

# EFFECTIVE IMPLEMENTATION OF GENERIC MARKET MODELS

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ABSTRACT. A number of standard market models are studied. For each one, algorithms of computational complexity equal to the number of rates times the number of factors to carry out the computations for each step is introduced. Two new classes of market models are developed and it is shown for them that similar results hold.

## 1. INTRODUCTION

Market models are an effective method of pricing interest-rate derivatives. These models rely on choosing processes for market observable rates rather than a hypothetical short rate. The rates evolved are typically LIBOR rates or swap-rates and one then has automatic calibration not just to the value of the rate but also to caplets and swaptions on that rate. The original cases studied were contiguous LIBOR rates and co-terminal swap rates, [1], [6], however, more general cases have recently been examined also, [3], [4], [11], including the concept of a *generic* market model. For general background on market models, see [2], [7] or [10].

In all of these models, rates are assumed to be log-normal (or some similar process such as displaced diffusion) and a single zero-coupon bond is chosen as numeraire. There will generally be  $n$  rates driven by an  $F$ -dimensional Brownian motion, and this will be said an  $F$ -factor model. Since the rates are not ratios of tradables to the numeraire, they are not martingales and turn out to have state-dependent drifts which are generally non-zero. This means that the implementation by Monte Carlo is tricky and computationally intensive. The problem of implementation of swap-rate market models by means other than Monte Carlo does not appear to have been addressed.

The implementation of such models requires several non-trivial computations. The first is the deduction of bond ratios from the observed rates, the second is the calculation of drifts, and thirdly the stochastic

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differential equation must be approximately solved. In this paper, we address how all of these can be done with a total of order  $nF$  computations for a wide range of cases including all the specific examples that have been studied.

In particular, we study the cases of co-terminal swap-rates, co-initial swap rates and the constant maturity market model. We also introduce two more general types of model: the incremental market model and the fully incremental market model, and establish similar results for them.

Specifically, we find order  $n$  algorithms for the deduction of the bond ratios in each of our specific cases, and for fully incremental models. We also show that the drifts computation is order  $nF$  in each of these cases. The simple Euler approximation (and also predictor corrector) is of order  $nF$  for solving the SDEs so the total computational order for a step in all these cases is therefore  $nF$ .

It is important to realize that it is not necessary to find a closed-form formula for the drift and the bond ratios, but merely to write down an algorithm that is efficiently implementable, and this is how we proceed.

The only papers to date where the issue of efficient algorithms for swap-rate drifts and bond ratios are discussed are [8], [9] and [11]. In [8], the LIBOR market model is studied and an algorithm for drift computation of order  $nF$  is presented. In [9], this result is extended to encompass the case of  $F$  common factors and  $n$  idiosyncratic factors. Note that a similar extension could easily be carried out in the cases studied but we do not do so for brevity. In [11], an order  $n^3$  algorithm is presented for deduction of bond ratios in the general case. An order  $nF$  algorithm is presented for computation of drifts in the co-terminal case, and an approximate algorithm for the drifts of order  $nF$  is presented for the constant maturity case.

The structure of this paper is as follows. In Section 2, we establish some notation and examine the computational order of the evolution of the SDE. We develop computational techniques in Section 3, which we apply in the rest of the paper. In Section 4, we show how to deduce the bond ratios from the swap rates in the co-terminal model, and the drift computation in that case is carried out in Section 5. The constant maturity market model is dealt with in Section 6. The arguments for the co-initial model are developed in Section 7. The concept of an incremental market model is introduced in Section 8. We present some numerical results in Section 9 and conclude in Section 10.

## 2. NOTATION AND MODEL SET-UP

We fix some notation. We have times  $0 < t_0 < t_1 < \dots < t_n$ , and  $\tau_j = t_{j+1} - t_j$ . We let  $P_j$  denote the price of the zero-coupon bond expiring at time  $t_j$ . We let  $\text{SR}_{\alpha,\beta}$  denote the swap-rate running from time  $t_\alpha$  to time  $t_\beta$ . We let  $A_{\alpha,\beta}$  be the value of the annuity of  $\text{SR}_{\alpha,\beta}$ , that is

$$A_{\alpha,\beta} = \sum_{j=\alpha}^{\beta-1} \tau_j P_{j+1}.$$

In all of our models, we will have  $n$  rates  $X_j = \text{SR}_{\alpha_j,\beta_j}$ . This is the number one would expect to be necessary to deduce  $n$  ratios  $P_j/P_N$  for some fixed  $N$ . The rates will be driven by  $F$  Brownian motions. We therefore have a system of stochastic differential equations,

$$dX_j = \tilde{\mu}_j(X)dt + X_j \sum_{k=1}^F \tilde{a}_{jk}(t)dW_k.$$

We will evolve the log across a discretized time-step  $(s, t)$ , and then will have

$$\log X_j(t) = \log X_j(s) + \mu_j - \frac{1}{2}C_{jj} + \sum_{k=1}^F a_{jk}Z_k,$$

where  $a_{jk}$  is the integral of  $\tilde{a}_{jk}$  across the step,  $C_{jj}$  is the variance of  $\log X_j$  and  $Z_k$  are  $F$  independent random variables. We have also set  $\mu_j$  to be the approximated drift across the step. In this paper, we will take it to be instantaneous drift at the start of the step with the covariance terms across the step integrated. Given the drift term, it is clear that the evolution across the step could be carried out with order  $nF$  computations. Similarly, if one decided to use a predictor-corrector type method as in [5], order  $nF$  computations once given the drifts would be sufficient.

## 3. THE CROSS-VARIATION DERIVATIVE

In what follows, it will be useful to work with the *cross-variation derivative* for two Ito processes. Given processes  $X_t$  and  $Y_t$ , we define

$$\langle X_t, Y_t \rangle$$

to be the coefficient of  $dt$  in  $dX_t.dY_t$ . (Note the cross-variation process is generally defined to be the process  $dX_t.dY_t$ , but the cross-variation

derivative will be more convenient for us.) This means that if we have

$$dX_t = \mu_X(t)dt + \sigma_X(X_t, Y_t, t)dW_t^X, \quad (3.1)$$

$$dY_t = \mu_Y(t)dt + \sigma_Y(X_t, Y_t, t)dW_t^Y, \quad (3.2)$$

with  $W^X, W^Y$  correlated (jointly normal) Brownian motions, then

$$\langle X_t, Y_t \rangle = \rho \sigma_X(X_t, Y_t, t) \sigma_Y(X_t, Y_t, t), \quad (3.3)$$

where  $\rho$  is the correlation between  $W^X$  and  $W^Y$ . Note that the drift terms do not appear in this expression.

The cross-variation derivative has some useful computational properties. First, it is linear in each term, i.e, if  $Y_j$  are a number of stochastic processes, and  $\alpha_j \in \mathbb{R}$ , then

$$\left\langle X, \sum_j \alpha_j Y_j \right\rangle = \sum_j \alpha_j \langle X, Y_j \rangle. \quad (3.4)$$

It is trivially symmetric in  $X$  and  $Y$ . We can also compute with products in a simple fashion

$$\langle X, YZ \rangle = \langle X, Y \rangle Z + \langle X, Z \rangle Y. \quad (3.5)$$

Note that trivially the cross-variation derivative with a constant is always zero.

We can deduce the value of  $\langle Y, X^{-1} \rangle$  :

$$\begin{aligned} \langle Y, 1 \rangle &= \langle Y, X.X^{-1} \rangle, \\ &= \langle Y, X \rangle X^{-1} + \langle Y, X^{-1} \rangle X, \end{aligned}$$

and therefore

$$\langle Y, X^{-1} \rangle = -X^{-2} \langle Y, X \rangle. \quad (3.6)$$

The cross-variation derivative will play an important role in our drift computations. Suppose we have a *rate*, by which we shall mean a quantity defined as the ratio of two assets; in other words, it is the exchange rate for converting one asset to the other.

So suppose  $X, Y$  and  $N$  are tradable assets, and we wish to compute the drift of  $R = X/Y$  when  $N$  is numeraire. We know that  $RY/N = X/N$  and  $Y/N$  are martingales. We have

$$d \frac{RY}{N} = \frac{Y}{N} dR + R d \frac{Y}{N} + dR. d \frac{Y}{N}.$$

Taking the drifts, and discarding martingale terms, we have that  $\mu_R$ , the drift of  $R$ , satisfies

$$\mu_R = -\frac{N}{Y} \left\langle R, \frac{Y}{N} \right\rangle. \quad (3.7)$$

So, in order to compute the drift of  $R$ , it is sufficient to compute the cross-variation derivative of  $R$  and  $\frac{Y}{N}$ .

The cross-variation will also be useful for assessing the impact of changing numeraire on a drift. Suppose we already know the drift,  $\mu_{R,N}$ , of  $R$  under the numeraire  $N$  and we want to compute the drift,  $\mu_{R,M}$ , with numeraire  $M$ . We have

$$\mu_{R,M} = - \left\langle R, \frac{Y}{M} \right\rangle \frac{M}{Y}, \quad (3.8)$$

$$= - \left\langle R, \frac{Y}{N} \frac{N}{M} \right\rangle \frac{M}{Y}, \quad (3.9)$$

$$= - \left\langle R, \frac{Y}{N} \right\rangle \frac{N}{M} \frac{M}{Y} - \left\langle R, \frac{N}{M} \right\rangle \frac{Y}{N} \frac{M}{Y}, \quad (3.10)$$

$$= \mu_{R,N} - \left\langle R, \frac{N}{M} \right\rangle \frac{M}{N}. \quad (3.11)$$

#### 4. DEDUCING THE BOND-RATIOS IN THE CO-TERMINAL CASE

In this section, we show how to compute the bond ratios in order  $n$  computations for the co-terminal swap-rate market model. In order to keep notation simple, in this section and the next, we let  $\text{SR}_j$  denote the swap-rate associated to times  $t_j, \dots, t_n$ . We also let  $A_j$  be the annuity of  $\text{SR}_j$ .

We first show to find the ratios  $P_j/P_n$ . Clearly, the ratio  $P_j/P_N$  is trivial for any  $N$  is then trivial to find. We work backwards. If  $j = n$ , we have  $P_j/P_n = 1$ , and we are done. For  $j < n$ , we assume that  $P_j/P_n$  has been found for larger  $j$ . We then have

$$\text{SR}_j = \frac{P_j - P_n}{A_j},$$

and it follows that

$$\frac{P_j}{P_n} = 1 + \text{SR}_j \frac{A_j}{P_n}.$$

The terms on the right hand side are already determined as  $A_j$  only involves bonds with maturity after  $t_j$ , the value of  $P_j/P_n$  follows and we are done.

Note that all the bond ratios can be deduced with order  $n$  computations.

## 5. CO-TERMINAL SWAP-RATE DRIFT COMPUTATIONS

We apply our results on cross-variation derivatives.<sup>1</sup> In the case of a swap-rate, we have

$$\text{SR}_j = \frac{P_j - P_n}{A_j}.$$

If we adopt  $P_N$  as numeraire, we conclude that the drift of  $\text{SR}_j$ ,  $\mu_j$  satisfies

$$\mu_j = -\frac{P_N}{A_j} \langle A_j/P_N, \text{SR}_j \rangle. \quad (5.1)$$

We therefore need to evaluate this cross-variation term.

We now specialize to the case where  $N = n$ , we will return to the general case further down. We can write

$$d\text{SR}_j = \text{SR}_j \sum_{k=1}^F a_{jk} dW_k + \text{drift}, \quad (5.2)$$

where the Brownian motions,  $W_k$ , are independent. Clearly, we have

$$\langle \text{SR}_j, A_j/P_n \rangle = \text{SR}_j \sum_{k=1}^F a_{jk} \langle W_k, A_j/P_n \rangle. \quad (5.3)$$

If we can compute  $\langle W_k, A_j/P_n \rangle$  for all  $j$  and  $k$  then we are done, and it will take  $\mathcal{O}(nF)$  computations to convert to drifts for  $\text{SR}_j$ .

We now address how to compute

$$\langle W_k, A_j/P_n \rangle$$

for a fixed  $k$  and all  $j$  with order  $n$  computations. We work backwards. The first case is  $j = n - 1$ , where  $A_{n-1}/P_n = \tau_{n-1}$  and the cross-variation is zero. Now suppose we have computed  $\langle W_k, A_{j+1}/P_n \rangle$ , we have

$$\langle W_k, A_j/P_n \rangle = \langle W_k, P_{j+1}/P_n \rangle \tau_j + \langle W_k, A_{j+1}/P_n \rangle, \quad (5.4)$$

the second term we already know. The first term we can rewrite:

$$\langle W_k, P_{j+1}/P_n \rangle = \langle W_k, 1 + \text{SR}_{j+1} A_{j+1}/P_n \rangle, \quad (5.5)$$

$$= \langle W_k, \text{SR}_{j+1} A_{j+1}/P_n \rangle, \quad (5.6)$$

$$= \langle W_k, \text{SR}_{j+1} \rangle A_{j+1}/P_n + \langle W_k, A_{j+1}/P_n \rangle \text{SR}_{j+1}. \quad (5.7)$$

The first angle bracket is  $\text{SR}_{j+1} a_{j+1,k}$  by definition and the second is already known. This means that we can deduce the  $j$ th term from the preceding computations with a fixed finite number of computations, and we are done.

<sup>1</sup>This section was heavily influenced by unpublished work of Jochen Theis.

Note that the above computations have computed the drift of  $\text{SR}_j$  whereas we would typically evolve  $\log(\text{SR}_j)$  instead, and we would therefore not carry out the final multiplication in (5.3), but subtract the standard  $-0.5C_{jj}$  for the transformation to log space.

We have computed the drift when  $P_n$  is numeraire. We may wish to use another numeraire, we can compute the new drift using (3.11). We have

$$\mu_{\text{SR}_j, P_N} = \mu_{\text{SR}_j, P_n} - \left\langle \text{SR}_j, \frac{P_n}{P_N} \right\rangle \frac{P_N}{P_n}.$$

We can expand  $\text{SR}_j$  in terms of  $W_k$  as before, and we therefore need to compute

$$\begin{aligned} \frac{P_N}{P_n} \left\langle W_k, \frac{P_n}{P_N} \right\rangle &= - \frac{P_N}{P_n} \left\langle W_k, \frac{P_N}{P_n} \right\rangle \frac{P_n^2}{P_N^2} \\ &= - \left\langle W_k, \frac{P_N}{P_n} \right\rangle \frac{P_n}{P_N}. \end{aligned}$$

As  $A_{N-1} - A_N = \tau_{N-1}P_N$ , this is easily rolled into our original computation, and we are still within order  $nF$  computations.

## 6. CONSTANT MATURITY MARKET MODELS

In this section, we examine constant maturity market models. For a constant maturity model, we consider the set of rates

$$\text{SR}_{\alpha, \alpha+r},$$

for a fixed  $r$ , and we make the convention that if  $\alpha+r \geq n$ , then we take it to equal  $n$ . We similarly let  $A_{\alpha, \alpha+r}$  denote the annuity of  $\text{SR}_{\alpha, \alpha+r}$ .

Note that we obtain a different rate for  $\alpha = 0, 1, \dots, n-1$ , and that for the last  $r$  rates we are working with co-terminal rates, and for those rates any analysis carries directly over from the co-terminal swap-rate market model. We need to compute drifts and find an algorithm for obtaining the bond ratios from the rates. We work with  $P_n$  as numeraire and work backwards.

We have

$$\text{SR}_{j, r+j} = \frac{P_j - P_{r+j}}{A_{j, r+j}},$$

by definition, (even when  $r+j > n$ ), which implies

$$\frac{P_j}{P_n} = \frac{P_{r+j}}{P_n} + \frac{A_{j, r+j}}{P_n} \text{SR}_{j, r+j}. \quad (6.1)$$

It is now clear that we can induct backwards computing  $A_{j, r+j}$  and  $P_j/P_n$  as we go.

If we are working in an  $F$  factor model, as before, we can write

$$d\text{SR}_{j,r+j} = \text{SR}_{j,r+j} \sum_{k=1}^F a_{jk} dW_k,$$

up to drift terms, and it follows that the drift of  $\text{SR}_{j,r+j}$  is equal to

$$- \sum_{k=1}^F a_{jk} \frac{P_n}{A_{j,r+j}} \text{SR}_{j,r+j} \left\langle W_k, \frac{A_{j,r+j}}{P_n} \right\rangle.$$

If we can compute the quadratic variation terms with order  $nF$  computations then that will be sufficient to show that we can compute all the drifts with that computational order. We work backwards. Suppose we know

$$\left\langle \frac{A_{j,j+r}}{P_n}, W_k \right\rangle \text{ for } j > l, \text{ and} \quad (6.2)$$

$$\left\langle \frac{P_r}{P_n}, W_k \right\rangle \text{ for } r > l + 1, \quad (6.3)$$

we show that we can find

$$\left\langle \frac{A_{l,l+r}}{P_n}, W_k \right\rangle \text{ and} \quad (6.4)$$

$$\left\langle \frac{P_{l+1}}{P_n}, W_k \right\rangle \quad (6.5)$$

with a fixed number of computations, which will be sufficient. With knowledge of the second term, the first follows immediately from linearity and the values of  $\left\langle \frac{A_{l+1,l+r+1}}{P_n}, W_k \right\rangle$ , and  $\left\langle \frac{P_{l+r+1}}{P_n}, W_k \right\rangle$ .

We compute

$$\left\langle \frac{P_{j+1}}{P_n}, W_k \right\rangle = \left\langle \text{SR}_{j+1,r+j+1} \frac{A_{j+1,r+j+1}}{P_n}, W_k \right\rangle + \left\langle \frac{P_{j+r+1}}{P_n}, W_k \right\rangle.$$

Using equation (6.1) and expanding, this is equal to

$$\text{SR}_{j+1,r+j+1} \left\langle \frac{A_{j+1,r+j+1}}{P_n}, W_k \right\rangle + \frac{A_{j+1,r+j+1}}{P_n} \langle \text{SR}_{j+1,r+j+1}, W_k \rangle + \left\langle \frac{P_{j+r+1}}{P_n}, W_k \right\rangle.$$

The first and third terms are known, and the second is trivial; we are done.



## 7. CO-INITIAL SWAP-RATES

Another case that can be analyzed is co-initial swap rates which was introduced by Gallucio and Hunter, [4]. We can solve it using similar techniques to the other cases we have discussed.

For this section, we use  $SR_j$  to denote the swap-rate which was  $SR_{0,j}$  in Section 2, similarly for  $A_j$ . Our class of swap-rates is  $SR_j$  for  $j = 1, \dots, n$ , and they therefore all start on the same date but finish on varying dates. As usual, we must first develop an algorithm to deduce the bond ratios from the swap-rates. We will work with  $P_0$  as numeraire in this section. In order to ease the notation, we shall also use a tilde to denote that a price has been divided by  $P_0$ . We thus have

$$\tilde{P}_k = \frac{P_k}{P_0}, \quad (7.1)$$

$$\tilde{A}_j = \frac{A_j}{P_0}. \quad (7.2)$$

As

$$SR_j = \frac{P_0 - P_j}{A_j},$$

we see that

$$\begin{aligned} \tilde{P}_j &= 1 - \tilde{A}_j SR_j \\ &= 1 - \tilde{A}_{j-1} SR_j - \tau_{j-1} \tilde{P}_j SR_j, \end{aligned}$$

and hence that

$$\tilde{P}_j = \frac{1 - \tilde{A}_{j-1} SR_j}{1 + \tau_{j-1} SR_j}. \quad (7.3)$$

Inducting on  $j$  increasing, it is clear how to deduce the bond ratios in order  $n$  steps.

By the usual arguments, to compute drifts for log-normal co-initial rates we need to find the cross-variation of  $W_k$  and  $\tilde{P}_j$ , which equals

$$\begin{aligned}
\langle W_k, \tilde{P}_j \rangle &= \left\langle W_k, \frac{1 - \tilde{A}_{j-1} \text{SR}_j}{1 + \tau_{j-1} \text{SR}_j} \right\rangle \\
&= \frac{\langle W_k, 1 - \tilde{A}_{j-1} \text{SR}_j \rangle}{1 + \tau_{j-1} \text{SR}_j} + \left\langle W_k, \frac{1}{1 + \tau_{j-1} \text{SR}_j} \right\rangle (1 - \tilde{A}_{j-1} \text{SR}_j), \\
&= - \frac{\langle W_k, \tilde{A}_{j-1} \rangle \text{SR}_j}{1 + \tau_{j-1} \text{SR}_j} - \langle W_k, \text{SR}_j \rangle \frac{\tilde{A}_{j-1}}{1 + \tau_{j-1} \text{SR}_j} \\
&\quad - \langle W_k, 1 + \tau_{j-1} \text{SR}_j \rangle \frac{1 - \tilde{A}_{j-1} \text{SR}_j}{(1 + \tau_{j-1} \text{SR}_j)^2}, \\
&= - \frac{\langle W_k, \tilde{A}_{j-1} \rangle \text{SR}_j}{1 + \tau_{j-1} \text{SR}_j} - \frac{\langle W_k, \text{SR}_j \rangle \tilde{A}_{j-1}}{1 + \tau_{j-1} \text{SR}_j} \\
&\quad - \langle W_k, \text{SR}_j \rangle \frac{\tau_{j-1} (1 - \tilde{A}_{j-1} \text{SR}_j)}{(1 + \tau_{j-1} \text{SR}_j)^2}.
\end{aligned}$$

If we make it our inductive hypothesis that we have already computed  $\langle W_k, \tilde{P}_{j-1} \rangle$ , and  $\langle W_k, \tilde{A}_{j-1} \rangle$ , it is clear that we can do the next term with a fixed finite number of computations, and the drifts follow as before.

## 8. INCREMENTAL MARKET MODELS

We have studied three cases: the the co-terminal swap-rate market model, the co-initial swap-rate market model, and the constant maturity market model. In addition, the fourth case of the LIBOR market model was studied in [8]. For each of these, we have seen that the bond ratios can be deduced in order  $n$  operations and the drifts computed in order  $nF$  operations. It is an interesting question whether we can formulate a general result. In this section, we introduce a new class of models for which we can compute the bond ratios with order  $n$  multiplications, and the drifts with order  $nF$  multiplications but both requiring order  $n^2$  additions and subtractions. Additions are much faster in most architectures than multiplications so this is still a worthwhile result. We then see how adding a further additional hypothesis can reduce the total number of computations to order  $nF$ .

Any market model is determined by picking a set of times  $t_0 < t_1 < \dots < t_n$ , and then choosing a subset of the swap-rates associated to (usually) contiguous subsets of those times. Let  $P_r$  be value of the

discount bond expiring at time  $t_r$ . We have

$$\text{SR}_{\alpha,\beta} = \frac{P_\alpha - P_\beta}{\sum_{r=\alpha}^{\beta-1} \tau_r P_{r+1}}.$$

Specifying a market model is therefore equivalent to specifying two sequences in  $\{0, 1, \dots, n-1\}$  :  $\alpha_0, \dots, \alpha_{n-1}$ , and  $\beta_0, \dots, \beta_{n-1}$ , such that  $\beta_j \geq \alpha_j + 1$ . Of course, for a given choice of the sequences, one needs to show that the bond ratios are uniquely determined.

**Definition 8.1.** *A market model is incremental if  $\beta_0 = \alpha_0 + 1$ , and for  $j > 0$ , either*

$$\alpha_j = \min_{r < j} \alpha_r, \quad \text{and} \quad \beta_j = 1 + \max_{r < j} \beta_r$$

or

$$\alpha_j = -1 + \min_{r < j} \alpha_r, \quad \text{and} \quad \beta_j = \max_{r < j} \beta_r.$$

In other words, in an incremental market model the introduction of each new rate causes dependency on exactly one more discount bond. Note that as it is really bond ratios we care about, this is true even of  $\text{SR}_{\alpha_0, \beta_0}$ , which depends purely on the ratio  $P_{\alpha_0}/P_{\beta_0}$ . If we fix a numeraire, there are  $n$  bond ratios, and  $n$  rates so when we get to the last rate we will have introduced dependency on all the bond ratios.

**Theorem 8.1.** *In an incremental market model, for any  $N$  the bond ratios  $P_j/P_N$  are determined by the swap-rates and can be deduced with order  $n$  multiplications and order  $n^2$  additions.*

*Proof.* We take  $N = \beta_0$ . Once  $P_j/P_{\beta_0}$  is known for all  $j$ , one simply writes

$$\frac{P_j}{P_N} = \frac{P_j}{P_{\beta_0}} \frac{P_{\beta_0}}{P_N},$$

to get the general case with an extra order  $n$  computations.

We have

$$\text{SR}_{\alpha_0, \beta_0} = \frac{P_{\beta_0} - P_{\alpha_0}}{\tau_{\alpha_0} P_{\beta_0}},$$

so the ratio  $\frac{P_{\alpha_0}}{P_{\beta_0}}$  is clearly determined.

We now show that given the bond ratios for the bonds underlying the first  $r-1$  rates, we can deduce the extra bond ratio underlying  $\text{SR}_{\alpha_r, \beta_r}$  from its value. There are two cases corresponding to whether the new bond is at the beginning or end.

If it is at the end, we have  $\beta_r = 1 + \max_{l < r} \beta_l$ , and

$$\text{SR}_{\alpha_r, \beta_r} = \frac{P_{\alpha_r} - P_{\beta_r}}{\sum_{l=\alpha_r}^{\beta_r-1} \tau_l P_{l+1}}.$$

Rearranging, we obtain

$$\frac{P_{\beta_r}}{P_{\beta_0}} = \frac{\frac{P_{\alpha_r}}{P_{\beta_0}} - \sum_{l=\alpha_r}^{\beta_r-2} \tau_l \frac{P_{l+1}}{P_{\beta_0}} \text{SR}_{\alpha_r, \beta_r}}{1 + \tau_{\beta_r-1} \text{SR}_{\alpha_r, \beta_r}}. \quad (8.1)$$

The ratio is therefore determined. Similarly, if the new bond is at the beginning, we have

$$\frac{P_{\alpha_r}}{P_{\beta_0}} = \frac{P_{\beta_r}}{P_{\beta_0}} + \text{SR}_{\alpha_r, \beta_r} \sum_{l=\alpha_r}^{\beta_r-1} \tau_l \frac{P_{l+1}}{P_{\beta_0}}, \quad (8.2)$$

and the first ratio is determined.

How many computations will this take? At each stage, we store each new bond ratio and its multiplication by the appropriate accrual,  $\tau_l$ , it is then clear that we only need a fixed number of multiplication per step and therefore order  $n$  in total. However, the sums will require up to  $n$  additions per step so we have order  $n^2$  additions in total. Note that in each of the four cases we studied in detail, there was extra structure that reduced the number of additions, but it seems unlikely that this will be possible in general without extra hypotheses.  $\square$

By the same arguments as in previous sections, if we take  $P_{\beta_0}$  as numeraire, we can deduce the drift of  $\text{SR}_{\alpha_j, \beta_j}$  for all  $j$  from the knowledge of

$$\left\langle W_k, \frac{A_{\alpha_j, \beta_j}}{P_{\beta_0}} \right\rangle,$$

with order  $nF$  operations. We proceed inductively on  $j$  as usual and each stage store the cross-variation of the swap-rate ratio of the new bond to the numeraire and its value multiplied by the appropriate accrual.

Just as with the deduction of the bond ratios, we have to proceed differently according to whether the introduction of the new bond is at the beginning or the end of the known cases. If it is at the start, using

(8.2), we have

$$\begin{aligned} \left\langle W_k, \frac{P_{\alpha_r}}{P_{\beta_0}} \right\rangle &= \left\langle W_k, \frac{P_{\beta_r}}{P_{\beta_0}} \right\rangle + \langle W_k, \text{SR}_{\alpha_r, \beta_r} \rangle \sum_{l=\alpha_r}^{\beta_r-1} \tau_l \frac{P_{l+1}}{P_{\beta_0}} \\ &\quad + \text{SR}_{\alpha_r, \beta_r} \sum_{l=\alpha_r}^{\beta_r-1} \tau_l \left\langle W_k, \frac{P_{l+1}}{P_{\beta_0}} \right\rangle. \end{aligned} \quad (8.3)$$

This is computable with order  $n$  additions and a fixed finite number of multiplications.

If at the end, using (8.1), we have

$$\begin{aligned} \left\langle W_k, \frac{P_{\beta_r}}{P_{\beta_0}} \right\rangle &= (1 + \tau_{\beta_r-1} \text{SR}_{\alpha_r, \beta_r})^{-1} \\ \left( \left\langle W_k, \frac{P_{\alpha_r}}{P_{\beta_0}} \right\rangle - \sum_{l=\alpha_r}^{\beta_r-2} \tau_l \left\langle W_k, \frac{P_{l+1}}{P_{\beta_0}} \right\rangle \text{SR}_{\alpha_r, \beta_r} - \sum_{l=\alpha_r}^{\beta_r-2} \tau_l \frac{P_{l+1}}{P_{\beta_0}} \langle W_k, \text{SR}_{\alpha_r, \beta_r} \rangle \right) \\ &\quad + \frac{\tau_{\beta_r-1} \langle W_k, \text{SR}_{\alpha_r, \beta_r} \rangle}{(1 + \tau_{\beta_r-1} \text{SR}_{\alpha_r, \beta_r})^2} \left( \frac{P_{\alpha_r}}{P_{\beta_0}} - \sum_{l=\alpha_r}^{\beta_r-2} \tau_l \frac{P_{l+1}}{P_{\beta_0}} \text{SR}_{\alpha_r, \beta_r} \right). \end{aligned} \quad (8.4)$$

This can also be computed with order  $n$  additions and a fixed finite number of multiplications.

Once we have the cross-variation derivative with each bond ratio, the cross-variation derivatives with the annuities are straightforward additions and we are done.

Studying the above proofs, one sees that the failure of the algorithm to attain order  $n$  operations for the deduction of bonds-ratios and order  $nF$  for the computation of drifts arises from the need to compute annuities. If we put an additional hypothesis on the annuities, we can attain these faster speeds.

**Definition 8.2.** *We shall say that a class of market models is fully incremental of order  $\theta$  if there exists  $\theta$  independent of  $n$  such that for each  $j$ , there exists  $i < j$ , such that  $\text{SR}_i$  differs from  $\text{SR}_j$  by at most  $\theta$  bonds.*

It is clear from studying the proofs above that the bond-ratios can be deduced in  $\mathcal{O}(n\theta)$ , operations and the drifts in  $\mathcal{O}(n(F+\theta))$  operations. The constant maturity market model is fully incremental of order 2, the other cases we have studied are fully incremental of order 1.

Rates	Time	Fit
5	10.11	10.17
10	31.97	31.77
15	65.36	65.11
20	110.68	110.19
25	165.65	167.01
30	236.24	235.57

TABLE 1. Timings for evolving a constant maturity swap market model of constant maturity 4 for a 3-factor model with varying numbers of rates

Rates	Time	Fit
5	12.41	13.02
10	41.19	40.51
15	85.63	85.01
20	146.6	146.54
25	223.33	225.09
30	321.58	320.66

TABLE 2. Timings for evolving a constant maturity swap model of constant maturity 4 for a 5-factor model with varying numbers of rates

## 9. NUMERICAL RESULTS

In this sections, we present timings using these techniques. The purpose of the modelling was to demonstrate the behaviour as a function of the number of rates,  $n$ , rather than to do the fastest possible implementation. For each of the constant maturity and co-terminal cases, we step all the rates that have not reset to each of the reset dates. We show timings for a fixed number of factors. Since we carry out an order  $nF$  algorithm for each of  $n$  steps, we obtain timings that are parabolic in  $n$ , and we display the values of a fitted parabola through the timings in each case. See tables 1 and 2 for the constant maturity case, and tables 3 and 4 for the co-terminal case.

In the co-initial case, we only evolve to the common initial time so we expect linear behaviour for speed. We display the timings and the best fit line through them in tables 5 and 6

Rates	Time	Fit
3	5.82	6.06
5	12.69	12.49
10	37.95	37.69
15	75.74	75.93
20	127.1	127.22
25	191.69	191.55

TABLE 3. Timings for evolving a co-terminal swap-rate market model for a 3-factor model with varying numbers of rates

Rates	Time	Fit
5	15.05	14.28
10	46.53	46.44
15	93.81	95.92
20	165.2	162.73
25	245.95	246.85

TABLE 4. Timings for evolving a co-terminal swap-rate model for a 5-factor model with varying numbers of rates

Rates	Time	Fit
3	1.76	1.78
5	2.14	2.14
10	3.07	3.04
15	3.96	3.95
20	4.84	4.85
25	5.76	5.75
30	6.64	6.65

TABLE 5. Timings for evolving a co-initial swap-rate market model for a 3-factor model with varying numbers of rates

## 10. CONCLUSION

We have examined a number of special cases: the co-terminal swap-rate model, the co-initial swap-rate model, the constant maturity market model, as well as the more general case of the incremental market model. For these cases, we have shown that efficient algorithms exist for the evolution of time steps. These models are therefore equally attractive to the LIBOR market model in terms of efficiency and one

Rates	Time	Fit
5	2.36	2.35
10	3.35	3.33
15	4.28	4.31
20	5.27	5.28
25	6.26	6.26
30	7.25	7.24

TABLE 6. Timings for evolving a co-initial swap-rate model for a 5-factor model with varying numbers of rates

should make model choice on the basis of other issues such as ease of calibration, and adaptation to the product being studied.

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