The Distribution of the Dividend Payments in the Compound Poisson Risk Model Perturbed by Diffusion

Shuanming Li
Center for Actuarial Studies, Department of Economics
The University of Melbourne, Australia

Abstract

We consider a diffusion perturbed classical compound Poisson risk model in the presence of a constant dividend barrier. An integro-differential equation with certain boundary conditions for the n-th moment of the discounted dividend payments prior to ruin is derived and solved. Its solution can be expressed in terms of the expected discounted penalty (Gerber-Shiu) functions due to oscillation in the corresponding perturbed risk model without a barrier. When the discount factor δ is zero, we show that all the results can be expressed in terms of the non-ruin probability in the perturbed risk model without a barrier.

Keywords: Compound Poisson process; Diffusion process; Discounted dividend payments; Gerber-Shiu function; Integro-differential equation; Time of ruin

1 Introduction

Consider the following classical surplus process perturbed by a diffusion

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i + \sigma B(t)\,, \qquad t \geq 0\,, \tag{1}$$

where $\{N(t);\, t \geq 0\}$ is a Poisson process with parameter λ, denoting the total number of claims from an insurance portfolio. $X_1, X_2, \ldots,$ independent of

$\{N(t); t \geq 0\}$, are positive i.i.d. random variables with common distribution function (df) $P(x) = 1 - \bar{P}(x) = P(X \leq x)$, density function $p(x)$, moments $\mu_j = \int_0^\infty x^j p(x)dx$, for $j = 0, 1, 2, \ldots$, and the Laplace transform $\hat{p}(s) = \int_0^\infty e^{-s\,x} p(x)dx$. $\{B(t); t \geq 0\}$ is a standard Wiener process that is independent of the aggregate claims process $S(t) := \sum_{i=1}^{N(t)} X_i$ and $\sigma > 0$ is the dispersion parameter. In the above model, $u = U(0) \geq 0$ is the initial surplus, $c = \lambda\,\mu_1(1+\theta)$ is the premium rate per unit time, and $\theta > 0$ is the relative security loading factor.

The classical risk model perturbed by a diffusion was first introduced by Gerber (1970) and has been further studied by many authors during the last few years; e.g., Dufresne and Gerber (1991), Gerber and Landry (1998), Wang and Wu (2000), Wang (2001), Tsai (2001, 2003), Tsai and Willmot (2002a, b), Zhang and Wang (2003), Chiu and Yin (2003), and the references therein.

In this paper, a barrier strategy is considered by assuming that there is a horizontal barrier of level $b \geq u$ such that when the surplus reaches level b, dividends are paid continuously such that the surplus stays at level b until it becomes less than b. Let $U_b(t)$ be the modified surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy. Li and Wu (2005) study some quantities related to time of ruin $T_b = \inf\{t : U_b(t) \leq 0\}$, such as the two types of ruin probabilities, the surplus before ruin, and the deficit at ruin, by analyzing the expected discounted penalty Gerber-Shiu functions (due to oscillation and caused by a claim).

Let $\delta > 0$ be the force of interest for valuation and define

$$D_{u,b} = \int_0^{T_b} e^{-\delta\,t} dD(t), \qquad 0 \leq u \leq b,$$

to be the present value of all dividends until time of ruin T_b, where $D(t)$ is the aggregate dividends paid by time t. Define the moment generating function (m.g.f.) of $D_{u,b}$ by

$$M(u, y; b) = E[e^{y\,D_{u,b}}], \qquad 0 \leq u \leq b,$$

where y is such that $M(u, y; b)$ exists, and n-th moment by

$$V_n(u; b) = E[D_{u,b}^n], \qquad 0 \leq u \leq b,\ n \in \mathbb{N},$$

with $V_0(u; b) = 1$.

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies for a compound Poisson risk process have been studied in a number of papers and books. These references include Bühlmann (1970), Segerdahl (1970), Gerber (1973, 1979, 1981), Gerber (1979), Paulsen and Gjessing (1997), Albrecher and Kainhofer (2002), Højgaard (2002), Lin et al. (2003), Dickson and Waters (2004), Li and Garrido (2004), and Albrecher et al. (2005). The main focus is on optimal dividend payouts and problems associated with time of ruin, under various barrier strategies and other economic conditions. For the risk model modeled by a Brownian motion, Gerber and Shiu (2004) give some very explicit calculations on the moments and distribution of the discounted dividends paid until ruin.

The main goal of this paper is to evaluate the moment generating function and the moments of the discounted sum of the total dividends until ruin $D_{u,b}$. This shows how the results in Dickson and Waters (2004) and Gerber and Shiu (2004) can be extend to the diffusion perturbed classical risk model.

2 An Integro-differential Equations and Its Solutions

In this section, we will show that $V_1(u;\, b)$ satisfies an integro-differential equation with certain boundary conditions as follows.

Theorem 1 Suppose $p(x)$ is continuously differentiable on $(0, \infty)$, then $V(u;\, b)$ satisfies the following homogenous integro-differential equation for $0 < u < b$:

$$\frac{\sigma^2}{2} V_1''(u;\, b) + c\, V_1'(u;\, b) = (\lambda + \delta) V_1(u;\, b) - \lambda \int_0^u V_1(u - x;\, b)\, p(x) dx\,, \tag{2}$$

with the boundary conditions

$$V_1(0;\, b) = 0, \tag{3}$$

$$V_1'(b;\, b) = 1\,. \tag{4}$$

Proof: Consider the infinitesimal interval from 0 to dt. Conditioning, one obtains that

$$\begin{aligned} V_1(u;\, b) = e^{-\delta dt} \Big\{ & P(W_1 > dt)\, E\left[V_1\big(u + cdt + \sigma B(dt);\, b\big)\right] \\ & + P(W_1 \le dt)\, E\left[V_1\big(u + cdt + \sigma B(dt) - X_1;\, b\big)\right] \Big\}\,. \end{aligned} \tag{5}$$

Since $e^{-\delta dt} = 1 - \delta dt + o(dt)$,

$$P(W_1 > dt) = 1 - \lambda\, dt + o(dt)\,, \qquad P(W_1 \le dt) = \lambda\, dt + o(dt),$$

Taylor's expansion (Theorem 3 in the Appendix shows that $V_1(u;\,b)$ is twice continuously differentiable in u) gives

$$E\left[V_1\big(u+cdt+\sigma B(dt);\,b\big)\right] = V_1(u;\,b) + \big[cV_1'(u;\,b) + \frac{\sigma^2}{2}V_1''(u;\,b)\big]dt + o(dt)\ ,$$

while

$$\begin{aligned}&E\left[V_1\big(u+cdt+\sigma B(dt)-X_1;\,b\big)\right]\\&= E\left[\int_0^{u+c\,dt+\sigma B(dt)} V_1\big(u+cdt+\sigma B(dt)-x;\,b\big)p(x)dx\right]\ ,\end{aligned}$$

then substituting these formulas into (5), subtracting $V_1(u;\,b)$ from both sides, interpreting dt and $o(dt)$ terms, canceling out common factors, and letting $dt \to 0$, we prove that the integro-differential equation (2) holds.

The boundary condition (3) is obvious: If $U(0) = 0$, ruin is immediate and no dividends are paid. To prove the boundary condition (4), let $\varepsilon > 0$ and $V_{1,\,\epsilon}(u;\,b)$ be the expected discounted dividends paid until ruin in the following risk model in the presence of the a dividend barrier b,

$$U_\epsilon(t) = u + (c + c_\epsilon)\,t - \sum_{i=1}^{N(t)} X_i - \epsilon\,N_\epsilon(t),$$

where $N_\epsilon(t)$ is a Poisson process with parameter $\lambda_\epsilon > 0$, and $c_\epsilon >$ is such that $c + c_\epsilon > \lambda\,\mu_1 + \epsilon\,\lambda_\epsilon$. It is well known that $\sum_{i=1}^{N(t)} X_i + \epsilon\,N_\epsilon(t)$ is also a compound Poisson process. Gerber and Shiu (1998, Eq. (7.4)) shows that $V'_{1,\,\epsilon}(b;\,b) = 1$. Now we choose ϵ, λ_ϵ, and c_ϵ such that $\mathrm{Var}[\epsilon N_1(t)] = \sigma^2\,t$ and $E[c_\epsilon\,t - \epsilon N_1(t)] = 0$. These two conditions yield $\lambda_\epsilon = \sigma^2/\epsilon^2$ and $c_\epsilon = \sigma^2/\epsilon$. It is easy to prove that, when $\epsilon \to 0^+$, $E[e^{z(c_\epsilon\,t-\epsilon N_1(t))}] \to e^{z^2\sigma^2\,t/2}$. This shows that the process $\{c_\epsilon\,t - \epsilon N_1(t);\,t \geq 0\}$ converges weakly to $\{\sigma\,B(t);\,t \geq 0\}$, therefore, the surplus process $\{U_\epsilon(t);\,t \geq 0\}$ converges weakly to the surplus process $\{U(t);\,t \geq 0\}$. Then we conclude that $\lim_{\epsilon\to 0^+} V_{1,\epsilon}(u;\,b) = V_1(u;\,b)$, and $\lim_{\epsilon\to 0^+} V'_{1,\,\epsilon}(b;\,b) = V_1'(b;\,b) = 1$. □

The solutions of above integro-differential equation with boundary conditions heavily depend on the solutions of the following homogenous integro-differential equation:

$$\frac{\sigma^2}{2}\,v''(u) + c\,v'(u) = (\lambda+\delta)v(u) - \lambda\int_0^u v(u-x)\,p(x)dx\,, \qquad u \geq 0\,. \tag{6}$$

The general solution of equation (6) is of the form

$$v(u) = \eta_1\,v_1(u) + \eta_2\,v_2(u), \qquad u \geq 0\,, \tag{7}$$

where $v_1(u)$ and $v_2(u)$ are two linearly independent particular solutions of (6), which will be discussed in the next section, and η_1, η_2 are any real numbers. Then the solution of the integro-differential equation (2) with boundary conditions (3) and (4) is

$$V_1(u;\, b) = \eta_1(b)\, v_1(u) + \eta_2(b)\, v_2(u), \qquad 0 \le u \le b\,, \tag{8}$$

where $\eta_1(b)$ and $\eta_2(b)$ can be determined by solving the following linear equation system

$$\begin{cases} \eta_1(b)\, v_1(0) + \eta_2(b)\, v_2(0) = 0\,, \\ \eta_1(b)\, v_1'(b) + \eta_2(b)\; v_2'(b) = 1\,. \end{cases}$$

3 Analysis of the Function $v(u)$

The solution of the homogenous equation (6) is uniquely determined by the initial conditions $v(0)$ and $v'(0)$ and can be solved by Laplace transforms. To begin with, we let $\rho = \rho(\delta)$ be the unique non-negative root of the following generalized Lundberg equation:

$$\lambda \hat{p}(s) = \lambda + \delta - c\, s - \sigma^2\, s^2/2\,, \tag{9}$$

with $\rho(0) = 0$. The proof of existence of ρ can be found in Gerber and Landry (1998). Then define

$$h(y) = \frac{2\,c}{\sigma^2} e^{-(\rho + \frac{2\,c}{\sigma^2})y},$$

$$\gamma(y) = \frac{\lambda}{c} \int_y^{\infty} e^{-\rho(x-y)} p(x) dx\,.$$

Now let $\hat{v}(s) = \int_0^{\infty} e^{-s\,x} v(x) dx$ be the Laplace transform of $v(u)$. Taking Laplace transforms on both sides of (6) gives

$$\left[\frac{1}{2}\sigma^2\, s^2 + c\, s - (\lambda + \delta) + \lambda\, \hat{p}(s)\right] \hat{v}(s) = \frac{\sigma^2}{2} v(0)\, s + c\, v(0) + \frac{\sigma^2}{2} v'(0). \tag{10}$$

Since $\sigma^2\, \rho^2/2 + c\, \rho - (\lambda + \delta) + \lambda\, \hat{p}(\rho) = 0$, then (10) can be rewritten as

$$\begin{aligned} &\left\{1 - \left(\frac{2\,\lambda/\sigma^2}{s + \rho + 2\, c/\sigma^2}\right) \left[\frac{\hat{p}(\rho) - \hat{p}(s)}{s - \rho}\right]\right\} \hat{v}(s) \\ &= \frac{v(0)}{s + \rho + 2\, c/\sigma^2} + \frac{v(0)(\rho + 2\, c/\sigma^2) + v'(0)}{(s - \rho)(s + \rho + 2\, c/\sigma^2)}. \end{aligned} \tag{11}$$

Inverting it yields

$$\begin{aligned} v(u) &= \int_0^u v(u-y)\, g(y)dy + \frac{\sigma^2\, v(0)}{2\, c} h(u) \\ &\quad + \frac{\sigma^2\, [v(0)(\rho + 2\, c/\sigma^2) + v'(0)]}{2\, c} e^{\rho\, u} * h(u)\,, \qquad u \geq 0, \end{aligned} \tag{12}$$

where $g(y) = h * \gamma(y)$, with $*$ denoting the convolution operation.

We remark that equation (12) is defective renewal equation, since $g(y)$ is a defective density function with $\int_0^\infty g(y)dy = (c\,\rho + \sigma^2\,\rho^2/2 - \delta)/(c\,\rho + \sigma^2\,\rho^2/2) < 1$, see Gerber and Landry (1998, eq. (16)).

One can find two linearly independent solutions $v_1(u)$ and $v_2(u)$ by specifying the initial conditions $v_i(0)$ and $v_i'(0)$ for $i = 1, 2$. For example, setting $v_1(0) = 1$ and $v_1'(0) = -(\rho + 2\, c/\sigma^2)$ yields

$$v_1(u) = \int_0^u v_1(u-y)\, g(y)dy + \frac{\sigma^2}{2\, c} h(u)\,, \qquad u \geq 0\,, \tag{13}$$

and setting $v_2(0) = 0$ and $v_2'(0) = 1$ yields

$$v_2(u) = \int_0^u v_2(u-y)\, g(y)dy + \frac{\sigma^2}{2\, c}\, e^{\rho\, u} * h(u)\,, \qquad u \geq 0\,. \tag{14}$$

To prove that $v_1(u)$ and $v_2(u)$ are linearly independent, we assume that there are constants c_1 and c_2 such that $c_1\, v_1(u) + c_2\, v_2(u) \equiv 0$, for any $u \geq 0$. Then we have $c_1\, v_1(0) + c_2\, v_2(0) = 0$ and $c_1\, v_1'(0) + c_2\, v_2'(0) = 0$. Solving these two equations gives $c_1 = c_2 = 0$. This proves that $v_1(u)$ and $v_2(u)$ are linearly independent.

As in Gerber and Landry (1998), define

$$\phi_d(u) = E[e^{-\delta\, T_\infty} I(T_\infty < \infty, U(T_\infty) = 0) \big| U(0) = u], \qquad u \geq 0,$$

to be the Laplace transform of ruin time T_∞ corresponding to risk model (1) without a barrier, if the ruin is due to oscillation. Gerber and Landry (1998, eq. (17)) have shown that $\phi_d(u)$ with $\phi_d(0) = 1$ also satisfies the defective renewal equation (13). By the uniqueness of the solution of the defective renewal equation (13), we have $v_1(u) = \phi_d(u)$. Further, by comparing (13) and (14), we can easily prove that

$$v_2(u) = e^{\rho\, u} * v_1(u) = e^{\rho\, u} * \phi_d(u) = \int_0^u \phi_d(u-x) e^{\rho\, x} dx, \quad u \geq 0\,.$$

Then eq. (8) gives for $0 \leq u \leq b$,

$$V_1(u;\, b) = \frac{v_2(u)}{v_2'(b)} = \frac{\int_0^u \phi_d(u-x) e^{\rho\, x} dx}{\rho \int_0^b \phi_d(b-x) e^{\rho\, x} dx + \phi_d(b)}\,. \tag{15}$$

The optimal value of dividend barrier b^* can be obtained as a solution of the equation

$$v_2''(b) = \rho^2 \int_0^b \phi_d(b-x) e^{\rho x} dx + \rho \phi_d(b) + \phi_d'(b) = 0, \tag{16}$$

provided that the solution is greater than u, otherwise, we set $b^* = u$.

In particular, if $\delta = 0$, then $\rho = 0$, and $\phi_d(u)$ simplifies to the ruin probabilities due to oscillations, $\Psi_d(u)$. Then the mean of total dividends paid until ruin is given by

$$V_1(u;\, b) = \frac{\int_0^u \Psi_d(x) dx}{\Psi_d(b)}, \qquad 0 \le u \le b\,.$$

Dufrensne and Gerber (1991, Eq. (4.7)) shows that

$$\Phi'(u) = \frac{2(c - \lambda\,\mu_1)}{\sigma^2} \Psi_d(u),$$

where $\Phi(u)$ is the non-ruin probability of the risk model (1). Therefore, when $\delta = 0$, $V_1(u;\, b)$ can be expressed as

$$V_1(u;\, b) = \frac{\Phi(u)}{\Phi'(b)}, \qquad 0 \le u \le b. \tag{17}$$

In this case, the optimal dividend barrier b^* is equal to the initial surplus u.

Properties of $\phi_d(u)$ and its applications have been studied extensively by Gerber and Landry (1998), Tsai (2001, 2003), Tsai and Willmot (2002a, 2002b), Chiu and Yin (2003), and Li and Garrido (2005) for $n = 1$. Therefore, we may use the properties of $\phi_d(u)$ to analyze $V_1(u;\, b)$.

Example 1 Suppose that the claim sizes are exponentially distributed with density function $p(x) = \kappa\, e^{-\kappa\, x}$, $x \ge 0$, and Laplace transform $\hat{p}(s) = \kappa/(s+\kappa)$. The equation

$$[\sigma^2\, s^2/2 + c\, s - (\lambda + \delta)](s + \kappa) + \lambda\, \kappa = 0 \tag{18}$$

has one positive root, say ρ, and two negative roots, say $-R_1, -R_2$. Then Li and Wu (2004) give

$$v_1(u) = \phi_d(u) = \frac{\kappa - R_1}{R_2 - R_1} e^{-R_1\, u} + \frac{\kappa - R_2}{R_1 - R_2} e^{-R_2\, u}, \qquad u \ge 0\,,$$

and

$$\begin{aligned} v_2(u) \quad &= \quad \frac{\rho + \kappa}{(\rho + R_1)(\rho + R_2)} e^{\rho\, u} + \frac{R_1 - \kappa}{(\rho + R_1)(R_2 - R_1)} e^{-R_1\, u} \\ &\quad + \frac{R_2 - \kappa}{(\rho + R_2)(R_1 - R_2)} e^{-R_2\, u}, \qquad u \ge 0\,. \end{aligned}$$

Then the optimal dividend barrier b^* is the solution of

$$v_2''(b) = \frac{\rho^2(\rho+\kappa)\,e^{\rho b}}{(\rho+R_1)(\rho+R_2)} + \frac{R_1^2(R_1-\kappa)\,e^{-R_1 b}}{(\rho+R_1)(R_2-R_1)} + \frac{R_2^2(R_2-\kappa)\,e^{-R_2 b}}{(\rho+R_2)(R_1-R_2)} = 0\,.$$

Like the classical risk model, b^* does not depend on u.

Now, let $c = 1.1$, $\lambda = 1$, $\kappa = 1$, $\sigma = 0.5$, $\delta = 0.05$, $b = 10$. The roots of equation (18) are: $\rho = 0.1812$, $-R_1 = -0.2264$, $-R_2 = -9.7548$. Then $b^* = 0.8305$.

4 Moment Generating Function and Higher Moments

In this section, we study the moment generating function $M(u, y; b)$, through which we can analyze the higher moments of $D_{u,b}$.

By the strong Markov property of $U_b(t)$, we have

$$\begin{aligned} M(u, y; b) &= E[M(U_b(dt), e^{-\delta\,dt}y;\, b)] \\ &= P(W_1 < dt)E[M(u + c\,dt + \sigma B(dt) - X_1, e^{-\delta\,dt}y;\, b)] \\ &\quad + P(W_1 \geq dt)E[M(u + c\,dt + \sigma B(dt), e^{-\delta\,dt}y;\, b)]. \end{aligned}$$

When $p(x)$ is continuously differentiable in x, Theorem 4 in the Appendix shows that $M(u, y\;\, b)$ is twice continuously differentiable in u and continuously differentiable in y. Then expanding the last expression and using the same arguments as that in the proof of Theorem 1, we obtain the following partial integro-differential equation for $M(u, y; b)$

$$\begin{aligned} &\frac{\sigma^2}{2}\frac{\partial^2 M(u,y\;,b)}{\partial u^2} + c\frac{\partial M(u,y\;,b)}{\partial u} - \delta\, y\frac{\partial M(u,y\;,b)}{\partial y} \\ &= \lambda M(u,y;\,b) - \lambda\int_0^u M(u-x,y;b)p(x)dx - \lambda\bar{P}(u)\,. \end{aligned} \tag{19}$$

Furthermore, the boundary conditions are

$$M(0, y;\, b) = 1, , \tag{20}$$

and

$$\frac{\partial M(u,y\;,b)}{\partial u}\Big|_{u=b} = y\,M(b, y;\, b). \tag{21}$$

Substituting $M(u,y;\,b) = 1 + \sum_{n=1}^{\infty}(y^n/n!)V_n(u;\,b)$ into (19) and comparing the coefficient of y^n yields the following integro-differential equation for $V_n(u;\,b)$,

$$\frac{\sigma^2}{2}V_n''(u;\,b) + c\,V_n'(u;\,b) - (\lambda + n\,\delta)V_n(u;\,b) = -\lambda\int_0^u V_n(u-x;\,b)p(x)dx\,. \quad (22)$$

It follows from (20) that

$$V_n(0;\,b) = 0, \qquad n = 1,2,\ldots, \quad (23)$$

and from (21) than

$$V_n'(u;\,b) = n\,V_{n-1}(b;\,b), \qquad n = 1,2,\ldots, \quad (24)$$

with $V_0(b;\,b) = 1$.

By the same argument as in Section 3, the solution of the integro-differential equation (22) with boundary conditions (23) and (24) can be expressed as

$$V_n(u;\,b) = c_n(b)\,g_n(u) = c_n(b)\int_0^u \phi_{d,n}(u-x)e^{\rho_n x}dx, \qquad 0 \le u \le b, \quad (25)$$

where

$$\phi_{d,\,n}(u) = E[e^{-n\,\delta\,T_\infty}I(T_\infty < \infty, U(T_\infty) = 0)\big|U(0) = u], \qquad u \ge 0, \quad (26)$$

with $\phi_{d,\,1}(u) = \phi_d(u)$, is the Laplace transform of the time of ruin T_∞ with respect to parameter $n\,\delta$, if the ruin is due to oscillation, and $\rho_n = \rho_n(\delta)$, with $\rho_n(0) = 0$ and $\rho_1 = \rho$, is the unique positive root of the following equation

$$\lambda + n\,\delta - c\,s - \frac{\sigma^2}{2}s^2 = \lambda\,\hat{p}(s).$$

Equation (15) gives

$$c_1(b) = \frac{1}{g_1'(b)} = \frac{1}{\rho_1\int_0^u \phi_{d,1}(u-x)e^{\rho_1 x}dx}.$$

Applying (25) to boundary condition (24) gives

$$c_n(b)\,g_n'(b) = n\,c_{n-1}(b)g_{n-1}(b), \quad k = 2,3,\ldots.$$

Recursively, we have

$$c_n(b) = n!\frac{g_1(b)g_2(b)\cdots g_{n-1}(b)}{g_1'(b)g_2'(b)\cdots g_n'(b)}, \qquad n = 1,2,\ldots,$$

therefore,

$$V_n(u;\,b) = n!\frac{g_1(b)g_2(b)\cdots g_{n-1}(b)g_n(u)}{g_1'(b)g_2'(b)\cdots g_n'(b)}, \qquad 0 \le u \le b,\ n = 1,2,\ldots. \quad (27)$$

5 The Distribution of $D(T_b)$

When $\delta = 0$, $\rho_n = 0$, $g_n(u) = \int_0^u \Psi_d(x)dx$, and $D_{u,b}$ simplifies to the total dividends paid until ruin $D(T_b)$, therefore, for $0 \le u \le b$,

$$V_n(u;\, b)\big|_{\delta=0} = E[D^n(T_b)] = n!\frac{\left[\int_0^b \Psi_d(x)dx\right]^{n-1}\int_0^u \Psi_d(x)dx}{[\Psi_d(b)]^n}, \tag{28}$$

$$= n!\frac{[\Phi(b)]^{n-1}\Phi(u)}{[\Phi'(b)]^n}. \tag{29}$$

and

$$\begin{aligned} M(u, y;\, b)\big|_{\delta=0} &= E\left[e^{y\,D(T_b)}\big|U(0) = u\right] \\ &= 1 + \sum_{n=1}^{\infty} y^n\,[V(b;\, b)]^{n-1}\,V(u;\, b) = 1 + \frac{V(u;\, b)y}{1 - V(b;\, b)y} \\ &= \left[1 - \frac{V(u;\, b)}{V(b;\, b)}\right] + \frac{V(u;\, b)}{V(b;\, b)}\frac{1}{1 - V(b;\, b)y}. \end{aligned} \tag{30}$$

This shows that the distribution of $D(T_b)$ is a mixture of the degenerate distribution at zero and the exponential distribution with mean

$$V(b;\, b) = \frac{\int_0^b \Psi_d(x)dx}{\Psi_d(b)} = \frac{\Phi(b)}{\Phi'(b)}.$$

The weight of this mixture are $p = 1 - \int_0^u \Psi_d(x)dx/\int_0^b \Psi_d(x)dx = 1 - \Phi(u)/\Phi(b)$ and $q = 1 - p$. Note that p is the probability that the surplus does not reach barrier b before ruin occurs.

By the same argument as in Gerber and Shiu (2004) or Dickson and Waters (2004), we can express $D(T_b)$ as

$$D(T_b) = \sum_{i=1}^{N} D_i,$$

where N denotes the total number of the streams of dividend payments which is geometric distribution distributed and D_i's are i.i.d. random variables denoting the dividends paid between streams i and $i+1$. N and D_i's are independent. To determine the distribution of N and D_i, we rewrite (30) as

$$M(u, y;\, b) = p\sum_{n=0}^{\infty}\left(\frac{q}{1 - p\,V(b; b)}\right)^n.$$

This gives $P(N = k) = pq^k$, $k = 0, 1, \ldots$, and the common distribution of D_i's is exponential with mean $p\,V(b;\, b) = [\Phi(b) - \Phi(u)]/\Phi'(b)$.

Appendix

In this Appendix, we study the conditions under which the moment generating function $M(u, y; , b)$ are twice continuously differentiable in u and continuously differentiable in y, and the moment $V_1(u; b)$ is twice continuously differentiable in u in $(0, b)$.

Let $a > 0$, define $\tau_a = \inf\{s : |B_s| = a\}$. For $x \in [-a, a]$, put

$$\begin{aligned} H(a,t,x) &= \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \Big\{ \exp\Big[-\frac{(x+4\,k\,a)^2}{2t}\Big] - \exp\Big[-\frac{(x-2\,a+4\,k\,a)^2}{2t}\Big]\Big\} \\ h(a,t) &= \frac{a}{2\sqrt{2\pi t^3}} \sum_{k=-\infty}^{\infty} \Big\{(4k+1)\exp\Big[-\frac{a^2(4k+1)^2}{2\,t}\Big] \\ &\quad +(4k-3)\exp\Big[-\frac{a^2(4k-3)^2}{2\,t}\Big] - (4k-1)\exp\Big[-\frac{a^2(4k-1)^2}{2\,t}\Big]\Big\}. \end{aligned}$$

It follows from Revuz and Yor (1991, pp. 105-106) that $P(B_s \in dx, \tau_a > s) = H(a, s, x)dx$ and $P(\tau_a \in ds) = h(a, s)ds$. Also it is easy to check that $h(a, t)$ is at least twice continuously differentiable in t and a, while $H(a, t, x)$ is at least twice continuously differentiable in a, t, and x.

Theorem 2 Let $0 < u < b$, then $V_1(u; b)$ satisfies the following integral equation:

$$\begin{aligned} &V_1(u;b) = e^{-(\lambda+\delta)t_0} \int_{-a}^{a} V_1(u + c\,t_0 + \sigma y;\, b) H(a, t_0, y) dy \\ &+ \int_0^{t_0} \lambda\, e^{-(\lambda+\delta)s} ds \int_{-a}^{a} H(a,s,y)dy \int_0^{u+c\,s+\sigma\,y} V_1(u + c\,s + \sigma\,y - z;\, b) p(z) dz \\ &+\frac{1}{2}\int_0^{t_0} \big[V_1(u + c\,t + \sigma\,a;\, b) + V_1(u + c\,t - \sigma\,a;\, b)\big] e^{-(\lambda+\delta)t} h(a,t) dt, \end{aligned} \tag{31}$$

where $0 \le t_0 \le (b-u)/(2\,c)$, $0 < a \le [(b-u) \wedge u]/(2\,\sigma)$.

Proof: Let $\tau = t_0 \wedge \tau_a \wedge W_1$, where W_1 is the occurences time of the first claim which is exponentially distributed with parameter λ. For $t \in (0, \tau)$, we have $0 < U_b(t) < b$. Conditioning on τ and using the total probability formula, one obtains that

$$\begin{aligned} V_1(u;\, b) &= E[e^{-\delta\tau} V_1(U_b(\tau);\, b)] \\ &= e^{-\delta t_0} E[V_1(u + c\,t_0 + \sigma\,B(t_0);\, b) I(t_0 < \tau_a \wedge W_1)] \\ &\quad +E[e^{-\delta\,\tau_a} V_1(u + c\,\tau_a + \sigma\,B(\tau_a);\, b) I(\tau_a \le t_0 \wedge W_1)] \\ &\quad +E[e^{-\delta\,W_1} V_1(u + c\,W_1 + \sigma\,B(W_1) - X_1;\, b) I(W_1 \le t_0, W_1 < \tau_a)] \\ &= I_1(u) + I_2(u) + I_3(u)\,. \end{aligned} \tag{32}$$

By the assumption of independence we have

$$
\begin{aligned}
I_1(u) &= e^{-\delta t_0} E[V_1(u + c\,t_0 + \sigma\,B(t_0);\, b) I(t_0 < \tau_a) I(t_0 < W_1)] \\
&= e^{-\delta t_0} E[I(W_1 > t_0)] E[V_1(u + c\,t_0 + \sigma\,B(t_0);\, b) I(\tau_a > t_0)] \\
&= e^{-(\delta+\lambda) t_0} \int_{-a}^{a} V_1(u + c\,t_0 + \sigma\,y;\, b) H(a, t_0, y) dy\,.
\end{aligned}
$$

By Proposition 2.8.3 of Port and Stone (1978) we have

$$
P(B(\tau_a) = a, \tau_a \in dt) = P(B(\tau_a) = -a, \tau_a \in dt) = \frac{1}{2} h(a,t) dt\,.
$$

Then

$$
\begin{aligned}
I_2(u) &= E[e^{-\delta\,\tau_a} V_1\big(u + c\,\tau_a + \sigma\,B(\tau_a);\, b\big) I(\tau_a \le t_0) I(\tau_a \le W_1)] \\
&= E[e^{-\delta\,\tau_a} V_1\big(u + c\,\tau_a + \sigma\,B(\tau_a);\, b\big) I(\tau_a \le t_0) I(\tau_a \le W_1) I(B(\tau_a) = a)] \\
&\quad + E[e^{-\delta\,\tau_a} V_1\big(u + c\,\tau_a + \sigma\,B(\tau_a);\, b\big) I(\tau_a \le t_0) I(\tau_a \le W_1) I(B(\tau_a) = -a)] \\
&= \int_0^{t_0} e^{-(\delta+\lambda)t} V_1(u + c\,t + \sigma\,a;\, b) P(B(\tau_a = a),\, \tau_a \in dt) \\
&\quad + \int_0^{t_0} e^{-(\delta+\lambda)t} V_1(u + c\,t - \sigma\,a;\, b) P(B(\tau_a = -a),\, \tau_a \in dt) \\
&= \frac{1}{2} \int_0^{t_0} e^{-(\delta+\lambda)t} \left[V_1(u + c\,t + \sigma\,a;\, b) + V_1(u + c\,t - \sigma\,a;\, b)\right] h(a,t) dt\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&I_3(u) = E\left[e^{-\delta\,W_1} V_1(u + c\,W_1 + \sigma\,B(W_1) - X_1;\, b) I(W_1 \le t_0) I(W_1 < \tau_a)\right] \\
&= \int_0^{t_0} \lambda\, e^{-(\lambda+\delta)s} ds\, E\Big[\int_0^{u+cs+\sigma B(s)} V_1(u + c\,s + \sigma B(s) - z;\, b) I(\tau_a > s) p(z) dz\Big] \\
&= \int_0^{t_0} \lambda\, e^{-(\lambda+\delta)s} ds \int_{-a}^{a} H(a,s,y) dy \int_0^{u+c\,s+\sigma y} V_1(u + c\,s + \sigma\,y - z;\, b) p(z) dz.
\end{aligned}
$$

This completes the proof. □

Theorem 3 If the density function $p(x)$ is continuously differentiable in $(0, \infty)$, then $V_1(u;\, b)$ is twice continuously differentiable in u in the interval $(0,\, b)$.

Proof: First, one can remove u in V_1 in the integrands by changing the variables of the integrations in (33), then $I_1(u), I_2(u)$, and $I_3(u)$ in (32) can be expressed as

$$\begin{aligned}
I_1(u) &= \frac{e^{-(\lambda+\delta)t_0}}{\sigma}\int_{u+c\,t_0-\sigma\,a}^{u+c\,t_0+\sigma\,a} V_1(x;\,b)\,H\big(a,t_0,\frac{x-u-c\,t_0}{\sigma}\big)\,dx\,,\\
I_2(u) &= \frac{1}{2\,c}\int_{u+\sigma\,a}^{u+\sigma\,a+c\,t_0} V_1(x;\,b)\,h\big(a,\frac{x-u-\sigma\,a}{c}\big)\,dx\\
&\quad+\frac{1}{2\,c}\int_{u-\sigma\,a}^{u-\sigma\,a+c\,t_0} V_1(x;\,b)\,h\big(a,\frac{x-u+\sigma\,a}{c}\big)\,dx\,,\\
I_3(u) &= \int_0^{t_0}\lambda\,e^{-(\lambda+\delta)s}ds\int_{-a}^{a}H(a,s,y)dy\int_0^{u+c\,s+\sigma y}V_1(x;\,b)p(u+c\,s+\sigma\,y-x)dx.
\end{aligned}$$

Then by the nice properties of h, H, and p, we can prove that all of $I_1(u), I_2(u)$, and $I_3(u)$ are continuously differentiable in interval $(0,\,b)$, and in particular, we have the following expression for $I_3'(u)$:

$$\begin{aligned}
I_3'(u) &= \int_0^{t_0}\lambda\,e^{-(\lambda+\delta)s}ds\int_{-a}^{a}H(a,s,y)dy\\
&\quad\times\left[\int_0^{u+c\,s+\sigma y}V_1(x;\,b)p'(u+c\,s+\sigma\,y-x)dx+V_1(u+c\,s+\sigma\,y;\,b)p(0)\right]\\
&= \int_0^{t_0}\lambda\,e^{-(\lambda+\delta)s}ds\int_{-a}^{a}H(a,s,y)dy\\
&\quad\times\left[\int_0^{u+c\,s+\sigma y}V_1(u+c\,s+\sigma\,y-x;\,b)p'(x)dx+V_1(u+c\,s+\sigma\,y;\,b)p(0)\right].
\end{aligned}$$

Then we prove that $V_1(u;\,b)$ is continuously differentiable in $(0,\,b)$. Secondly, Since both of h and H are twice continuously differentiable in their variables, and both p and V_1 are continuously differentiable, we can further prove than $I_1(u), I_2(u)$ and $I_3(u)$ are twice continuously differentiable in u in $(0,\,b)$ and in particular,

$$\begin{aligned}
I_3''(u) &= \int_0^{t_0}\lambda\,e^{-(\lambda+\delta)s}ds\int_{-a}^{a}H(a,s,y)dy\times\Big[V_1'(u+c\,s+\sigma\,y;\,b)p(0)\\
&\quad+\int_0^{u+c\,s+\sigma y}V_1'(u+c\,s+\sigma\,y-x;\,b)p'(x)dx\Big].
\end{aligned}$$

Then we have that $V_1(u;\,b)$ are twice continuously differentiable in u in $(0,\,b)$. □

Using the same arguments as in Theorem 2, we can show that $M(u,y;\,b)$ satisfies the following integral equation.

Theorem 4 $M(u, y;\, b)$ satisfies the following integral equation for $0 < u < b$,

$$M(u, y;\, b) = e^{-\lambda\, t_0} \int_{-a}^{a} M(u + c\, t_0 + \sigma x, e^{-\delta\, t_0} y;\, b) H(a, t_0, y) dy$$
$$+ \int_0^{t_0} \lambda\, e^{-\lambda\, s} ds \int_{-a}^{a} H(a, s, x) dx \int_0^{u+c\, s+\sigma\, x} M(u + c\, s + \sigma\, x - z, e^{-\delta\, s} y;\, b) p(z) dz$$
$$+ \frac{1}{2} \int_0^{t_0} \big[M(u + c\, t + \sigma\, a, e^{-\delta t} y;\, b) + M(u + c\, t - \sigma\, a, e^{-\delta t} y;\, b) \big] e^{-\lambda\, t} h(a, t) dt,$$

where a and t_0 are described in Theorem 2.

From the above integral equation, we can prove, by using the same arguments as in the proof of Theorem 3, that when $p(x)$ is continuously differentiable, $M(u, y;\, b)$ is twice continously differentiable in u in $(0, b)$, through which, we can further prove that $M(u, y;\, b)$ is continously differentiable in y.

References

[1] Albrecher, H. and Kainhofer, R. (2002). Risk theory with a nonlinear dividend barrier. *Computing,* **68,** 289–311.

[2] Albrecher, H., Hartinger, J. and Tichy, R.F. (2005). On the distribution of dividend payments and the discounted penalty function in a risk model with linear dividend barrier. *Scandinavian Actuarial Journal,* to appear.

[3] Bühlmann, H. (1970). *Mathematical Methods in Risk Theory.* Springer-Verlag, New York.

[4] Chiu, S.N. and Yin, C.C. (2003). The time of ruin, the surplus prior to ruin and the deficit at ruin for the classical risk process perturbed by diffusion. *Insurance: Mathematics and Economics*, **33**(1), 59-66.

[5] De Finetti, B. (1957). Su un'impostazione alternativa dell teoria colletiva del rischio. *Transactions of the XV International Congress of Actuaries,* **2,** 433–443.

[6] Dickson, D.C.M. and Waters, H. (2004). Some optimal dividends problems. *ASTIN Bulletin*, **34**(1), 49-74.

[7] Dufresne, F. and Gerber, H.U. (1991). Risk theory for the compound Poisson process that is perturbed by diffusion. *Insurance: Mathematics and Economics*, **10**, 51-59.

[8] Gerber, H.U. (1970). An extension of the renewal equation and its application in the collective theory of risk. *Skandinavisk Aktuarietidskrift*, 205-210.

[9] Gerber, H.U. (1973). Martingales in risk theory. *Mitteilungen der Schweizer Vereinigung der Versicherungsmathematiker,* **73**, 205–206.

[10] Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory.* Huebner Foundation, Monograph Series 8, Philadelphia.

[11] Gerber. H.U. (1981). On the probability of ruin in the presence of a linear dividend barrier. *Scandinavian Actuarial Journal*, (2), 105–115.

[12] Gerber, H.U. and Landry, B. (1998). On the discounted penalty at ruin in a jump–diffusion and the perpetual put option. *Insurance: Mathematics and Economics*, **22**, 263-276.

[13] Gerber, H.U. and Shiu, E.S.W. (2004). Optimal dividends: analysis with Brownian motion. *North American Actuarial Journal*, **8**(1), 1-20.

[14] Højgaard, B. (2002). Optimal dynamic premium control in non-life insurance: maximizing dividend payouts. *Scandinavian Actuarial Journal*, 225–245.

[15] Li, S. and Garrido, J. (2004). On a class of renewal risk models with a constant dividend barrier. *Insurance: Mathematics and Economics*, **35**, 691-701.

[16] Li, S. and Garrido, J. (2005). On the Gerber-Shiu function for a Sparre Andersen risk process perturbed by diffusion. *Scandinavian Actuarial Journal*, forthcoming.

[17] Li, S. and Wu, B., 2005, The diffusion perturbed compound Poisson risk model with a dividend barrier. *Stochastic Processes and Their Applications*, submitted for publication.

[18] Lin, X.S., Willmot, G.E. and Drekic, S. (2003). The classical risk model with a constant dividend barrier: Analysis of the Gerber-Shiu discounted penalty function. *Insurance: Mathematics and Economics*, **33**, 551–566.

[19] Paulsen, J. and Gjessing, H. (1997). Optimal choice of dividend barriers for a risk process with stochastic return on investments. *Insurance: Mathematics and Economics*, **20**, 215–223.

[20] Port, S. and Stone, C. (1978). *Brownian Motion and Classical Potential Theory.* Academic Press, New York.

[21] Revuz, D. and Yor, M. (1991). *Continuous Martingales and Brownian Motion.* Springer, Berlin.

[22] Segerdahl, C. (1970). On some distributions in time-connected with the collective theory of risk. *Scandinavian Actuarial Journal,* 167-192.

[23] Tsai, C.C.L. (2001). On the discounted distribution functions of the surplus process perturbed by diffusion. *Insurance: Mathematics and Economics*, **28**, 401-419.

[24] Tsai, C.C.L. (2003). On the expectations of the present values of the time of ruin perturbed by diffusion. *Insurance: Mathematics and Economics*, **32**, 413-429.

[25] Tsai, C.C.L. and Willmot, G.E. (2002a). A generalized defective renewal equation for the surplus process perturbed by diffusion. *Insurance: Mathematics and Economics*, **30**, 51-66.

[26] Tsai, C.C.L. and Willmot, G.E. (2002b). On the moments of the surplus process perturbed by diffusion. *Insurance: Mathematics and Economics*, **31**, 327-350.

[27] Wang, G. (2001). A decomposition of the ruin probability for the risk process perturbed by diffusion. *Insurance: Mathematics and Economics*, **28**, 49-59.

[28] Wang, G. and Wu, R. (2000). Some distributions for classical risk processes that is perturbed by diffusion. *Insurance: Mathematics and Economics*, **26**, 15-24.

[29] Zhang, C. and Wang, G. (2003). The joint density function of three characteristics on jump-diffusion risk process. *Insurance: Mathematics and Economics*, **32**, 445-455.

Address for correspondence:

Shuanming Li
Centre for Actuarial Studies
Department of Economics
The University of Melbourne
Parkville 3052, Victoria
Australia
E-mail: shli@unimelb.edu.au
Fax: 61+3 83446899