Some finite time ruin problems

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Abstract

In the classical risk model, we use probabilistic arguments to write down expressions in terms of the density function of aggregate claims for joint density functions involving the time to ruin, the deficit at ruin and the surplus prior to ruin. We give some applications of these formulae in the cases when the individual claim amount distribution is exponential and Erlang(2).

Key words: finite time ruin; joint density; deficit at ruin

1 Introduction

We consider the classical risk model in which the surplus process $\{U(t)\}_{t\geq 0}$ is given by

$$U(t) = u + ct - S(t)$$

where u is the insurer's initial surplus, c is the rate of premium income per unit time, and $S(t)$ represents the aggregate amount of claims up to time t. The process $\{S(t)\}_{t\geq 0}$ is a compound Poisson process, with $S(t) = \sum_{i=1}^{N(t)} X_i$, where $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, with X_i representing the amount of the ith claim, and $N(t)$ follows a Poisson distribution with parameter λt. We assume that the individual claim amounts have a continuous distribution with distribution function $F = 1 - \bar{F}$, density function f and mean m_1, and we adopt the notation $G(x,t) = \Pr(S(t) \leq x)$, with $g(x,t) = \frac{\partial}{\partial x} G(x,t)$.

The finite time ruin probability is defined as

$$\psi(u,t) = \Pr(U(s) < 0 \text{ for some } s,\ 0 < s \leq t)$$

and let $\delta(u,t) = 1 - \psi(u,t)$. Prahbu's (1961) formula for $\delta(u,t)$ is

$$\delta(u,t) = G(u+ct,t) - c\int_0^t g(u+cs,s)\delta(0,t-s)ds, \qquad (1)$$

with

$$\delta(0,t) = \frac{1}{ct}\int_0^{ct} G(x,t)dx$$

which can be written as

$$\delta(0,t) = e^{-\lambda t} + \int_0^{ct} \delta(0,t,x)dx$$

where

$$\delta(0,t,x) = \frac{x}{ct}g(ct-x,t)$$

is the density associated with non-ruin from initial surplus 0 over $[0,t]$ and a surplus of x at time t. See Gerber (1979, p.112).

Although extensive numerical work has been done using Prahbu's formula, much of which is summarised in Seal (1978), few explicit solutions for $\delta(u,t)$ have arisen from this formula, presumably because the use of Prahbu's formula requires explicit expressions for G and g. The notable exception is when F is the exponential distribution function.

Dickson and Willmot (2005) derived an expression for the density of the time to ruin, T, defined as

$$T = \inf\{t\colon\, U(t) < 0\}$$

with $T = \infty$ if $U(t) > 0$ for all $t > 0$. Their expression was found by inverting the Laplace transform of T. It is not a concise solution, but is suitable for computational purposes, as illustrated by Dickson and Willmot (2005) and Willmot and Woo (2007). Our main purpose in this paper is to identify alternative expressions for the density of the time to ruin. We do this in Section 2 by applying a probabilistic argument, and we confirm the validity of the argument by differentiating equation (1). Based on the probabilistic argument, we then give formulae in Section 3 for some joint distributions relating to the time to ruin, the deficit at ruin and the surplus prior to ruin. Some illustrations of the application of these formulae are given in Sections 4 and 5, and conclusions are presented in Section 6.

2 The density of the time to ruin

Let $w(u,t)$ denote the (defective) density of the time to ruin, so that $w(u,t) = \frac{\partial}{\partial t}\psi(u,t)$. We know from Dickson and Willmot (2005) that

$$\begin{aligned} w(0,t) &= \lambda e^{-\lambda t}\bar{F}(ct) + \lambda\int_0^{ct} \frac{x}{ct}g(ct-x,t)\bar{F}(x)dx & (2) \\ &= \lambda e^{-\lambda t}\bar{F}(ct) + \lambda\int_0^{ct} \delta(0,t,x)\bar{F}(x)dx, & (3) \end{aligned}$$

and we can interpret this as follows. The expression $w(0,t)dt$ is interpreted as the probability of ruin occurring in the interval $(t, t+dt)$. The term $\lambda e^{-\lambda t}\bar{F}(ct)dt$ is interpreted as the probability that the first claim occurs in the interval $(t, t+dt)$, and that this claim exceeds ct, causing ruin. Similarly, for a fixed value of x, $\lambda\delta(0,t,x)\bar{F}(x)dxdt$ is interpreted as the probability of

(a) non-ruin over $[0,t]$ with $U(t)=x$, and

(b) a claim occurring in the interval $(t, t+dt)$ of amount greater than x, causing ruin.

Extending this interpretation to an initial surplus greater than 0, we get

$$w(u,t) = \lambda e^{-\lambda t}\bar{F}(u+ct) + \lambda \int_0^{u+ct} \delta(u,t,x)\bar{F}(x)dx \tag{4}$$

where $\delta(u,t,x)$ is the density associated with non-ruin from initial surplus u over $[0,t]$ and a surplus of x at time t. Extending formula (3.13) of Dickson and Waters (2006) from $t=1$ to a general value for t, we obtain

$$\begin{aligned}\delta(u,t,x) &= g(u+ct-x,t) - cI(t>x/c)\int_0^{t-x/c} g(u+cs,s)\delta(0,t-s,x)ds \\ &\quad -I(t>x/c)g(u+ct-x,t-x/c)e^{-\lambda x/c}\end{aligned}$$

where I is the indicator function. Formula (4) is not the most useful formula for computing $w(u,t)$, but its structure allows us to write down a joint density function, and this will be done in the next section.

Starting from Prahbu's formula, we now give a formal proof of formula (4). Using equation (1), we obtain

$$\begin{aligned}w(u,t) &= -\frac{\partial}{\partial t}\delta(u,t) \\ &= -\frac{\partial}{\partial t}G(u+ct,t) + cg(u+ct,t)\delta(0,0) \\ &\quad -c\int_0^t g(u+cs,s)w(0,t-s)ds.\end{aligned}$$

Now

$$G(u+ct,t) = e^{-\lambda t} + \sum_{n=1}^{\infty} p_n(t)F^{n*}(u+ct)$$

where $p_n(t) = e^{-\lambda t}(\lambda t)^n/n!$, and h^{n*} denotes the n-fold convolution of a function h with itself. Thus,

$$
\begin{aligned}
\frac{\partial}{\partial t}G(u+ct,t) &= -\lambda e^{-\lambda t} + c\sum_{n=1}^{\infty} p_n(t) f^{n*}(u+ct) + \sum_{n=1}^{\infty} p_n'(t) F^{n*}(u+ct) \\
&= -\lambda e^{-\lambda t} + cg(u+ct,t) - \lambda \sum_{n=1}^{\infty} p_n(t) F^{n*}(u+ct) \\
&\quad + \sum_{n=1}^{\infty} e^{-\lambda t} \frac{\lambda^n t^{n-1}}{(n-1)!} F^{n*}(u+ct) \\
&= -\lambda G(u+ct,t) + cg(u+ct,t) + \lambda \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} F^{n*}(u+ct),
\end{aligned}
$$

giving

$$
w(u,t) = \lambda G(u+ct,t) - \lambda \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} F^{n*}(u+ct) - c\int_0^t g(u+cs,s) w(0,t-s) ds.
$$

Now

$$
\begin{aligned}
&\lambda G(u+ct,t) - \lambda \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} F^{n*}(u+ct) \\
&= \lambda \left(G(u+ct,t) - \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n}}{(n)!} F^{(n+1)*}(u+ct) \right) \\
&= \lambda \left(G(u+ct,t) - e^{-\lambda t}F(u+ct) - \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n}}{n!} F^{(n+1)*}(u+ct) \right) \\
&= \lambda \left(G(u+ct,t) - e^{-\lambda t}F(u+ct) - \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^{n}}{n!} \int_0^{u+ct} f^{n*}(u+ct-x)F(x)dx \right) \\
&= \lambda \left(e^{-\lambda t} + \int_0^{u+ct} g(x,t)dx - e^{-\lambda t}F(u+ct) - \int_0^{u+ct} g(u+ct-x,t)F(x)dx \right) \\
&= \lambda \left(e^{-\lambda t}\bar{F}(u+ct) + \int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)dx \right).
\end{aligned}
$$

This gives us

$$
\begin{aligned}
w(u,t) &= \lambda e^{-\lambda t}\bar{F}(u+ct) + \lambda \int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)dx \\
&\quad - c\int_0^t g(u+cs,s)w(0,t-s)ds. \qquad (5)
\end{aligned}
$$

Hence, using formula (3),

$$
\begin{aligned}
w(u,t) &= \lambda\left(e^{-\lambda t}\bar{F}(u+ct)+\int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)dx\right) \\
&\quad -c\int_0^t g(u+cs,s)\left(\lambda e^{-\lambda(t-s)}\bar{F}(c(t-s))+\lambda\int_0^{c(t-s)}\delta(0,t-s,x)\bar{F}(x)dx\right)ds \\
&= \lambda e^{-\lambda t}\bar{F}(u+ct)+\lambda\int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)dx \\
&\quad -c\int_0^t g(u+cs,s)\lambda e^{-\lambda(t-s)}\bar{F}(c(t-s))ds \\
&\quad -\lambda c\int_0^{ct}\int_0^{t-x/c} g(u+cs,s)\delta(0,t-s,x)ds\bar{F}(x)dx.
\end{aligned}
$$

By substituting $x=c(t-s)$, the middle integral can be written as

$$
-\int_0^{ct} g(u+ct-x,t-x/c)\lambda e^{-\lambda x/c}\bar{F}(x)dx
$$

so that

$$
\begin{aligned}
w(u,t) &= \lambda e^{-\lambda t}\bar{F}(u+ct)+\lambda\int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)dx \\
&\quad -\int_0^{ct} g(u+ct-x,t-x/c)\lambda e^{-\lambda x/c}\bar{F}(x)dx \\
&\quad -\lambda c\int_0^{ct}\int_0^{t-x/c} g(u+cs,s)\delta(0,t-s,x)ds\bar{F}(x)dx \\
&= \lambda e^{-\lambda t}\bar{F}(u+ct)+\lambda\int_0^{u+ct}\delta(u,t,x)\bar{F}(x)dx,
\end{aligned}
$$

which is equation (4).

We remark that equation (5) can be used to compute $w(u,t)$ for certain individual claim amount distributions, as illustrated in Sections 4 and 5.

3 Joint densities

Now let Y denote the deficit at ruin and let $w_{Y,T}(u,y,t)$ denote the (defective) joint density of Y and T. Dickson and Willmot (2005, Section 3) obtain formula (3) by inverting the Laplace transform of T. If we apply the same approach to the bivariate Laplace transform of Y and T, straightforward manipulations give

$$w_{Y,T}(0,y,t) = \lambda e^{-\lambda t} f(ct+y) + \lambda \int_0^{ct} \delta(0,t,x) f(x+y) dx. \tag{6}$$

The interpretation here is as for formula (3), except that for $x \leq ct$, $\bar{F}(x)$ in formula (3) is replaced by $f(x+y)$, i.e. rather than requiring a claim above x to cause ruin, we require a claim of amount $x+y$ to cause ruin with a deficit of y. By adapting this interpretation to an initial surplus $u > 0$, we obtain

$$w_{Y,T}(u,y,t) = \lambda e^{-\lambda t} f(u+ct+y) + \lambda \int_0^{u+ct} \delta(u,t,x) f(x+y) dx. \tag{7}$$

Inserting for $\delta(u,t,x)$, similar manipulations to those in the previous section yield

$$\begin{aligned} w_{Y,T}(u,y,t) &= \lambda e^{-\lambda t} f(u+ct+y) + \lambda \int_0^{u+ct} g(u+ct-x,t) f(x+y) dx \\ &\quad -c \int_0^t g(u+c(t-s),t-s) w_{Y,T}(0,y,s) ds. \end{aligned} \tag{8}$$

Formula (8) provides an alternative means to the transform inversion method of calculating the joint density of Y and T presented in Dickson (2007). Its main advantage is that it is a simple integral expression in terms of g which makes it suitable for direct application for certain forms of f, or for numerical integration. Its main disadvantage is that the third term is of a similar form to the integral term in Prahbu's formula.

Further, if we let X denote the surplus prior to ruin and define $w_{X,Y,T}(u,x,y,t)$ to be the (defective) joint density of X, Y and T, we see that for $0 < x < u+ct$,

$$w_{X,Y,T}(u,x,y,t) = \lambda \delta(u,t,x) f(x+y)$$

with density of $\lambda e^{-\lambda t} f(u+ct+y)$ for $x = u+ct$. This argument is given by Wu et al (2003), but they use a different formula for $\delta(u,t,x)$.

Integrating out y, we obtain the joint density of the surplus prior to ruin and the time to ruin as

$$w_{X,T}(u,x,t) = \lambda \delta(u,t,x) \bar{F}(x)$$

for $0 < x < u+ct$, with density of $\lambda e^{-\lambda t} \bar{F}(u+ct)$ for $x = u+ct$. This leads to the relationship stated in Gerber and Shiu (1997):

$$w_{X,Y,T}(u,x,y,t) = w_{X,T}(u,x,t) \frac{f(x+y)}{\bar{F}(x)}.$$

4 Exponential Claims

Let us first consider the case when $\bar{F}(x) = e^{-\alpha x}$, where $\alpha > 0$, so that $f(x) = \alpha\bar{F}(x)$. Then inserting for f in equation (7) gives

$$\begin{aligned} w_{Y,T}(u,y,t) &= \lambda e^{-\lambda t}\alpha e^{-\alpha(u+ct+y)} + \lambda\int_0^{u+ct} \delta(u,t,x)\alpha e^{-\alpha(x+y)}dx \\ &= \alpha e^{-\alpha y}\left(\lambda e^{-\lambda t}\bar{F}(u+ct) + \lambda\int_0^{u+ct} \delta(u,t,x)\bar{F}(x)dx\right) \\ &= \alpha e^{-\alpha y}w(u,t), \end{aligned}$$

which is a well-known result – see, for example, Gerber (1979).

Consider next equation (5) for $w(u,t)$. The first two terms give

$$\begin{aligned} &\lambda e^{-\lambda t}\bar{F}(u+ct) + \lambda\int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)dx \\ &= \lambda e^{-\lambda t-\alpha u-\alpha ct} + \lambda\sum_{m=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^m}{m!}\int_0^{u+ct} f^{m*}(x)\bar{F}(u+ct-x)dx \\ &= \lambda e^{-\lambda t-\alpha u-\alpha ct} + \lambda\sum_{m=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^m}{m!}\int_0^{u+ct} \frac{\alpha^m x^{m-1}e^{-\alpha x}}{\Gamma(m)}e^{-\alpha(u+ct-x)}dx \\ &= \lambda e^{-\lambda t-\alpha u-\alpha ct}\left(1+\sum_{m=1}^{\infty}\frac{(\alpha\lambda t)^m}{m!\Gamma(m)}\int_0^{u+ct} x^{m-1}dx\right) \\ &= \lambda e^{-\lambda t-\alpha u-\alpha ct}\left(1+\sum_{m=1}^{\infty}\frac{(\alpha\lambda t)^m(u+ct)^m}{m!m!}\right) \\ &= \lambda e^{-\lambda t-\alpha u-\alpha ct}I_0\left(\sqrt{4\alpha\lambda t(u+ct)}\right), \end{aligned}$$

where

$$I_v(t) = \sum_{n=0}^{\infty}\frac{(t/2)^{2n+v}}{n!(n+v)!}$$

is the modified Bessel function of order v.

To obtain the third term of equation (5) we first find $w(0,t)$ from equation (2) as

$$\begin{aligned} w(0,t) &= \lambda e^{-\lambda t}\bar{F}(ct) + \lambda\int_0^{ct}\frac{x}{ct}g(ct-x,t)\bar{F}(x)dx \\ &= \lambda e^{-\lambda t}e^{-\alpha ct} + \lambda\sum_{m=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^m}{m!}\int_0^{ct}\frac{x}{ct}\frac{\alpha^m(ct-x)^{m-1}e^{-\alpha(ct-x)}}{\Gamma(m)}e^{-\alpha x}dx \end{aligned}$$

$$
\begin{aligned}
&= \lambda e^{-\lambda t} e^{-\alpha c t} + \lambda \sum_{m=1}^{\infty} e^{-\lambda t} e^{-\alpha c t} \frac{(\alpha \lambda t)^m}{m!\Gamma(m)} \frac{\Gamma(2)\Gamma(m)}{\Gamma(m+2)} (ct)^m \\
&= \lambda e^{-\lambda t} e^{-\alpha c t} \sum_{m=0}^{\infty} \frac{(\alpha \lambda c t^2)^m}{m!(m+1)!} \\
&= \lambda e^{-\lambda t} e^{-\alpha c t} \eta_1(t),
\end{aligned}
$$

where

$$
\eta_1(t) = \sum_{m=0}^{\infty} \frac{(\alpha \lambda c t^2)^m}{m!(m+1)!} = \frac{1}{\sqrt{\alpha \lambda c t^2}} I_1\left(\sqrt{4 \alpha \lambda c t^2}\right).
$$

Thus, (minus) the third term in equation (5) is

$$
\begin{aligned}
& c \int_0^t g(u + c(t-s), t-s) \lambda e^{-(\lambda + \alpha c)s} \eta_1(s) ds \\
&= \lambda c \int_0^t \sum_{n=1}^{\infty} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!} \frac{\alpha^n (u + c(t-s))^{n-1}}{(n-1)!} e^{-\alpha(u + c(t-s))} e^{-(\lambda+\alpha c)s} \eta_1(s) ds \\
&= \lambda c e^{-\alpha u - (\lambda + \alpha c)t} \int_0^t \sum_{n=1}^{\infty} \frac{(\lambda(t-s))^n}{n!} \frac{\alpha^n (u + c(t-s))^{n-1}}{(n-1)!} \eta_1(s) ds. \qquad (9)
\end{aligned}
$$

We can evaluate (9) by observing that the integral is the convolution of

$$
\eta_2(t) = \sum_{n=1}^{\infty} \frac{(\alpha \lambda t)^n}{n!} \frac{(u + ct)^{n-1}}{(n-1)!} = \frac{\alpha \lambda t}{\sqrt{\alpha \lambda t (u + ct)}} I_1\left(\sqrt{4 \alpha \lambda t (u + ct)}\right)
$$

and η_1, and we can find this from the following result, taken from Erdélyi (1954, p.201).

Result 4.1: *Let*

$$
\phi(t) = \frac{t^{v/2}}{(t+\beta)^{v/2}} I_v\left(A\sqrt{t^2 + \beta t}\right).
$$

Then

$$
\tilde{\phi}(s) = \int_0^{\infty} e^{-st} \phi(t) dt = \frac{A^v}{\sqrt{s^2 - A^2}} \frac{1}{\left(s + \sqrt{s^2 - A^2}\right)^v} \exp\left\{\frac{\beta}{2}\left(s - \sqrt{s^2 - A^2}\right)\right\}.
$$

To apply this result we note that we can write

$$
\eta_2(t) = \frac{\sqrt{\alpha \lambda / c}\sqrt{t}}{\sqrt{\frac{u}{c} + t}} I_1\left(\sqrt{4 \alpha \lambda c}\sqrt{\frac{u}{c} t + t^2}\right).
$$

Then by Result 4.1, with $v = 1$, $\beta = u/c$ and $A = \sqrt{4\alpha\lambda c}$,

$$\tilde{\eta}_2(s) = \sqrt{\alpha\lambda/c}\frac{A}{\sqrt{s^2 - A^2}}\frac{1}{s + \sqrt{s^2 - A^2}} \exp\left\{\frac{u}{2c}\left(s - \sqrt{s^2 - A^2}\right)\right\},$$

and from Gradshteyn and Ryzhik (1994, p.1182) we have

$$\tilde{\eta}_1(s) = \frac{2}{A}\frac{A}{s + \sqrt{s^2 - A^2}},$$

which we can write as $\tilde{\eta}_1(s) = \left(s - \sqrt{s^2 - A^2}\right)/(2\alpha\lambda c)$. Thus, we have

$$\begin{aligned}
\tilde{\eta}_1(s)\tilde{\eta}_2(s) &= \frac{\sqrt{\alpha\lambda/c}}{2\alpha\lambda c}\frac{A}{\sqrt{s^2 - A^2}}\frac{s - \sqrt{s^2 - A^2}}{s + \sqrt{s^2 - A^2}} \exp\left\{\frac{u}{2c}\left(s - \sqrt{s^2 - A^2}\right)\right\} \\
&= \frac{1}{2c\sqrt{\alpha\lambda c}}\frac{A}{\sqrt{s^2 - A^2}}\frac{A^2}{\left(s + \sqrt{s^2 - A^2}\right)^2} \exp\left\{\frac{u}{2c}\left(s - \sqrt{s^2 - A^2}\right)\right\},
\end{aligned}$$

which, by Result 4.1, is the Laplace transform of

$$\frac{A}{2c\sqrt{\alpha\lambda c}}\frac{t}{t + u/c}I_2\left(\sqrt{4\alpha\lambda c}\sqrt{t^2 + \frac{u}{c}t}\right) = \frac{t}{ct + u}I_2\left(\sqrt{4\alpha\lambda t(u + ct)}\right).$$

Hence formula (9) becomes

$$\lambda e^{-\alpha u - (\lambda + \alpha c)t}\frac{ct}{ct + u}I_2\left(\sqrt{4\alpha\lambda t(u + ct)}\right),$$

and we obtain

$$w(u, t) = \lambda e^{-\alpha u - (\lambda + \alpha c)t}\left(I_0\left(\sqrt{4\alpha\lambda t(u + ct)}\right) - \frac{ct}{ct + u}I_2\left(\sqrt{4\alpha\lambda t(u + ct)}\right)\right),$$

a result derived by Dickson et al (2005) by complex inversion of the Laplace transform with respect to t of $w(u, t)$.

5 Erlang(2) claims

Let us now consider the situation when $\bar{F}(x) = e^{-\alpha x}(1 + \alpha x)$, so that

$$g(x, t) = \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\alpha^{2n}x^{2n-1}e^{-\alpha x}}{\Gamma(2n)}.$$

We know from Cheung et al (2006) that

$$w_{Y,T}(u, y, t) = k(u, t)\alpha e^{-\alpha y} + h(u, t)\alpha^2 y e^{-\alpha y}, \tag{10}$$

and the analysis below provides formulae for $h(u,t)$ and $k(u,t)$. We remark that solutions for $h(u,t)$ and $k(u,t)$ were obtained by Dickson (2007), using a very different approach. The formulae we obtain below are quite different for $u>0$, but the same for $u=0$. However, the approach presented here also leads to alternative formulae for $h(0,t)$ and $k(0,t)$. These formulae are stated without proof at the end of this section.

We start by finding $w_{Y,T}(0,y,t)$. From equation (6) we have

$$w_{Y,T}(0,y,t)=\lambda e^{-\lambda t}f(y+ct)+\frac{\lambda}{c}\int_0^{ct}\frac{x}{t}g(ct-x,t)f(y+x)dx,$$

and we will write

$$\int_0^{ct}\frac{x}{t}g(ct-x,t)f(y+x)dx=\int_0^{ct}\frac{ct-x}{t}g(x,t)f(y+ct-x)dx.$$

Now

$$\begin{aligned}
& c\int_0^{ct} g(x,t)f(y+ct-x)dx \\
= \;& c\int_0^{ct}\sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\alpha^{2n}x^{2n-1}e^{-\alpha x}}{\Gamma(2n)}\alpha^2(y+ct-x)e^{-\alpha(y+ct-x)}dx \\
= \;& c\sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\alpha^{2(n+1)}}{\Gamma(2n)}e^{-\alpha(y+ct)}\int_0^{ct}x^{2n-1}(y+ct-x)dx \\
= \;& c\sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\alpha^{2(n+1)}}{\Gamma(2n)}e^{-\alpha(y+ct)}\left[\left((y+ct)\frac{(ct)^{2n}}{2n}-\frac{(ct)^{2n+1}}{2n+1}\right)\right] \\
= \;& c\sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\alpha^{2(n+1)}}{\Gamma(2n)}e^{-\alpha(y+ct)}\left[y\frac{(ct)^{2n}}{2n}+\frac{(ct)^{2n+1}}{2n(2n+1)}\right], \qquad (11)
\end{aligned}$$

and similar manipulations give

$$\begin{aligned}
& \int_0^{ct}\frac{x}{t}g(x,t)f(y+ct-x)dx \\
= \;& \sum_{n=1}^{\infty}e^{-\lambda t}\frac{\lambda^n t^{n-1}}{n!}\frac{\alpha^{2(n+1)}}{\Gamma(2n)}e^{-\alpha(y+ct)}\left[y\frac{(ct)^{2n+1}}{2n+1}+\frac{(ct)^{2n+2}}{(2n+1)(2n+2)}\right]. \qquad (12)
\end{aligned}$$

Combining the terms in y in (11) and (12) gives

$$\begin{aligned}
& \sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\alpha^{2(n+1)}}{\Gamma(2n)}e^{-\alpha(y+ct)}\left[cy\frac{(ct)^{2n}}{2n}-\frac{y}{t}\frac{(ct)^{2n+1}}{2n+1}\right] \\
= \;& c\alpha^2 ye^{-\alpha y}\sum_{n=1}^{\infty}e^{-(\lambda+c\alpha)t}\frac{(\lambda t)^n}{n!}\frac{(c\alpha t)^{2n}}{(2n+1)!},
\end{aligned}$$

and combining the remaining terms gives

$$
\begin{aligned}
& \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{\alpha^{2(n+1)}}{\Gamma(2n)} e^{-\alpha(y+ct)} \left[\frac{c(ct)^{2n+1}}{2n(2n+1)} - \frac{(ct)^{2n+2}}{t(2n+1)(2n+2)} \right] \\
= & \ 2c\alpha e^{-\alpha y} \sum_{n=1}^{\infty} e^{-(\lambda+c\alpha)t} \frac{(\lambda t)^n}{n!} \frac{(\alpha ct)^{2n+1}}{(2n+2)!}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
w_{Y,T}(0,y,t) &= \lambda e^{-\lambda t} \alpha^2 (y+ct) e^{-\alpha(y+ct)} \\
& \quad + \frac{\lambda}{c} c\alpha^2 y e^{-\alpha y} \sum_{n=1}^{\infty} e^{-(\lambda+c\alpha)t} \frac{(\lambda t)^n}{n!} \frac{(c\alpha t)^{2n}}{(2n+1)!} \\
& \quad + \frac{\lambda}{c} 2c\alpha e^{-\alpha y} \sum_{n=1}^{\infty} e^{-(\lambda+c\alpha)t} \frac{(\lambda t)^n}{n!} \frac{(\alpha ct)^{2n+1}}{(2n+2)!} \\
&= \lambda \alpha^2 y e^{-\alpha y} \sum_{n=0}^{\infty} e^{-(\lambda+c\alpha)t} \frac{(\lambda t)^n}{n!} \frac{(c\alpha t)^{2n}}{(2n+1)!} \\
& \quad + 2\lambda\alpha e^{-\alpha y} \sum_{n=0}^{\infty} e^{-(\lambda+c\alpha)t} \frac{(\lambda t)^n}{n!} \frac{(\alpha ct)^{2n+1}}{(2n+2)!} \qquad (13)
\end{aligned}
$$

Noting that

$$
(2n+1)! = 4^n (1)_n (\tfrac{3}{2})_n
$$

and

$$
(2n+2)! = 2\left[4^n (\tfrac{3}{2})_n (2)_n\right],
$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol, we can write

$$
\begin{aligned}
w_{Y,T}(0,y,t) &= \alpha^2 y e^{-\alpha y} \lambda e^{-(\lambda+c\alpha)t} \ {}_0F_2\left(1, \frac{3}{2}; \frac{\lambda(\alpha c)^2 t^3}{4}\right) \\
& \quad + \alpha e^{-\alpha y} \lambda \alpha c t e^{-(\lambda+c\alpha)t} \ {}_0F_2\left(\frac{3}{2}, 2; \frac{\lambda(\alpha c)^2 t^3}{4}\right), \qquad (14)
\end{aligned}
$$

where

$$
{}_0F_q(C_1, C_2, \ldots C_q; Z) = \sum_{m=0}^{\infty} \frac{1}{(C_1)_m (C_2)_m \ldots (C_q)_m} \frac{Z^m}{m!}
$$

is a generalised hypergeometric function. This form of solution is convenient for computational purposes.

Consider now $w_{Y,T}(u,y,t)$ and the first two terms in formula (8):

$$
\lambda e^{-\lambda t} f(u+ct+y) + \lambda \int_0^{u+ct} g(u+ct-x,t) f(x+y) dx
$$

$$
\begin{aligned}
&= \lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\alpha^2(u+ct+y) \\
&\quad +\lambda\sum_{m=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^m}{m!}\int_0^{u+ct} f^{m*}(x)f(u+ct+y-x)dx \\
&= \lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\alpha^2(u+ct+y) \\
&\quad +\lambda\sum_{m=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^m}{m!}\int_0^{u+ct} \frac{\alpha^{2m}x^{2m-1}e^{-\alpha x}}{\Gamma(2m)}\alpha^2(u+ct+y-x)e^{-\alpha(u+ct+y-x)}dx \\
&= \lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\alpha^2(u+ct+y) \\
&\quad +\lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=1}^{\infty}\frac{(\lambda t)^m}{m!}\frac{\alpha^{2m+2}}{\Gamma(2m)}\int_0^{u+ct} x^{2m-1}(u+ct+y-x)dx \\
&= \lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\alpha^2(u+ct+y) \\
&\quad +\lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=1}^{\infty}\frac{(\lambda t)^m}{m!}\frac{\alpha^{2m+2}}{\Gamma(2m)}\left[(u+ct+y)\frac{(u+ct)^{2m}}{2m}-\frac{(u+ct)^{2m+1}}{2m+1}\right] \\
&= \lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\alpha^2(u+ct+y) \\
&\quad +\lambda y e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=1}^{\infty}\frac{(\lambda t)^m}{m!}\frac{\alpha^{2m+2}}{\Gamma(2m)}\frac{(u+ct)^{2m}}{2m} \\
&\quad +\lambda e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=1}^{\infty}\frac{(\lambda t)^m}{m!}\frac{\alpha^{2m+2}}{\Gamma(2m)}\frac{(u+ct)^{2m+1}}{2m(2m+1)} \\
&= \lambda\alpha^2 y e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}\frac{[\alpha(u+ct)]^{2m}}{(2m)!} \\
&\quad +\lambda\alpha e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}\frac{[\alpha(u+ct)]^{2m+1}}{(2m+1)!} \\
&= \alpha^2 y e^{-\alpha y}\lambda e^{-\alpha u-(\lambda+\alpha c)t}\ {}_0F_2\left(\tfrac{1}{2},1;\lambda t(\alpha(u+ct))^2/4\right) \\
&\quad +\alpha e^{-\alpha y}\alpha(u+ct)\lambda e^{-\alpha u-(\lambda+\alpha c)t}\ {}_0F_2\left(1,\tfrac{3}{2};\lambda t(\alpha(u+ct))^2/4\right). \qquad (15)
\end{aligned}
$$

Next, (minus) the third term of formula (8) is

$$
\begin{aligned}
&\quad c\int_0^t g(u+c(t-s),t-s)w_{Y,T}(0,y,s)ds \\
&= c\int_0^t g(u+c(t-s),t-s)\lambda\alpha^2 y e^{-\alpha y}\sum_{n=0}^{\infty}e^{-(\lambda+c\alpha)s}\frac{(\lambda s)^n}{n!}\frac{(c\alpha s)^{2n}}{(2n+1)!}ds \\
&\quad +c\int_0^t g(u+c(t-s),t-s)2\lambda\alpha e^{-\alpha y}\sum_{n=0}^{\infty}e^{-(\lambda+c\alpha)s}\frac{(\lambda s)^n}{n!}\frac{(\alpha cs)^{2n+1}}{(2n+2)!}ds
\end{aligned}
$$

$$
\begin{aligned}
&= c\lambda\alpha^2 y e^{-\alpha y}\int_0^t \sum_{m=1}^{\infty} e^{-\lambda(t-s)}\frac{(\lambda(t-s))^m}{m!} f^{m*}(u+c(t-s))\sum_{n=0}^{\infty} e^{-(\lambda+c\alpha)s}\frac{(\lambda s)^n}{n!}\frac{(c\alpha s)^{2n}}{(2n+1)!}ds \\
&\quad +2c\lambda\alpha e^{-\alpha y}\int_0^t \sum_{m=1}^{\infty} e^{-\lambda(t-s)}\frac{(\lambda(t-s))^m}{m!} f^{m*}(u+c(t-s))\sum_{n=0}^{\infty} e^{-(\lambda+c\alpha)s}\frac{(\lambda s)^n}{n!}\frac{(\alpha c s)^{2n+1}}{(2n+2)!}ds \\
&= c\lambda\alpha^2 y e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=1}^{\infty}\frac{\lambda^m\alpha^{2m}}{m!\Gamma(2m)}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{(\alpha c)^{2n}}{(2n+1)!}\int_0^t (t-s)^m(u+c(t-s))^{2m-1}s^{3n}ds \\
&\quad +2c\lambda\alpha e^{-\alpha(u+y)-(\lambda+\alpha c)t}\sum_{m=1}^{\infty}\frac{\lambda^m\alpha^{2m}}{m!\Gamma(2m)}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{(\alpha c)^{2n+1}}{(2n+2)!}\int_0^t (t-s)^m(u+c(t-s))^{2m-1}s^{3n+1}ds.
\end{aligned}
$$

Using a binomial expansion of $(u+c(t-s))^m$ it is straightforward to show that for $r \geq 0$ and $m, n \geq 1$,

$$
\int_0^t (t-s)^n(u+c(t-s))^m s^r ds = \sum_{i=0}^{m}\binom{m}{i}c^i u^{m-i}t^{n+i+r+1}\frac{r!(n+i)!}{(n+r+i+1)!}.
$$

Thus, we can use expression (15) and the above development to write $h(u,t)$ and $k(u,t)$ from formula (10) as

$$
\begin{aligned}
h(u,t) &= \lambda e^{-\alpha u-(\lambda+\alpha c)t}\Bigg({}_0F_2\left(\tfrac{1}{2},1;\lambda t(\alpha(u+ct))^2/4\right) \\
&\quad -\frac{ct}{u}\sum_{m=1}^{\infty}\frac{(\lambda t\alpha^2u^2)^m}{m!\Gamma(2m)}\sum_{n=0}^{\infty}\frac{(\lambda\alpha^2c^2t^3)^n}{n!(2n+1)!}\sum_{i=0}^{2m-1}\binom{2m-1}{i}\frac{(ct/u)^i(3n)!(m+i)!}{(m+3n+i+1)!}\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
k(u,t) &= \lambda e^{-\alpha u-(\lambda+\alpha c)t}\Bigg(\alpha(u+ct)\ {}_0F_2\left(1,\tfrac{3}{2};\lambda t(\alpha(u+ct))^2/4\right) \\
&\quad -\frac{2\alpha\,(ct)^2}{u}\sum_{m=1}^{\infty}\frac{(\lambda t\alpha^2u^2)^m}{m!\Gamma(2m)}\sum_{n=0}^{\infty}\frac{(\lambda\alpha^2c^2t^3)^n}{n!(2n+2)!}\sum_{i=0}^{2m-1}\binom{2m-1}{i}\frac{(ct/u)^i(3n+1)!(m+i)!}{(m+3n+i+2)!}\Bigg).
\end{aligned}
$$

Although these formulae for $h(u,t)$ and $k(u,t)$ are not particularly elegant, their implementation is straightforward provided we truncate the infinite sums at a suitable point. Given the form of the solution for $w(u,t)$ in the previous section, it would appear possible that the solutions for $h(u,t)$ and $k(u,t)$ could be expressed entirely in terms of generalised hypergeometric functions. Indeed, it can be shown that by setting $u=0$ in the third term of formula (8) we can obtain an alternative to formula (14), namely

$$
w_{Y,T}(0,y,t) = h(0,t)\alpha^2 y e^{-\alpha y} + k(0,t)\alpha e^{-\alpha y},
$$

where

$$h(0,t) = \lambda e^{-(\lambda+\alpha c)t} \left({}_0F_2\left(\frac{1}{2}, 1; \frac{\lambda\alpha^2c^2t^3}{4}\right) - \frac{\lambda\alpha^2c^2t^3}{3}\, {}_0F_2\left(2, \frac{5}{2}; \frac{\lambda\alpha^2c^2t^3}{4}\right)\right)$$

and

$$k(0,t) = \alpha ct\lambda e^{-(\lambda+\alpha c)t} \left({}_0F_2\left(1, \frac{3}{2}; \frac{\lambda\alpha^2c^2t^3}{4}\right) - \frac{\lambda\alpha^2c^2t^3}{12}\, {}_0F_2\left(\frac{5}{2}, 3; \frac{\lambda\alpha^2c^2t^3}{4}\right)\right).$$

6 Concluding remarks

Liu and Zhao (2007) consider the joint distribution of the surplus prior to ruin, the deficit at ruin and the time to ruin in the compound binomial model. If we instead consider a discrete time compound Poisson risk model where the premium income per unit time is 1 and claims are distributed on the integers, as described in Dickson and Waters (1992), then there are counterparts of formulae in Sections 2 and 3 for this model. However, from a computational point of view, the recursive calculation scheme of Dickson and Waters (1992) (which is for the joint distribution of the deficit at ruin and the time to ruin, but can be extended to include the surplus prior to ruin) seems preferable.

Dickson and Willmot (2005) derived an expression for $w(u,t)$ when the individual claim amount distribution is an infinite mixture of Erlang distributions with the same scale parameter. Each of the individual claim amount distributions in Sections 4 and 5 is a special case of this distribution, and hence it is not surprising that the formulae derived above are simpler to program than Dickson and Willmot's general expression. However, it seems that the application of formulae (5) and (8) for other claim size distributions is not necessarily as simple, but is feasible.

References

[1] Abramowitz, M. and Stegun, I.A. (1965) Handbook of Mathematical Functions. *Dover, New York.*

[2] Cheung, E.C.K., Dickson, D.C.M. and Drekic, S. (2006) *Optimal dividend threshhold levels for two classes of claim size distributions.* Unpublished manuscript.

[3] Dickson, D.C.M. (2007) *Some explicit solutions for the joint density of the time of ruin and the deficit at ruin.* Centre for Actuarial Studies Research Paper Series No. 150, University of Melbourne.

[4] Dickson, D.C.M. and Waters, H.R. (1992) *The probability and severity of ruin in finite and in infinite time.* ASTIN Bulletin 22, 177-190.

[5] Dickson, D.C.M. and Waters, H.R. (2006) *Optimal dynamic reinsurance.* ASTIN Bulletin 36, 415-432.

[6] Dickson, D.C.M. and Willmot, G.E. (2005) *The density of the time to ruin in the classical Poisson risk model.* ASTIN Bulletin 35, 45-60.

[7] Dickson, D.C.M., Hughes, B.D. and Lianzeng, Z. (2005) *The density of the time to ruin for a Sparre Andersen process with Erlang arrivals and exponential claims.* Scandinavian Actuarial Journal, 2005, 5, 358-376.

[8] Erdélyi, A. (ed.) (1954) Tables of Integral Transforms, Volume 1. *McGraw-Hill, New York.*

[9] Gerber, H.U. (1979) An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation, Philadelphia, PA.*

[10] Gerber, H.U. and Shiu, E.S.W. (1997) *The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin.* Insurance: Mathematics & Economics 21, 129-137.

[11] Gradshteyn, I.S. and Ryzhik, I.M. (1994). Table of Integrals, Series, and Products (Fifth Edition). *Academic Press, San Diego.*

[12] Liu, G. and Zhao, J. (2007) *Joint distributions of some actuarial random vectors in the compound binomial model.* Insurance: Mathematics & Economics 40, 95-103.

[13] Prabhu, N.U. (1961) *On the ruin problem of collective risk theory.* Annals of Mathematical Statistics 32, 757-764.

[14] Seal, H.L. (1978) Survival probabilities. *John Wiley & Sons, New York.*

[15] Willmot, G.E. and Woo, J.K. (2007) *On the class of Erlang mixtures with risk theoretic applications.* North American Actuarial Journal 11(2), 99-115.

[16] Wu, R., Wang, G., and Wei, L. (2003) *Joint distributions of some actuarial random vectors containing the time of ruin.* Insurance: Mathematics & Economics 33, 147-161.

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