

Finite time ruin problems for the Erlang(2) risk model

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Abstract

We consider the Erlang(2) risk model and derive expressions for the density of the time to ruin and the joint density of the time to ruin and the deficit at ruin when the individual claim amount distribution is (i) an exponential distribution and (ii) an Erlang(2) distribution. We also consider the special case when the initial surplus is zero. We illustrate our results by plotting density functions.

1 Introduction

In this paper we use results given by Dickson and Hipp (2001) and ideas given in Cheung et al (2008) and Dickson (2008) to study finite time ruin problems for the Erlang(2) risk model. In particular we aim to find formulae for the density of the time to ruin and for joint densities of the time to ruin and deficit at ruin.

In the literature there are very few exact formulae for finite time ruin probabilities. For the classical risk model, Dickson and Willmot (2005) give a formula for the finite time ruin probability in the case when the individual claims have a distribution that is an infinite mixture of Erlang distributions. Willmot and Woo (2007) explain why this formula covers a range of individual claim amount distributions and covers all cases for which formulae for the finite time ruin probability exist. See also Drekić and Willmot (2003) and Garcia (2004). In the case of Sparre Andersen risk models formulae for finite time ruin probabilities exist only in the case of exponential claims – see Dickson et al (2005) and Borovkov and Dickson (2008).

Dickson (2008) found formulae for the joint density of the time to ruin and the deficit at ruin in the classical risk model when the distribution of individual claims was either Erlang(2) or a mixture of two exponential distributions. However, no such results presently exist for Sparre Andersen risk

models. Here we apply the methodology in Dickson (2008) to derive such results for the Erlang(2) risk model.

The outline of this paper is as follows. In Section 2 we set out the mathematical preliminaries and give transform relationships that are central to our derivations in subsequent sections. In Section 3 we discuss the special case when $u = 0$ and find that the derivation of exact results is somewhat complicated in general, but less so when the individual claim amount distribution has a particular form. The cases of individual claim amounts having (i) an exponential distribution and (ii) an Erlang(2) distribution are discussed in Sections 4 and 5 respectively. We make some concluding remarks in Section 6.

2 Preliminaries

We adopt the model of Dickson and Hipp (2001). Thus, we are dealing with a Sparre Andersen risk model under which claim inter-arrival times are distributed as Erlang(2) with scale parameter β . We denote by p the density function of individual claim amounts, and denote the k th moment as m_k . Further, the Laplace transform of p is denoted \tilde{p} where

$$\tilde{p}(s) = \int_0^{\infty} e^{-sx} p(x) dx.$$

Let $P = 1 - \bar{P}$ denote the distribution function. We denote by c the insurer's premium income per unit time and assume that $2c/\beta > m_1$.

Let $\{U(t)\}_{t \geq 0}$ denote the surplus process, let T denote the time to ruin, and let $Y = |U(T)|$ denote the deficit at ruin. Define

$$\phi(u) = E [e^{-\delta T - sY} I(T < \infty) | U(0) = u]$$

to be the bivariate Laplace transform of T and Y . Let $w(u, t)$ denote the (defective) density of T , and let $w(u, y, t)$ denote the (defective) joint density of T and Y so that

$$\phi(u) = \int_0^{\infty} \int_0^{\infty} e^{-\delta t - sy} w(u, y, t) dy dt.$$

For this risk model, Lundberg's fundamental equation is

$$s^2 - 2\frac{\beta + \delta}{c}s + \left(\frac{\beta + \delta}{c}\right)^2 = \frac{\beta^2}{c^2}\tilde{p}(s)$$

and Dickson and Hipp (2001) show that this equation has two positive solutions r_1 and r_2 such that $r_1 < (\beta + \delta)/c < r_2$. We easily deduce that

$$r_1 = \frac{\beta + \delta}{c} - \frac{\beta}{c} \sqrt{\tilde{p}(r_1)}, \quad (1)$$

$$r_2 = \frac{\beta + \delta}{c} + \frac{\beta}{c} \sqrt{\tilde{p}(r_2)}. \quad (2)$$

For the remainder of this paper we assume that $\tilde{p}(s) = \tilde{q}(s)^2$ for some density function q . An easy way of thinking of this assumption is that an Erlang(2) risk process is a classical risk process modified so that the first two claims are paid together at the time of the second claim, claims three and four are paid together at the time of the fourth claim and so on. Indeed, such a transform relationship holds if the individual claim amount distribution is infinitely divisible. Rewriting expressions (1) and (2) as

$$\begin{aligned} \beta + \delta - cr_1 &= \beta \tilde{q}(r_1), \\ \beta + \delta - cr_2 &= -\beta \tilde{q}(r_2), \end{aligned} \quad (3)$$

we see that equation (3) is identical in form to Lundberg's fundamental equation for the classical risk model, $\lambda + \delta - c\rho = \lambda \tilde{p}(\rho)$ (see Gerber and Shiu (1998)). Thus we can apply the ideas of Dickson and Willmot (2005, page 49) to obtain the following transform relationship. If

$$\tilde{f}(r_1) = \int_0^\infty e^{-r_1 t} f(t) dt = \tilde{g}(\delta) = \int_0^\infty e^{-\delta t} g(t) dt$$

then

$$g(t) = ce^{-\beta t} f(ct) + \sum_{n=1}^{\infty} \frac{\beta^n t^{n-1} e^{-\beta t}}{n!} \int_0^{ct} y q^{n*}(ct - y) f(y) dy \quad (4)$$

where q^{n*} denotes the n -fold convolution of q . The derivation is exactly as in Dickson and Willmot (2005) – all that is different is the notation. The same arguments give a further relationship involving r_2 . If

$$\tilde{f}(r_2) = \int_0^\infty e^{-r_2 t} f(t) dt = \tilde{h}(\delta) = \int_0^\infty e^{-\delta t} h(t) dt$$

then

$$h(t) = ce^{-\beta t} f(ct) + \sum_{n=1}^{\infty} \frac{(-\beta)^n t^{n-1} e^{-\beta t}}{n!} \int_0^{ct} y q^{n*}(ct - y) f(y) dy. \quad (5)$$

In the following sections we obtain Laplace transforms with transform parameters r_1 and r_2 . Our approach is to invert these Laplace transforms using relationships (4) and (5).

3 The case $u = 0$

3.1 A general approach to finding $w(0, t)$

It follows from Dickson and Hipp (2001) (see also Li and Garrido (2004)) that

$$\int_0^\infty e^{-\delta t} w(0, t) dt = \frac{c^2 r_1 r_2 - (\beta + \delta)^2 + \beta^2}{c^2 r_1 r_2}$$

and if we use equations (1) and (2) for r_1 and r_2 we obtain

$$\begin{aligned} \int_0^\infty e^{-\delta t} w(0, t) dt &= \frac{\beta^2 [1 - \tilde{q}(r_1) + \tilde{q}(r_2) - \tilde{q}(r_1)\tilde{q}(r_2)] - \beta\delta[\tilde{q}(r_1) - \tilde{q}(r_2)]}{c^2 r_1 r_2} \\ &= \frac{\beta^2}{c^2} \left[\frac{1 - \tilde{q}(r_1)}{r_1} \right] \left[\frac{1 + \tilde{q}(r_2)}{r_2} \right] - \frac{\beta\delta[\tilde{q}(r_1) - \tilde{q}(r_2)]}{c^2 r_1 r_2}. \end{aligned} \quad (6)$$

Now let

$$\begin{aligned} \tilde{U}_1(\delta) &= \int_0^\infty e^{-\delta t} U_1(t) dt = \frac{1 - \tilde{q}(r_1)}{r_1} = \int_0^\infty e^{-r_1 x} \left(1 - \int_0^x q(y) dy \right) dx, \\ \tilde{U}_2(\delta) &= \int_0^\infty e^{-\delta t} U_2(t) dt = \frac{1 + \tilde{q}(r_2)}{r_2} = \int_0^\infty e^{-r_2 x} \left(1 + \int_0^x q(y) dy \right) dx, \\ \tilde{U}_3(\delta) &= \int_0^\infty e^{-\delta t} U_3(t) dt = \frac{1}{r_1} = \int_0^\infty e^{-r_1 t} dt, \\ \tilde{U}_4(\delta) &= \int_0^\infty e^{-\delta t} U_4(t) dt = \frac{1}{r_2} = \int_0^\infty e^{-r_2 t} dt. \end{aligned}$$

Then by the transform relationships (4) and (5), we have

$$\begin{aligned} U_3(t) &= ce^{-\beta t} + \sum_{n=1}^\infty \frac{\beta^n t^{n-1} e^{-\beta t}}{n!} \int_0^{ct} y q^{n*}(ct - y) dy, \\ U_4(t) &= ce^{-\beta t} + \sum_{n=1}^\infty \frac{(-\beta)^n t^{n-1} e^{-\beta t}}{n!} \int_0^{ct} y q^{n*}(ct - y) dy, \\ U_1(t) &= U_3(t) - ce^{-\beta t} \int_0^{ct} q(y) dy \\ &\quad - \sum_{n=1}^\infty \frac{\beta^n t^{n-1} e^{-\beta t}}{n!} \int_0^{ct} y q^{n*}(ct - y) \left(\int_0^y q(s) ds \right) dy, \\ U_2(t) &= U_4(t) + ce^{-\beta t} \int_0^{ct} q(y) dy \\ &\quad + \sum_{n=1}^\infty \frac{(-\beta)^n t^{n-1} e^{-\beta t}}{n!} \int_0^{ct} y q^{n*}(ct - y) \left(\int_0^y q(s) ds \right) dy. \end{aligned}$$

With this notation, the first term in (6) is $(\beta/c)^2 \tilde{U}_1(\delta) \tilde{U}_2(\delta)$ and inverts to $(\beta/c)^2 U_1 * U_2(t)$.

To invert the second term in (6), consider

$$\tilde{m}(\delta) = \int_0^\infty e^{-\delta x} m(x) dx = \frac{\delta}{r_1 r_2} (\tilde{q}(r_1) - \tilde{q}(r_2)).$$

Then

$$\begin{aligned} \frac{\tilde{m}(\delta)}{\delta} &= \int_0^\infty e^{-\delta x} \int_0^x m(y) dy dx \\ &= \frac{1 - \tilde{q}(r_2)}{r_2} \frac{1}{r_1} - \frac{1 - \tilde{q}(r_1)}{r_1} \frac{1}{r_2} \\ &= \tilde{U}_5(\delta) \tilde{U}_3(\delta) - \tilde{U}_1(\delta) \tilde{U}_4(\delta) \end{aligned}$$

where $\tilde{U}_5(\delta) = (1 - \tilde{q}(r_2))/r_2$. This gives

$$\int_0^t m(y) dy = U_5 * U_3(t) - U_1 * U_4(t)$$

and so

$$w(0, t) = \frac{\beta^2}{c^2} U_1 * U_2(t) - \frac{\beta}{c^2} \frac{d}{dt} (U_5 * U_3(t) - U_1 * U_4(t)).$$

This formula does not seem particularly attractive in terms of finding elegant formulae for $w(0, t)$ for specific individual claim amount distributions. However, for certain individual claim amount distributions, it is possible to find solutions both for $w(0, t)$ and $w(0, y, t)$ as shown in the next section.

3.2 A specific claim amount distribution

In this section we assume that the individual claim amount density is such that

$$p(x + y) = \sum_{j=1}^m \eta_j(x) \tau_j(y) \tag{7}$$

for some functions $\{\eta_j, \tau_j\}_{j=1}^m$. This factorisation was introduced by Willmot (2007), and he shows that if the individual claim amount distribution is an infinite mixture of Erlang distributions with the same scale parameter then this factorisation applies. Results in Willmot and Woo (2007) show that this infinite mixture of Erlangs contains many well-known distributions as special cases.

We make the further assumption that the functions $\{\tau_j\}$ are in fact density functions (as is the case when p is an infinite mixture of Erlang densities). Then

$$\begin{aligned}\bar{P}(x+y) &= \int_y^\infty p(x+v)dv \\ &= \int_y^\infty \sum_{j=1}^m \eta_j(x)\tau_j(v)dv \\ &= \sum_{j=1}^m \eta_j(x)\bar{T}_j(y)\end{aligned}$$

where $T_j = 1 - \bar{T}_j$ is the distribution function associated with τ_j .

From Dickson and Hipp (2001) we know that

$$\int_0^\infty e^{-\delta t}w(0,t)dt = \frac{\beta^2}{c^2} \int_0^\infty \int_0^\infty e^{-r_1x}e^{-r_2y}\bar{P}(x+y)dydx,$$

so if p satisfies equation (7) we obtain

$$\begin{aligned}\int_0^\infty e^{-\delta t}w(0,t)dt &= \frac{\beta^2}{c^2} \int_0^\infty \int_0^\infty e^{-r_1x}e^{-r_2y} \sum_{j=1}^m \eta_j(x)\bar{T}_j(y)dydx \\ &= \frac{\beta^2}{c^2} \sum_{j=1}^m \int_0^\infty e^{-r_1x}\eta_j(x)dx \int_0^\infty e^{-r_2y}\bar{T}_j(y)dy.\end{aligned}\quad (8)$$

The significance of equation (8) is that each of the integrals is a Laplace transform – with respect to either r_1 or r_2 , and hence by the transform relationships (4) and (5) they are Laplace transforms with respect to δ . By inverting each transform, we can obtain the inverse of the product of two transforms by finding the convolution of the two inverses.

For example, the simplest case to satisfy equation (7) is $p(x) = \alpha e^{-\alpha x}$ giving $\eta_1(x) = e^{-\alpha x}$ and $\tau_1(y) = \alpha e^{-\alpha y}$, with $\bar{T}_1(y) = e^{-\alpha y}$. Then equation (8) gives the already known equation

$$\int_0^\infty e^{-\delta t}w(0,t)dt = \frac{\beta^2}{c^2(\alpha+r_1)(\alpha+r_2)},$$

(see Dickson and Hipp (2001)), and we show how to invert this Laplace transform in the next section. The same techniques apply to other individual claim amount distributions whose densities satisfy equation (7).

This approach can be extended if the functions $\{\eta_j\}$ satisfy the same type of factorisation as equation (7). As Willmot (2007) shows, this is the

case when the individual claim amount distribution is an infinite mixture of Erlang distributions with the same scale parameter. Specifically, let us suppose that

$$\eta_j(x+y) = \sum_{i=1}^n \xi_{ij}(x)\zeta_{ij}(y). \quad (9)$$

From equation (4) of Sun (2005), we know that

$$\int_0^\infty \int_0^\infty e^{-\delta t - sy} w(0, y, t) dy dt = \frac{\beta^2}{c^2} \int_0^\infty \int_0^\infty e^{-r_1 x} e^{-r_2 y} \varpi(x+y) dy dx$$

where

$$\varpi(u) = \int_0^\infty e^{-sy} p(u+y) dy.$$

Applying (7) in the above expression we obtain

$$\varpi(u) = \int_0^\infty e^{-sy} \sum_{j=1}^m \eta_j(u) \tau_j(y) dy = \sum_{j=1}^m \eta_j(u) \tilde{\tau}_j(s),$$

and if we now apply (9) we obtain

$$\varpi(x+y) = \sum_{j=1}^m \sum_{i=1}^n \xi_{ij}(x) \zeta_{ij}(y) \tilde{\tau}_j(s),$$

so that

$$\int_0^\infty \int_0^\infty e^{-\delta t - sy} w(0, y, t) dy dt = \frac{\beta^2}{c^2} \sum_{j=1}^m \sum_{i=1}^n \tilde{\xi}_{ij}(r_1) \tilde{\zeta}_{ij}(r_2) \tilde{\tau}_j(s).$$

Thus, the bivariate Laplace transform factorises in terms of transforms with transform parameters r_1 , r_2 and s .

For example, when $p(x) = \alpha^2 x e^{-\alpha x}$, $x > 0$, we get

$$\eta_1(x) = \alpha x e^{-\alpha x}, \quad \tau_1(y) = \alpha e^{-\alpha y}, \quad \eta_2(x) = e^{-\alpha x}, \quad \tau_2(y) = \alpha^2 y e^{-\alpha y},$$

and we can write

$$\begin{aligned} \eta_1(x+y) &= \xi_{11}(x)\zeta_{11}(y) + \xi_{21}(x)\zeta_{21}(y), \\ \eta_2(x+y) &= \xi_{12}(x)\zeta_{12}(y), \end{aligned}$$

where

$$\begin{aligned} \xi_{11}(x) &= \alpha x e^{-\alpha x}, \quad \zeta_{11}(y) = \zeta_{12}(y) = e^{-\alpha y}, \\ \xi_{21}(x) &= \xi_{12}(x) = e^{-\alpha x}, \quad \zeta_{21}(y) = \alpha y e^{-\alpha y}. \end{aligned}$$

Then

$$\begin{aligned} & \int_0^\infty \int_0^\infty e^{-\delta t - sy} w(0, y, t) dy dt \\ &= \frac{\beta^2}{c^2} \left(\frac{\alpha}{(\alpha + r_1)^2 (\alpha + r_2)} + \frac{\alpha}{(\alpha + r_1) (\alpha + r_2)^2} \right) \frac{\alpha}{\alpha + s} \\ & \quad + \frac{\beta^2}{c^2} \frac{1}{(\alpha + r_1) (\alpha + r_2)} \left(\frac{\alpha}{\alpha + s} \right)^2. \end{aligned}$$

As r_1 and r_2 are both functions of δ , the above equation tells us that

$$w(0, y, t) = f_1(t) \alpha e^{-\alpha y} + f_2(t) \alpha^2 y e^{-\alpha y},$$

and we show in Section 5 how the functions f_1 and f_2 can be identified.

4 Exponential claims

In this section we consider the case $p(x) = \alpha e^{-\alpha x}$, $x > 0$, so that $q(x) = (\alpha/2)e^{-\alpha x/2}$. Although formulae for $w(u, t)$ are known for this case (see Dickson et al (2005) and Borovkov and Dickson (2008)), it is interesting to explore this case for two reasons. First, it introduces the techniques needed for the more complicated case of Erlang(2) claims discussed in the next section, and, second, we end up with a new formula for $w(u, t)$ which parallels the formula for the density of the time to ruin in a classical risk model with exponential claims obtained by Drekić and Willmott (2003).

Dickson (2008) shows that

$$w(u, t) = \int_0^t \int_0^u w(0, y, \tau) w(u - y, t - \tau) dy d\tau + \int_u^\infty w(0, y, t) dy, \quad (10)$$

and it is well-known that by the memoryless property of the exponential distribution

$$w(u, y, t) = w(u, t) \alpha e^{-\alpha y}.$$

Following the approach in Section 3 of Dickson (2008), we insert this expression in equation (10) then take the Laplace transform of the resulting equation. Defining

$$\tilde{w}(s, \delta) = \int_0^\infty \int_0^\infty e^{-su - \delta t} w(u, t) dt du \quad \text{and} \quad \tilde{w}(0, \delta) = \int_0^\infty e^{-\delta t} w(0, t) dt,$$

we obtain

$$\tilde{w}(s, \delta) = \frac{\tilde{w}(0, \delta) \frac{1}{\alpha + s}}{1 - \tilde{w}(0, \delta) \frac{\alpha}{\alpha + s}} = \frac{1}{\alpha} \sum_{n=1}^{\infty} (\tilde{w}(0, \delta))^n \left(\frac{\alpha}{\alpha + s} \right)^n$$

which inverts to

$$w(u, t) = \sum_{n=1}^{\infty} w^{n*}(0, t) \frac{(\alpha u)^{n-1} e^{-\alpha u}}{\Gamma(n)}. \quad (11)$$

We remark that the derivation of this result does not depend on the claim inter-arrival distribution being Erlang(2) – formula (11) is a general result.

We saw in Section 3.2 that when the claim inter-arrival distribution is Erlang(2) with scale parameter β ,

$$\tilde{w}(0, \delta) = \frac{\beta^2}{c^2(\alpha + r_1)(\alpha + r_2)}$$

so that $w^{n*}(0, t)$ can be found by inverting

$$\frac{\beta^{2n}}{c^{2n}(\alpha + r_1)^n(\alpha + r_2)^n}.$$

Now let $\tilde{V}_n(\delta) = (\alpha + r_1)^{-n}$ (where r_1 depends on δ) be the Laplace transform of a function $V_n(t)$, and similarly let $\tilde{W}_n(\delta) = (\alpha + r_2)^{-n}$. Then inversion of $\tilde{V}_n(\delta)$ with respect to r_1 yields $t^{n-1}e^{-\alpha t}/\Gamma(n)$ and by (4) inversion with respect to δ yields

$$ce^{-\beta t} \frac{(ct)^{n-1} e^{-\alpha ct}}{\Gamma(n)} + \sum_{m=1}^{\infty} \frac{\beta^m t^{m-1} e^{-\beta t}}{m!} \int_0^{ct} y q^{m*}(ct - y) \frac{y^{n-1} e^{-\alpha y}}{\Gamma(n)} dy.$$

We find that

$$\int_0^{ct} y q^{m*}(ct - y) \frac{y^{n-1} e^{-\alpha y}}{\Gamma(n)} dy = \frac{n\alpha^{m/2} e^{-\alpha ct} (ct)^{n+m/2}}{\Gamma(m/2 + n + 1)}$$

giving

$$V_n(t) = ne^{-(\beta+\alpha)t} \sum_{m=0}^{\infty} \frac{\alpha^{m/2} \beta^m}{m!} \frac{c^{n+m/2} t^{n+3m/2-1}}{\Gamma(m/2 + n + 1)}.$$

Similarly, by applying (5) we obtain

$$W_n(t) = ne^{-(\beta+\alpha)t} \sum_{m=0}^{\infty} \frac{\alpha^{m/2} (-1)^m \beta^m}{m!} \frac{c^{n+m/2} t^{n+3m/2-1}}{\Gamma(m/2 + n + 1)}.$$

Thus,

$$w^{n*}(0, t) = \frac{\beta^2}{c^2} V_n * W_n(t)$$

We can evaluate the convolution $V_n * W_n(t)$ by taking a term by term approach. We write

$$V_n(t) = \sum_{m=0}^{\infty} a_m \frac{t^{n+3m/2-1} e^{-(\beta+\alpha)t}}{\Gamma(n+3m/2)}$$

where

$$a_m = \frac{n\alpha^{m/2}\beta^m c^{n+m/2}\Gamma(n+3m/2)}{m! \Gamma(m/2+n+1)},$$

and so we can write

$$W_n(t) = \sum_{m=0}^{\infty} (-1)^m a_m \frac{t^{n+3m/2-1} e^{-(\beta+\alpha)t}}{\Gamma(n+3m/2)}.$$

If we take the convolution of the i th term in V_n with the j th term in W_n we get

$$(-1)^j a_i a_j \frac{e^{-(\beta+\alpha)t} t^{2n+\frac{3}{2}(i+j)-1}}{\Gamma(2n+\frac{3}{2}(i+j))}$$

so that

$$w^{n*}(0, t) = \frac{\beta^{2n}}{c^{2n}} \sum_{r=0}^{\infty} k_r \frac{e^{-(\beta+\alpha)t} t^{2n+\frac{3}{2}r-1}}{\Gamma(2n+\frac{3}{2}r)}$$

where

$$k_r = \sum_{i=0}^r (-1)^{r-i} a_i a_{r-i}.$$

By symmetry, $k_r = 0$ if r is odd, so we obtain

$$w^{n*}(0, t) = \frac{\beta^{2n} t^{2n-1} e^{-(\beta+\alpha)t}}{c^{2n}} \sum_{r=0}^{\infty} k_{2r} \frac{t^{3r}}{\Gamma(2n+3r)}.$$

Tidying up gives

$$k_{2r} = \alpha^r \beta^{2r} c^{2n+r} \sigma(2r, n)$$

where

$$\begin{aligned} \sigma(2r, n) &= n^2 \sum_{i=0}^{2r} (-1)^i \frac{\Gamma(n+3i/2)}{i! \Gamma(i/2+n+1)} \frac{\Gamma(n+3(2r-i)/2)}{(2r-i)! \Gamma((2r-i)/2+n+1)} \\ &= \frac{\Gamma(2n+3r)}{4^r r! (2n-1)! (n+1/2)_r (n+1)_r} \end{aligned} \quad (12)$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol. (We do not have a proof of result (12). We have conjectured this identity based on a study of

$\sigma(2r, n)$ for a range of values of r , and readers can easily test the conjecture numerically.) Thus we can write

$$w^{n*}(0, t) = \frac{\beta^{2n} t^{2n-1} e^{-(\beta+\alpha)t}}{\Gamma(2n)} {}_0F_2 \left(n + \frac{1}{2}, n + 1; \frac{\alpha c \beta^2 t^3}{4} \right)$$

where

$${}_pF_q(B_1, B_2, \dots, B_p, C_1, C_2, \dots, C_q; Z) = \sum_{m=0}^{\infty} \frac{(B_1)_m (B_2)_m \dots (B_p)_m}{(C_1)_m (C_2)_m \dots (C_q)_m} \frac{Z^m}{m!}$$

is the generalised hypergeometric function. This gives

$$w(u, t) = e^{-\alpha u - (\beta + \alpha)t} \sum_{n=1}^{\infty} \frac{(\alpha u)^{n-1} \beta^{2n} t^{2n-1}}{\Gamma(n) \Gamma(2n)} {}_0F_2 \left(n + \frac{1}{2}, n + 1; \frac{\alpha c \beta^2 t^3}{4} \right).$$

Whilst this expression is easily computed, it is not as compact as solutions for $w(u, t)$ presented in Dickson et al (2005) and Borovkov and Dickson (2008). However, it is the counterpart of Drekić and Willmot's (2003) formula for the density of the time to ruin in the classical risk model with exponentially distributed claims. Their formula can also be obtained by applying formula (11) to their model, and their formula, which is expressed in terms of Bessel functions, can be written in terms of ${}_0F_1$ functions using the relationship between these functions and Bessel functions.

5 Erlang(2) claims

We now consider the situation when $p(x) = \alpha^2 x e^{-\alpha x}$, $x > 0$, so that $q(x) = \alpha e^{-\alpha x}$, $x > 0$. In this case we have

$$w(u, y, t) = h(u, t) \alpha^2 y e^{-\alpha y} + k(u, t) \alpha e^{-\alpha y} \quad (13)$$

where $h(u, t)$ and $k(u, t)$ are functions that we will identify. For the sake of brevity we omit the details of why we know $w(u, y, t)$ is of this form. The approach to showing this is essentially that given by Cheung et al (2008), but adapted to our risk model.

5.1 Main results

As the form of the solution for $w(u, y, t)$ is exactly the same as in the classical risk model with Erlang(2, α) claims, we can apply the approach given in Dickson (2008, Section 3). The reason we can do this is that we are simply

inserting formula (13) into equation (1) of Dickson (2008), and that equation applies to any Sparre Andersen model. Let

$$\tilde{h}(s, \delta) = \int_0^\infty \int_0^\infty e^{-su-\delta t} h(u, t) dt du$$

and let

$$\tilde{h}(0, \delta) = \int_0^\infty e^{-\delta t} h(0, t) dt \quad \text{and} \quad \tilde{k}(0, \delta) = \int_0^\infty e^{-\delta t} k(0, t) dt.$$

Exactly as in Dickson (2008) we have

$$\tilde{h}(s, \delta) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \sum_{r=0}^n \binom{n}{r} \tilde{h}(0, \delta)^{r+1} \tilde{k}(0, \delta)^{n-r} \left(\frac{\alpha}{\alpha + s} \right)^{n+r+1}$$

and – this is where we now differ from Dickson (2008) – using results from Section 3.2,

$$\begin{aligned} & \tilde{h}(0, \delta)^{r+1} \tilde{k}(0, \delta)^{n-r} \\ &= \left(\frac{\beta^2}{c^2(\alpha + r_1)(\alpha + r_2)} \right)^{r+1} \left(\frac{\beta^2 \alpha}{c^2(\alpha + r_1)^2(\alpha + r_2)} + \frac{\beta^2 \alpha}{c^2(\alpha + r_1)(\alpha + r_2)^2} \right)^{n-r} \\ &= \left(\frac{\beta^2}{c^2(\alpha + r_1)(\alpha + r_2)} \right)^{r+1} \left(\frac{\beta^2 \alpha}{c^2(\alpha + r_1)(\alpha + r_2)} \right)^{n-r} \left(\frac{1}{\alpha + r_1} + \frac{1}{\alpha + r_2} \right)^{n-r} \\ &= \frac{\beta^{2n+2} \alpha^{n-r}}{c^{2n+2}} \left(\frac{1}{(\alpha + r_1)(\alpha + r_2)} \right)^{n+1} \sum_{x=0}^{n-r} \binom{n-r}{x} \frac{1}{(\alpha + r_1)^x} \frac{1}{(\alpha + r_2)^{n-r-x}} \\ &= \frac{\beta^{2n+2} \alpha^{n-r}}{c^{2n+2}} \sum_{x=0}^{n-r} \binom{n-r}{x} \frac{1}{(\alpha + r_1)^{n+1+x}} \frac{1}{(\alpha + r_2)^{2n+1-r-x}}. \end{aligned}$$

Now by equation (4), $1/(\alpha + r_1)^{k+1}$ inverts to

$$ce^{-\beta t} \frac{(ct)^k e^{-\alpha ct}}{\Gamma(k+1)} + \sum_{m=1}^{\infty} \frac{\beta^m t^{m-1} e^{-\beta t}}{m!} \int_0^{ct} y q^{m*}(ct-y) \frac{y^k e^{-\alpha y}}{\Gamma(k+1)} dy. \quad (14)$$

As q^{m*} is the Erlang(m, α) density, the integral becomes

$$\int_0^{ct} y \frac{\alpha^m (ct-y)^{m-1} e^{-\alpha(ct-y)}}{\Gamma(m)} \frac{y^k e^{-\alpha y}}{\Gamma(k+1)} dy = \frac{(k+1) \alpha^m e^{-\alpha ct} (ct)^{m+k+1}}{\Gamma(k+2+m)}$$

and expression (14) becomes

$$\begin{aligned}
& ce^{-\beta t} \frac{(ct)^k e^{-\alpha ct}}{\Gamma(k+1)} + (k+1) \sum_{m=1}^{\infty} \frac{\beta^m t^{m-1} e^{-\beta t}}{m!} \frac{\alpha^m e^{-\alpha ct} (ct)^{m+k+1}}{\Gamma(k+2+m)} \\
= & ce^{-\beta t} \frac{(ct)^k e^{-\alpha ct}}{\Gamma(k+1)} + c(k+1) \sum_{m=1}^{\infty} \frac{\beta^m t^m e^{-\beta t}}{m!} \frac{\alpha^m e^{-\alpha ct} (ct)^{m+k}}{\Gamma(k+2+m)} \\
= & \frac{c(ct)^k e^{-(\beta+\alpha)t}}{k!} \sum_{m=0}^{\infty} \frac{1}{(k+2)_m} \frac{(\alpha\beta ct^2)^m}{m!} \\
= & \frac{c(ct)^k e^{-(\beta+\alpha)t}}{k!} {}_0F_1(k+2; \alpha\beta ct^2).
\end{aligned}$$

Hence, $1/(\alpha + r_1)^{n+1+x}$ inverts to

$$\frac{c(ct)^{n+x} e^{-(\beta+\alpha)t}}{(n+x)!} {}_0F_1(n+2+x; \alpha\beta ct^2) = A_{n,x}(t), \text{ say.}$$

Next, by the symmetry of (4) and (5), $1/(\alpha + r_2)^{n+1}$ inverts to

$$\frac{c(ct)^n e^{-(\beta+\alpha)t}}{n!} {}_0F_1(n+2; -\alpha\beta ct^2)$$

and hence $1/(\alpha + r_2)^{2n+1-r-x}$ inverts to

$$\frac{c(ct)^{2n-r-x} e^{-(\beta+\alpha)t}}{(2n-r-x)!} {}_0F_1(2n-r-x+2; -\alpha\beta ct^2) = B_{n,x,r}(t), \text{ say.}$$

Thus, if we define $C_{n,x,r}$ to be the convolution of $A_{n,x}$ and $B_{n,x,r}$, then

$$h(u, t) = e^{-\alpha u} \frac{\beta^2}{c^2} \sum_{n=0}^{\infty} \frac{(\alpha\beta)^{2n}}{c^{2n}} \sum_{r=0}^n \binom{n}{r} \frac{u^{n+r}}{(n+r)!} \sum_{x=0}^{n-r} \binom{n-r}{x} C_{n,x,r}(t). \quad (15)$$

To find $C_{n,x,r}$ let us write

$$A_{n,x}(t) = \sum_{m=0}^{\infty} a_m \frac{t^{2m+n+x} e^{-(\beta+\alpha)t}}{\Gamma(2m+n+x+1)}$$

where

$$a_m = \frac{\Gamma(2m+n+x+1)}{(n+2+x)_m} \frac{c^{n+x+1} (\alpha\beta c)^m}{m! (n+x)!},$$

and let us write

$$B_{n,x,r}(t) = \sum_{k=0}^{\infty} b_k \frac{t^{2k+2n-r-x} e^{-(\beta+\alpha)t}}{\Gamma(2k+2n-r-x+1)}$$

where

$$b_k = \frac{\Gamma(2k + 2n - r - x + 1) c^{2n-r-x+1} (-\alpha\beta c)^k}{(2n - r + 2 - x)_k k! (2n - r - x)!}.$$

We recall that n and r are fixed. The convolution of the m th term of $A_{n,x}$ with the k th term of $B_{n,x,r}$ has Laplace transform with transform parameter s

$$\frac{a_m b_k}{(\beta + \alpha c + s)^{2m+3n-r+2k+2}}$$

which inverts to

$$\frac{a_m b_k t^{2m+3n-r+2k+1} e^{-(\beta+\alpha c)t}}{\Gamma(2m + 3n - r + 2k + 2)}.$$

Then for $l = 0, 1, 2, \dots$ the coefficient of

$$\frac{t^{3n-r+2l+1} e^{-(\beta+\alpha c)t}}{\Gamma(3n - r + 2l + 2)}$$

is τ_l where

$$\begin{aligned} \tau_l &= \sum_{i=0}^l a_i b_{l-i} \\ &= \sum_{i=0}^l \frac{\Gamma(2i + n + x + 1) c^{n+x+1} (\alpha\beta c)^i \Gamma(2(l-i) + 2n - r - x + 1) c^{2n-r-x+1} (-\alpha\beta c)^{l-i}}{(n+2+x)_i i! (n+x)! (2n-r+2-x)_{l-i} (l-i)! (2n-r-x)!} \\ &= \frac{c^{3n-r+2} (\alpha\beta c)^l}{(n+x)! (2n-r-x)! l!} \\ &\quad \times \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} \frac{\Gamma(2i + n + x + 1) \Gamma(2(l-i) + 2n - r - x + 1)}{(n+2+x)_i (2n-r+2-x)_{l-i}} \\ &= \frac{c^{3n-r+2}}{(n+x)! (2n-r-x)!} \frac{(\alpha\beta c)^l}{l!} \sigma_{l,n,r,x} \end{aligned}$$

and

$$\sigma_{l,n,r,x} = \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} \frac{\Gamma(2i + n + x + 1) \Gamma(2(l-i) + 2n - r - x + 1)}{(n+2+x)_i (2n-r+2-x)_{l-i}}.$$

Thus,

$$\begin{aligned} C_{n,x,r}(t) &= \sum_{l=0}^{\infty} \frac{c^{3n-r+2}}{(n+x)! (2n-r-x)!} \frac{(\alpha\beta c)^l}{l!} \sigma_{l,n,r,x} \frac{t^{3n-r+2l+1} e^{-(\beta+\alpha c)t}}{\Gamma(3n-r+2l+2)} \\ &= \frac{c(ct)^{3n-r+1} e^{-(\beta+\alpha c)t}}{(n+x)! (2n-r-x)!} \sum_{l=0}^{\infty} \frac{(\alpha\beta ct^2)^l}{l!} \frac{\sigma_{l,n,x,r}}{\Gamma(3n-r+2l+2)} \quad (16) \end{aligned}$$

and

$$\begin{aligned}
h(u, t) &= e^{-\alpha u - (\beta + \alpha c)t} \frac{\beta^2}{c} \sum_{n=0}^{\infty} \frac{(\alpha\beta)^{2n}}{c^{2n}} \sum_{r=0}^n \binom{n}{r} \frac{u^{n+r}}{(n+r)!} \\
&\quad \times \sum_{x=0}^{n-r} \binom{n-r}{x} \frac{(ct)^{3n-r+1}}{(n+x)!(2n-r-x)!} \sum_{l=0}^{\infty} \frac{(\alpha\beta ct^2)^l}{l!} \frac{\sigma_{l,n,x,r}}{\Gamma(3n-r+2l+2)}
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
k(u, t) &= e^{-\alpha u - (\beta + \alpha c)t} \frac{\alpha\beta^2}{c} \sum_{n=0}^{\infty} \frac{(\alpha\beta)^{2n}}{c^{2n}} \sum_{r=0}^n \binom{n}{r} \frac{u^{n+r+1}}{(n+r+1)!} \\
&\quad \times \sum_{x=0}^{n-r} \binom{n-r}{x} \frac{(ct)^{3n-r+1}}{(n+x)!(2n-r-x)!} \sum_{l=0}^{\infty} \frac{(\alpha\beta ct^2)^l}{l!} \frac{\sigma_{l,n,x,r}}{\Gamma(3n-r+2l+2)} \\
&\quad + \alpha\beta^2 ct^2 e^{-\alpha u} \sum_{n=0}^{\infty} \frac{(\alpha\beta)^{2n}}{c^{2n}} \sum_{r=0}^n \binom{n}{r} \frac{u^{n+r}}{(n+r)!} \\
&\quad \times \sum_{x=0}^{n-r+1} \binom{n-r+1}{x} \frac{(ct)^{3n-r} e^{-(\beta+\alpha c)t}}{(n+x)!(2n-r-x+1)!} \sum_{l=0}^{\infty} \frac{(\alpha\beta ct^2)^l}{l!} \frac{\sigma_{l,n,r,x}^*}{\Gamma(2l+3n-r+3)}
\end{aligned}$$

where

$$\sigma_{l,n,r,x}^* = \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} \frac{\Gamma(2i+n+x+1)}{(n+2+x)_i} \frac{\Gamma(2(l-i)+2n-r-x+2)}{(2n-r+3-x)_{l-i}}.$$

Finally, we remark that the density of the time to ruin is obtained as $w(u, t) = h(u, t) + k(u, t)$.

5.2 Hypergeometric solutions

From a computational point of view, it would be useful if we could write the above solutions for $h(u, t)$ and $k(u, t)$ in terms of hypergeometric functions. Whilst we believe this may be possible, we are unable to do this in a systematic way. The route to finding such solutions is to find closed form solutions for $\sigma_{l,n,x,r}$ and $\sigma_{l,n,r,x}^*$, and we can do this using techniques described in Graham et al (1994, Chapter 5). However, we are unable to find general expressions in terms of n , x and r , and the solutions we obtained for specific values of n , x and r do not point to general expressions.

An exception is the case when $u = 0$. We find that

$$\sigma_{l,0,0,0} = l! \sum_{i=0}^l (-1)^{l-i} \binom{2i+1}{i} \frac{1}{2i+1} \binom{2(l-i)+1}{l-i} \frac{1}{2(l-i)+1}$$

is zero when l is odd (by symmetry), and equals

$$\frac{l!4^{l+1}}{\Gamma(\frac{1}{2})} \left(\frac{\Gamma(\frac{l+1}{2})}{2\Gamma(\frac{l}{2} + 2)} - \frac{\Gamma(l + \frac{3}{2})}{\Gamma(l + 3)} \right)$$

when l is even, leading to

$$h(0, t) = e^{-(\beta+\alpha c)t} \beta^2 t \left(2 {}_0F_3 \left(\left\{ \frac{3}{4}, \frac{5}{4}, 2 \right\}; \frac{\alpha^2 \beta^2 c^2 t^4}{4^2} \right) - {}_0F_3 \left(\left\{ \frac{1}{2}, \frac{3}{2}, 2 \right\}; \frac{\alpha^2 \beta^2 c^2 t^4}{4^2} \right) \right).$$

Similarly, we can show that

$$k(0, t) = e^{-(\beta+\alpha c)t} \alpha c \beta^2 t^2 {}_0F_3 \left(\left\{ \frac{3}{2}, \frac{3}{2}, 2 \right\}; \frac{\alpha^2 \beta^2 c^2 t^4}{4^2} \right).$$

5.3 Examples

Figures 1 to 3 show plots of $h(u, t)$, $k(u, t)$ and $w(u, t)$ for $u = 0, 1$ and 5 respectively. The plots for $u > 0$ are similar to those in Dickson (2008) for the classical risk model with Erlang(2) claims, but the plot of $h(0, t)$ is different in this case.

Whilst it was straightforward to perform calculations in Mathematica by truncating infinite sums, the time required to produce a single value of either $h(u, t)$ or $k(u, t)$ often exceeded 24 hours.

6 Concluding remarks

6.1 Other quantities of interest

We note that the solutions for $h(u, t)$ and $k(u, t)$ in Section 5.1 can be written as infinite sums including Erlang densities. Thus, we can express the finite time ruin probability as an infinite mixture of Erlang distributions. Similarly, we can use this observation to obtain expressions for the moments of the time to ruin. For the sake of brevity we do not include these expressions here.

Similar comments apply in the case of exponential claims from Section 4.

6.2 A dual risk model

According to Mazza and Rullière (2004), since we know the density of the time to ruin in a Sparre Andersen model with Erlang(2, β) claim inter-arrival times, Erlang(2, α) claim amounts and premium rate c per unit time, we can

compute the density of the time to ruin in a dual Sparre Andersen model with Erlang($2, \alpha$) gain inter-arrival times, Erlang($2, \beta$) gains and expenses rate $E = 1/c$. Our formulae for $h(u, t)$ and $k(u, t)$ in Section 5.1 allow us to apply Mazza and Rullière's results to obtain a formula for the density of the time to ruin in this dual model. We omit the details, again for the sake of brevity.

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