APRA GENERAL INSURANCE RISK MARGINS

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Summary

It is common for actuaries to estimate percentile-based risk margins on the assumption of a log normal distribution of liability, together with an estimate of coefficient of variation ("CoV"). This can yield seemingly anomalous results, with percentage risk margin decreasing as CoV increases. The mathematics of this type of risk margin is explored.

An APRA risk margin is the maximum of this type and a multiple of CoV. Such risk margins are studied in a more general setting than APRA's, with both percentile p and CoV multiple k free. The APRA risk margins form a special case within this setting.

Particular attention is paid to **risk margin transition points**, values of the log normal dispersion parameter at which the risk margin changes from one form to the other as that parameter increases. For given values of p and k, the existence, uniqueness and location of transition points is investigated. The direction of change of a transition point in the presence of increasing p or k is also investigated.

Various numerical examples are given.

Keywords: APRA, risk margin, risk margin transition point.

1. Introduction

General insurance is regulated in Australia by the Australian Prudential Regulation Authority ("APRA") under the Insurance Act 1973. Associated Prudential Standards govern certain financial aspects of the regulation.

Prudential Standard GPS 210 (soon to be replaced by GPS 310) stipulates that provisions for insurance liabilities must include a **risk margin** over and above the mean, or **central estimate**, of those liabilities. The risk margin in respect of a particular liability must be the greater of:

- half of the estimated standard deviation of the liability; and
- that margin which would give the provision a 75% probability of adequacy to meet the liability.

This will be referred to here as the **"APRA risk margin"**. It is common for actuaries to estimate risk margins of this sort under an assumption that the amount of liability, considered as a random variable, is log normally distributed. This gives rise to some properties of the risk margin that have, from time to time, been considered anomalous.

The purpose of the present note to is to explore some of the mathematical properties of the APRA risk margin. It will be of some interest to do so in a slightly more general setting in which the half standard deviation condition is replaced by a general multiple *k* of standard deviation, and the 75% probability of adequacy is generalised to p%.

2. Definition of risk margin

Let L denote the amount of the liability under consideration, and assume that

$$L \sim \log N\left(\mu, \sigma^2\right) \tag{2.1}$$

Then

$$E[L] = \exp\left(\mu + \tfrac{1}{2}\sigma^2\right) \tag{2.2}$$

$$Var[L] = \exp\left(2\mu + \sigma^2\right)\left[\exp\sigma^2 - 1\right] \tag{2.3}$$

$$CoV[L] = \left[\exp\sigma^2 - 1\right]^{\frac{1}{2}} \tag{2.4}$$

Let

$\alpha_p(L)$ denote the 100p-percentile of L, ie

$$\text{Prob}\left[L \leq \alpha_p(L)\right] = p \tag{2.5}$$

Then

$$\begin{aligned} p &= \text{Prob}\left[\log L \leq \log\alpha_p(L)\right] \\ &= \Phi\left(\left[\log\alpha_p(L) - \mu\right]/\sigma\right) \end{aligned} \tag{2.6}$$

where $\Phi(.)$ is the unit normal d.f.

It follows that

$$\alpha_p(L) = \exp\left(\mu + z_p\sigma\right) \tag{2.7}$$

where z_p is the normal standard score associated with probability p.

Note that, by (2.2), $\alpha_p(L)$ may be expressed in the form

$$\alpha_p(L) = E[L]\exp\left(z_p\sigma - \tfrac{1}{2}\sigma^2\right) \tag{2.8}$$

which is independent of the log normal location parameter μ.

Suppose the provision made in the company accounts for the liability L is

$$P(L;p,k) = \max\left\{\alpha_p(L), E[L]\left[1 + kCoV[L]\right]\right\} \tag{2.9}$$

for some constant $k > 0$.

Substitution of (2.4) and (2.8) into (2.9) gives

$$P(L;p,k) = E[L]\,f(L;p,k) \tag{2.10}$$

where

$$f(L;p,k) = \max\left[f_1(\sigma;p), f_2(\sigma;k)\right] \tag{2.11}$$

$$f_1(\sigma;p) = \exp\left(z_p\sigma - \tfrac{1}{2}\sigma^2\right) \tag{2.12}$$

$$f_2(\sigma;k) = 1 + k\left(\exp\sigma^2 - 1\right)^{\frac{1}{2}} \tag{2.13}$$

Result 1. The quantity $f(L;p,k)$ given by (2.11) – (2.13) is the multiplier that converts the central estimate $E[L]$ to the provision $P(L;p,k)$. □

It is of interest to examine how this multiplier varies with σ.

3. Variation of log normal coefficient of variation with dispersion parameter

By (2.13),

$$\partial f_2 / \partial\sigma = k\sigma\left(\exp\sigma^2\right)\left(\exp\sigma^2 - 1\right)^{-\frac{1}{2}} \tag{3.1}$$

$$\partial^2 f_2 / \partial\sigma^2 = k\left(\exp\sigma^2\right)\left(\exp\sigma^2 - 1\right)^{-\frac{3}{2}}\left[\left(1+\sigma^2\right)\exp\sigma^2 - \left(1+2\sigma^2\right)\right] \tag{3.2}$$

$$\begin{aligned}\partial^3 f_2 / \partial\sigma^3 = {} & k\sigma\left(\exp\sigma^2\right)\left(\exp\sigma^2 - 1\right)^{-\frac{5}{2}} \\ & \text{x}\left[\left(3+\sigma^2\right)\left(\exp 2\sigma^2\right) - \left(9+2\sigma^2\right)\left(\exp\sigma^2\right) + 2\left(3+2\sigma^2\right)\right]\end{aligned} \tag{3.3}$$

For small σ, $\exp\sigma^2 - 1$ is approximated by σ^2, so that as $\sigma \to 0$

$$\partial f_2 / \partial\sigma \sim k\exp\sigma^2 \to k \tag{3.4}$$

$$\partial^2 f_2 / \partial\sigma^2 \sim k\exp\sigma^2\left[\left(1+\sigma^2\right)^2 - \left(1+2\sigma^2\right)\right] / \sigma^3 \sim k\sigma \to 0 \tag{3.5}$$

$$\partial^3 f_2 / \partial\sigma^3 \sim k\left[\left(3+\sigma^2\right)\left(1+2\sigma^2\right) - \left(9+2\sigma^2\right)\left(1+\sigma^2\right) + 2\left(3+2\sigma^2\right) + O\left(\sigma^6\right)\right] / \sigma^4 \to 0 \tag{3.6}$$

By (3.1) and (3.4),

$$\partial f_2 / \partial \sigma > 0 \text{ for all } \sigma \tag{3.7}$$

Moreover, it is shown in Appendix A that

$$\partial^2 f_2 / \partial \sigma^2 > 0 \tag{3.8}$$

$$\partial^3 f_2 / \partial \sigma^3 > 0 \tag{3.9}$$

for all $\sigma > 0$.

Hence the following result.

Result 2. The function $f_2(\sigma;k)$ has positive gradient k at $\sigma = 0$, and positive curvature everywhere except at $\sigma = 0$, where the curvature is zero. Its gradient therefore is positive everywhere, increasing without limit as $\sigma \to \infty$. □

4. Variation of log normal quantile with dispersion parameter

The function $f_1(\sigma;p)$ may be recognized as representing a Gaussian curve (in variable σ) with mean z_p, unit variance, and scaled to assume a value of unity at $\sigma = 0$.

By (2.12),

$$\partial f_1 / \partial \sigma = (z_p - \sigma)\exp\left(z_p\sigma - \tfrac{1}{2}\sigma^2\right) \tag{4.1}$$

$$\partial^2 f_1 / \partial \sigma^2 = \left[(z_p - \sigma)^2 - 1\right]\exp\left(z_p\sigma - \tfrac{1}{2}\sigma^2\right) \tag{4.2}$$

$$\partial^3 f_1 / \partial \sigma^3 = \left[(z_p - \sigma)^3 - 3(z_p - \sigma)\right]\exp\left(z_p\sigma - \tfrac{1}{2}\sigma^2\right) \tag{4.3}$$

It follows that $f_1(\sigma;p)$ has a maximum of $\exp\frac{1}{2}z_p^2$ at $\sigma = z_p$.

Further

$$\begin{aligned} \partial^2 f_1 / \partial \sigma^2 &< 0 \text{ for } z_p - 1 < \sigma < z_p + 1 \\ &> 0 \text{ for } \sigma < z_p - 1 \text{ or } \sigma > z_p + 1 \end{aligned} \tag{4.4}$$

$$\partial^3 f_1 / \partial \sigma^3 < 0 \text{ if } z_p - \sqrt{3} < \sigma < z_p \tag{4.5}$$

Note also, from (4.1) to (4.3), that, at $\sigma = 0$,

$$\partial f_1 / \partial \sigma = z_p \tag{4.6}$$

$$\partial^2 f_1 / \partial\sigma^2 = z_p^2 - 1 \tag{4.7}$$

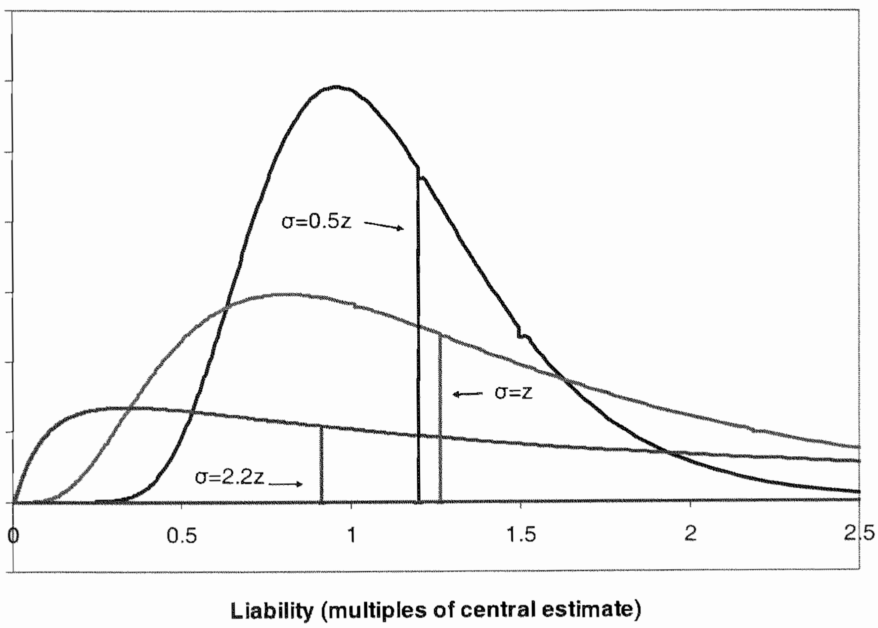
$$\partial^3 f_1 / \partial\sigma^3 = z_p \left(z_p^2 - 3\right) \tag{4.8}$$

Equations (2.12) and (4.1) imply the following.

Result 3. The function $f_1(\sigma; p)$ increases as σ increases from 0 to z_p, and then decreases as σ increases from z_p to ∞. The latter occurs because, for $\sigma > z_p$, the right-hand tail of the distribution of L lengthens sufficiently rapidly that, with its mean held constant, the probability mass below a fixed quantile must compensate by concentrating more to the left. □

Figure 4.1 plots the p.d.f of L for the three cases $\sigma = \frac{1}{2} z_p, z_p$ and $2.2\, z_p$, where p = 75%, and shows the value of $f_1(\sigma; p)$ in each case. Consistently with Result 3, it shows that $f_1(\sigma; p)$ increases as σ goes from $\frac{1}{2} z_p$ to z_p, but then decreases as σ goes from z_p to $2.2\, z_p$. Note also that in the last case $f_1(\sigma; p) < 1$.

Figure 4.1
Variation of 75-percentile with log normal dispersion parameters



These are not properties of the log normal distribution particularly, nor properties of heavy tailed distributions, of which the log normal is one. They are rather properties of the boundedness of L below, and its right skewness. As will be seen in Section 7, they persist in the case of the gamma distribution, which is short tailed.

5. Risk margin transition points

Define the function

$$g(\sigma;p,k) = f_1(\sigma;p) - f_2(\sigma;k) \tag{5.1}$$

Note that

$$f = f_1 \text{ when } g \geq 0$$
$$= f_2 \text{ when } g \leq 0$$

Call $\sigma = \sigma^* > 0$ a **risk margin transition point** if $g(\sigma;p,k)$ changes sign at $\sigma = \sigma^*$. This means that $f = f_1$ for σ on one side of σ^*, and $f = f_2$ on the other side.

At $\sigma = \sigma^*$

$$g(\sigma) = 0 \tag{5.2}$$

5.1 Existence and uniqueness of transition points

It is of interest to enquire into the existence and uniqueness of transition points for various values of p,k. A few basic properties of g are obtainable from Sections 2 to 4.

By (2.12) and (2.13)

$$g(0) = 0 \tag{5.3}$$

By (4.6) – (4.8) and (3.4) – (3.6), the derivatives of g take the following values at $\sigma = 0$:

$$\partial g / \partial \sigma = z - k \tag{5.4}$$
$$\partial^2 g / \partial \sigma^2 = z^2 - 1 \tag{5.5}$$
$$\partial^3 g / \partial \sigma^3 = z(z^2 - 3) \tag{5.6}$$

where, for brevity, the subscript p has been suppressed.

These relations create 5 cases according to which the behaviour of $g(\sigma)$ is investigated. These are labelled Cases Ia, Ib, IIa, IIb, III, which are combinations of those set out in Table 5.1.

Table 5.1
Cases of behaviour of $g(\sigma)$

Case	Description
I	$z \le 1$
II	$1 < z \le \sqrt{3}$
III	$z > \sqrt{3}$
a	$z \le k$
b	$z > k$

Note that the APRA case ($z = 0.67, k = \frac{1}{2}$) is Case Ib. Note also that, by (2.12) and (2.13),

$$g(\infty) < 0 \qquad (5.7)$$

and, by (4.1) and (3.1),

$$\partial g / \partial\sigma < 0 \text{ for } \sigma \ge z \qquad (5.8)$$

The behaviour of $g(\sigma)$, and the solutions of (5.2), are studied in Appendix B, whose results are summarised in Table 5.2.

In the table, the function $\xi(z)$ is defined as

$$\xi(z) = \frac{\exp\frac{1}{2}(z^2 - 3) - 1}{\left[\exp\left(z - \sqrt{3}\right)^2 - 1\right]^{\frac{1}{2}}} \qquad (5.9)$$

Table 5.2
Existence of transition points

Case	Transition points
Ia: $z \le 1, z < k$	None
Ib and IIb: $z \le \sqrt{3}, z > k$	Unique
IIa: $1 < z \le \sqrt{3}, z \le k$	None or 2
III: $z > \sqrt{3}$	Indeterminate for values $< z - \sqrt{3}$
	For values $> z - \sqrt{3}$:
	0 or 2 if $k > \xi(z)$
	1 or 3 if $k < \xi(z)$
	≤ 3 if $k = \xi(z)$

Note that, as $z \downarrow \sqrt{3}$, $\xi(z) \to \frac{1}{2}\left(z + \sqrt{3}\right) \to z$. Thus, the conditions $k > (<) \xi(z)$ in Table 5.2 merge with the conditions $z <$ (>) k as $z \downarrow \sqrt{3}$.

Figures 5.1 to 5.8 illustrate the cases listed in Table 5.2

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Figure 5.1
Case Ia: z = 0.44, k = 0.5

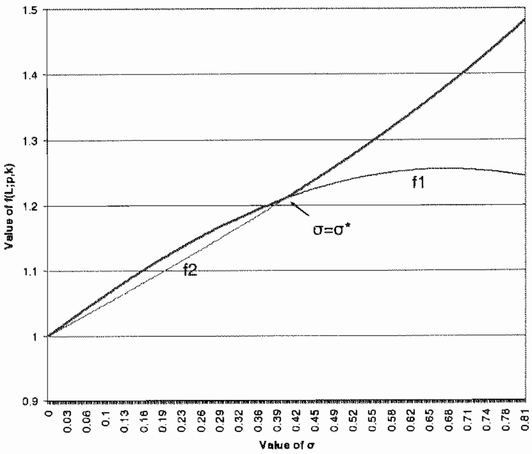


Figure 5.2
Case Ib: z = 0.67, k = 0.5

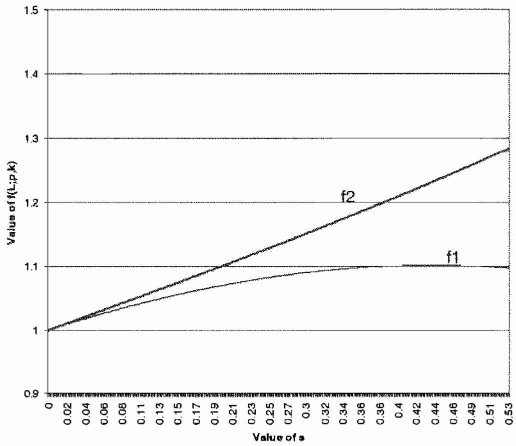


Figure 5.3
Case IIa: z = 1.04, k = 1.5

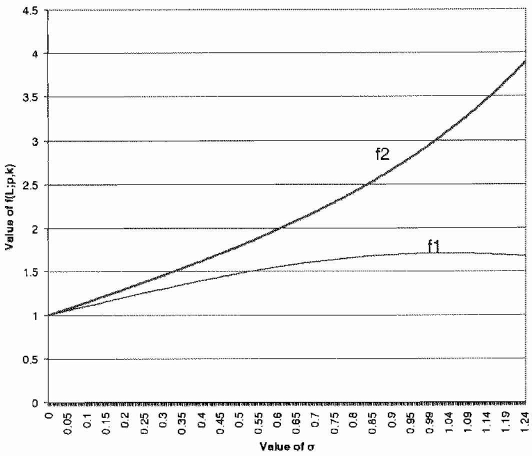


Figure 5.4
Case IIb: z = 1.28, k = 1.1

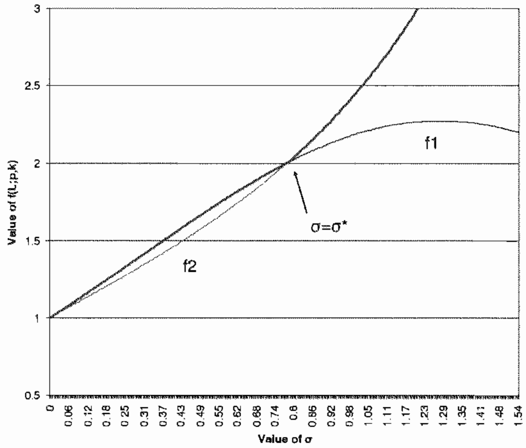


Figure 5.5
Case III: z = 1.75, ($\xi(z)$ = 1.77), k = 2.2

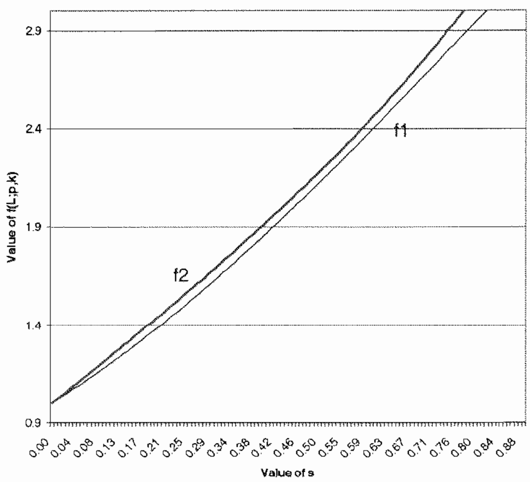


Figure 5.6
Case III: z = 1.75, ($\xi(z)$ = 1.77), k = 2.05

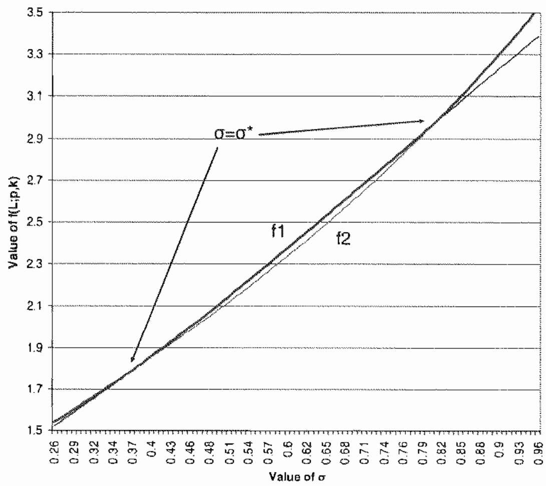


Figure 5.7
Case III: z = 1.75, ($\xi(z)$ = 1.77), k = 1.5

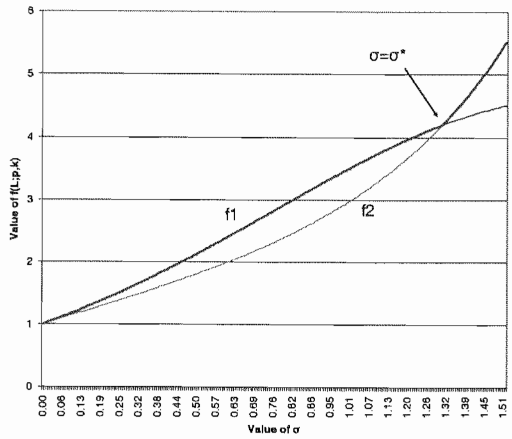
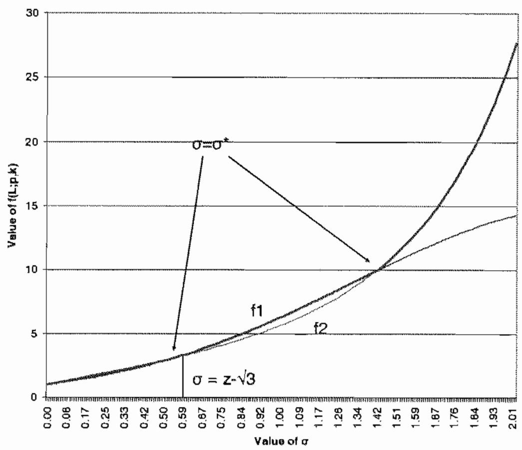


Figure 5.8
Case III: z = 2.32, ($\xi(z)$ = 3.60), k = 3.5



The case of no transition point is illustrated in Figure 5.3. The alternative Case IIa of 2 transition points occurs for z = 1.28 (p = 90%), k = 1.32. The transition points are 0.21 and 0.26, but the functions f_1 and f_2 are visually indistinguishable in this region, so have not been illustrated here.

In Figure 5.8 two transition points occur, but only one $> z-\sqrt{3}$, as predicted by Table 5.2.

Numerical testing of Case III with $k \le \xi(z)$ found no examples of 3 transition points, and such examples may not exist despite the failure here to exclude them.

5.2 Location of transition points

The equation characterising a transition point $\sigma = \sigma^*$ is $g(\sigma) = 0$, or

$$\exp\left(z\sigma - \tfrac{1}{2}\sigma^2\right) = 1 + k\left(\exp\sigma^2 - 1\right)^{\frac{1}{2}} \qquad (5.10)$$

It is shown in Appendix C that this implies

$$\tfrac{1}{2}k\sigma^2 - \left(kz - \tfrac{1}{2}\right)\sigma - \left(z - k\right) < 0 \qquad (5.11)$$

This inequality can be satisfied only if the discriminant of the quadratic, $\left(kz + \tfrac{1}{2}\right)^2 - 2k^2$ is non-negative. A sufficient condition for this is that $z \ge k$ (which includes Case b from Section 5.1). Hence the following result.

Result 4. Suppose that $z + 1/2k \ge \sqrt{2}$, for which a sufficient condition is $z \ge k$. Then any transition point

$$\sigma^* < (z - 1/2k) + \left[(z + 1/2k)^2 - 2\right]^{\frac{1}{2}} \qquad (5.12)$$

□

It is evident from the development of (5.11) in Appendix C that it is tight for small σ. Note that the right side of (5.12) is small for z,k in the vicinity of $z = k = 1/\sqrt{2}$. Hence Result 4 will be tight in this vicinity.

Table 5.3 gives a few sample values of σ^*, in each case compared with the upper bound (5.12). Blanks in the table indicate non-existent values. Note that there are a couple of examples in which the transition point does not exist, but the upper bound (5.12) does.

As predicted, the upper bound performs better for small σ^* than for large.

Table 5.3
Values of transition points for various *z* and *k*, compared with upper bounds (in parenthesis)

$z =$	Value of transition point for $k =$					
	0.45	0.70	0.95	1.20	1.45	1.70
0.5	0.121 (0.161)					
0.75	0.633 (0.849)	0.167 (0.415)				
1.00	1.057 (1.456)	0.734 (1.255)	0.290 (1.048)			
1.25	1.422 (2.030)	1.157 (1.899)	0.893 (1.799)	0.538 (1.715)	(1.642)	(1.576)
1.50	1.740 (2.584)	1.512 (2.490)	1.311 (2.425)	1.102 (2.377)	0.851 (2.340)	(2.310)
1.75	2.030 (3.126)	1.822 (3.054)	1.653 (3.007)	1.497 (2.975)	1.340 (2.951)	1.169 (2.932)

5.3 Variation of transition point with *p* and *k*

For the present sub-section, let σ^* denote the **maximal** transition point, when at least one transition point exists.

The parameters z and k will be referred to as **control parameters**. Let c denote either of them and consider how σ^* varies with c while the other control parameter is held constant.

To recognise the dependency of σ^* on c, write $\sigma^*(c)$. Then

$$g\left(\sigma^*(c)\right)=0 \qquad (5.13)$$

whenever a transition point exists.

It is shown in Appendix D that

$$\operatorname{sgn}\left(\partial\sigma^*/\partial c\right)=\operatorname{sgn}\left(\partial g/\partial c\right) \text{ evaluated at } \sigma=\sigma^*(c) \qquad (5.14)$$

Variation with *k*

By (5.1), (2.12) and (2.13),

$$\partial g/\partial k=-\partial f_2/\partial k=-\left(\exp\sigma^2-1\right)^{\frac{1}{2}}<0 \text{ for } \sigma>0 \qquad (5.15)$$

Then (5.14) gives

$$\partial\sigma^*/\partial k<0 \qquad (5.16)$$

Variation with *p*

$$\partial g/\partial z=\partial f_1/\partial z=\sigma\exp\left(z\sigma-\tfrac{1}{2}\sigma^2\right)>0 \text{ for } \sigma>0 \qquad (5.17)$$

Then (5.14) gives

$$\partial\sigma^*/\partial z>0 \qquad (5.18)$$

Since p and z_p increase and decrease in sympathy, it follows that

$$\partial\sigma^*/\partial p>0 \qquad (5.19)$$

Relations (5.16) and (5.19) yield the following.

Result 5. The maximal transition point σ^* increases as p (or z_p) increases, and as k decreases. □

This result is illustrated by Table 5.3.

6. Relative transition points

Since Result 5 shows that z_p and maximal σ^* increase and decrease in sympathy, it is of interest to study the behaviour of the ratio σ^*/z_p. For fixed k, and with $\sigma^*\left(z_p\right)$ having the same meaning as in Section 5.3, define the **relative transition point**

$$\bar{\sigma}\left(z_p\right)=\sigma^*\left(z_p\right)/z_p \qquad (6.1)$$

By (5.10), it follows that

$$\frac{1}{\bar{\sigma}(z)} = \frac{1}{2} + \frac{\log(1+kv)}{\log(1+v^2)} \qquad (6.2)$$

Where the subscript p has been suppressed again, and

$$v = \left(\exp \sigma^{*2} - 1\right)^{\frac{1}{2}} \qquad (6.3)$$

It follows immediately that

$$\bar{\sigma}(z) = \tfrac{2}{3} \text{ for } v = k \qquad (6.4)$$

$$\bar{\sigma}(z) \to \left(\tfrac{1}{2} + k/v\right)^{-1} \to 0 \text{ as } \sigma^* \to 0 \qquad (6.5)$$

To examine the limiting behaviour of $\bar{\sigma}(z)$ for large v (equivalently large σ^*), note that

$$1/\bar{\sigma}(z) \to \tfrac{1}{2} + \log kv / \log v^2 = 1 + \tfrac{1}{2}\log k / \log v$$

Hence

$$\bar{\sigma}(z) \uparrow 1 \text{ as } \sigma^* \to \infty \text{ if } k > 1 \qquad (6.6)$$

$$\bar{\sigma}(z) \downarrow 1 \text{ as } \sigma^* \to \infty \text{ if } k < 1 \qquad (6.7)$$

Together, (6.5) and (6.6) also imply that $\bar{\sigma}(z)$ is **not monotone** in σ^*, therefore **not monotone** in z (see Result 5), in the case (6.7).

In fact, Appendix E refines this result by calculating the gradient of $1/\bar{\sigma}(z)$. Result E.2, combined with the earlier results of the present section, yields the following.

Result 6. In the case $k \geq 1$, $\bar{\sigma}(z_p)$ is monotone increasing in z_p from 0 at $z_p = 0$ to 1 as $z_p \to \infty$. In the case $k < 1$, $\bar{\sigma}(z_p)$ is monotone increasing from 0 at $z_p = 0$ to $\frac{2}{3}$ at the value of z_p corresponding to $v = k$. For larger values of z_p, $\bar{\sigma}(z_p)$ passes through at least one stationary point, and is eventually monotone decreasing to 1 as $z_p \to \infty$.

Figure 6.1 plots $\bar{\sigma}(z_p)$ as a function of z_p for k = 0.5, 0.7, 1.0, 1.3, 1.5. Note the non-monotonicity for the cases where $k < 1$. It appears that there is a single stationary point in each case.

Figure 6.2 is the corresponding plot of $\sigma^*(z_p)$ against z_p.

Figure 6.1
Relative transition point as a function of z_p

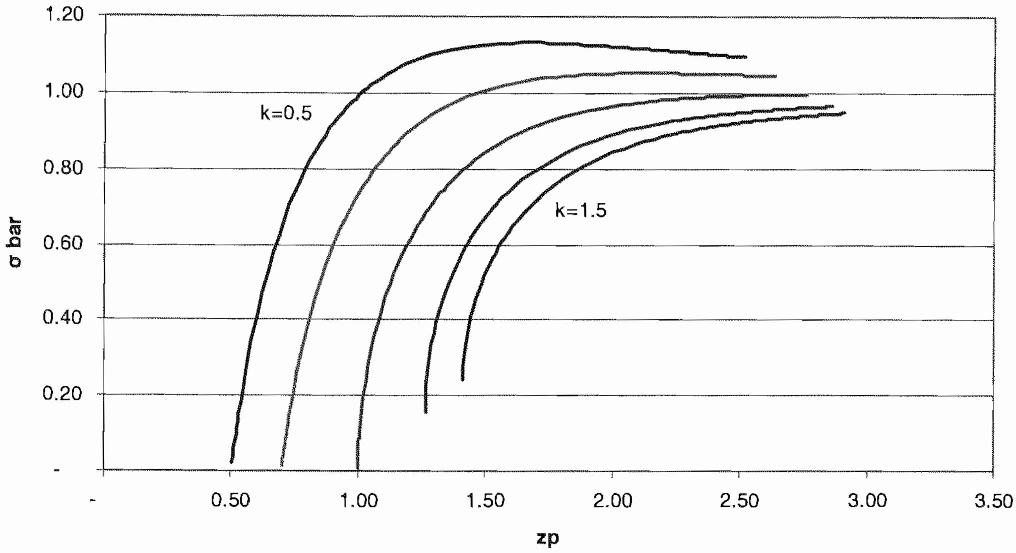
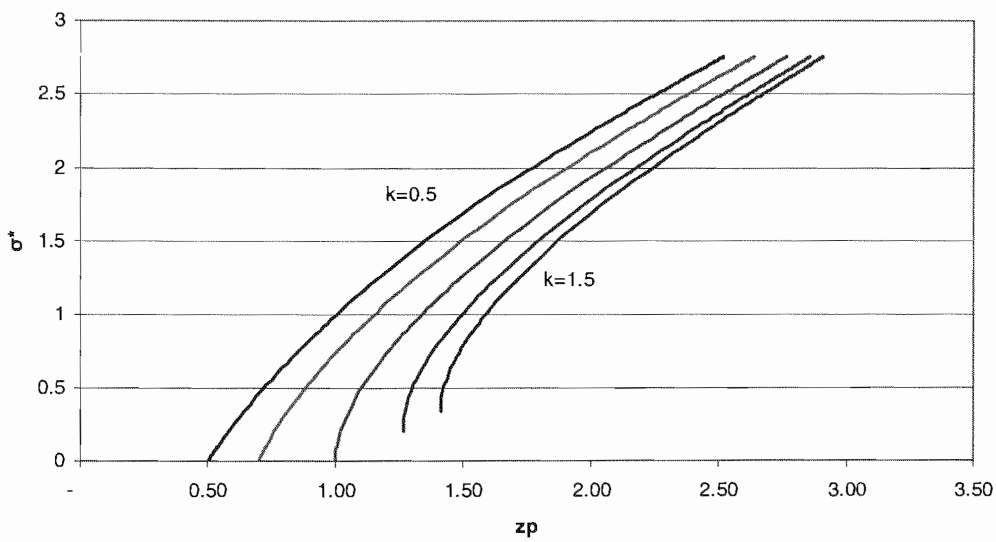


Figure 6.2
Maximal transition point as a function of z_p



7. Gamma distribution of liabilities

It was noted at the end of Section 4 that a particular log normal percentile increased as the dispersion parameter increased from zero but, for higher values of the dispersion parameter, decreased as that parameter increased. This was noted to be a property of right skewness rather than specific to log normal.

Figure 7.1 illustrates this in the case of the gamma distribution. The three gamma distributions displayed have the same CoVs as the log normals in Figure 4.1.

Denote the gamma p.d.f by

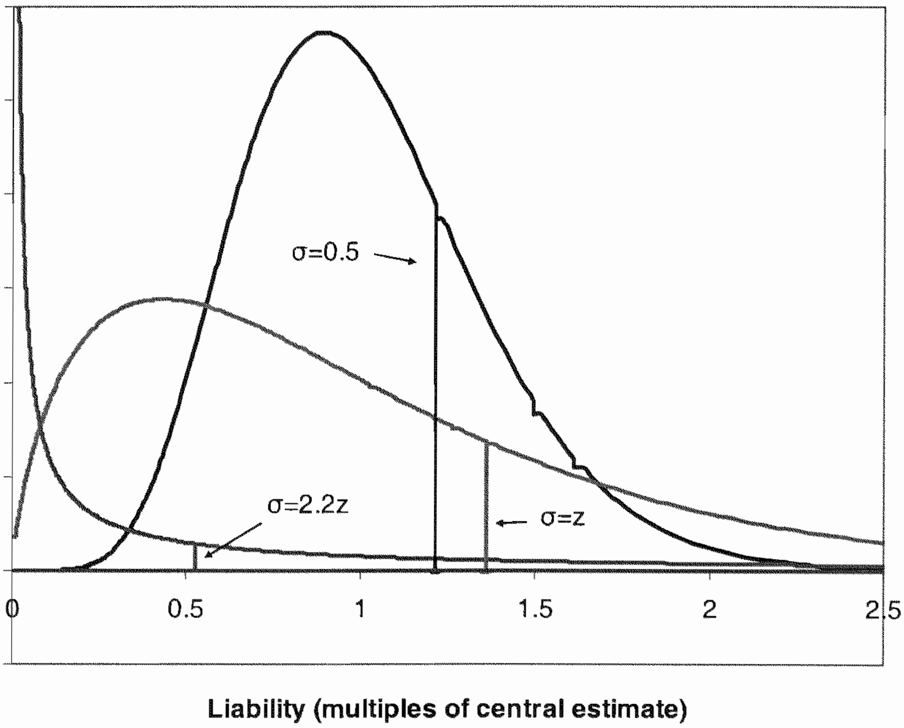
$$(cx)^{\gamma-1} e^{-cx} / \Gamma(\gamma) \tag{7.1}$$

whose CoV is $1/\gamma^{\frac{1}{2}}$. The log normal CoV in Section 4 is $\left(\exp\sigma^2 - 1\right)^{\frac{1}{2}}$. Hence

$$\gamma = \left(\exp\sigma^2 - 1\right)^{-1} \tag{7.2}$$

where $\sigma = \frac{1}{2} z_p, z_p$ and 2.2 z_p for $p = 75\%$.

Figure 7.1
Variation of 75-percentile with gamma dispersion parameter



Appendix A. Derivatives of f_2 From (3.2),

$$\begin{aligned} \partial^2 f_2 / \partial\sigma^2 &> k\left(\exp\sigma^2\right)\left(\exp\sigma^2 - 1\right)^{-\frac{3}{2}}\left[\left(1+\sigma^2\right)^2 - \left(1+2\sigma^2\right)\right] \\ &= k\sigma^4\left(\exp\sigma^2\right)\left(\exp\sigma^2 - 1\right)^{-\frac{3}{2}} \\ &> 0 \text{ for } \sigma > 0 \end{aligned} \tag{A.1}$$

From (3.3)

$$\partial^3 f_2 / \partial\sigma^3 = k\sigma\left(\exp\sigma^2\right)\left(\exp\sigma^2 - 1\right)^{-\frac{5}{2}} \zeta(\sigma) \tag{A.2}$$

where

$$\begin{aligned}\zeta(\sigma) &= \left(3+\sigma^2\right)\left(\exp 2\sigma^2\right)-\left(9+2\sigma^2\right)\left(\exp\sigma^2\right)+2\left(3+2\sigma^2\right)\\ &= \left(3+\sigma^2\right)\sum_{j=0}^{\infty}\left(2\sigma^2\right)^j / j! - \left(9+2\sigma^2\right)\sum_{j=0}^{\infty}\left(\sigma^2\right)^j / j! + 2\left(3+2\sigma^2\right) \qquad (A.3)\\ &= 6+4\sigma^2+\sum_{j=0}^{\infty}\left(3 \text{x} 2^j-9\right)\sigma^{2j} / j! + \sum_{j=0}^{\infty}\left(2^j-2\right)\sigma^{2j+2} / j!\end{aligned}$$

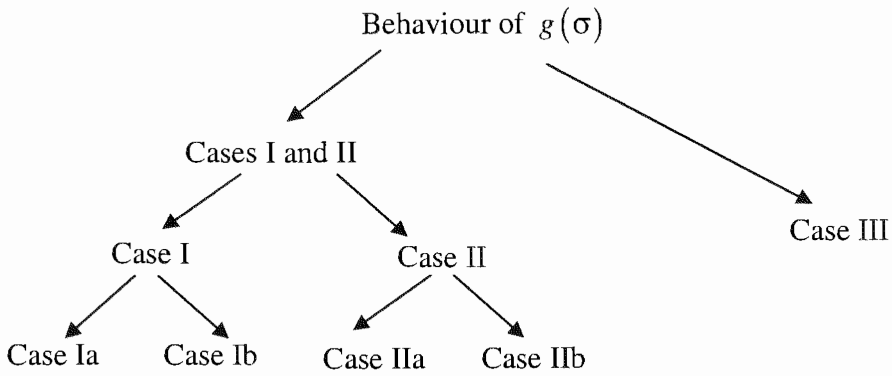
With like powers collected,

$$\begin{aligned}\zeta(\sigma) &= \sum_{j=2}^{\infty}\sigma^{2j}\left[\left(3 \text{x} 2^j-9\right)/ j! + \left(2^{j-1}-2\right)/(j-1)!\right] \qquad (A.4)\\ &> 0 \text{ for } \sigma > 0\end{aligned}$$

since all bracketed terms in the summands are non-negative for $j \geq 2$, and all are strictly positive except $\left(2^{j-1}-2\right)$, for the case $j = 2$.

Appendix B. Behaviour of $g(\sigma)$. The cases to be considered are set out in Table 5.1. They will be considered according to the hierarchy depicted in Figure B.1.

Figure B.1
Hierarchy of cases



Cases I and II: $z \leq \sqrt{3}$ Relation (4.5) yields

$$\partial^3 f_1 / \partial\sigma^3 < 0 \text{ for } 0 < \sigma < z \qquad (B.1)$$

Then (3.9) yields

$$\partial^3 g / \partial\sigma^3 < 0 \text{ for } 0 < \sigma < z \qquad (B.2)$$

Case I: $z \leq 1$. By (4.4) and (3.8),

$$\partial^2 g / \partial\sigma^2 < 0 \text{ for } 0 < \sigma < z \qquad (B.3)$$

Case Ia: $z \leq 1, z \leq k$. By (5.4),

$$\partial g / \partial \sigma \leq 0 \text{ at } \sigma = 0 \tag{B.4}$$

Then (B.3) implies that

$$\partial g / \partial \sigma < 0 \text{ for } 0 < \sigma \leq z \tag{B.5}$$

Combining this with (5.8) gives

$$\partial g / \partial \sigma < 0 \text{ for all } \sigma > 0 \tag{B.6}$$

This, combined with (5.3), implies that

$$g(\sigma) < 0 \text{ for all } \sigma > 0 \tag{B.7}$$

and so $g(\sigma) = 0$ has no positive solution.

Case Ib: $z \leq 1, z > k$. By (5.4),

$$\partial g / \partial \sigma > 0 \text{ at } \sigma = 0 \tag{B.8}$$

This and (5.8) imply that $\partial g / \partial \sigma$ has at least one sign change for $0 < \sigma < z$. By (B.3), there cannot be more than one, and so there is exactly one. Moreover, by (5.8), there are no further sign changes for $\sigma \geq z$.

By (5.3) and (B.8), $g(0+) > 0$ and by (5.7) $g(\infty) < 0$, so $g(\sigma) = 0$ has at least one positive solution. Then the behaviour of $\partial g / \partial \sigma$ implies exactly one solution.

Case II: $\sqrt{3} \geq z > 1$. By (5.5),

$$\partial^2 g / \partial \sigma^2 > 0 \text{ at } \sigma = 0 \tag{B.9}$$

By (4.2) and (3.8),

$$\partial^2 g / \partial \sigma^2 < 0 \text{ at } \sigma = z \tag{B.10}$$

Thus, $\partial^2 g / \partial \sigma^2$ has at least one sign change for $0 < \sigma < z$ and, by (B.2), there is exactly one.

Case IIa: $\sqrt{3} \geq z > 1, z \leq k$. Relation (B.4) holds, just as for Case Ia. The inequality is strict unless $z = k$. Taken together with (5.8), this implies an even number of sign changes of $\partial g / \partial \sigma$ for $0 < \sigma < z$. By (5.8), there are no further changes for $\sigma \geq z$.

It follows from the behaviour of $\partial^2 g / \partial\sigma^2$ that $\partial g / \partial\sigma$ has 0 or 2 sign changes and, in the latter case, both occur for $0 < \sigma < z$.

In the case of 0 sign changes, (B.4) implies that $g(\sigma) < 0$ for all $\sigma > 0$, and so $g(\sigma) = 0$ has no solution. In the case of 2 sign changes, (B.4) and (5.7) imply either 0 or 2 positive solutions of $g(\sigma) = 0$.

Case IIb: $\sqrt{3} \geq z > 1, z > k$. By the same reasoning as in Case Ib, $\partial g / \partial\sigma$ has at least one sign change for $0 < \sigma < z$. By the behaviour of $\partial^2 g / \partial\sigma^2$, there must be exactly one. The remainder of the reasoning of Case Ib applies to the present case, implying that $g(\sigma) = 0$ has a unique positive solution.

Case III: $z > \sqrt{3}$. Relations (B.1) and (B.2) no longer necessarily hold. Instead, (4.5) and (3.9) yield

$$\partial^3 g / \partial\sigma^3 < 0 \text{ for } z - \sqrt{3} < \sigma < z \qquad (B.11)$$

This implies that $\partial g / \partial\sigma$ at most 2 changes of sign over this range of σ, and so, by (5.8) has at most 2 changes of sign over the entire range $\sigma > z - \sqrt{3}$.

Hence $g(\sigma) = 0$ has at most 3 solutions for $\sigma > z - \sqrt{3}$. The behaviour of $g(\sigma)$ for $0 < \sigma < z - \sqrt{3}$ is undetermined.

By (5.7), $g(\infty) < 0$. If $g(z - \sqrt{3}) > 0$, there must be an odd number of solutions for $\sigma > z - \sqrt{3}$, and therefore 1 or 3. The condition $g(z - \sqrt{3}) > 0$ is equivalent to $f_1(z - \sqrt{3}) > f_2(z - \sqrt{3})$. By (2.12) and (2.13), this is

$$k < \frac{\exp \frac{1}{2}(z^2 - 3) - 1}{\left[\exp\left(z - \sqrt{3}\right)^2 - 1\right]^{\frac{1}{2}}} \qquad (B.12)$$

Appendix C. Location of transition points. Note that $\exp \sigma^2 > 1 + \sigma^2$, and so (5.10) implies

$$\exp\left(z\sigma - \tfrac{1}{2}\sigma^2\right) > 1 + k\sigma \qquad (C.1)$$

Take logs and apply the inequality $\log(1 + x) > x / (1 + x)$ for $x > 0$ (Abramowitz and Stegun, 1972, (4.1.33)) to obtain

$$z\sigma - \tfrac{1}{2}\sigma^2 > k\sigma / (1 + k\sigma) \qquad (C.2)$$

A small amount of manipulation yields the result

$$\tfrac{1}{2}k\sigma^2 - \left(kz - \tfrac{1}{2}\right)\sigma - (z-k) < 0 \qquad (C.3)$$

Appendix D. The derivative $\partial\sigma^* / \partial c$

Consider $\sigma^*(c)$ defined by (5.13). If this is the maximal transition point, then $g(\sigma)$ undergoes no sign change for $\sigma > \sigma^*(c)$. Then, by (5.7),

$$g\left(\sigma^*(c)+0\right) < 0 \qquad (D.1)$$

Since a sign change in g occurs at a transition point,

$$g\left(\sigma^*(c)-0\right) > 0 \qquad (D.2)$$

Hence

$$\partial g / \partial\sigma < 0 \text{ at } \sigma=\sigma^*(c) \qquad (D.3)$$

Now differentiate (5.13) with respect to c:

$$\frac{\partial g}{\partial c} + \frac{\partial g}{\partial \sigma}\frac{\partial \sigma^*}{\partial c} = 0 \text{ at } \sigma = \sigma^*$$

and so

$$\frac{\partial \sigma^*}{\partial c} = -\frac{\partial g}{\partial c} / \frac{\partial g}{\partial \sigma} \qquad (D.4)$$

By (D.3) and (D.4)

$$\operatorname{sgn}\left(\partial\sigma^* / \partial c\right) = \operatorname{sgn}\left(\partial g / \partial c\right) \text{ evaluated at } \sigma = \sigma^*(c) \qquad (D.5)$$

Appendix E. Derivative of $1/\bar{\sigma}(z)$. From (6.2),

$$\frac{d\left(1/\bar{\sigma}(z_p)\right)}{dv} = \frac{k\left(1+v^2\right)\log\left(1+v^2\right) - 2v(1+kv)\log(1+kv)}{(1+kv)\left(1+v^2\right)\left[\log\left(1+v^2\right)\right]^2} \qquad (E.1)$$

Let y denote the numerator of this expression, and re-write it in the form

$$y = v(1+kv)\left[\log\left(1+v^2\right) - 2\log(1+kv)\right]$$
$$\quad + \left[k\left(1+v^2\right) - v(1+kv)\right]\log\left(1+v^2\right)$$
$$= -v(1+kv)\log\frac{(1+kv)^2}{1+v^2} - (v-k)\log\left(1+v^2\right) \qquad (E.2)$$

Case I: $v > k$. Here (E.2) yields

$$y < -v(1+kv)\log\frac{(1+kv)^2}{1+v^2}$$

Then

$$y/v(1+kv) < -\log\frac{(1+v)^2}{1+v^2} < -\log 1 = 0 \text{ if } k \geq 1 \qquad (E.3)$$

Case II: $v \leq k$. Write (E.2) in the form

$$y = (k-v)\log\left(1+v^2\right) - v(1+kv)\log\frac{(1+kv)^2}{1+v^2} \qquad (E.4)$$

From Abramowitz and Stegun (1972, (4.1.33))

$$x/(1+x) < \log(1+x) < x \text{ for } x > -1, x \neq 0 \qquad (E.5)$$

The first half of this inequality may be re-expressed as

$$(x-1)/x \leq \log x \text{ for } x > 0 \qquad (E.6)$$

Now apply the second half of (E.5) to the first member on the right side of (E.4), and apply (E.6) to the second member, yielding

$$y < (k-v)v^2 - v(1+kv)\frac{v\left[2k+\left(k^2-1\right)v\right]}{(1+kv)^2}$$

Then

$$y(1+kv)/v^2 < (k-v)(1+kv) - \left[2k+\left(k^2-1\right)v\right]$$
$$= -k\left(1+v^2\right) \qquad (E.7)$$
$$< 0$$

Result E.1 From (E.3) and (E.7),

$$\frac{d\left(1/\bar{\sigma}\left(z_p\right)\right)}{dv} < 0 \text{ for all } v > 0 \text{ if } k \geq 1 \text{, and for } v \leq k \text{ if } k < 1. \qquad \square$$

By (6.3) and (5.18), v, σ^* and z_p are all increasing functions of one another. This yields the following.

Result E.2. Result E1 continues to hold if *d/dv* is replaced by $d/d\sigma^*$ or d/dz_p. $\square$

Reference

Abramowitz M and Stegun IA (eds) (1972). **Handbook of mathematical functions** (8th printing). Dover Publications, Inc, New York.