STOCHASTIC LIFE ANNUITIES

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Abstract

This paper aims at finding explicit approximations to the distribution of a stochastic life annuity. It is assumed that returns follow a geometric Brownian motion (lognormal process). The distribution of the stochastic annuity may be used to answer questions such as "What is the probability that an amount *F* is sufficient to fund a pension with annual amount *y* to a pensioner aged *x*?" The main idea is to approximate the future lifetime distribution with a combination of exponentials, and then apply a known formula related to the integral of geometric Brownian motion.

Keywords: Stochastic life annuities; Integral of geometric Brownian motion; Sums of lognormals

1. Introduction

Pollard (1971) was apparently the first to consider the problem of finding the distribution of a life annuity, when both mortality and the discount rates are random. Several authors have studied the same problem since, including Boyle (1976), Waters (1978), Wilkie (1978), Panjer & Bellhouse (1980), Bellhouse & Panjer (1981), Westcott (1981), Dufresne (1989, 1990, 1992), Frees (1991), Parker (1994a, 1994b). Most, if not all, of the contributions focused on calculating the moments of the annuity, one goal being that the distribution could be reconstructed (or approximated) from its moments. This paper proceeds differently. The distribution of a stochastic life annuity with an exponential lifetime distribution is already known explicitly, in the case where rates of discount are independent over time (more details below). It can then be seen that the same holds if the lifetime has a density given by a combinations of exponentials. Next, combinations of exponentials are dense in the set of distributions on $\mathbb{R}_+$; in other words, any lifetime distribution may be approximated to any degree of precision by a combination of exponentials. Consequently, the distribution of a stochastic life annuity, with a given (arbitrary) lifetime distribution, may be approximated by the distributions of stochastic life annuities with lifetimes distributed as combinations of exponentials, and the approximating distributions become exact as the approximating lifetime distributions converge to the true lifetime distribution. Each approximating stochastic life annuity distribution is known exactly, and will be identified. Numerical illustrations will be given in a subsequent paper.

Many of the papers cited above assumed that rates of discount form an autoregressive process, probably to reflect the well-known serial dependence of interest rates. This paper is based on a possibly different idea, which goes back to the classical definition of discounting. Let U_t be the time-t value of one dollar invested at time 0. This dollar may be invested in stocks, bonds, and so on. The amount required at time 0 to fund a c dollar cash-flow at time t is thus c divided by U_t. The amount initially required to fund several cash flows is then the weighted sum

$$S = \sum_j \frac{c_j}{U_{t_j}}.$$

This amount is of course random, and is not a "price" in the sense of no-arbitrage pricing (as described in Harrison & Pliska (1981)). Nevertheless, there is an interest in knowing the probability distribution of S, for instance when comparing investment strategies, or, possibly, when informing insurance contractholders of the likely amounts of their benefit payments, if there is a "variable" component in their contracts.

It will be assumed that the rates of return on the investments chosen are serially independent. This is the usual assumption for stocks, adopted by Black & Scholes (1973), though their "lognormal" model goes back to Osborne (1959); the first appearance of Brownian motion was also as part of a financial model (Bachelier, 1900), and also featured returns which are independent over time, though it is the asset values themselves (and not their logarithm) which were normally distributed. Bond yields, however, are highly correlated over time. Nevertheless, it can be argued that a managed portfolio of bonds would often have returns which are less correlated over time.

Let $\{W_t; t \geq 0\}$ be a standard Brownian motion, and define

$$U_t \; = \; e^{mt+\sigma W_t}.$$

The process $\{U_t; t \geq 0\}$ is called "geometric Brownian motion". As indicated above, U_t will represent the time-t value of of one dollar invested at time 0. The distribution of U_t is always lognormal, but the precise parameters of the distribution depend on the investment strategy followed. The variable S defined above is a sum of lognormals, a topic studied from a different point of view in Dufresne (2004b).

There is another source of randomness in a stochastic life annuity. The variable T will represent the future lifetime of an insured or pensioner. We will always assume that investment results and survival do not affect one another, or, expressed mathematically, that T and the Brownian motion W are stochastically independent. The amount required at time 0 to fund an annuity paid continuously while the pensioner is alive is

$$\int_0^T U_s^{-1}\, ds \; = \; \int_0^T e^{-ms-\sigma W_s}\, ds. \tag{1.1}$$

This formula corresponds to a level annuity, but note that obvious modifications allow for (1) payments that increase exponentially (say, with inflation), by including an exponential factor e^{yt} in the integral; and (2) guarantee periods, by replacing T with $\max(T, n)$. It does not seem possible to apply our results to discretely paid annuities. The difference between the discounted values of annuities paid monthly and continuously is not very great (see Bowers *et al*, 1997). Observe that the distribution of T does not have to be continuous (see Sections 3 and 4).

Section 2 recalls some known results about the integral of Brownian motion which are used in the sequel. Section 3 deals with the approximation of distributions on $\mathbb{R}_+$ by combinations of exponentials, and Section 4 proves the main result, that the distribution of a stochastic life annuity can be approximated to any degree of precision by a stochastic annuity for a duration T that has a combination of exponentials as density. Section 5 looks at transforms and moments of stochastic life annuities, and shows that the distribution of a stochastic life annuity is in general NOT determined by its moments (but it is determined by its reciprocal moments). Finally, Section 6 looks at how the distribution of a stochastic annuity certain (*i.e.* with fixed duration but stochastic discount rates) may be obtained from the result in Section 4; two apparently new techniques to approximate the distribution are described.

2. Fundamental results on the integral of Brownian motion

This section collects some know results about the integral of Brownian motion. More details can be found in the references; in particular, Yor (2001) contains several papers on these topics, while Dufresne (2004a) gives a concise account of the results, with complete references.

The expression in (1.1) has three parameters. Using the scaling property of Brownian motion, it is possible to transform it in such a way that one of the parameters has a fixed value. The connection with Bessel processes (Yor, 1992) makes the choice $\sigma = 2$ more convenient in many developments. We thus follow Yor's notation and define

$$A_t^{(\mu)} \;=\; \int_0^t e^{2(\mu s + W_s)}\,ds, \quad t \geq 0, \tag{2.1}$$

where W is a standard Brownian motion.The conversion rule between (1.1) and (2.1) is:

$$\int_0^T e^{-ms-\sigma W_s} \text{ has the same distribution as } \frac{4}{\sigma^2}A_t^{(\mu)}, \text{ where } t \;=\; \frac{\sigma^2 T}{4},\ \mu \;=\; -\frac{2m}{\sigma^2}. \tag{2.2}$$

The process $A_t^{(\mu)}$ has been studied in several contexts, notably in relation to Asian options (for instance Geman & Yor (1993)).

Moments. The distribution of $A_t^{(\mu)}$ has all moments (positive and negative) finite, that is

$$\mathsf{E}(A_t^{(\mu)})^r \;<\; \infty \qquad \forall\, r \in \mathbb{R}.$$

Ramakrishnan (1954) found that

$$\mathsf{E}(A_t^{(\mu)})^n \;=\; \sum_{k=0}^n b_{n,k}e^{a_k t}, \quad a_k \;=\; 2k\mu + 2k^2, \quad b_{n,k} \;=\; n!\Big/\prod_{\substack{j=0\\ j\neq k}}^n (a_j - a_k). \tag{2.3}$$

This result was rederived by Dufresne (1989) and (independently) by Yor (1992). The conversion rule (2.2) is used in an obvious way to find corresponding formulas for

$$\mathsf{E}\Big(\int_0^T e^{\mu s+\sigma W_s}\,ds\Big)^n, \qquad n = 1, 2 \ldots$$

The distribution of $A_t^{(\mu)}$ sampled at an independent exponential time. The result below first appeared in Yor (1992). Some steps of the proof are given, as background for Sections 4 and 6.

Let $S_\lambda \sim$ **Exponential**(λ), that is, let S_λ be a random variable with density

$$f_{S_\lambda}(x) \;=\; \lambda e^{-\lambda x}1_{\{x>0\}}.$$

Suppose moreover that S_λ is independent of the Brownian motion W in (2.1). The idea of the proof is to find a function $h_\lambda(\cdot,\cdot)$ such that for any non-negative functions $f(\cdot)$ and $g(\cdot)$,

$$\mathsf{E}[f(e^{W^{(\mu)}_{S_\lambda}})g(A^{(\mu)}_{S_\lambda})] \;=\; \int_0^\infty\int_0^\infty f(r)g(u)h_\lambda(r,u)\,dr\,du. \tag{2.4}$$

(Here $W_t^{(\mu)} = \mu t + W_t$.) Then $h_\lambda(\cdot,\cdot)$ is the joint density function of $(e^{W_{S_\lambda}^{(\mu)}}, A_{S_\lambda}^{(\mu)})$.

By the Cameron-Martin theorem,

$$\mathsf{E}[f(e^{W_t^{(\mu)}})g(A_t^{(\mu)})] = e^{-\mu^2 t/2}\mathsf{E}[e^{\mu W_t} f(e^{W_t})g(A_t)],$$

and so

$$\mathsf{E}[f(e^{W_{S_\lambda}^{(\mu)}})g(A_{S_\lambda}^{(\mu)})] \;=\; \mathsf{E}\int_0^\infty \lambda e^{-\lambda t-\mu^2 t/2} e^{\mu W_t} f(e^{W_t})g(A_t)\,dt.$$

Yor then used a stochastic time change and the properties of Bessel processes to find that, if $\gamma = \sqrt{2\lambda+\mu^2}$,

$$h_\lambda(r,u) \;=\; \frac{\lambda}{u} r^{\mu-1} e^{-(1+r^2)/2u} I_\gamma(\tfrac{r}{u}) 1_{\{r,u>0\}}, \tag{2.5}$$

where $I_\cdot(\cdot)$ is the modified Bessel function of the first kind:

$$I_p(z) \;=\; \sum_{n=0}^\infty \frac{(z/2)^{p+2n}}{n!\Gamma(n+p+1)}.$$

The density of $A_{S_\lambda}^{(\mu)}$ may be found by integrating out r in (2.5); this yields

$$f_\lambda(u) \;=\; \lambda(2u)^{(\mu-\gamma)/2-1}\frac{\Gamma\big(\frac{\mu+\gamma}{2}\big)}{\Gamma(\gamma+1)}\,{}_1F_1\big(\tfrac{\gamma-\mu}{2}+1, \gamma+1, -\tfrac{1}{2u}\big)1_{\{u>0\}}, \tag{2.6}$$

where ${}_1F_1$ is the confluent hypergeometric function:

$${}_1F_1(a,b;z) \;=\; \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{z^n}{n!}.$$

Now, observe that if $B \sim$ **Beta**(c,d) and $G \sim$ **Gamma**$(f,1)$ are independent, then the density of B/G is

$$g(x) \;=\; \frac{x^{-f-1}\Gamma(c+d)}{\Gamma(c)\Gamma(d)\Gamma(f)}\int_0^1 e^{-y/x}y^{c+f-1}(1-y)^{d-1}\,dy\,1_{\{x>0\}}.$$

From the formulas (Lebedev, 1972, pp.266-267),

$$\begin{aligned}
{}_1F_1(a,b;z) \;&=\; \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}\,dt, \qquad \mathsf{Re}\,b > \mathsf{Re}\,a > 0\\
e^{-z}{}_1F_1(a,b;z) \;&=\; {}_1F_1(b-a,b;-z),
\end{aligned}$$

the density of B/G can also be expressed as

$$g(x) \;=\; \frac{x^{-f-1}\Gamma(c+d)\Gamma(c+f)}{\Gamma(c)\Gamma(f)\Gamma(c+d+f)}\,{}_1F_1(c+f, c+d+f; -\tfrac{1}{x})\,1_{\{x>0\}}.$$

This proves the following result.

Theorem 2.1 (Yor). *The distribution of $A^{(\mu)}_{S_\lambda}$ has density (2.6), and is the same as that of $B_{1,\alpha}/2G_\beta$, where $B_{1,\alpha}$ and G_β are independent, $B_{1,\alpha} \sim$ **Beta**$(1,\alpha)$, $G_\beta \sim$ **Gamma**$(\beta, 1)$, with*

$$\alpha \;=\; \frac{\gamma+\mu}{2}, \qquad \beta \;=\; \frac{\gamma-\mu}{2}, \qquad \gamma \;=\; \sqrt{2\lambda+\mu^2}.$$

Laplace transform of the distribution of $A^{(\mu)}_t$. It has been proved that the distribution of $A^{(\mu)}_t$ is not determined by its moments (Hörfelt, 2004); this makes sense intuitively, as the same situation prevails for the lognormal distribution. Moreover, $\mathsf{E}\exp(sA^{(\mu)}_t) = \infty$ for all $s > 0$, as for the lognormal distribution. By contrast, and a little surprisingly, $\mathsf{E}\exp(s/(2A^{(\mu)}_t)) < \infty$ if $s < 1$, which implies that the distribution of $1/A^{(\mu)}_t$ is determined by its moments (Dufresne, 2001). The same reference shows formulas for the moments of $1/A^{(\mu)}_t$. No simple formula is known for the Laplace transform of $A^{(\mu)}_t$.

The distribution of $A^{(\mu)}_t$. The distribution of the integral of geometric Brownian motion at a fixed time is known, but the expressions found so far for its density are more complicated than in the case of the process sampled at an independent exponential time (see (2.6)).

The first expression for the density of $A^{(\mu)}_t$ is due to Wong (1964). Monthus and Comtet (1994) write Wong's eigenfunction expansion as

$$\begin{aligned} f(t,x) \;&= e^{-\frac{1}{2x}}\Bigg[\sum_{0\le n<-\mu/2} e^{2tn(\mu+n)}\tfrac{(-1)^{n+1}(\mu+2n)}{\Gamma(1-\mu-n)}\left(\tfrac{1}{2x}\right)^{1-\mu-n} L_n^{-\mu-2n}\left(\tfrac{1}{2x}\right) \\ &\qquad + \tfrac{1}{2\pi^2}\int_0^\infty ds\, e^{-\frac{t}{2}(\mu^2+s^2)} s\sinh(\pi s)\left|\Gamma\left(\tfrac{\mu+is}{2}\right)\right|^2\left(\tfrac{1}{2x}\right)^{\frac{1-\mu}{2}} W_{(1-\mu)/2,is/2}\left(\tfrac{1}{2x}\right)\Bigg], \end{aligned}$$

where $x > 0$, $W_{a,b}$ is Whittaker's function, and L^a_b is the generalized Laguerre polynomial.

The second expression for the density of $A^{(\mu)}_t$ is due to Yor (1992), who found it to be

$$f(t,x) \;=\; \frac{e^{-\mu^2 t/2}}{x}\int_{-\infty}^{\infty} e^{\mu u-\frac{1}{2x}(1+e^{2u})}\theta_{e^u/x}(t)\,du,$$

where

$$\theta_r(t) \;=\; \frac{re^{\frac{\pi^2}{2t}}}{\sqrt{2\pi^3 t}}\int_0^\infty \exp(-y^2/2t)\exp(-r\cosh y)\sinh(y)\sin\left(\tfrac{\pi y}{t}\right)dy.$$

A third expression for the density of $A^{(\mu)}_t$ is (Dufresne, 2000)

$$f(t,x) \;=\; 2^{-b-1}c^{a+1}x^{-b-2}e^{-c/2x}\sum_{n=0}^{\infty} a_n(t)L^a_n(c/2x),$$

where $a > -1, b \in \mathbb{R}, 0 < c < 1$, and

$$\begin{aligned} a_n(t) \;&=\; \frac{n!}{\Gamma(n+a+1)}\mathsf{E}\,L^a_n\big(c/2A^{(\mu)}_t\big) \\ &=\; \sum_{k=0}^{n}\frac{n!(-c)^k}{\Gamma(k+a+1)k!(n-k)!}\mathsf{E}(2A^{(\mu)}_t)^{-(a-b+k)}. \end{aligned}$$

A fourth expression for the distribution of $A_t^{(\mu)}$ is given in Dufresne (2001). If

$$q(y,t) \;=\; \frac{e^{\frac{\pi^2}{8t}-\frac{y^2}{2t}}}{\pi\sqrt{2t}}\cosh y,$$

then the density of $1/(2A_t^{(\mu)})$ is

$$2^{-\mu}x^{-\frac{\mu+1}{2}}e^{-\mu^2t/2}\int_{-\infty}^{\infty}e^{-x\cosh^2 y}q(y,t)\cos\left(\tfrac{\pi}{2}(\tfrac{y}{t}-\mu)\right)H_\mu(\sqrt{x}\sinh y)\,dy.$$

Here H_μ is the Hermite function (Lebedev, 1972, Chapter 10). This expression reduces to a single integral when $\mu = 0, 1, 2, \ldots$, but is otherwise a double integral. Other equivalent expressions are given in Dufresne (2001).

The distribution of $A_\infty^{(\mu)}$. For any $\mu < 0$,

$$\frac{1}{2A_\infty^{(\mu)}} \;\sim\; \textbf{Gamma}(-\mu, 1) \tag{2.7}$$

(Dufresne, 1990). If $\mu \geq 0$, then

$$\lim_{t\to\infty} A_t^{(\mu)} \;=\; \infty \quad \text{a.s.}$$

N.B. From the scaling property of Brownian motion, this means that

$$\frac{2}{\sigma^2}\left(\int_0^\infty e^{-mt-\sigma W_t}\,dt\right)^{-1} \;\sim\; \textbf{Gamma}\big(\tfrac{2m}{\sigma^2}, 1\big) \qquad \forall\, m, \sigma > 0.$$

3. Approximating distributions on $\mathbb{R}_+$ by combinations of exponentials

By "combination of exponentials" we mean a function of the form

$$f(x) \;=\; \sum_{j=1}^n a_j\lambda_j e^{-\lambda_j x}1_{\{x>0\}}, \tag{3.1}$$

where $\{a_j\}, \{\lambda_j\}$ are constants. This function is a probability density function if

$$(1)\quad \sum_{j=1}^n a_j = 1; \quad (2)\quad \lambda_j > 0 \quad \forall\, j; \quad (3)\quad f(x) \;\geq\; 0 \quad \forall\, x \geq 0.$$

Conditions (1) and (2) imply that the function $f(\cdot)$ integrates to one over $\mathbb{R}_+$, but they do not imply (3); for example, consider

$$f(x) \;=\; e^{-x} - 8\cdot 2e^{-2x} + 8\cdot 3e^{-3x}.$$

If $a_j > 0$ for all j, then (3.1) is called a "mixture of exponentials". The following results are well-known. (*N.B*. A sequence of random variables $\{X_n\}$ is said to *converge in distribution* to a random variable X if $\mathsf{P}(X_n \leq x)$ converges to $\mathsf{P}(X \leq x)$ for all x where the distribution function of X is continuous; convergence in distribution is also called *weak convergence*. See Billingsley (1986).)

Theorem 3.1. *(a) Suppose T is a non-negative random variable. Then there exists a sequence $\{T_n\}$ of random variables each with a density given by a combination of exponentials and such that T_n converges in distribution to T.*

(b) If the distribution of T has no atom (i.e. a point t_0 with $\mathsf{P}(T = t_0) > 0$), then

$$\lim_{n\to\infty} \sup_{0\leq t<\infty} |F_T(t) - F_{T_n}(t)| \;=\; 0.$$

Proof. (a) A distribution function $F(\cdot)$ can be approximated to any degree of precision (at all points t) by some discrete distribution. For instance, one may define a new distribution by first giving it the same atoms as $F(\cdot)$, and then, betweens those atoms, making it a step function close to $F(\cdot)$. The problem then reduces to approximating discrete distributions that have a finite number of possible values; the latter are convex combinations of degenerate distributions (*i.e* that take only one value). Part (a) will then be proved if we find how to approximate a degenerate distribution at a point $t_0 \geq 0$ by combinations of exponentials. It is not difficult to see that **Gamma**(m, β) distributions converge to the degenerate distribution at $t_0 \geq 0$, if m tends to infinity and m/β tends to t_0. Finally, combinations of exponentials can approximate a **Gamma**(m, β) distribution to any degree of precision, for any positive integer m and $\beta > 0$.

(b) Suppose the limit distribution $F_T(\cdot)$ is continuous, and let $\{T_n\}$ be as in (a). Let $\epsilon > 0$, and find points $t_1 < \cdots < t_m$ such that

$$F(t_1) \;<\; \epsilon, \quad F_T(t_{j+1}) - F_T(t_j) \;<\; \epsilon \quad \forall\, j = 1, \ldots, m-1, \quad F(t_m) \;>\; 1-\epsilon.$$

Then there exists n_0 such that

$$|F_{T_n}(t_j) - F_T(t_j)| \;<\; \epsilon, \quad j = 1, \ldots, m, \quad \forall\, n \geq n_0.$$

Finally,

$$\sup_{0\leq t<\infty} |F_{T_n}(t) - F_T(t)| \;<\; 2\epsilon. \qquad \square$$

4. The distribution of a stochastic life annuity

Given that T is independent of the Brownian motion W, the distribution of $A_T^{(\mu)}$ can be found by conditioning on T:

$$\mathsf{P}(A_T^{(\mu)} \leq x) \;=\; \int \mathsf{P}(A_t^{(\mu)} \leq x)\, dF_T(t). \tag{4.1}$$

However, in light of the known expressions for the density of $A_t^{(\mu)}$ (see Section 2), it appears that (4.1) may not be easy to use in numerical applications. The approach suggested in this paper is to approximate the distribution of T by a combination of exponentials, and then apply Theorem 2.1.

The following result says that the error made in approximating the distribution of $A_{T_1}^{(\mu)}$ by $A_{T_2}^{(\mu)}$ is never larger than the error made in approximating T_1 by T_2, which is a good thing numerically.

Theorem 4.1. *Let $T_1, T_2 \geq 0$ be random variables with distribution functions $F_{T_1}(\cdot), F_{T_2}(\cdot)$, respectively, and that are independent of W. Then*

$$|\mathsf{P}(A_{T_1}^{(\mu)} \leq x) - \mathsf{P}(A_{T_2}^{(\mu)} \leq x)| \;\leq\; \sup_{0 \leq t < \infty} |F_{T_1}(t) - F_{T_2}(t)| \quad \forall\, x. \tag{4.2}$$

If $\mu < 0$, then

$$|\mathsf{P}(A_{T_1}^{(\mu)} \leq x) - \mathsf{P}(A_{T_2}^{(\mu)} \leq x)| \;\leq\; C \sup_{0 \leq t < \infty} |F_{T_1}(t) - F_{T_2}(t)|,$$

where

$$C \;=\; \frac{1}{\Gamma(-\mu)} \int_0^{\frac{1}{2x}} x^{-\mu-1} e^{-x}\, dx \;<\; 1.$$

Proof. Applying the integration by parts formula to (4.1), we get, for $x > 0$,

$$\begin{aligned} \mathsf{P}(A_T^{(\mu)} \leq x) \;&=\; -\int \mathsf{P}(A_t^{(\mu)} \leq x)\, d[1 - F_T(t)] \\ &=\; 1 - \int [1 - F_T(t)]\, d_t \mathsf{P}(A_t^{(\mu)} > x). \end{aligned}$$

The function $t \mapsto \mathsf{P}(A_t^{(\mu)} > x)$ is non-decreasing. Hence

$$\begin{aligned} |\mathsf{P}(A_{T_1}^{(\mu)} \leq x) - \mathsf{P}(A_{T_2}^{(\mu)} \leq x)| \;&\leq\; \int |F_{T_1}(t) - F_{T_2}(t)|\, d_t \mathsf{P}(A_t^{(\mu)} > x) \\ &\leq\; \Big(\sup_{0 \leq t < \infty} |F_{T_1}(t) - F_{T_2}(t)| \Big) \mathsf{P}(\lim_{t \to \infty} A_t^{(\mu)} > x). \end{aligned}$$

This proves (4.2). If $\mu < 0$, we may then apply (2.7), which says that

$$\mathsf{P}(A_\infty^{(\mu)} > x) \;=\; \mathsf{P}(G_{-\mu} < \tfrac{1}{2x}) \;=\; C. \qquad \square$$

Remark. This theorem does not rest on the particular definition of the process $\{A_t^{(\mu)}\}$, as the same proof holds more generally: *if T_1, T_2 are independent of a non-decreasing process $\{X_t\}$ with $X_t = 0$ for $t < 0$, then, for all x,*

$$|\mathsf{P}(X_{T_1} \leq x) - \mathsf{P}(X_{T_2} \leq x)| \;\leq\; \mathsf{P}(\lim_{t \to \infty} X_t > x) \sup_{0 \leq t < \infty} |F_{T_1}(t) - F_{T_2}(t)|.$$

A question arises, however, in cases where the distribution function of $A_T^{(\mu)}$ is approximated by a function which is not a true distribution function. Specifically, the approximation to $F_T(\cdot)$, call it $G(\cdot)$ may be smaller than 0 or greater than 1 in places, or may decrease in places. The approximation for $\mathsf{P}(A^{(\mu)}_{T_1} \leq x)$ is then

$$H(x) \;=\; \int \mathsf{P}(A^{(\mu)}_t \leq x)\, dG(t).$$

Does the result of Theorem 4.1 still hold? A review of the proof of Theorem 4.1 shows that the result becomes:

Theorem 4.2. *Suppose $G(\cdot)$ has bounded variation, with $G(t) = 0$ for $t < 0$, and suppose also that*

$$\lim_{t\to\infty} G(t) \;=\; G(\infty)$$

exists. Then

$$\begin{aligned}|\mathsf{P}(A^{(\mu)}_T \leq x) - H(x)| \;&\leq\; |1-G(\infty)|\mathsf{P}(\lim_{t\to\infty} A_t \leq x) + \sup_{0\leq t<\infty} |F_T(t) - G(t)|\mathsf{P}(\lim_{t\to\infty} A_t > x)\\ &\leq\; \sup_{0\leq t<\infty} |F_T(t) - G(t)| \quad \forall\, x.\end{aligned}$$

An immediate consequence of Theorems 3.1 and 4.1 is that if $T, \{T_n\}$ are independent of W, if $F_T(\cdot)$ is continuous, and if T_n converges in distribution to T, then $A^{(\mu)}_{T_n}$ converges in distribution to $A^{(\mu)}_T$. A more general result is easily obtained.

Theorem 4.3. *If $T, \{T_n\}$ are independent of W, and if T_n converges in distribution to T, then $(e^{W^{(\mu)}_{T_n}}, A^{(\mu)}_{T_n})$ converges in distribution to $(e^{W^{(\mu)}_T}, A^{(\mu)}_T)$.*

Proof. Write the joint characteristic function of the pair $(e^{W^{(\mu)}_{T_n}}, A^{(\mu)}_{T_n})$ as

$$\int \mathsf{E}\, e^{is_1 e^{W^{(\mu)}_t} + is_2 A^{(\mu)}_t}\, dF_{T_n}(t), \qquad s_1, s_2 \in \mathbb{R}.$$

The function $f(t) = \mathsf{E}\,\exp(is_1 e^{W^{(\mu)}_t} + is_2 A^{(\mu)}_t)$ is continuous and bounded, and thus the weak convergence of $\{T_n\}$ to T implies that $\mathsf{E}\, f(T_n)$ converges to $\mathsf{E}\, f(T)$ (this is a classical result, see for instance Billingsley (1986), p.344, Theorem 25.8). □

Let $g(t,x)$ be the the density function of $A^{(\mu)}_t$. If $S_\lambda \sim$ **Exponential**(λ) is independent of W, then the density of $A^{(\mu)}_{S_\lambda}$ is (see (2.6))

$$f_\lambda(x) \;=\; \int_0^\infty \lambda e^{-\lambda t} f(t,x)\, dt \;=\; \lambda (2x)^{(\mu-\gamma)/2-1} \frac{\Gamma\big(\frac{\mu+\gamma}{2}\big)}{\Gamma(\gamma+1)}\, {}_1F_1\big(\tfrac{\gamma-\mu}{2}+1, \gamma+1, -\tfrac{1}{2x}\big) 1_{\{x>0\}}.$$

Recall that $\gamma = \sqrt{2\lambda + \mu^2}$ in this expression. Next, suppose that T has an arbitrary distribution on $\mathbb{R}_+$, that it is independent of W, and that $F_T(\cdot)$ is approximated by $G(\cdot)$, a combination of exponentials, with

$$\frac{dG(t)}{dt} \;=\; \sum_{j=1}^n a_j \lambda_j e^{-\lambda_j t}, \qquad t > 0. \tag{4.3}$$

The density of $A_t^{(\mu)}$ is then approximated by

$$f_G(x) \;=\; \sum_{j=1}^{n} a_j f_{\lambda_j}(x). \tag{4.4}$$

It is possible to find the approximating distribution function for $A_T^{(\mu)}$ in explicit form. Theorem 2.1 implies that, if $x > 0$,

$$\begin{aligned}
\mathsf{P}(A_{S_\lambda}^{(\mu)} > x) \;&=\; \mathsf{P}\Big(\frac{B_{1,\alpha}}{G_\beta} > 2x\Big) \\
&=\; -\int_0^1 \mathsf{P}\Big(\frac{u}{G_\beta} > 2x\Big)\, d_u(1-u)^\alpha \\
&=\; -\int_0^1 \int_0^{u/2x} \frac{v^{\beta-1}}{\Gamma(\beta)} e^{-v}\, dv\, d_u(1-u)^\alpha \\
&=\; \frac{1}{(2x)^\beta \Gamma(\beta)} \int_0^1 u^{\beta-1}(1-u)^\alpha e^{-\frac{u}{2x}}\, d_u \\
&=\; (2x)^{-\beta} \frac{\Gamma(\alpha+1)}{\Gamma(\gamma+1)}\, {}_1F_1\big(\beta, \gamma+1, -\tfrac{1}{2x}\big).
\end{aligned}$$

We have thus proved the following result.

Theorem 4.4. *If G is as in (4.3), then*

$$\int \mathsf{P}(A_t^{(\mu)} > x)\, dG(t) \;=\; \sum_{j=1}^{n} a_j (2x)^{\frac{\mu-\gamma_j}{2}} \frac{\Gamma(\frac{\mu+\gamma_j}{2}+1)}{\Gamma(\gamma_j+1)}\, {}_1F_1\big(\tfrac{\gamma_j-\mu}{2}, \gamma_j+1, -\tfrac{1}{2x}\big),$$

where $\gamma_j = \sqrt{2\lambda_j + \mu^2}$.

Theorem 4.5. *Suppose $\{T_n\}$ converges in distribution to T. Then*

$$\lim_{n\to\infty} \mathsf{P}(A_{T_n}^{(\mu)} \le x) \;=\; \mathsf{P}(A_T^{(\mu)} \le x) \quad \forall\, x.$$

Proof. Since the distribution of $A_T^{(\mu)}$ is continuous, this is a direct consequence of Theorem 4.3. □

5. Moments and Laplace transform of stochastic life annuities

The proof of the following result is a consequence of (2.3).

Theorem 5.1. *Let $a_k \;=\; 2k\mu + 2k^2$. If T is independent of W, and if $\mathsf{E}e^{a_k T} < \infty$ for $k = 1, \ldots, n$, then*

$$\mathsf{E}\left(A_T^{(\mu)}\right)^n \;=\; \sum_{k=0}^{n} b_{n,k}\, \mathsf{E}e^{a_k T}, \quad b_{n,k} \;=\; n!\Big/\prod_{\substack{j=0\\ j\neq k}}^{n} (a_j - a_k).$$

Note that $\mathsf{E}\, e^{a_k T}$ is the actuarial value of an insurance of one unit payable at the moment of death (usually written $\overline{A}_x$), with a geometric rate of interest equal to $-a_k$. In practical applications it would be finite for any a_k (observe that a_k may be positive or negative depending on k, μ and σ).

The Laplace transform of $A_T^{(\mu)}$ may also be obtained from that of $A_t^{(\mu)}$ by conditioning on T,

$$\mathsf{E}\, e^{-sA_T^{(\mu)}} \;=\; \int \mathsf{E}\, e^{-sA_t^{(\mu)}}\, dF_T(t),$$

and is finite for all $s > 0$.

Theorem 5.2. *(a) If* $\mathsf{P}(T > 0) > 0$*, then*

$$\mathsf{E} e^{sA_T^{(\mu)}} \;=\; \infty \quad \forall\, s > 0.$$

(b) If the distribution of T *has one or more atoms, then the distribution of* $A_T^{(\mu)}$ *is not determined by its moments.*

(c) If T *takes a finite number of strictly positive values, then*

$$\mathsf{E} e^{s/(2A_T^{(\mu)})} \;<\infty \quad \forall\, s < 1,$$

and the distribution of $1/A_T^{(\mu)}$ *is determined by its moments.*

Proof. (a) Condition on T, and use the result for fixed t (Section 2).

(b) Hörfelt (2004) has shown that the distribution of $A_t^{(\mu)}$ is not determined by its moments, for any fixed $t > 0$. Suppose t_0 is an atom of the distribution of T. Then there are two distinct distributions with the same moments as $A_{t_0}^{(\mu)}$. We can then produce two distinct distributions with the same moments as $A_T^{(\mu)}$, by changing the distribution of $A_T^{(\mu)}$ on the set $T = t_0$.

(c) Same as (a). □

It is plausible that parts (b) and (c) of the theorem above hold for any distribution for T.

6. Two other techniques for approximating the density of $A_t^{(\mu)}$ (t fixed)

Theorem 4.5 applies to all distributions $F_T(\cdot)$, in particular to the degenerate distribution at a fixed $t > 0$. Hence, if (4.3) is an approximation to this distribution, then (4.4) is an approximation to the distribution of $A_t^{(\mu)}$.

Another way to look at this is to go back to the derivation of the distribution of $A_{S_\lambda}^{(\mu)}$ in Section 2. Suppose $S_{n,\lambda} \sim$ **Gamma**(n, λ). Then $S_{n,\lambda}$ converges in distribution to the degenerate distribution at t if $n \to \infty$ and $n/\lambda \to t$. Replacing S_λ with $S_{n,\lambda}$ in (2.4), we get

$$\begin{aligned}\mathsf{E}[f(e^{W^{(\mu)}_{S_{n,\lambda}}})g(A^{(\mu)}_{S_{n,\lambda}})] \;&=\; \mathsf{E}\int_0^\infty \frac{\lambda^n t^{n-1}}{\Gamma(n)} e^{-\lambda t-\mu^2 t/2} e^{\mu W_t} f(e^{W_t}) g(A_t)\, dt \\ &=\; \frac{(-1)^{n-1}\lambda^n}{(n-1)!}\frac{\partial^{n-1}}{\partial\lambda^{n-1}} \mathsf{E}\int_0^\infty e^{-\lambda t-\mu^2 t/2} e^{\mu W_t} f(e^{W_t}) g(A_t)\, dt.\end{aligned}$$

Then the joint density function of $(e^{W^{(\mu)}_{S_{n,\lambda}}}, A^{(\mu)}_{S_{n,\lambda}})$ is

$$h_{n,\lambda}(r,u) \;=\; \frac{1}{u} r^{\mu-1} e^{-(1+r^2)/2u} \frac{(-1)^{n-1}\lambda^n}{(n-1)!} \frac{\partial^{n-1}}{\partial\lambda^{n-1}} I_\gamma(\tfrac{r}{u}) 1_{\{r,u>0\}}$$

(where $\gamma = \sqrt{2\lambda+\mu^2}$), and the density of $A^{(\mu)}_{S_{n,\lambda}}$ is

$$f_{n,\lambda}(u) \;=\; \frac{(-1)^{n-1}\lambda^n}{(n-1)!} \frac{\partial^{n-1}}{\partial\lambda^{n-1}} \left[(2u)^{(\mu-\gamma)/2-1} \frac{\Gamma\big(\frac{\mu+\gamma}{2}\big)}{\Gamma(\gamma+1)}\, {}_1F_1\big(\tfrac{\gamma-\mu}{2}+1, \gamma+1, -\tfrac{1}{2u}\big) \right].$$

Letting $n \to \infty$ we get the following result, which is really an application of a classical inversion theorem for Laplace transforms (see Feller, 1971, p.233).

Theorem 6.1. *The density of $A^{(\mu)}_t$ is*

$$f(t,x) = \lim_{n\to\infty} \frac{(-1)^{n-1}}{(n-1)!} \Big(\frac{n}{t}\Big)^n \frac{\partial^{n-1}}{\partial\lambda^{n-1}} \left[(2u)^{(\mu-\gamma)/2-1} \frac{\Gamma\big(\frac{\mu+\gamma}{2}\big)}{\Gamma(\gamma+1)}\, {}_1F_1\big(\tfrac{\gamma-\mu}{2}+1, \gamma+1, -\tfrac{1}{2x}\big) \right]_{\lambda=\frac{n}{t}}.$$

Conclusion

Some possible applications of the foregoing results are:

(1) Given particular values for m and σ (which represent the type of investment strategy chosen), find the probability that a pension will be paid in full by the initial amount plus its investment returns.

(2) Find the effect on the distribution of the discounted annuity of introducing an n-year guarantee.

(3) Among the set of pairs (m,σ) available in the market, find the ones which maximise the probability of a certain amount being sufficient to fund a pension.

A subsequent paper will give some numerical illustrations.

References

Bachelier, L. (1900). Théorie de la spéculation. *Annales scientifiques de l'École Normale Supérieure* III-17: 21–86. Doctoral thesis. Reprinted by Éditions Jacques Gabay, Paris (1995). English translation in Cootner, P. (ed.) (1964). *The random character of stockmarket prices*. MIT Press, Cambridge, pp.17–78.

Bellhouse, D.R., and Panjer, H.H. (1981). Stochastic modelling of interest rates with applications to life contingencies – Part II. *Journal of Risk and Insurance* **48**: 628-637.

Billingsley, P. (1986). *Probability and Measure*. (Second Edition.) Wiley, New York.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**: 637-654.

Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. (1997). *Actuarial Mathematics (Second Edition)*. Society of Actuaries, Itasca, Illinois.

Boyle, P. (1976). Rates of return as random variables. *Journal of Risk and Insurance*. **43**: 693-713.

Dufresne, D. (1989). Weak convergence of random growth processes with applications to insurance. *Insurance: Mathematics and Economics* **8**: 187-201.

Dufresne, D. (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuarial J.* **1990**: 39-79, 1990.

Dufresne, D. (1992). On discounting when rates of return are random. *Transactions of the Twenty-Fourth International Actuarial Congress*, Montreal,1992, **1**: 27-44.

Dufresne, D. (2000). Laguerre series for Asian and other options. *Mathematical Finance* **10**: 407-428.

Dufresne, D. (2001). The integral of geometric Brownian motion. *Advances in Applied Probability* **33**: 223-241.

Dufresne, D. (2004a). Bessel processes and Asian options. To appear in: BenAmeur, H., and Breton, M. *Numerical Methods in Finance*. Kluwer Academic Publishers

Dufresne, D. (2004b). The log-normal approximation in financial and other computations. *Advances in Applied Probability* **36**: 747-773.

Feller, W. (1971). *An Introduction to Probability Theory and its Applications, II.* (Second Edition.) Wiley, New York.

Frees, E.W. (1991). Stochastic life contingencies with solvency considerations. *Transactions of the Society of Actuaries* **42**: 91-148.

Geman, H. et Yor, M. (1992). Quelques relations entre processus de Bessel, options asiatiques et fonctions confluentes hypergéométriques. *C. R. Acad. Sci. Paris, Série I* **314**: 471-474.

Geman, H., and Yor, M. (1993). Bessel processes, Asian options and perpetuities. *Mathematical Finance* **3**: 349-375.

Harrison, J.M., and Pliska, S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Proc. Appl.* **11**: 215-260.

Hörfelt, P. (2004). The integral of a geometric brownian motion is indeterminate by its moments. Preprint. Centre for Industrial Mathematics, Göteborg, Sweden. Presented at the Bachelier Finance Society Third World Congress, Chicago, July 2004.

Lebedev, N.N. (1972). *Special Functions and their Applications*. Dover, New York.

Monthus, C., and Comtet, A. (1994). On the flux in a one-dimensional disordered system. *J. Phys. I (France)* **4**: 635-653.

Osborne, M.F.M. (1959) Brownian motion in the stock market. *Operations research* **9**: 145-193.

Panjer, H.H. and Bellhouse, D.R. (1980). Stochastic modelling of interest rates with applications to life contingencies. *Journal of Risk and Insurance* **47**: 91-110.

Parker, G. (1994a). Two stochastic approaches for discounting actuarial functions. *ASTIN Bulletin* **24**: 167-181.

Parker, G. (1994b). Moments of the present value of a portfolio of policies. *Scandinavian Actuarial Journal* **1994**: 53-67.

Pollard, J. (1971). On fluctuating interest rates. *Bulletin de l'Association royale des actuaires belges* **66**: 68-94.

Ramakrishnan, A. (1954). A stochastic model of a fluctuating density field. *Astrophys. J.* **119**: 682-685.

Waters, H.H, (1978). The moments and distributions of actuarial functions. *Journal of the Institute of Actuaries* **105**: 61-75.

Wilkie, A.D. (1978). The rate of interest as a stochastic process. *Proceedings of the 20th Actuarial Congress of Actuaries* **1**: 325-338.

Wong, E. (1964). The construction of a class of stationary Markoff processes. In: Bellman, R. (Ed.), *Sixteenth Symposium in Applied Mathematics – Stochatic Processess in Mathematical Physics and Engineering*, American Mathematical Society, Providence, RI, pp. 264-276, 1964.

Westcott, D.A. (1981). Moments of compound interest functions under fluctuating interest rates. *Scandinavian Actuarial Journal* **1981**: 237-244.

Yor, M. (1992) On some exponential functionals of Brownian motion. *Adv. Appl. Prob.* **24**: 509-531. (Reproduced in Yor (2001).)

Yor, M. (2001). *Exponential Functionals of Brownian Motion and Related Processes*. Springer-Verlag, New York. (Contains ten papers by Marc Yor and co-authors.)

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