \underset{3\times1}{\lambda} \tag{8.43}$$

with

$$\theta^T = [\theta_1(0),...,\theta_1(16),\theta_2(0),...,\theta_2(16)] \tag{8.44}$$

$$\lambda^T = \left[\lambda_1^{(1)},\lambda_2^{(1)},\lambda_1^{(2)}\right] \tag{8.45}$$

$$U = \text{diag}\left[U^{(1)},U^{(2)}\right] \tag{8.46}$$

$U^{(1)}$ is the 17 x 2 matrix with j-th row $\left(j^{-2},j^{-3}\right)$ and $U^{(2)}$ is the 17 x 1 matrix with j-th entry $(0.8)^{2(j-1)}$.

Just as in Section 7,

$$E\left[Y_{ij}\mid\theta\right]=\theta(j) \tag{8.47}$$

and so (8.8) yields

$$\underset{2\times34}{B_{ij}} = \begin{bmatrix} 0...0 & 1 & 0...0 & 0 & 0...0 \\ 0...0 & 0 & 0...0 & 1 & 0...0 \end{bmatrix}. \tag{8.48}$$

$$\begin{matrix} \text{column} & \text{column} \\ j+1 & j+18 \end{matrix}$$

Then, by (8.10),

$$\underset{2\times3}{C_{ij}} = \begin{bmatrix} (j+1)^{-2} & (j+1)^{-3} & 0 \\ 0 & 0 & (0.8)^{2j} \end{bmatrix}, \tag{8.49}$$

and, by (8.14),

$$C_i = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2^{-2} & 2^{-3} & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (J_i+1)^{-2} & (J_i+1)^{-3} & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & (0.8)^2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & (0.8)^{2J_i} \end{array}\right]. \tag{8.50}$$

Results (8.29) – (8.31) hold for the present example. The covariance matrix $V_h^{(r)}$ appearing there can be estimated as follows. By (8.18) and (8.24),

$$V_h^{(1)} = E_\lambda \, V\left[X(h) \mid \lambda\right] \tag{8.51}$$

where

$$X(h) = \left[X(h,0), X(h,1),...\right]^T . \tag{8.52}$$

Combine (8.36) and (8.39) with (8.51) to obtain

$$\begin{aligned} V_h^{(1)} &= E_\lambda \, \lambda_1^{(2)} \text{ diag } \left[1, 0.8, (0.8)^2, ...\right] \\ &= \beta_1^{(2)} \text{ diag } \left[1, 0.8, (0.8)^2, ...\right]. \end{aligned} \tag{8.53}$$

It is assumed that $V_h^{(2)}$ is diagonal with (j,j)-element equal to $\left[0.25 + 0.5(j+1)\right]^2$ times the corresponding element of $V_h^{(1)}$.

From (8.29), $\beta_1^{(2)}$ is estimated, without bias, by

$$\hat{\beta}_1^{(2)} = \left[1 - Z^{(2)T}\right] \beta^{(2)} + Z^{(2)T} \, \bar{\lambda}^{(2)}, \tag{8.54}$$

and so $V_h^{(1)}$ is estimated by

$$\hat{\beta}_1^{(2)} \text{ diag } \left[1, 0.8, (0.8)^2, ...\right]. \tag{8.55}$$

Substitution of (8.55) in place of $V_h^{(1)}$ in (8.28) produces an estimated credibility matrix parallel to (5.16).

The above calculations are carried out for the sequence of data triangles $X'(k), k = 2,3,...,16$. Since the $X(i,j)$ are logged age-to-age factors, this sequence consists of 3 x 3, 4 x 4, ..., 18 x 18 triangles of incurred losses from Table 7.1.

The prior parameter values are:

$$\beta^{(1)} = \begin{bmatrix} 0.97 \\ -0.37 \end{bmatrix} \qquad \beta^{(2)} = (0.19 \text{x} 0.8)^2$$

$$\Gamma^{(1)} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \qquad \Gamma^{(2)} = (0.02)^2 .$$

Excerpts of the results are given below at the same dates as in Figure 7.1.

At end 1980:

$$Z^{(1)} = \begin{bmatrix} 0.53 & 0.43 \\ 0.43 & 0.50 \end{bmatrix} \quad Z^{(2)} = 0.12$$

At end 1981:

$$Z^{(1)} = \begin{bmatrix} 0.60 & 0.39 \\ 0.39 & 0.57 \end{bmatrix} \quad Z^{(2)} = 0.26$$

At end 1983:

$$Z^{(1)} = \begin{bmatrix} 0.71 & 0.30 \\ 0.30 & 0.68 \end{bmatrix} \quad Z^{(2)} = 0.46$$

At end 1988:

$$Z^{(1)} = \begin{bmatrix} 0.84 & 0.16 \\ 0.16 & 0.82 \end{bmatrix} \quad Z^{(2)} = 0.69$$

At end 1995:

$$Z^{(1)} = \begin{bmatrix} 0.92 & 0.08 \\ 0.08 & 0.91 \end{bmatrix} \quad Z^{(2)} = 0.81$$

Table 8.1 **Credibility estimates $\hat{Y}_h^{(1)}$**

At end of	Estimate for j = 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1980	0.65	0.19															
1981	0.56	0.17	0.08														
1982	0.53	0.16	0.07	0.042													
1983	0.54	0.20	0.10	0.057	0.038												
1984	0.64	0.20	0.10	0.057	0.037	0.026											
1985	0.65	0.19	0.09	0.052	0.034	0.024	0.018										
1986	0.63	0.18	0.08	0.048	0.031	0.022	0.016	0.012									
1987	0.67	0.19	0.09	0.049	0.032	0.022	0.016	0.013	0.010								
1988	0.69	0.19	0.09	0.048	0.031	0.022	0.016	0.012	0.010	0.008							
1989	0.69	0.18	0.08	0.047	0.030	0.021	0.016	0.012	0.009	0.008	0.006						
1990	0.71	0.19	0.09	0.048	0.031	0.022	0.016	0.012	0.010	0.008	0.007	0.005					
1991	0.71	0.18	0.08	0.047	0.030	0.021	0.015	0.012	0.009	0.008	0.006	0.005	0.004				
1992	0.72	0.19	0.09	0.048	0.031	0.022	0.016	0.012	0.010	0.008	0.006	0.005	0.005	0.004			
1993	0.72	0.19	0.09	0.049	0.032	0.022	0.016	0.013	0.010	0.008	0.007	0.006	0.005	0.004	0.004		
1994	0.72	0.19	0.09	0.049	0.032	0.022	0.016	0.012	0.010	0.008	0.007	0.006	0.005	0.004	0.004	0.003	
1995	0.72	0.18	0.08	0.047	0.030	0.021	0.015	0.012	0.009	0.008	0.006	0.005	0.004	0.004	0.003	0.003	0.003

Table 8.2 **Credibility estimates $\hat{Y}_h^{(2)}$**

At end of	Estimate for j = 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1980	0.022															
1981	0.020	0.013														
1982	0.018	0.011	0.007													
1983	0.021	0.013	0.008	0.0054												
1984	0.034	0.022	0.014	0.0090	0.0058											
1985	0.032	0.020	0.013	0.0083	0.0053	0.0034										
1986	0.031	0.020	0.013	0.0081	0.0052	0.0033	0.0021									
1987	0.032	0.020	0.013	0.0083	0.0053	0.0034	0.0022	0.0014								
1988	0.031	0.020	0.013	0.0082	0.0053	0.0034	0.0022	0.0014	0.0009							
1989	0.029	0.018	0.012	0.0076	0.0048	0.0031	0.0020	0.0013	0.0008	0.0005						
1990	0.027	0.018	0.011	0.0072	0.0046	0.0029	0.0019	0.0012	0.0008	0.0005	0.0003					
1991	0.027	0.017	0.011	0.0070	0.0045	0.0029	0.0018	0.0012	0.0007	0.0005	0.0003	0.0002				
1992	0.028	0.018	0.012	0.0074	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005	0.0003	0.0002	0.0001			
1993	0.027	0.017	0.011	0.0072	0.0046	0.0029	0.0019	0.0012	0.0008	0.0005	0.0003	0.0002	0.0001	0.0001		
1994	0.026	0.017	0.011	0.0069	0.0044	0.0028	0.0018	0.0012	0.0007	0.0005	0.0003	0.0002	0.0001	0.0001	0.0001	
1995	0.025	0.016	0.010	0.0066	0.0042	0.0027	0.0017	0.0011	0.0007	0.0005	0.0003	0.0002	0.0001	0.0001	0.0000	0.0000

Table 8.1 gives estimates $\hat{Y}_h^{(1)}$ in which $V_h^{(1)}$ has been replaced by (8.55), as discussed above. These may be used to estimate ultimate claims incurred for each accident year on the basis of any particular data triangle.

For example, consider triangle $X'(k)$, yielding estimates $\hat{Y}_h^{(1)}$, $h = 0, 1, \ldots, k$, where $\hat{Y}_h^{(1)}$ is a $(k+1-h)-$vector. Extend each of these vectors to a 17-vector $\hat{Y}_h^{*(1)}$ by inserting prior estimates for the missing values:

$$\hat{Y}_h^{*(1)}(k) = \begin{bmatrix} \hat{Y}_h^{(1)} \\ C_{h,k+1-h}\,\lambda \\ \cdot \\ \cdot \\ C_{h,16}\,\lambda \end{bmatrix}. \tag{8.56}$$

Write w_h for the 17-vector:

$$w_h^T = \Big[\underbrace{0, \ldots, 0,}_{k+1-h \text{ terms}} \quad \underbrace{1, \ldots, 1}_{16-k+h \text{ terms}}\Big]. \tag{8.57}$$

Then

$$R_h^{(1)}(k) = w_h^T\,\hat{Y}_h^{*(1)}(k), h = 0,1,\ldots,k, \tag{8.58}$$

which is the estimated logged age-to-ultimate factor for underwriting year h on the basis of data triangle $X'(k)$.

The MSEP of the $R_h^{(1)}(k)$ may also be estimated. The estimates are given by

$$R_h^{(2)}(k) = w_h^T \underset{17\text{x}17}{MSEP_h^*}\, w_h,\; h = 0,1,\ldots,k, \tag{8.59}$$

with

$$MSEP_h^* = \begin{bmatrix} MSEP\left[\hat{Y}_h^{(1)}\right] & 0 \\ 0 & * \end{bmatrix} \tag{8.60}$$

where the upper left block is given by (8.35) and the lower right block consists of the prior estimate of covariance matrix. This latter may be obtained by setting $Z^{(1)} = 0$ in (8.35) and replacing $V_h^{(1)}$ by its estimate $\hat{V}_h^{(1)}$, and selecting the relevant sub-matrix.

Equations (8.58) and (8.59) generate triangles of $R_h^{(1)}(k)$ and $R_h^{(2)}(k)$. These are set out in Tables 8.3 and 8.4, where the factor shown for development year j applies to the case where this is the latest development year in the data triangle for the accident year concerned, ie $h + j = k$.

Table 8.3 Estimated logged age-to-ultimate factors

Accident year	Estimated logged age-to-ultimate factor at end of development year j= 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1978	0.300	0.206	0.151	0.115	0.090	0.071	0.057	0.045	0.036	0.028	0.022	0.016	0.011	0.007	0.003
1979	0.286	0.193	0.153	0.116	0.089	0.069	0.055	0.044	0.034	0.027	0.020	0.015	0.011	0.006	0.003
1980	0.266	0.210	0.153	0.113	0.085	0.068	0.053	0.042	0.034	0.026	0.020	0.015	0.010	0.006	.
1981	0.307	0.210	0.147	0.107	0.084	0.066	0.052	0.041	0.032	0.025	0.019	0.014	0.009	.	.
1982	0.307	0.199	0.138	0.107	0.082	0.063	0.051	0.040	0.032	0.025	0.019	0.013	.	.	.
1983	0.290	0.186	0.139	0.104	0.079	0.064	0.049	0.040	0.032	0.024	0.017	.	.	.	.
1984	0.270	0.188	0.135	0.100	0.080	0.061	0.049	0.040	0.031	0.023	.	.	.	.	.
1985	0.274	0.183	0.130	0.101	0.076	0.061	0.050	0.039	0.029	.	.	.	.	.	.
1986	0.268	0.178	0.133	0.097	0.077	0.062	0.049	0.036	.	.	.	.	.	.	.
1987	0.261	0.181	0.127	0.099	0.079	0.061	0.046	.	.	.	.	.	.	.	.
1988	0.266	0.174	0.130	0.101	0.077	0.058	.	.	.	.	.	.	.	.	.
1989	0.257	0.178	0.133	0.099	0.073	.	.	.	.	.	.	.	.	.	.
1990	0.263	0.182	0.131	0.094	.	.	.	.	.	.	.	.	.	.	.
1991	0.269	0.180	0.124	.	.	.	.	.	.	.	.	.	.	.	.
1992	0.266	0.171	.	.	.	.	.	.	.	.	.	.	.	.	.
1993	0.254	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Prior	0.300	0.206	0.151	0.115	0.090	0.071	0.057	0.045	0.036	0.028	0.022	0.016	0.011	0.007	0.003

Table 8.4 Estimated MSEP of logged age-to-ultimate factors

Accident	Estimated logged age-to-ultimate factor at end of development year j =														
year	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1978	0.049	0.025	0.014	0.009	0.008	0.005	0.003	0.002	0.001	0.001	0.000	0.000	0.000	0.000	0.000
1979	0.034	0.018	0.013	0.012	0.007	0.004	0.003	0.002	0.001	0.001	0.000	0.000	0.000	0.000	0.000
1980	0.026	0.018	0.018	0.011	0.007	0.004	0.003	0.002	0.001	0.001	0.000	0.000	0.000	0.000	.
1981	0.028	0.028	0.016	0.010	0.007	0.004	0.002	0.001	0.001	0.001	0.000	0.000	0.000	.	.
1982	0.043	0.025	0.015	0.010	0.006	0.004	0.002	0.001	0.001	0.001	0.000	0.000	.	.	.
1983	0.039	0.024	0.016	0.010	0.006	0.003	0.002	0.001	0.001	0.001	0.000	.	.	.	.
1984	0.038	0.024	0.015	0.009	0.005	0.003	0.002	0.001	0.001	0.000	.	.	.	.	.
1985	0.039	0.024	0.014	0.008	0.005	0.004	0.002	0.001	0.001	.	.	.	.	.	.
1986	0.038	0.022	0.013	0.008	0.006	0.003	0.002	0.001	.	.	.	.	.	.	.
1987	0.035	0.021	0.013	0.009	0.005	0.003	0.002	.	.	.	.	.	.	.	.
1988	0.033	0.020	0.014	0.008	0.005	0.003	.	.	.	.	.	.	.	.	.
1989	0.032	0.021	0.013	0.008	0.005	.	.	.	.	.	.	.	.	.	.
1990	0.034	0.021	0.013	0.008	.	.	.	.	.	.	.	.	.	.	.
1991	0.033	0.020	0.012	.	.	.	.	.	.	.	.	.	.	.	.
1992	0.031	0.019	.	.	.	.	.	.	.	.	.	.	.	.	.
1993	0.030	.	.	.	.	.	.	.	.	.	.	.	.	.	.

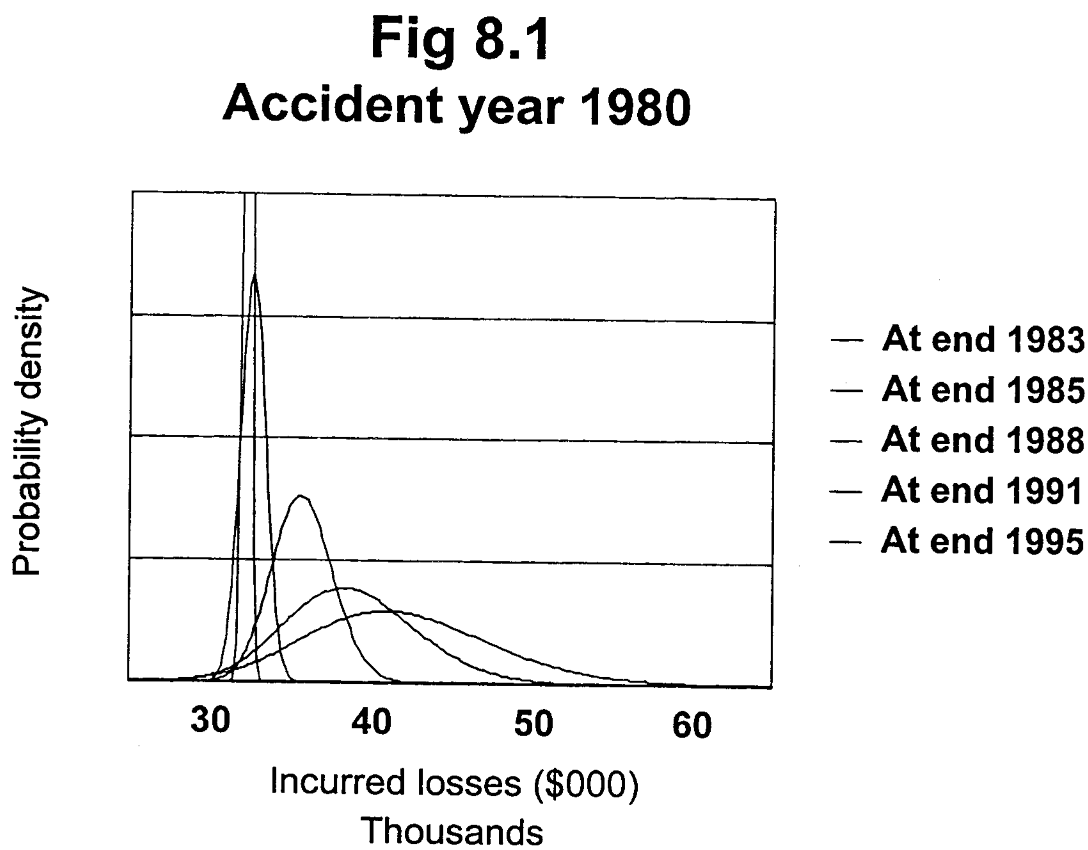
Table 8.5 Estimates of ultimate incurred losses

Accident year	Ultimate incurred losses as estimated at end of development year j=														
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000
1978	27,917	27,845	26,673	29,697	28,751	28,183	26,980	26,956	26,665	26,186	26,021	25,835	25,673	25,700	25,730
1979	23,142	22,698	24,073	24,350	25,466	24,034	22,841	22,386	21,599	21,750	20,910	20,279	20,172	20,076	20,044
1980	32,984	41,399	39,150	38,604	37,168	36,491	35,751	35,420	34,733	32,722	32,853	32,828	32,575	32,349	
1981	55,127	53,095	49,965	47,820	46,321	45,273	44,375	43,400	41,775	41,478	41,127	40,911	40,458		
1982	56,764	47,987	45,720	44,583	43,940	43,080	42,272	41,528	41,202	41,179	40,588	40,723			
1983	48,518	43,553	44,748	43,937	42,571	42,373	39,707	39,450	38,592	38,274	37,727				
1984	55,745	55,129	54,525	57,414	56,618	54,486	54,629	53,076	53,099	52,567					
1985	48,371	46,104	45,944	45,077	43,672	43,547	42,716	42,970	43,635						
1986	46,553	45,652	47,439	49,685	49,687	48,408	47,660	46,056							
1987	35,498	39,145	40,299	44,964	47,463	47,499	46,309								
1988	35,125	39,676	41,637	41,154	42,975	42,857									
1989	42,171	47,023	52,708	50,047	49,063										
1990	43,620	52,019	53,831	54,360											
1991	44,909	46,759	46,841												
1992	49,035	52,782													
1993	44,779														

Just as in Sections 5 to 7, all distributions of logged age-to-age or age-to-ultimate factors are normal. Thus, estimates of logged ultimate incurred losses, derived from Tables 7.1, 8.3 and 8.4, are also normal. This leads to the estimates of (unlogged) ultimate incurred losses in Table 8.5.

Figure 8.1 corresponds to Figure 7.1, giving a plot of the evolving distribution of estimated ultimate incurred losses for accident year 1980.

Figure 8.1



9. Acknowledgment

I am grateful to my colleague Steven Lim, who programmed calculations for the numerical example in Section 8.

Appendix A

Derivation of inhomogeneous credibility formula

Section 4.1 required minimisation of:

$$\Phi_{ir} = E\left[\alpha_0 + Y^T\alpha - A_{ir}^T\theta\right]^2. \quad (4.10)$$

Optimise on α_0 by setting $\partial\Phi_{ir} / \partial\alpha_0 = 0$:

$$E\left[\alpha_0 + Y^T\alpha - A_{ir}^T\theta\right] = 0. \quad (A.1)$$

Substitute this in (4.10):

$$\Phi_{ir} = E\left\{\left[Y^T - E\left[Y^T\right]\right]\alpha - A_{ir}^T(\theta-\beta)\right\}^2. \quad (A.2)$$

Optimise on α by setting $\partial\Phi_{ir} / \partial\alpha = 0$:

$$E\{Y - E[Y]\}\left\{\left[Y^T - E\left[Y^T\right]\right]\alpha - A_{ir}^T(\theta-\beta)\right\} = 0. \quad (A.3)$$

ie

$$V[Y]\alpha = Cov\left[Y, A_{ir}^T\theta\right]. \quad (A.4)$$

By (4.6),

$$\underset{nm\times 1}{Cov\left[Y, A_{ir}^T\theta\right]} = \begin{bmatrix} Cov\left[Y_1, A_{ir}^T\theta\right] \\ \vdots \\ Cov\left[Y_n, A_{ir}^T\theta\right] \end{bmatrix} \quad (A.5)$$

and

$$\begin{aligned} Cov\left[Y_h, A_{ir}^T\theta\right] &= E\left\{\left[Y_h - A_h\theta\right] + A_h(\theta-\beta)\right\} \mathrm{x} \left\{(\theta-\beta)^T A_{ir}\right\} \\ &= A_h \Gamma A_{ir}, \end{aligned} \quad (A.6)$$

by the independence assumptions set out in Section 4.1.

Similarly, $V[Y]$ takes the block diagonal from

$$\underset{nm \times nm}{V[Y]} = \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ V_{21} & & & \\ \vdots & & & \vdots \\ V_{n1} & & \cdots & V_{nn} \end{bmatrix} \tag{A.7}$$

with

$$\underset{m \times m}{V_{hi}} = \delta_{hi} V_i + A_h \Gamma A_i^T, \tag{A.8}$$

and δ_{hi} denoting the Kronesker delta.

When $V[Y]$ is put in the block form (A.7), it enables (A.4) to be expressed as n equations, of which the h-th is:

$$\sum_g V_{hg} \alpha_g = Cov\left[Y_h, A_{ir}^T \theta\right], \qquad h = 1, 2, \ldots, n, \tag{A.9}$$

where

$$\alpha^T = \left(\underset{1 \times m}{\alpha_1^T}, \ldots, \alpha_n^T \right). \tag{A.10}$$

Substitute (A.6) and (A.8) into (A.9):

$$V_h \alpha_h + A_h \Gamma \left(\sum_g A_g^T \alpha_g \right) = A_h \Gamma A_{ir}. \tag{A.11}$$

Pre-multiply by $A_h^T V_h^{-1}$ and sum the result over h:

$$(1 + M)\left(\sum_g A_g^T \alpha_g \right) = M A_{ir}, \tag{A.12}$$

with

$$M = \sum_{h=1}^{n} M_h \tag{A.13}$$

$$M_h = A_h^T V_h^{-1} A_h \Gamma. \tag{A.14}$$

Then

$$\sum_g A_g^T \alpha_g = Z A_{ir}, \tag{A.15}$$

with

$$Z = (1+M)^{-1} M = M(1+M)^{-1}. \tag{A.16}$$

Substitute (A.15) into (A.11) to obtain:

$$\alpha_h = V_h^{-1} A_h \Gamma (1-Z) A_{ir}. \tag{A.17}$$

Define

$$\hat{\theta}_h = \left(A_h^T V_h^{-1} A_h\right)^{-1} A_h^T V_h^{-1} Y_h, \tag{A.18}$$

which is the weighted least squares regression estimate of θ based on data Y_h.

Also define

$$Z_h = (1+M_h)^{-1} M_h = M_h (1+M_h)^{-1}. \tag{A.19}$$

Then

$$\begin{aligned} \hat{\theta}_h^T Z_h &= Y_h^T V_h^{-1} A_h \left(A_h^T V_h^{-1} A_h\right)^{-1} M_h (1+M_h)^{-1} \\ &= Y_h^T V_h^{-1} A_h \Gamma (1+M_h)^{-1} \\ &= Y_h^T V_h^{-1} A_h \Gamma (1-Z_h), \end{aligned} \tag{A.20}$$

where use has been made of the fact that

$$1 - Z_h = (1+M_h)^{-1}, \tag{A.21}$$

by (A.19).

By (A.20),

$$\hat{\theta}_h^T Z_h (1-Z_h)^{-1} (1-Z) A_{ir} = Y_h^T V_h^{-1} A_h \Gamma (1-Z) A_{ir} = Y_h^T \alpha_h. \tag{A.22}$$

Then

$$Y^T \alpha = \sum_h Y_h^T \alpha_h = \sum_h \hat{\theta}_h^T Z_h (1-Z_h)^{-1} (1-Z) A_{ir}. \tag{A.23}$$

It remains to evaluate α_0. By (A.1)

$$\begin{aligned}\alpha_0 &= A_{ir}^T\beta - E\left[Y^T\right]\alpha \\ &= \beta^T A_{ir} - \beta^T A^T\alpha \qquad \text{[by (4.1a), (4.6) and (4.7)]} \\ &= \beta^T(1-Z)A_{ir},\end{aligned} \tag{A.24}$$

by (A.15).

Substitute (A.23) and (A.24) in (4.8):

$$\hat{Y}_{ir} = \left[\beta^T + \sum_h \hat{\theta}_h^T Z_h (1-Z_h)^{-1}\right](1-Z)A_{ir}.$$

Equivalently,

$$\hat{Y}_{ir} = A_{ir}^T(1-Z^T)\left[\beta + \sum_h (1-Z_h^T)^{-1} Z_h^T\hat{\theta}_h\right],$$

and so

$$\hat{Y}_i = A_i(1-Z^T)\left[\beta + \sum_h (1-Z_h^T)^{-1} Z_h^T\hat{\theta}_h\right], \tag{A.25}$$

where $\hat{Y}_i$ is the m-vector with $\hat{Y}_{ir}$ as its r-th component.

Appendix B

Derivation of homogeneous credibility formula

The estimator (4.8) of μ_{ir} is modified to the following homogeneous form:

$$\hat{Y}_{ir} = Y^T \alpha, \tag{B.1}$$

subject to the **unbiasedness** constraint

$$E\left[\hat{Y}_{ir}\right] = E_\theta\left[\hat{Y}_{ir} \mid \theta\right]. \tag{B.2}$$

This condition reduces to:

$$A^T \alpha = A_{ir} \tag{B.3}$$

by (B.1) (4.1a), (4.6) and (4.7).

The argument below largely follows Taylor (1977).

The loss function to be minimised is still (4.9), but now subject to constraints (B.1) and (B.3). Therefore, define, in place of (4.10),

$$\Phi_{ir} \quad = E\left[Y^T \alpha - A_{ir}^T \theta\right]^2 - \underset{1 \times p}{\lambda^T} \underset{p \times 1}{\left(A^T \alpha - A_{ir}\right)}, \tag{B.4}$$

where λ is a Lagrange multiplier.

By (4.3), (4.6) and (4.7)

$$\begin{aligned} E\left[Y^T \alpha\right] &= \beta^T A^T \alpha \\ &= A_{ir}^T \beta, \end{aligned} \tag{B.5}$$

by (B.3).

Substitute (B.5) in (B.4):

$$\Phi_{ir} \quad = E\left\{\left[Y^T - E\left[Y^T\right]\right]\alpha - A_{ir}^T(\theta - \beta)\right\}^2 - \lambda^T\left(A^T \alpha - A_{ir}\right). \tag{B.6}$$

Differentiate with respect to α and set the result to zero:

$$E\{Y - E[Y]\}\left\{\left[Y^T - E\left[Y^T\right]\right]\alpha - A_{ir}^T(\theta - \beta)\right\} - A\lambda = 0. \tag{B.7}$$

This result replaces (A.3) in the inhomogeneous case. The earlier results (A.6) to (A.8) still hold. Substitute these in (B.7) to obtain n equations of which the h-th is:

$$V_h\alpha_h + A_h\Gamma\left(\sum_g A_g^T\alpha_g\right) - A_h\Gamma A_{ir} - A_h\lambda = 0, \tag{B.8}$$

which replaces (A.11).

Now follow the same procedure as led from (A.11) to (A.17), obtaining:

$$\alpha_h = V_h^{-1}A_h\Gamma(1-Z)A_{ir} + V_h^{-1}A_h\left(1-\Gamma Z\Gamma^{-1}\right)\lambda. \tag{B.9}$$

Pre-multiply by A_h^T:

$$\begin{aligned} A_h^T\alpha_h &= A_h^TV_h^{-1}A_h\Gamma(1-Z)A_{ir} + \left(A_h^TV_h^{-1}A_h - A_h^TV_h^{-1}A_h\Gamma Z\Gamma^{-1}\right)\lambda \\ &= M_h(1-Z)A_{ir} + \left(A_h^TV_h^{-1}A_h - M_hZ\Gamma^{-1}\right)\lambda, \end{aligned} \tag{B.10}$$

by (A.14).

Sum over h and apply constraint (B.3):

$$\begin{aligned} A_{ir} &= M(1-Z)A_{ir} + \left(M\Gamma^{-1} - MZ\Gamma^{-1}\right)\lambda && \text{[by (A.13) and (A.14)]} \\ &= ZA_{ir} + \left[1-M(1+M)^{-1}\right]M\Gamma^{-1}\lambda && \text{[by (A.16)]} \\ &= ZA_{ir} + (1+M)^{-1}M\Gamma^{-1}\lambda \\ &= ZA_{ir} + Z\Gamma^{-1}\lambda, \end{aligned} \tag{B.11}$$

by (A.16).

Solve for λ:

$$\lambda = \Gamma Z^{-1}(1-Z)A_{ir}. \tag{B.12}$$

Substitute (B.12) in (B.9):

$$\begin{aligned} \alpha_h &= V_h^{-1}A_h\Gamma(1-Z)A_{ir} + V_h^{-1}A_h\left(1-\Gamma Z\Gamma^{-1}\right)\Gamma Z^{-1}(1-Z)A_{ir} \\ &= V_h^{-1}A_h\Gamma Z^{-1}(1-Z)A_{ir} \\ &= V_h^{-1}A_h\Gamma M^{-1}A_{ir}, \end{aligned} \tag{B.13}$$

by (4.15).

Then

$$
\begin{aligned}
Y_h^T \alpha_h &= Y_h^T V_h^{-1} A_h \Gamma M^{-1} A_{ir} \\
&= \hat{\theta}_h^T \left(A_h^T V_h^{-1} A_h \right) \Gamma M^{-1} A_{ir} && \text{[by (4.12)]} \\
&= \hat{\theta}_h^T M_h M^{-1} A_{ir}, && \text{(B.14)}
\end{aligned}
$$

by (4.14).

Thus, by (B.1),

$$
\begin{aligned}
\hat{Y}_{ir} &= \sum_h Y_h^T \alpha_h = \left[\sum_h \hat{\theta}_h^T M_h \right] M^{-1} A_{ir} \\
&= A_{ir}^T (M^T)^{-1} \sum_h M_h^T \hat{\theta}_h \\
&= A_{ir}^T \overline{\theta}, && \text{(B.15)}
\end{aligned}
$$

by (4.17).

So

$$\hat{Y}_i = A_i \overline{\theta}. \qquad \text{(B.16)}$$

Appendix C
Example of Section 7

Table C.1 applies (5.10) for the case r=1 to the data of Table 7.2.

Table C.1
Summary statistics: means

Experience year	Mean logged age to age factor from development year j to j+1 as measured at end of experience year at left																
	development year j=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.678																
1980	0.586	0.100															
1981	0.549	0.079	0.104														
1982	0.543	0.088	0.092	0.018													
1983	0.644	0.154	0.157	0.060	0.145												
1984	0.661	0.175	0.133	0.040	0.097	-0.007											
1985	0.638	0.158	0.096	0.032	0.075	0.034	0.000										
1986	0.674	0.166	0.081	0.029	0.056	0.020	-0.018	-0.028									
1987	0.694	0.193	0.080	0.038	0.046	0.013	-0.012	-0.032	0.011								
1988	0.694	0.201	0.077	0.039	0.042	0.013	-0.010	-0.023	0.001	-0.001							
1989	0.706	0.208	0.077	0.041	0.049	0.010	-0.008	-0.019	0.002	-0.014	-0.010						
1990	0.699	0.221	0.088	0.046	0.044	0.010	-0.005	-0.016	-0.002	-0.013	0.002	0.000					
1991	0.715	0.244	0.100	0.050	0.049	0.008	-0.007	-0.022	-0.002	-0.017	-0.016	-0.016	-0.001				
1992	0.705	0.251	0.107	0.054	0.058	0.009	-0.004	-0.017	-0.001	-0.013	-0.012	-0.007	-0.013	-0.001			
1993	0.711	0.253	0.119	0.063	0.054	0.016	-0.005	-0.015	-0.004	-0.013	-0.008	-0.006	-0.007	-0.001	0.005		
1994	0.705	0.252	0.121	0.065	0.049	0.021	-0.002	-0.014	-0.001	-0.010	-0.007	-0.007	-0.006	-0.002	0.002	0.005	
1995	0.699	0.250	0.124	0.065	0.049	0.020	-0.001	-0.013	-0.004	-0.006	-0.006	-0.007	-0.003	-0.003	0.001	0.004	-0.007

Table C.2 applies (5.14) to the data of Table 7.2. The bold cells are those for which only one observation X(i,j) is available, and hence (5.14) is undefined. The entry in these cases is based on the value v1 (Table 7.3).

Table C.2
Summary statistics: unbiased standard deviations

Experience year	Standard deviation of logged age to age factor from development year j to j+1 as measured at end of experience year at left																
	development year j=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	**0.152**																
1980	0.131	**0.122**															
1981	0.113	0.029	**0.097**														
1982	0.093	0.025	0.016	**0.078**													
1983	0.239	0.135	0.113	0.060	**0.062**												
1984	0.218	0.126	0.104	0.054	0.069	**0.050**											
1985	0.208	0.120	0.122	0.047	0.062	0.057	**0.040**										
1986	0.217	0.112	0.116	0.041	0.063	0.047	0.026	**0.032**									
1987	0.212	0.128	0.106	0.042	0.059	0.041	0.021	0.006	**0.026**								
1988	0.200	0.122	0.098	0.039	0.054	0.035	0.018	0.016	0.014	**0.020**							
1989	0.194	0.117	0.092	0.036	0.052	0.033	0.016	0.016	0.010	0.018	**0.016**						
1990	0.186	0.120	0.093	0.037	0.050	0.030	0.017	0.015	0.010	0.013	0.017	**0.013**					
1991	0.187	0.139	0.097	0.037	0.048	0.028	0.016	0.019	0.009	0.013	0.033	0.023	**0.010**				
1992	0.183	0.135	0.096	0.038	0.054	0.027	0.016	0.022	0.009	0.014	0.028	0.022	0.017	**0.008**			
1993	0.178	0.130	0.102	0.048	0.053	0.033	0.015	0.021	0.010	0.012	0.026	0.018	0.016	0.001	**0.007**		
1994	0.174	0.126	0.098	0.047	0.054	0.035	0.016	0.020	0.012	0.014	0.023	0.016	0.013	0.001	0.004	**0.005**	
1995	0.169	0.121	0.095	0.045	0.052	0.033	0.017	0.019	0.013	0.018	0.021	0.014	0.013	0.002	0.004	0.002	**0.004**

Table C.3 applies (5.16) to the parameters set out in Table 7.3. Note that it also draws on the credibility estimates in Table C.6.

Table C.3
Credibility factors z1

Experi-ence year	Credibility factor z1 for logged age to age factor from development year j to j+1 as measured at end of experience year at left															
	development year j=															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.333																
1980	0.519	0.333															
1981	0.643	0.578	0.333														
1982	0.735	0.701	0.581	0.333													
1983	0.590	0.644	0.570	0.531	0.333												
1984	0.656	0.708	0.652	0.651	0.485	0.333											
1985	0.699	0.753	0.660	0.736	0.601	0.478	0.333										
1986	0.709	0.794	0.710	0.796	0.664	0.609	0.544	0.333									
1987	0.736	0.790	0.760	0.830	0.726	0.700	0.672	0.580	0.333								
1988	0.770	0.818	0.798	0.862	0.778	0.768	0.756	0.676	0.557	0.333							
1989	0.794	0.840	0.828	0.886	0.809	0.814	0.811	0.750	0.688	0.519	0.333						
1990	0.816	0.849	0.841	0.899	0.837	0.849	0.845	0.804	0.761	0.661	0.490	0.333					
1991	0.826	0.831	0.847	0.911	0.858	0.874	0.873	0.821	0.816	0.731	0.405	0.380	0.333				
1992	0.840	0.847	0.859	0.920	0.857	0.893	0.891	0.833	0.853	0.777	0.513	0.464	0.408	0.333			
1993	0.854	0.863	0.859	0.914	0.873	0.890	0.908	0.860	0.873	0.823	0.587	0.581	0.505	0.583	0.333		
1994	0.866	0.877	0.874	0.924	0.879	0.894	0.918	0.880	0.884	0.835	0.656	0.666	0.609	0.703	0.548	0.333	
1995	0.877	0.889	0.885	0.932	0.893	0.907	0.927	0.898	0.895	0.821	0.711	0.730	0.655	0.772	0.663	0.571	0.333

Table C.4 applies (5.13) to the parameters set out in Table 7.3.

Table C.4
Credibility factors z2

Experience year	Credibility factor z2 for logged age to age factor from development year j to j+1 as measured at end of experience year at left																
	development year j=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.167																
1980	0.286	0.167															
1981	0.375	0.286	0.167														
1982	0.444	0.375	0.286	0.167													
1983	0.500	0.444	0.375	0.286	0.167												
1984	0.545	0.500	0.444	0.375	0.286	0.167											
1985	0.583	0.545	0.500	0.444	0.375	0.286	0.167										
1986	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167									
1987	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167								
1988	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167							
1989	0.688	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167						
1990	0.706	0.688	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167					
1991	0.722	0.706	0.688	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167				
1992	0.737	0.722	0.706	0.688	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167			
1993	0.750	0.737	0.722	0.706	0.688	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167		
1994	0.762	0.750	0.737	0.722	0.706	0.688	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167	
1995	0.773	0.762	0.750	0.737	0.722	0.706	0.688	0.667	0.643	0.615	0.583	0.545	0.500	0.444	0.375	0.286	0.167

Tables C.5 and C.6 apply (5.8) for r=1,2, respectively, taking into account the parameters set out in Table 7.3 and the results of Tables C.1 to C.4.

Table C.5
Credibility estimates of parameters: means

Experience year	Credibility estimate of mean logged age to age factor from development year j to j+1 as measured at end of experience year at left																
	development year j=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.626																
1980	0.593	0.167															
1981	0.567	0.130	0.101														
1982	0.558	0.121	0.096	0.039													
1983	0.626	0.171	0.133	0.055	0.068												
1984	0.640	0.182	0.121	0.044	0.062	0.011											
1985	0.627	0.168	0.097	0.037	0.057	0.027	0.000										
1986	0.652	0.173	0.086	0.033	0.047	0.020	-0.010	-0.009									
1987	0.669	0.195	0.085	0.040	0.042	0.015	-0.008	-0.019	0.004								
1988	0.673	0.201	0.081	0.040	0.039	0.014	-0.007	-0.016	0.001	-0.000							
1989	0.684	0.207	0.081	0.042	0.045	0.012	-0.006	-0.014	0.001	-0.007	-0.003						
1990	0.681	0.218	0.090	0.046	0.042	0.011	-0.004	-0.013	-0.001	-0.008	0.001	0.000					
1991	0.695	0.237	0.100	0.050	0.046	0.009	-0.006	-0.018	-0.002	-0.012	-0.006	-0.006	-0.000				
1992	0.688	0.243	0.106	0.054	0.054	0.010	-0.004	-0.014	-0.001	-0.010	-0.006	-0.003	-0.005	-0.000			
1993	0.695	0.245	0.117	0.062	0.051	0.016	-0.004	-0.013	-0.003	-0.011	-0.005	-0.004	-0.004	-0.001	0.002		
1994	0.691	0.245	0.118	0.064	0.046	0.021	-0.002	-0.012	-0.001	-0.008	-0.005	-0.004	-0.003	-0.001	0.001	0.002	
1995	0.687	0.245	0.122	0.064	0.047	0.020	-0.000	-0.012	-0.003	-0.005	-0.004	-0.005	-0.002	-0.002	0.000	0.002	-0.002

Table C.6
Credibility estimates of parameters: standard deviations

Experience year	Credibility estimate of s.d. of logged age to age factors from development year j to j+1 as measured at end of experience year at left																
	development year j=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.152																
1980	0.146	0.122															
1981	0.139	0.104	0.097														
1982	0.129	0.097	0.083	0.078													
1983	0.200	0.128	0.104	0.073	0.062												
1984	0.191	0.124	0.101	0.070	0.064	0.050											
1985	0.187	0.121	0.110	0.066	0.062	0.052	0.040										
1986	0.195	0.116	0.108	0.062	0.063	0.049	0.036	0.032									
1987	0.193	0.126	0.102	0.061	0.060	0.046	0.034	0.027	0.026								
1988	0.186	0.122	0.098	0.058	0.058	0.043	0.032	0.027	0.023	0.020							
1989	0.182	0.119	0.094	0.056	0.057	0.041	0.030	0.026	0.021	0.020	0.016						
1990	0.177	0.120	0.095	0.055	0.055	0.039	0.030	0.025	0.020	0.018	0.017	0.013					
1991	0.178	0.134	0.097	0.054	0.054	0.038	0.028	0.026	0.019	0.017	0.024	0.017	0.010				
1992	0.175	0.132	0.097	0.054	0.057	0.037	0.028	0.027	0.018	0.017	0.022	0.017	0.013	0.008			
1993	0.172	0.128	0.101	0.058	0.056	0.039	0.027	0.026	0.018	0.016	0.022	0.016	0.013	0.007	0.007		
1994	0.169	0.125	0.098	0.057	0.057	0.040	0.027	0.025	0.018	0.017	0.020	0.015	0.012	0.007	0.006	0.005	
1995	0.166	0.121	0.096	0.055	0.055	0.039	0.026	0.024	0.019	0.019	0.019	0.014	0.012	0.006	0.006	0.005	0.004

Table C.7 repeats the credibility estimates of Table C.5 as forecasts of future age-to-age factors, based on data up to the end of the development year shown.

The bold cells are those in which forecasts need to be made with no data. Consider experience year 1981, for example, at the end of which accident years 1978 to1981 will have been observed over various development years, ranging up to 3. Hence, there are no data for forecasts beyond the case j=2 in the table. These cells adopt the prior means (Table 7.3) as forecasts.

Table C.7
Credibility forecasts of age-to-age factors

Experi-ence year	Credibility forecast of logged age to age factor from development year j to j+1 as measured at end of experience year at left															
	development year j=															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.626	**0.200**	**0.100**	**0.050**	**0.030**	**0.020**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1980	0.593	0.167	**0.100**	**0.050**	**0.030**	**0.020**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1981	0.567	0.130	0.101	**0.050**	**0.030**	**0.020**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1982	0.558	0.121	0.096	0.039	**0.030**	**0.020**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1983	0.626	0.171	0.133	0.055	0.068	**0.020**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1984	0.640	0.182	0.121	0.044	0.062	0.011	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1985	0.627	0.168	0.097	0.037	0.057	0.027	0.000	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1986	0.652	0.173	0.086	0.033	0.047	0.020	-0.010	-0.009	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1987	0.669	0.195	0.085	0.040	0.042	0.015	-0.008	-0.019	0.004	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1988	0.673	0.201	0.081	0.040	0.039	0.014	-0.007	-0.016	0.001	-0.000	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1989	0.684	0.207	0.081	0.042	0.045	0.012	-0.006	-0.014	0.001	-0.007	-0.003	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1990	0.681	0.218	0.090	0.046	0.042	0.011	-0.004	-0.013	-0.001	-0.008	0.001	0.000	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**
1991	0.695	0.237	0.100	0.050	0.046	0.009	-0.006	-0.018	-0.002	-0.012	-0.006	-0.006	-0.000	**0.000**	**0.000**	**0.000**	**0.000**
1992	0.688	0.243	0.106	0.054	0.054	0.010	-0.004	-0.014	-0.001	-0.010	-0.006	-0.003	-0.005	-0.000	**0.000**	**0.000**	**0.000**
1993	0.695	0.245	0.117	0.062	0.051	0.016	-0.004	-0.013	-0.003	-0.011	-0.005	-0.004	-0.004	-0.001	0.002	**0.000**	**0.000**
1994	0.691	0.245	0.118	0.064	0.046	0.021	-0.002	-0.012	-0.001	-0.008	-0.005	-0.004	-0.003	-0.001	0.001	0.002	**0.000**
1995	0.687	0.245	0.122	0.064	0.047	0.020	-0.000	-0.012	-0.003	-0.005	-0.004	-0.005	-0.002	-0.002	0.000	0.002	-0.002

Table C.8 uses (5.23) to calculate the RMSEP associated with Table C.7. Again, the bold cells adopt prior estimates, as given by (5.22) with z1=0.

Table C.8
RMSEP of credibility forecasts of age-to-age factors

Experi-ence year	MSEP of logged age to age factor from development year j to j+1 as measured at end of experience year at left																
	development year j=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.176	**0.149**	**0.119**	**0.095**	**0.076**	**0.061**	**0.049**	**0.039**	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1980	0.164	0.140	**0.119**	**0.095**	**0.076**	**0.061**	**0.049**	**0.039**	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1981	0.153	0.118	0.112	**0.095**	**0.076**	**0.061**	**0.049**	**0.039**	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1982	0.140	0.108	0.094	0.090	**0.076**	**0.061**	**0.049**	**0.039**	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1983	0.212	0.138	0.113	0.082	0.072	**0.061**	**0.049**	**0.039**	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1984	0.201	0.132	0.108	0.077	0.071	0.058	**0.049**	**0.039**	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1985	0.196	0.128	0.118	0.072	0.068	0.058	0.046	**0.039**	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1986	0.203	0.122	0.114	0.067	0.068	0.054	0.041	0.037	**0.031**	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1987	0.201	0.132	0.108	0.065	0.065	0.050	0.038	0.031	0.029	**0.025**	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1988	0.193	0.127	0.103	0.062	0.061	0.046	0.035	0.030	0.026	0.024	**0.020**	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1989	0.188	0.123	0.098	0.059	0.060	0.044	0.033	0.028	0.023	0.022	0.019	**0.016**	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1990	0.183	0.125	0.098	0.058	0.058	0.042	0.032	0.027	0.022	0.020	0.019	0.015	**0.013**	**0.010**	**0.008**	**0.007**	**0.005**
1991	0.184	0.139	0.101	0.057	0.056	0.040	0.030	0.027	0.021	0.019	0.026	0.018	0.012	**0.010**	**0.008**	**0.007**	**0.005**
1992	0.181	0.136	0.100	0.056	0.059	0.038	0.029	0.028	0.020	0.019	0.024	0.018	0.014	0.010	**0.008**	**0.007**	**0.005**
1993	0.177	0.132	0.104	0.061	0.058	0.041	0.028	0.027	0.019	0.017	0.023	0.017	0.014	0.008	0.008	**0.007**	**0.005**
1994	0.173	0.128	0.101	0.059	0.059	0.042	0.028	0.026	0.019	0.018	0.022	0.016	0.013	0.007	0.007	0.006	**0.005**
1995	0.170	0.125	0.099	0.057	0.057	0.040	0.027	0.025	0.019	0.020	0.020	0.015	0.013	0.007	0.006	0.005	0.005

Table C.9 combines the age-to-age factors of Table C.7.

Table C.9
Credibility forecasts of age-to-ultimate factors

Experi-ence	Credibility forecast of logged age to ultimate factor from end of development year j as measured at end of experience year at left																
	development year j=																
year	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	1.026	0.400	0.200	0.100	0.050	0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1980	0.959	0.367	0.200	0.100	0.050	0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1981	0.898	0.331	0.201	0.100	0.050	0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1982	0.864	0.306	0.185	0.089	0.050	0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1983	1.073	0.447	0.276	0.144	0.088	0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1984	1.061	0.421	0.239	0.117	0.074	0.011	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1985	1.013	0.386	0.218	0.120	0.084	0.027	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1986	0.993	0.341	0.167	0.081	0.048	0.001	-0.019	-0.009	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1987	1.022	0.353	0.158	0.073	0.034	-0.008	-0.023	-0.015	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1988	1.026	0.353	0.153	0.071	0.031	-0.008	-0.023	-0.015	0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1989	1.041	0.357	0.150	0.069	0.027	-0.018	-0.030	-0.023	-0.009	-0.010	-0.003	0.000	0.000	0.000	0.000	0.000	0.000
1990	1.063	0.382	0.164	0.074	0.028	-0.014	-0.025	-0.021	-0.008	-0.007	0.001	0.000	0.000	0.000	0.000	0.000	0.000
1991	1.086	0.391	0.154	0.055	0.005	-0.041	-0.051	-0.045	-0.027	-0.025	-0.013	-0.007	-0.000	0.000	0.000	0.000	0.000
1992	1.111	0.423	0.180	0.074	0.020	-0.034	-0.044	-0.041	-0.027	-0.026	-0.015	-0.009	-0.006	-0.000	0.000	0.000	0.000
1993	1.144	0.449	0.204	0.087	0.025	-0.027	-0.043	-0.039	-0.025	-0.022	-0.011	-0.006	-0.003	0.001	0.002	0.000	0.000
1994	1.151	0.460	0.215	0.097	0.032	-0.014	-0.034	-0.032	-0.020	-0.019	-0.011	-0.006	-0.002	0.002	0.003	0.002	0.000
1995	1.151	0.464	0.219	0.097	0.033	-0.013	-0.033	-0.033	-0.021	-0.017	-0.013	-0.008	-0.004	-0.002	0.000	-0.000	-0.002

Table C.10 combines the RMSEP of Table C.8, making use of the assumption of stochastic independence between the observations of different. development years.

Table C.10
RMSEP of credibility forecasts of age-to-ultimate factors

Experi-ence year	RMSEP of logged age to ultimate factor from end of development year j as measured at end of experience year at left development year j=																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1979	0.304	0.248	0.198	0.159	0.127	0.101	0.081	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1980	0.293	0.243	0.198	0.159	0.127	0.101	0.081	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1981	0.274	0.227	0.194	0.159	0.127	0.101	0.081	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1982	0.254	0.211	0.182	0.155	0.127	0.101	0.081	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1983	0.314	0.232	0.187	0.149	0.124	0.101	0.081	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1984	0.301	0.224	0.181	0.145	0.122	0.099	0.081	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1985	0.297	0.223	0.182	0.139	0.120	0.098	0.079	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1986	0.295	0.213	0.175	0.133	0.115	0.093	0.076	0.063	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1987	0.292	0.212	0.166	0.126	0.108	0.086	0.070	0.059	0.051	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1988	0.279	0.202	0.157	0.119	0.102	0.081	0.066	0.056	0.048	0.040	0.033	0.026	0.020	0.016	0.012	0.008	0.005
1989	0.271	0.194	0.150	0.114	0.097	0.077	0.063	0.053	0.045	0.039	0.032	0.026	0.020	0.016	0.012	0.008	0.005
1990	0.266	0.193	0.147	0.110	0.093	0.073	0.060	0.051	0.043	0.037	0.031	0.025	0.020	0.016	0.012	0.008	0.005
1991	0.274	0.203	0.148	0.109	0.093	0.074	0.062	0.054	0.047	0.042	0.037	0.027	0.020	0.016	0.012	0.008	0.005
1992	0.270	0.201	0.148	0.109	0.093	0.072	0.061	0.053	0.045	0.041	0.037	0.028	0.021	0.015	0.012	0.008	0.005
1993	0.267	0.201	0.151	0.110	0.092	0.071	0.058	0.051	0.043	0.039	0.034	0.026	0.020	0.014	0.011	0.008	0.005
1994	0.262	0.196	0.148	0.109	0.092	0.070	0.057	0.049	0.042	0.037	0.032	0.024	0.018	0.013	0.011	0.008	0.005
1995	0.256	0.191	0.145	0.106	0.089	0.069	0.056	0.048	0.041	0.037	0.031	0.023	0.017	0.012	0.010	0.007	0.005

Tables C.11 and C.12 give the forecast parameters of the normal distributions associated with each accident year (note that rows now denote accident years rather than experience years as in Tables C.1 to C.10) as it develops from its start to the end of experience year 1995.

These results are based directly on Tables C.9 and C.10. In the case of Table C.11, the forecast is obtained by taking the logged incurred claims for the cell concerned (Table 7.1) and adding the relevant forecast logged age-to-ultimate factor.

In each table two bold figures appear. That at development year 0 (end of 1978) is a forecast which precedes the data on which the above tables are based, and the prior logged age-to-ultimate factor and RMSEP have been used. The bold cell at the extreme right is that at which all development is assumed to be complete.

Table C.11
Credibility forecasts of mean logged incurred claims

Accident year	Credibility forecast of mean logged incurred claims at end of development year j																	
	development year j=																	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000
1978	**10.13**	10.21	10.11	10.12	10.08	10.20	10.17	10.17	10.14	10.16	10.15	10.14	10.14	10.14	10.14	10.15	10.15	**10.15**
1979	10.22	10.05	9.95	9.92	10.02	9.99	10.05	10.01	9.98	9.97	9.94	9.96	9.93	9.90	9.90	9.90	9.90	
1980	10.51	10.35	10.31	10.56	10.49	10.47	10.42	10.42	10.43	10.42	10.42	10.36	10.37	10.39	10.38	10.38		
1981	10.61	10.55	10.87	10.77	10.75	10.66	10.63	10.64	10.64	10.63	10.59	10.60	10.60	10.61	10.60			
1982	10.18	10.81	10.86	10.69	10.63	10.59	10.58	10.58	10.59	10.57	10.58	10.59	10.59	10.60				
1983	10.73	10.83	10.70	10.56	10.60	10.57	10.55	10.57	10.51	10.52	10.52	10.52	10.52					
1984	10.98	10.81	10.81	10.79	10.79	10.84	10.84	10.80	10.83	10.82	10.84	10.84						
1985	10.20	10.45	10.65	10.61	10.63	10.60	10.55	10.58	10.59	10.61	10.64							
1986	10.33	10.55	10.61	10.61	10.66	10.67	10.69	10.68	10.70	10.68								
1987	10.26	10.28	10.35	10.46	10.48	10.58	10.64	10.67	10.68									
1988	10.03	10.19	10.35	10.46	10.52	10.49	10.55	10.57										
1989	10.29	10.26	10.53	10.64	10.76	10.70	10.69											
1990	10.24	10.47	10.58	10.75	10.79	10.79												
1991	10.65	10.57	10.63	10.66	10.66													
1992	10.60	10.73	10.73	10.79														
1993	10.74	10.67	10.66															
1994	10.58	10.51																
1995	10.33																	

Table C.12
RMSEP of credibility forecasts of logged incurred claims

Accident year	RMSEP of credibility forecast of logged incurred claims at end of development year j development year j=																	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000
1978	**0.310**	0.248	0.198	0.159	0.127	0.101	0.081	0.065	0.052	0.041	0.033	0.026	0.020	0.016	0.012	0.008	0.005	**0.000**
1979	0.304	0.243	0.194	0.155	0.124	0.099	0.079	0.063	0.051	0.040	0.032	0.025	0.020	0.015	0.011	0.008	0.005	
1980	0.293	0.227	0.182	0.149	0.122	0.098	0.076	0.059	0.048	0.039	0.031	0.027	0.021	0.014	0.011	0.007		
1981	0.274	0.211	0.187	0.145	0.120	0.093	0.070	0.056	0.045	0.037	0.037	0.028	0.020	0.013	0.010			
1982	0.254	0.232	0.181	0.139	0.115	0.086	0.066	0.053	0.043	0.042	0.037	0.026	0.018	0.012				
1983	0.314	0.224	0.182	0.133	0.108	0.081	0.063	0.051	0.047	0.041	0.034	0.024	0.017					
1984	0.301	0.223	0.175	0.126	0.102	0.077	0.060	0.054	0.045	0.039	0.032	0.023						
1985	0.297	0.213	0.166	0.119	0.097	0.073	0.062	0.053	0.043	0.037	0.031							
1986	0.295	0.212	0.157	0.114	0.093	0.074	0.061	0.051	0.042	0.037								
1987	0.292	0.202	0.150	0.110	0.093	0.072	0.058	0.049	0.041									
1988	0.279	0.194	0.147	0.109	0.093	0.071	0.057	0.048										
1989	0.271	0.193	0.148	0.109	0.092	0.070	0.056											
1990	0.266	0.203	0.148	0.110	0.092	0.069												
1991	0.274	0.201	0.151	0.109	0.089													
1992	0.270	0.201	0.148	0.106														
1993	0.267	0.196	0.145															
1994	0.262	0.191																
1995	0.256																	

References

Bühlmann, H (1967). Experience rating and credibility. **Astin Bulletin**, 4, 199-207.

De Vylder, F (1981). Practical credibility theory with emphasis on optimal parameter estimation. **Astin Bulletin**, 12, 115-132.

Hachemeister, CA (1975). Credibility for regression models with application to trend. Appears in Kahn (1975), 129-163.

Jewell, WS (1974). The credible distribution. **Astin Bulletin**, 7, 237-269.

Kahn, PM (ed) (1975). **Credibility: theory and applications**. Academic Press, New York USA.

Taylor, GC (1977). Abstract credibility. **Scandinavian Actuarial Journal**, 149-168.

Taylor, GC (1999). **Loss reserving: an actuarial perspective**. Kluwer Academic Publishers, Boston USA (in press).

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28	FEB 96	THE AUSTRALIAN GOVERNMENT SUPERANNUATION CO-CONTRIBUTIONS: ANALYSIS AND COMPARISON	Margaret E Atkinson
29	MAR 96	A SURVEY OF VALUATION ASSUMPTIONS AND FUNDING METHODS USED BY AUSTRALIAN ACTUARIES IN DEFINED BENEFIT SUPERANNUATION FUND VALUATIONS	Des Welch Shauna Ferris
30	MAR 96	THE EFFECT OF INTEREST ON NEGATIVE SURPLUS	David C M Dickson Alfredo D Egídio dos Reis

31	MAR 96	RESERVING CONSECUTIVE LAYERS OF INWARDS EXCESS-OFF-LOSS REINSURANCE	Greg Taylor
32	AUG 96	EFFECTIVE AND ETHICAL INSTITUTIONAL INVESTMENT	Anthony Asher
33	AUG 96	STOCHASTIC INVESTMENT MODELS: UNIT ROOTS, COINTEGRATION, STATE SPACE AND GARCH MODELS FOR AUSTRALIA	Michael Sherris Leanna Tedesco Ben Zehnwirth
34	AUG 96	THREE POWERFUL DIAGNOSTIC MODELS FOR LOSS RESERVING	Ben Zehnwirth
35	SEPT 96	KALMAN FILTERS WITH APPLICATIONS TO LOSS RESERVING	Ben Zehnwirth
36	OCT 96	RELATIVE REINSURANCE RETENTION LEVELS	David C M Dickson Howard R Waters
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38	OCT 96	GEOGRAPHIC PREMIUM RATING BY WHITTAKER SPATIAL SMOOTHING	Greg Taylor
39	OCT 96	RISK, CAPITAL AND PROFIT IN INSURANCE	Greg Taylor
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42	DEC 96	VIDEO-CONFERENCING IN ACTUARIAL STUDIES – A THREE YEAR CASE STUDY	David M Knox
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47	NOV 97	ON NUMERICAL EVALUATION OF FINITE TIME RUIN PROBABILITIES	David C M Dickson
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49	JAN 98	A DECOMPOSITION OF ACTUARIAL SURPLUS AND APPLICATIONS	Daniel Dufresne
50	JAN 98	PARTICIPATION PROFILES OF AUSTRALIAN WOMEN	M. E. Atkinson Roslyn Cornish

51	MAR 98	PRICING THE STOCHASTIC VOLATILITY PUT OPTION OF BANKS' CREDIT LINE COMMITMENTS	J.P. Chateau Daniel Dufresne
52	MAR 98	ON ROBUST ESTIMATION IN BÜHLMANN STRAUB'S CREDIBILITY MODEL	José Garrido Georgios Pitselis
53	MAR 98	AN ANALYSIS OF THE EQUITY IMPLICATIONS OF RECENT TAXATION CHANGES TO AUSTRALIAN SUPERANNUATION	David M Knox M. E. Atkinson Susan Donath
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55	APR 98	SUPER BENEFITS? ESTIMATES OF THE RETIREMENT INCOMES THAT AUSTRALIAN WOMEN WILL RECEIVE FROM SUPERANNUATION	Susan Donath
56	APR 98	A UNIFIED APPROACH TO THE STUDY OF TAIL PROBABILITIES OF COMPOUND DISTRIBUTIONS	Jun Cai José Garrido
57	MAY 98	THE DE PRIL TRANSFORM OF A COMPOUND R_k DISTRIBUTION	Bjørn Sundt Okechukwu Ekuma
58	MAY 98	ON MULTIVARIATE PANJER RECURSIONS	Bjørn Sundt
59	MAY 98	THE MULTIVARIATE DE PRIL TRANSFORM	Bjørn Sundt
60	JUNE 98	ON ERROR BOUNDS FOR MULTIVARIATE DISTRIBUTIONS	Bjørn Sundt
61	JUNE 98	THE EQUITY IMPLICATIONS OF CHANGING THE TAX BASIS FOR PENSION FUNDS	M E Atkinson John Creedy David Knox
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63	JUNE 98	AN AFFINE PROPERTY OF THE RECIPROCAL ASIAN OPTION PROCESS	Daniel Dufresne
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