On the generalized Gerber-Shiu function for surplus processes with interest

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Abstract

In this paper, we study the generalized expected discounted penalty (Gerber-Shiu) function in a risk process with credit and debit interests. We define $T_{u,z}$ to be the first time that the surplus process drops below a certain level z from the initial surplus $u(> z)$. The time of ruin and the time of absolute ruin are special cases of this stopping time. The generalized Gerber-Shiu function is defined on three random variables: the first time that the surplus drops below z from u, $T_{u,z}$, the surplus prior to $T_{u,z}$, and the amount by which the surplus is below z.

An explicit expression for the Gerber-Shiu function when $u = z$ is obtained when credit and debit interest rates are equal, and explicit results for the Gerber-Shiu function under exponential claims are then obtained. Using these results, we investigate the probability that the surplus reaches an upper level without dropping below a lower level and the distribution of the maximum severity of ruin.

Keywords: Risk model with interest; The generalized Gerber-Shiu function; Ruin probability; Absolute ruin; The maximum surplus before ruin; The maximum deficit after ruin.

1 Introduction

Consider the classical compound Poisson risk model, in which the insurer's surplus process $\{U(t);\, t \geq 0\}$ is given by

$$U(t) = u + ct - S(t)\,, \qquad t \geq 0\,,$$

where c denotes the constant premium income rate per unit of time and $u(\geq 0)$ is the initial surplus. The counting process $\{N(t);\, t \geq 0\}$ is assumed a Poisson process with parameter λ, representing the number of claims that have occurred before time t, and

$$S(t) = \sum_{n=1}^{N(t)} X_n\,, \qquad t \geq 0\,,$$

is the aggregate claim amounts up to time t, where the sequence of random claim amounts $\{X_n;\, n \geq 0\}$ is assumed to have a common probability density function $p(x)$, for $x \geq 0$. Suppose further that $\{X_n;\, n \geq 0\}$ is also independent of $\{N(t);\, t \geq 0\}$ and $c > \lambda\mu$ to have a positive loading condition with $\mu = \mathbb{E}[X_1]$.

Assume that the insurer receives credit interest continuously at a constant force of interest $\delta > 0$ per unit of time while its surplus is positive, and borrows money which is equal to the deficit continuously at a constant force of debit interest $r > 0$ per unit of time while its surplus is negative. Meanwhile, the insurer repays the debts continuously from its premium income. Then the corresponding surplus process, denoted by $\{U_{\delta,r}(t);\, t \geq 0\}$, can be described as

$$dU_{\delta,r}(t) = \begin{cases} (c + \delta U_{\delta,r}(t))dt - dS(t)\,, & U_{\delta,r}(t) > 0 \\ (c + rU_{\delta,r}(t))dt - dS(t)\,, & U_{\delta,r}(t) < 0 \end{cases}. \tag{1.1}$$

We further reasonably assume that $\delta \leq r$. When $\delta = r$, we call (1.1) the classical risk model with constant interest, and when $\delta = 0$ and $r > 0$, we call (1.1) the classical risk model with debit interest.

The risk model (1.1) has attracted a fair amount of attention in the literature. Among other quantities studied, the absolute ruin problem is of particular interest to the researchers. The so-called absolute ruin probability is the probability when the surplus becomes less than $-c/r$. Note that when the surplus level becomes less than $-c/r$, the surplus will never become positive again and thus we call that the absolute ruin occurs. The classical risk model with constant interest ($\delta = r$) was first treated in 1940s; see, Sundt and Teugels (1995) for some earlier references in the literature. For example, Gerber (1971) studied the absolute ruin probability, and Sundt and Teugels (1995) investigated the probability of ruin, namely, the probability that the surplus becomes negative. The ruin probability when the initial surplus is zero was obtained and the explicit expression of the ruin probability was derived when claims are exponentially distributed.

Dassios and Embrechts (1989) analyzed some insurance risk processes including (1.1) by using the martingale approach along with the theory of piecewise-deterministic Markov processes (PDMP); the ruin probability when $\delta = r$, and the absolute ruin probability when $\delta = 0$ were derived for exponential distributed claim amounts. Embrechts and Schmidli (1994) also used the theory of PDMP to investigate a class of general insurance risk models which includes (1.1) as a special case, and discussed the absolute ruin probability. Dickson and Egídio dos Reis (1997) examined the effect of interest on negative surplus. Recently, Zhu and Yang (2008) studied the asymptotic properties of the absolute ruin probability of model (1.1). In addition, surplus process (1.1) is a special case of the piecewise-deterministic compound Poisson risk model considered in Cai et al. (2009), where the expectation of total discounted operating costs up to the time of default was intensively investigated.

Model (1.1) attracts more attention in recent years since the introduction of the expected discounted penalty function by Gerber and Shiu (1998), which provides a unified approach for studying the ruin-related quantities. Cai and Dickson (2002) studied the expected discounted penalty function at ruin for this risk model with constant interest; an integral equation was derived and the exact solution was obtained when the initial surplus is zero (see, also, Liu and Mao (2006), for a more general result). Cai (2007) considered the expected discounted penalty function at the time of absolute ruin for model (1.1) when $\delta = 0$; the integro-differential equations were derived and the explicit solutions were obtained for exponential claims. See also Yuen et al. (2007), Yang et al. (2008) and Wang et al. (2010) for results in modified surplus processes (1.1) with dividend strategies. Recently, Mitric et al. (2012) further considered a Sparre Andersen risk model with constant interest.

In addition to the classical risk models raised above, it is worth mentioning that Garrido (1989) studied a family of diffusion models for risk reserves which take into account the investment earns and the effect of the inflation. Gerber and Yang (2007) studied absolute ruin probabilities in a compound Poisson insurance risk model perturbed by diffusion with investment.

In the above mentioned literature, one is interested in either the time of ruin, defined by $T_1(u) = \inf\{t \geq 0 : U_{\delta,r}(t) < 0\}$, or the time of absolute ruin, defined by $T_2(u) = \inf\{t \geq 0 : U_{\delta,r}(t) < -c/r\}$, for $u \geq 0$. In this paper, we introduce a new stopping time which on one hand includes the time of ruin and the time of absolute ruin as special cases, and on the other hand can be used to study other ruin-related quantities such as the maximum severity of ruin. For this purpose, we define

$$T_{u,z} = \inf\{t \geq 0 : U_{\delta,r}(t) < z\}, \qquad -\frac{c}{\delta} \leq z < 0,\ z \leq u,$$

to be the first time that the surplus process drops below z from level $u(\geq z)$ with $T_{u,z} = \infty$ if $U_{r,\delta}(t) \geq z, \forall t \geq 0$.

Now define

$$\psi_z(u) = \mathbb{P}\big(T_{u,z} < \infty\big)\,, \qquad -\frac{c}{r} \leq z \leq 0,\, z \leq u \tag{1.2}$$

to be the probability that the surplus will ever drop below z and then $\phi_z(u) = 1-\psi_z(u)$ is the probability that the surplus never drops below z. Further, define

$$G_z(u;y) = \mathbb{P}\big(z - U_{\delta,r}(T_{u,z}) \leq y, T_{u,z} < \infty\big)\,, \qquad -\frac{c}{r} \leq z \leq 0,\, z \leq u,\; y \geq 0 \tag{1.3}$$

to be the distribution function of the amount by which the surplus is below z with $g_z(u,y) = \frac{\partial}{\partial y} G_z(u,y)$ being its density function.

Moreover, we define a corresponding expected discounted penalty (Gerber-Shiu) function at $T_{u,z}$. Let $w(x,y)$, for $x > -c/r$, $y > 0$, be a bivariate non-negative penalty function and define for $r \geq \delta > 0, \alpha \geq 0$, $-(c/r) \leq z \leq 0$, and $z \leq u$

$$\Phi_{\alpha,\delta,r}(u;z) = \mathbb{E}\Big[e^{-\alpha T_{u,z}} w\big(U_\delta(T_{u,z}-), z - U_\delta(T_{u,z})\big) I(T_{u,z} < \infty)\Big]\,, \tag{1.4}$$

where u is the initial surplus. Apparently, $\Phi_{\alpha,\delta,r}(u;z)$ can be seen as a generalized expected discounted penalty function at $T_{u,z}$ for the surplus prior to $T_{u,z}$, and the amount by which the surplus is below level z. In particular, when $u > 0$ and $\delta = r$, $\Phi_{\alpha,\delta,r}(u;0)$ is the usual Gerber-Shiu function studied, for example, in Cai and Dickson (2002), and when $\delta = 0$, $\Phi_{\alpha,\delta,r}(u;-c/r)$ is the Gerber-Shiu function related to the time of absolute ruin considered in Cai (2007). When $\alpha = 0$ and $w(x,y) = 1$, $\Phi_{\alpha,\delta,r}(u;z)$ reduces to $\psi_z(u)$, which has been studied in Zhu and Yang (2008). Finally, we remark that (1.4) can be seen as a special case the quantity defined by (1.4) in Cai et al. (2009) by choosing a specific expression of ϖ in their (1.3). Due to different focuses, most of the results obtained in this paper are not overlapped with theirs.

To provide deep understanding of the newly introduced quantities and to see the links to the quantities that have been studied in the literature, we give the following further remarks.

Remarks:

1. $T_{u,u}$ is the first time that the surplus drops below the initial level u. $T_{u,0}$ is the time of ruin and $T_{u,-c/r}$ is the time of absolute ruin for $u \geq 0$.

2. $\psi_0(u)$ is the probability of ruin for $u \geq 0$, $\psi_{-c/r}(u)$ is the probability of absolute ruin, and $\psi_u(u)$ is the probability that the surplus will ever drop below its initial level $u(< 0)$; they are special cases of Gerber-Shiu function when $\alpha = 0$ and $w(x,y) = 1$, and $z = 0, -c/r$, and u, respectively.

3. For the classical risk model, i.e., $\delta = r = 0$, $T_{u,z}$ has the same distribution as $T_{u-z,0}$ and $\psi_z(u) = \psi_0(u-z)$; however, that is not the case for $r \geq \delta > 0$.

4. $G_0(u; y)$ is the distribution of the deficit at ruin if ruin occurs and $G_{-c/r}(u; y)$ is the distribution of the deficit at absolute ruin if absolute ruin occurs; both are special cases of the Gerber-Shiu function when $\alpha = 0$ and $w(x_1, x_2) = I(x_2 \leq y)$, and $z = 0$ and $z = -c/r$, respectively.

The propose of this paper is to study in general the first time that the surplus process drops below a certain level, which includes the time of ruin and the time of absolute ruin as special cases. As mentioned in remarks above, it is of particular important to explore this random variable for the risk models with interest, because in the classical risk model case it can be studied by homogeneity. The introduction of the generalized Gerber-Shiu function provides a unified approach to analyze quantities related to the time that the surplus process first drops below a certain level, which are more general than those ruin-related ones. With the help of this generalized Gerber-Shiu function, we can obtain the probability of hitting an upper level without dropping below a lower level and the distribution of the maximum severity of ruin.

We aim to provide complementary results to the existing ones for the surplus process (1.1), especially those developed in Cai (2007), Zhu and Yang (2008) and Cai et al. (2009). We want to show that some ruin-related quantities can also be analyzed for the classical risk model with interest as for that without interest. It should be noted that there are additional difficulties in studying the general stopping time other than the time of ruin and the time of absolute ruin. Hence, some results are derived only in the special cases of model (1.1).

In this paper, we first obtain in Section 2 an integro-differential equation for $\Phi_{\alpha,\delta,r}(u, z)$, and then in Section 3 derive an explicit expression for $\Phi_{\alpha,\delta,r}(z; z)$ when $\delta = r$, a quantity that is equally important to the value of usual Gerber-Shiu function for risk models when the initial surplus is zero. When the claim amounts are exponentially distributed and the penalty function depends only on the amount incurred while the surplus drops below level z, in Section 4 the integro-differential equation is completely solved and the explicit expression of $\Phi_{\alpha,\delta,r}(u, z)$ is obtained for models with constant interest. We then are able to express in Section 5 the probability of hitting an upper level without dropping below a lower level in terms of $\psi_z(u)$, a special case of the generalized Gerber-Shiu function, defined in (1.2). Finally, we study in Section 6 the distribution of the maximum severity of ruin; this distribution is fully determined by special cases of the generalized Gerber-Shiu functions.

2 Integro-differential equations for $\Phi_{\alpha,\delta,r}(u;z)$

In this section we derive integro-differential equations satisfied by $\Phi_{\alpha,\delta,r}(u;z)$. For notation convenience, following Zhu and Yang (2008) we define

$$\Phi_{\alpha,\delta,r}(u;z) = \Phi^{+}_{\alpha,\delta,r}(u;z)I(u \geq 0) + \Phi^{-}_{\alpha,\delta,r}(u;z)I(u<0)\,, \qquad -(c/r) \leq z \leq 0, z \leq u\,,$$

and let

$$\begin{aligned} s_{+}(t) &= ue^{\delta t} + c\bar{s}_{\overline{t}|\delta}\,, \qquad u \geq 0\,, \\ s_{-}(t) &= ue^{rt} + c\bar{s}_{\overline{t}|r} I(t < t(u)) + c\bar{s}_{\overline{t}|\delta} I(t \geq t(u))\,, \qquad u < 0\,, \end{aligned}$$

where

$$t(u) = \frac{1}{r}\ln\left(\frac{c}{c+ru}\right)\,, \qquad u<0\,.$$

By conditioning on the time of the first claim, t, and the size of the first claim, x, we get for $-(c/r) \leq z \leq 0 \leq u$,

$$\begin{aligned} \Phi^{+}_{\alpha,\delta,r}(u;z) = \int_0^{\infty} \lambda e^{-(\lambda+\alpha)t} \Bigg[& \int_0^{s_{+}(t)} \Phi^{+}_{\alpha,\delta,r}(s_{+}(t)-x;z)p(x)dx \\ & + \int_{s_{+}(t)}^{s_{+}(t)-z} \Phi^{-}_{\alpha,\delta,r}(s_{+}(t)-x;z)p(x)dx \\ & + \int_{s_{+}(t)-z}^{\infty} w(s_{+}(t), z+x-s_{+}(t))p(x)dx \Bigg] dt\,, \end{aligned} \tag{2.1}$$

and for $-(c/r) \leq z \leq u < 0$,

$$\begin{aligned} \Phi^{-}_{\alpha,\delta,r}(u;z) = \int_0^{t(u)} \lambda e^{-(\lambda+\alpha)t} \Bigg[& \int_0^{s_{-}(t)-z} \Phi^{-}_{\alpha,\delta,r}(s_{-}(t)-x;z)p(x)dx \\ & + \int_{s_{-}(t)-z}^{\infty} w(s_{-}(t), z+x-s_{-}(t))p(x)dx \Bigg] dt\,, \\ + \int_{t(u)}^{\infty} \lambda e^{-(\lambda+\alpha)t} \Bigg[& \int_0^{s_{-}(t)} \Phi^{+}_{\alpha,\delta,r}(s_{-}(t)-x;z)p(x)dx \\ & + \int_{s_{-}(t)}^{s_{-}(t)-z} \Phi^{-}_{\alpha,\delta,r}(s_{-}(t)-x;z)p(x)dx \\ & + \int_{s_{-}(t)-z}^{\infty} w(s_{-}(t), z+x-s_{-}(t))p(x)dx \Bigg] dt\,. \end{aligned} \tag{2.2}$$

Note that the first term in (2.1) corresponds to the case where the amount of the first claim is no more than $s_{+}(t)$ so that the level of the new surplus keeps non-negative, and in the second term the amount of the first claim results in a negative new surplus and it does not drop below z, while the third term corresponds to the case where the surplus drops below level z after the first claim with the amount below z being $z - x + s_+(t)$. The terms in (2.2) can be interpreted analogously.

Substituting $y = s_+(t)$ in (2.1) and $y = s_-(t)$ in (2.2) after a time transformation, we have

$$\Phi^+_{\alpha,\delta,r}(u;z) = \lambda(\delta u + c)^{\frac{\lambda+\alpha}{\delta}} \int_u^\infty (\delta y + c)^{-\frac{\lambda+\alpha}{\delta}-1} \left[\int_0^y \Phi^+_{\alpha,\delta,r}(y-x;z)p(x)dx \right.$$
$$\left. + \int_y^{y-z} \Phi^-_{\alpha,\delta,r}(y-x;z)p(x)dx + \omega(y;z) \right] dy \,, \tag{2.3}$$
$$\Phi^-_{\alpha,\delta,r}(u;z) = \lambda(ru + c)^{\frac{\lambda+\alpha}{r}} \int_u^0 (ry + c)^{-\frac{\lambda+\alpha}{r}-1} \left[\int_0^{y-z} \Phi^-_{\alpha,\delta,r}(y-x;z)p(x)dx + \omega(y;z) \right] dy$$
$$+ \lambda c^{\frac{\lambda+\alpha}{\delta}} e^{-(\lambda+\alpha)t(u)} \int_0^\infty (\delta y + c)^{-\frac{\lambda+\alpha}{\delta}-1} \left[\int_0^y \Phi^+_{\alpha,\delta,r}(y-x;z)p(x)dx \right.$$
$$\left. + \int_y^{y-z} \Phi^-_{\alpha,\delta,r}(y-x;z)p(x)dx + \omega(y;z) \right] dy, \tag{2.4}$$

where

$$\omega(u;z) = \int_{u-z}^\infty w(u, z + x - u)p(x)dx \,. \tag{2.5}$$

Now differentiating (2.3) and (2.4) with respect to u and rearranging yield

$$(\lambda+\alpha)\Phi^+_{\alpha,\delta,r}(u;z) = (\delta u + c)\Phi^{+\,\prime}_{\alpha,\delta,r}(u;z) + \lambda \int_0^u \Phi^+_{\alpha,\delta,r}(u-x;z)p(x)dx$$
$$+ \lambda \int_u^{u-z} \Phi^-_{\alpha,\delta,r}(u-x;z)p(x)dx + \lambda\omega(u;z) \,, \qquad -\frac{c}{r} \le z \le 0 \le u \,, \tag{2.6}$$
$$(\lambda+\alpha)\Phi^-_{\alpha,\delta,r}(u;z) = (ru + c)\Phi^{-\,\prime}_{\alpha,\delta,r}(u;z) + \lambda \int_0^{u-z} \Phi^-_{\alpha,\delta,r}(u-x;z)p(x)dx$$
$$+ \lambda\omega(u;z) \,, \qquad -\frac{c}{r} \le z \le u < 0 \,. \tag{2.7}$$

Let $\delta^*(u) = \delta I(u \ge 0) + rI(u < 0)$. Then equations (2.6) and (2.7) can be combined, for $-(c/r) \le z \le 0$ and $z \le u$, as

$$(\lambda+\alpha)\Phi_{\alpha,\delta,r}(u;z) = (\delta^*(u)u + c)\Phi'_{\alpha,\delta,r}(u;z) + \lambda \int_0^{u-z} \Phi_{\alpha,\delta,r}(u-x;z)p(x)dx + \lambda\omega(u;z) \,. \tag{2.8}$$

Remarks:

1. When $z=0$ and $\delta=r$, equation (2.6) is (2.3) in Cai and Dickson (2002).
2. When $z=0$ and $\delta=r$, $\alpha=0$ and $w(x,y)=1$, $\Phi_{\alpha,\delta,r}(u;z)$ reduces to $\psi_0(u)$. Replacing $\psi_0(u)=1-\phi_0(u)$ in (2.6), the resulting equation on the non-ruin probability $\phi_0(u)$ is (1) in Sundt and Teugels (1995).
3. When $z=-c/r$, $\alpha=0$ and $w(x,y)=1$, $\Phi_{\alpha,\delta,r}(u;z)$ reduces to $\psi_{-c/r}(u)$. Replacing $\psi_{-c/r}(u)=1-\phi_{-c/r}(u)$ in (2.6) and (2.7), the resulting equations on the non-bankruptcy probability $\psi^{+}_{-c/r}(u)$ and $\psi^{-}_{-c/r}(u)$ are (2.3) and (2.4) in Zhu and Yang (2008).
4. Integro-differential equations (2.6) and (2.7) can be obtained by (3.3) in Cai et al. (2009) by setting special functions of g, ϖ and l in their paper.

3 Explicit expression of $\Phi_{\alpha,\delta}(z;z)$

As mentioned in Section 1, our focus is to find the solutions of (2.8) in general and use them to study other ruin-related quantities in Sections 5 and 6. We want to show in this section that our methodology can be used to overcome additional difficulties raised while working on a general stopping time other than the time of ruin and the time of absolute ruin, that applies to the case when $\delta=r$. For notational simplicity, throughout this and subsequent sections we denote surplus process $U_{\delta,r}$ as U_δ and $\Phi_{\alpha,\delta,r}(u;z)$ as $\Phi_{\alpha,\delta}(u;z)$, and we remark that z can be positive such that $z\le u$. Then (2.8) simplifies to

$$(\lambda+\alpha)\Phi_{\alpha,\delta}(u;z)=(\delta u+c)\Phi'_{\alpha,\delta}(u;z)+\lambda\int_0^{u-z}\Phi_{\alpha,\delta}(u-x;z)p(x)dx+\lambda\omega(u;z)\,, \tag{3.1}$$

where $-c/\delta\le z\le u$. Replacing u by t in (3.1) and integrating both sides from z to u with respect to t, we get

$$\begin{aligned}
&(\lambda+\alpha)\int_z^u\Phi_{\alpha,\delta}(t;z)dt\\
&=\int_z^u(\delta t+c)d\Phi_{\alpha,\delta}(t;z)+\lambda\int_z^u\int_0^{t-z}\Phi_{\alpha,\delta}(t-x;z)p(x)dxdt+\lambda\int_z^u\omega(t;z)dt\\
&=(\delta u+c)\Phi_{\alpha,\delta}(u;z)-(\delta z+c)\Phi_{\alpha,\delta}(z;z)-\delta\int_z^u\Phi_{\alpha,\delta}(t;z)dt\\
&\quad+\lambda\int_0^{u-z}p(x)\left(\int_z^{u-x}\Phi_{\alpha,\delta}(y;z)dy\right)dx+\lambda\int_z^u\omega(t;z)dt
\end{aligned}$$

which can be further rewritten as

$$(\delta u+c)\Phi_{\alpha,\delta}(u;z)=(\delta z+c)\Phi_{\alpha,\delta}(z;z)-\lambda\int_z^u \omega(t;z)dt+\int_z^u [\delta+\alpha+\lambda]\,\Phi_{\alpha,\delta}(t;z)dt$$
$$-\lambda\int_0^{u-z} p(x)\left(\int_z^{u-x}\Phi_{\alpha,\delta}(y;z)dy\right)dx\,. \tag{3.2}$$

Note that the quantity $\Phi_{\alpha,\delta}(z;z)$ is essential for the expression of $\Phi_{\alpha,\delta}(u;z)$. In the section below, we obtain an explicit expression for $\Phi_{\alpha,\delta}(z;z)$.

Similar to the method in Sundt and Teugels (1995), for $z>-c/\delta$, we define an auxiliary function

$$Y_{\alpha,\delta}(u;z)=\begin{cases}\frac{\Phi_{\alpha,\delta}(z;z)-\Phi_{\alpha,\delta}(u;z)}{\Phi_{\alpha,\delta}(z;z)}\,, & u\geq z\\ 0\,, & u<z\end{cases}\,,$$

which can be rewritten as

$$\Phi_{\alpha,\delta}(u;z)=\Phi_{\alpha,\delta}(z;z)-\Phi_{\alpha,\delta}(z;z)Y_{\alpha,\delta}(u;z)\,. \tag{3.3}$$

Clearly, $Y_{\alpha,\delta}(z;z)=0$. In the sequel, we assume that $\lim_{u\to\infty}\Phi_{\alpha,\delta}(u;z)=0$ which implies $\lim_{u\to\infty}Y_{\alpha,\delta}(u;z)=1$. A sufficient condition for $\lim_{u\to\infty}\Phi_{\alpha,\delta}(u;z)=0$ is that w is bounded, i.e., $w(x,y)\leq L$, for some $L>0$. To show this, we note that

$$\begin{aligned}&\lim_{u\to\infty}\Phi_{\alpha,\delta}(u;z)\leq L\lim_{u\to\infty}\mathbb{E}\big[e^{-\alpha T_{u,z}}I(T_{u,z}<\infty)\big]\\ &\leq L\lim_{u\to\infty}\mathbb{P}(T_{u,z}<\infty)=L\lim_{u\to\infty}\psi_z(u)=0.\end{aligned}$$

Substituting (3.3) into (3.2), we obtain

$$\begin{aligned}&(\delta u+c)\left[\Phi_{\alpha,\delta}(z;z)-\Phi_{\alpha,\delta}(z;z)Y_{\alpha,\delta}(u;z)\right]\\ &\quad=(\delta z+c)\Phi_{\alpha,\delta}(z;z)-\lambda\int_z^u\omega(t;z)dt\\ &\qquad+(\delta+\alpha+\lambda)\int_z^u\left[\Phi_{\alpha,\delta}(z;z)-\Phi_{\alpha,\delta}(z;z)Y_{\alpha,\delta}(t;z)\right]dt\\ &\qquad-\lambda\int_0^{u-z}p(x)\left(\int_z^{u-x}\left[\Phi_{\alpha,\delta}(z;z)-\Phi_{\alpha,\delta}(z;z)Y_{\alpha,\delta}(y;z)\right]dy\right)dx\,,\end{aligned}$$

and it can be further rearranged, for $u>z$, as

$$\begin{aligned}(\delta u+c)Y_{\alpha,\delta}(u;z)=&(\delta+\alpha+\lambda)\int_z^u Y_{\alpha,\delta}(t;z)dt-(\alpha+\lambda)(u-z)\\ &+\lambda\int_0^{u-z}(u-z-x)p(x)dx+\frac{\lambda}{\Phi_{\alpha,\delta}(z;z)}\int_z^u\omega(t;z)dt\\ &-\lambda\int_0^{u-z}p(x)\left(\int_z^{u-x}Y_{\alpha,\delta}(y;z)dy\right)dx\,.\end{aligned} \tag{3.4}$$

Let $\tilde{y}_{\alpha,\delta}(s;z)$ be the Laplace transform of $Y_{\alpha,\delta}(u;z)$ with respect to u on the entire real line, defined by

$$\tilde{y}_{\alpha,\delta}(s;z) = \int_{-\infty}^{\infty} e^{-su} Y_{\alpha,\delta}(u;z)du = \int_{z}^{\infty} e^{-su} Y_{\alpha,\delta}(u;z)du \,,$$

and $\tilde{p}(s)$ and $\tilde{\omega}(r;z)$ be the corresponding Laplace transforms of $p(x)$ and $\omega(x;z)$. Now, taking the Laplace transforms of both sides of (3.4) with respect to u, we get

$$\begin{aligned} -\delta\tilde{y}'_{\alpha,\delta}(s;z) + c\tilde{y}_{\alpha,\delta}(s;z) &= \frac{\delta+\alpha+\lambda}{s}\tilde{y}_{\alpha,\delta}(s;z) - \frac{\alpha+\lambda}{s^2}e^{-sz} + \frac{\lambda}{s^2}e^{-sz}\tilde{p}(s) \\ &\quad + \frac{\lambda}{\Phi_{\alpha,\delta}(z;z)}\frac{\tilde{\omega}(s;z)}{s} - \frac{\lambda}{s}\tilde{p}(s)\tilde{y}_{\alpha,\delta}(s;z)\,. \end{aligned} \tag{3.5}$$

Equation (3.5) can be rearranged as

$$\begin{aligned} &\delta\tilde{y}'_{\alpha,\delta}(s;z) + \left(\frac{\delta+\alpha+\lambda[1-\tilde{p}(s)]}{s} - c\right)\tilde{y}_{\alpha,\delta}(s;z) \\ &= \frac{\alpha+\lambda[1-\tilde{p}(s)]}{s^2}e^{-zs} - \frac{\lambda}{\Phi_{\alpha,\delta}(z;z)}\frac{\tilde{\omega}(s;z)}{s}\,. \end{aligned} \tag{3.6}$$

Equation (3.6) is a first-order linear ordinary differential equation for $\tilde{y}_{\alpha,\delta}(s;z)$. It is not difficult to see that

$$\begin{aligned} &\frac{d}{ds}\left[s^{\frac{\delta+\alpha}{\delta}}\tilde{y}_{\alpha,\delta}(s;z)e^{-\frac{1}{\delta}\int_0^s\left(c-\frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt}\right] \\ &= \frac{1}{\delta}s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{1}{\delta}\int_0^s\left(c-\frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt}\left(\frac{\alpha+\lambda[1-\tilde{p}(s)]}{s^2}e^{-zs} - \frac{\lambda}{\Phi_{\alpha,\delta}(z;z)}\frac{\tilde{\omega}(s;z)}{s}\right)\,. \end{aligned} \tag{3.7}$$

Now, since $\lim_{s\to\infty}\tilde{y}_{\alpha,\delta}(s;z) = 0$ and

$$0 < s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{1}{\delta}\int_0^s\left(c-\frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt} \le s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{1}{\delta}\int_0^s(c-\lambda\mu)dt} = \frac{s^{\frac{\delta+\alpha}{\delta}}}{e^{\frac{1}{\delta}(c-\lambda\mu)s}},$$

we have

$$\lim_{s\to\infty} s^{\frac{\delta+\alpha}{\delta}}\tilde{y}_{\alpha,\delta}(s;z)e^{-\frac{1}{\delta}\int_0^s\left(c-\frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt} = 0.$$

Then integrating both sides of (3.7) from s to ∞ yields

$$\begin{aligned} &s^{\frac{\delta+\alpha}{\delta}}\tilde{y}_{\alpha,\delta}(s;z)e^{-\frac{1}{\delta}\int_0^s\left(c-\frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt} \\ &= \frac{1}{\delta}\int_s^{\infty} v^{\frac{\delta+\alpha}{\delta}}e^{-\frac{1}{\delta}\int_0^v\left(c-\frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt}\left(\frac{\lambda}{\Phi_{\alpha,\delta}(z;z)}\frac{\tilde{\omega}(v;z)}{v} - \frac{\alpha+\lambda[1-\tilde{p}(v)]}{v^2}e^{-vz}\right)dv\,. \end{aligned} \tag{3.8}$$

Let $s = 0$ in (3.8), implying

$$0 = \frac{1}{\delta}\int_0^\infty v^{\frac{\delta+\alpha}{\delta}} e^{-\frac{1}{\delta}\int_0^v \left(c - \frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt} \left(\frac{\lambda}{\Phi_{\alpha,\delta}(z;z)} \frac{\tilde{\omega}(v;z)}{v} - \frac{\alpha+\lambda[1-\tilde{p}(v)]}{v^2} e^{-vz} \right) dv \, ,$$

and from which we obtain the following explicit expression for $\Phi_{\alpha,\delta}(z;z)$:

$$\begin{aligned}
\Phi_{\alpha,\delta}(z;z) &= \frac{\lambda \int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\frac{1}{\delta}\int_0^v \left(c - \frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt} \tilde{\omega}(v;z)dv}{\int_0^\infty v^{\frac{\alpha}{\delta}-1} e^{-\frac{1}{\delta}\int_0^v \left(c - \frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt} \left(\alpha + \lambda[1-\tilde{p}(v)]\right) e^{-vz} dv} \\
&= \frac{\lambda \int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\frac{1}{\delta}\left(cv - \lambda \int_0^v \frac{1-\tilde{p}(t)}{t}dt\right)} \tilde{\omega}(v;z)dv}{\int_0^\infty v^{\frac{\alpha}{\delta}-1} e^{-\left(z+\frac{c}{\delta}\right)v + \frac{\lambda}{\delta}\int_0^v \frac{1-\tilde{p}(t)}{t}dt} \left(\alpha + \lambda[1-\tilde{p}(v)]\right) dv} \, , \qquad z > -\frac{c}{\delta} \, . \qquad (3.9)
\end{aligned}$$

Note that from (3.8), we can obtain an expression of $\tilde{y}_{\alpha,\delta}(s;z)$:

$$\tilde{y}_{\alpha,\delta}(s;z) = \frac{1}{\delta s}\int_s^\infty \left(\frac{v}{s}\right)^{\frac{\alpha}{\delta}} e^{-\frac{1}{\delta}\int_s^v \left(c - \frac{\lambda[1-\tilde{p}(t)]}{t}\right)dt} \left(\frac{\lambda \tilde{\omega}(v;z)}{\Phi_{\alpha,\delta}(z;z)} - \frac{\alpha+\lambda[1-\tilde{p}(v)]}{v} e^{-zv} \right) dv \, , \qquad (3.10)$$

with $\Phi_{\alpha,\delta}(z;z)$ given by (3.9). It is in general very difficult to find the Laplace inversion of $\tilde{y}_{\alpha,\delta}(s;z)$ from (3.10). In Section 4, we discuss the case when claims are exponentially distributed, where an explicit expression of $\Phi_{\alpha,\delta}(u;z)$ in term of some special functions is obtained.

3.1 Some special cases

We now illustrate some special cases of expression (3.9).

1. When $z = 0$, (3.9) reduces to

$$\Phi_{\alpha,\delta}(0;0) = \frac{\lambda \int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\frac{1}{\delta}\left(cv - \lambda \int_0^v \frac{1-\tilde{p}(t)}{t}dt\right)} \tilde{\omega}(v;0)dv}{\int_0^\infty v^{\frac{\alpha}{\delta}-1} e^{-\frac{c}{\delta}v + \frac{\lambda}{\delta}\int_0^v \frac{1-\tilde{p}(t)}{t}dt} \left(\alpha + \lambda[1-\tilde{p}(v)]\right) dv} \, , \qquad (3.11)$$

   which is the expression for the Gerber-Shiu function at zero in the classical risk model with constant interest force. Note that (3.11) is (3.6) in Liu and Mao (2006).

2. When $\alpha = 0$ and $z = 0$, (3.9) simplifies to

$$\Phi_{0,\delta}(0;0) = \frac{\int_0^\infty e^{-\frac{1}{\delta}\left(cv - \lambda \int_0^v \frac{1-\tilde{p}(t)}{t}dt\right)} \tilde{\omega}(v;0)dv}{\int_0^\infty v^{-1} e^{-\frac{c}{\delta}v + \frac{\lambda}{\delta}\int_0^v \frac{1-\tilde{p}(t)}{t}dt} \left[1-\tilde{p}(v)\right] dv} \, ,$$

   which can be verified after few variable changes being the same as (3.8), derived in Cai and Dickson (2002).

3. Consider that claim amounts are exponentially distributed with $p(x) = \beta e^{-\beta x}$, and $w(x, y) = w(y)$. In this case, $\tilde{p}(s) = \beta/(s + \beta)$, and the Laplace transform of function ω in (2.5) is

$$\tilde{\omega}(s; z) = \int_z^\infty e^{-su} \left[\int_{u-z}^\infty w(z + x - u)\beta e^{-\beta x} dx \right] du = \frac{\beta \tilde{w}(\beta)}{\beta + s} e^{-zs},$$

where $\tilde{w}(s)$ is the Laplace transform of $w(y)$. Then (3.9) becomes to

$$\begin{aligned} \Phi_{\alpha,\delta}(z; z) &= \frac{\lambda \int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\frac{1}{\delta}\left(cv - \lambda \int_0^v \frac{1-\tilde{p}(t)}{t} dt\right)} e^{-zv} \frac{\beta \tilde{w}(\beta)}{\beta + v} dv}{\int_0^\infty v^{\frac{\alpha}{\delta}-1} e^{-\left(z+\frac{c}{\delta}\right)v + \frac{\lambda}{\delta} \int_0^v \frac{1-\tilde{p}(t)}{t} dt} \left(\alpha + \frac{\lambda v}{\beta + v}\right) dv} \\ &= \frac{\lambda \tilde{w}(\beta) \int_0^\infty v^{\frac{\alpha}{\delta}} \left(\frac{\beta+v}{\beta}\right)^{\frac{\lambda}{\delta}-1} e^{-\left(z+\frac{c}{\delta}\right)v} dv}{\int_0^\infty v^{\frac{\alpha}{\delta}-1} \left(\frac{\beta+v}{\beta}\right)^{\frac{\lambda}{\delta}} e^{-\left(z+\frac{c}{\delta}\right)v} \left(\alpha + \frac{\lambda v}{\beta+v}\right) dv}, \qquad z > -\frac{c}{\delta}. \qquad (3.12) \end{aligned}$$

By using the confluent hypergeometric function of the second kind $U(a, b, x)$, defined by

$$U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1 + t)^{b-a-1} dt, \qquad (3.13)$$

and the relationship (see Abramowitz and Stegun (1970, 13.2.5 and 13.4.18)) that

$$xU(a, b + 1, x) = (b - a)U(a, b, x) + U(a - 1, b, x),$$

it is not difficult to see that (3.12) can be expressed as

$$\begin{aligned} \Phi_{\alpha,\delta}(z; z) &= \frac{\frac{\lambda}{\delta} \beta \tilde{w}(\beta) U\left(1 + \frac{\alpha}{\delta}, 1 + \frac{\lambda+\alpha}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right)}{U\left(\frac{\alpha}{\delta}, 1 + \frac{\lambda+\alpha}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right) + \frac{\lambda}{\delta} U\left(1 + \frac{\alpha}{\delta}, 1 + \frac{\lambda+\alpha}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right)} \\ &= \frac{\lambda \tilde{w}(\beta)}{\delta\left(z + \frac{c}{\delta}\right)} \frac{U\left(1 + \frac{\alpha}{\delta}, 1 + \frac{\lambda+\alpha}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right)}{U\left(1 + \frac{\alpha}{\delta}, 2 + \frac{\lambda+\alpha}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right)}, \qquad z > -\frac{c}{\delta}. \end{aligned}$$

3.2 The moment of $T_{z,z}$

In this subsection, we study a special case of $\Phi_{\alpha,\delta}(z; z)$ when $w(x, y) = 1$. Recall that $T_{z,z}$ is the first time that the surplus process drops below level z, starting from the same level z. When z is the initial surplus, $T_{z,z}$ is the first time that the surplus process drops below its initial level. Hence, with $w(x, y) = 1$, $\Phi_{\alpha,\delta}(z; z)$ is the Laplace transform of the duration that the surplus is above its initial level z with parameter α. Let P_e be the equilibrium distribution of P, namely,

$$P_e(x) = \frac{1}{\mu}\int_0^x \bar{P}(y)dy\,,$$

where $p_e(x) = \bar{P}(x)/\mu$ its probability density function with $\bar{P}(y) = 1 - P(y) = \int_y^\infty p(x)dx$. In this case, we have $\tilde{p}_e(s) = [1-\tilde{p}(s)]/(\mu s)$ and $\tilde{\omega}(s;z) = \mu e^{-zs}\tilde{p}_e(s)$. Now $\Phi_{\alpha,\delta}(z;z)$ in (3.9) can be re-expressed as

$$\Phi_{\alpha,\delta}(z;z) = \frac{\lambda\mu\int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda\mu}{\delta}\int_0^v \tilde{p}_e(t)dt}\tilde{p}_e(v)dv}{\int_0^\infty v^{\frac{\alpha}{\delta}-1} e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda}{\delta}\int_0^v \tilde{p}_e(t)dt}\left[\alpha+\lambda\mu v\tilde{p}_e(v)\right]dv}\,. \tag{3.14}$$

Furthermore, we let

$$\pi\left(z;\frac{\alpha}{\delta}\right) = \int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda\mu}{\delta}\int_0^v \tilde{p}_e(t)dt}dv\,. \tag{3.15}$$

By the integral by parts, we obtain the following equation:

$$\frac{\lambda\mu}{\delta}\int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda\mu}{\delta}\int_0^v \tilde{p}_e(t)dt}\tilde{p}_e(v)dv = \left(z+\frac{c}{\delta}\right)\pi\left(z;\frac{\alpha}{\delta}\right) - \frac{\alpha}{\delta}\pi\left(z;\frac{\alpha}{\delta}-1\right)\,. \tag{3.16}$$

Using the notation (3.15) and the relationship (3.16), (3.14) can be written as

$$\begin{aligned}\Phi_{\alpha,\delta}(z;z) &= \frac{\lambda\mu\int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda\mu}{\delta}\int_0^v \tilde{p}_e(t)dt}\tilde{p}_e(v)dv}{\alpha\pi\left(z;\frac{\alpha}{\delta}-1\right)+\lambda\mu\int_0^\infty v^{\frac{\alpha}{\delta}} e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda\mu}{\delta}\int_0^v \tilde{p}_e(t)dt}\tilde{p}_e(v)dv} \\ &= 1 - \frac{\alpha}{c+\delta z}\,\frac{\pi\left(z;\frac{\alpha}{\delta}-1\right)}{\pi\left(z;\frac{\alpha}{\delta}\right)}\,. \end{aligned} \tag{3.17}$$

From the Laplace transform expression (3.17), we can further get the expected time that the surplus process stays above level z, starting from its initial level z, $\mathbb{E}[T_{z,z}]$, which is given by

$$\begin{aligned}\mathbb{E}[T_{z,z}] &= -\frac{\partial}{\partial\alpha}\Phi_{\alpha,\delta}(z;z)\Big|_{\alpha=0} = \frac{1}{c+\delta z}\,\frac{\pi\left(z;\frac{\alpha}{\delta}-1\right)}{\pi\left(z;\frac{\alpha}{\delta}\right)}\Big|_{\alpha=0} \\ &= \frac{1}{c+\delta z}\,\frac{\int_0^\infty v^{-1}e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda\mu}{\delta}\int_0^v \tilde{p}_e(t)dt}dv}{\int_0^\infty e^{-\left(z+\frac{c}{\delta}\right)v+\frac{\lambda\mu}{\delta}\int_0^v \tilde{p}_e(t)dt}dv}\,, \qquad z > -\frac{c}{\delta}\,. \end{aligned} \tag{3.18}$$

When claim amounts are exponentially distributed with $p(x) = \beta e^{-\beta x}$, by the integral definition of function $U(a, b; x)$ given in (3.13), we can write $\pi(z; \alpha/\delta)$ as

$$\pi\left(z; \frac{\alpha}{\delta}\right) = \beta^{\frac{\alpha}{\delta}+1}\Gamma\left(1 + \frac{\alpha}{\delta}\right) U\left(1 + \frac{\alpha}{\delta}, 2 + \frac{\lambda + \alpha}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right),$$

and hence (3.18) gives

$$\mathbb{E}[T_{z,z}] = \frac{1}{\delta[\beta\left(z + \frac{c}{\delta}\right)]} \frac{U\left(0, 1 + \frac{\lambda}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right)}{U\left(1, 2 + \frac{\lambda}{\delta}, \beta\left(z + \frac{c}{\delta}\right)\right)}, \qquad z > -\frac{c}{\delta}.$$

4 Exponentially distributed claims

Now we consider a special case when $p(x) = \beta e^{-\beta x}$. Then, with a variable change of $y = u - x$ for the integral in (2.6), it becomes

$$(\lambda + \alpha)\Phi_{\alpha,\delta}(u; z) = (\delta u + c)\Phi'_{\alpha,\delta}(u; z) + \lambda \int_z^u \Phi_{\alpha,\delta}(y, z)\beta e^{-\beta(u-y)}dy + \lambda\omega(u; z). \tag{4.1}$$

Differentiating (4.1) with respect to u gives

$$(\lambda + \alpha - \delta)\Phi'_{\alpha,\delta}(u; z) = (\delta u + c)\Phi''_{\alpha,\delta}(u; z) - \lambda \int_z^u \Phi_{\alpha,\delta}(y, z)\beta^2 e^{-\beta(u-y)}dy + \lambda\beta\Phi_{\alpha,\delta}(u; z) + \lambda\omega'(u; z). \tag{4.2}$$

Putting the integral expression in (3.2) to (4.2), we further obtain

$$[\lambda + \alpha - \delta - \beta(\delta u + c)]\Phi'_{\alpha,\delta}(u; z) = (\delta u + c)\Phi''_{\alpha,\delta}(u; z) - \alpha\beta\Phi_{\alpha,\delta}(u; z) + \lambda[\omega'(u; z) + \beta\omega(u; z)], \tag{4.3}$$

with the boundary condition $\Phi_{\alpha,\delta}(\infty, z) = 0$ which was assumed in Section 3. Equation (4.3) is a non-homogeneous second order linear differential equation whose coefficients a linear function of the independent variable.

Consider the case where $w(x, y) = w(y)$, that is, the penalty function depends only on the "deficit" below level z when the surplus dropping below z occurs. In this case, it can be easily verified that $\omega'(u; z) + \beta\omega(u; z) = 0$, and then (4.3) reduces to the following homogeneous second order linear differential equation:

$$[\lambda + \alpha - \delta - \beta(\delta u + c)]\Phi'_{\alpha,\delta}(u; z) = (\delta u + c)\Phi''_{\alpha,\delta}(u; z) - \alpha\beta\Phi_{\alpha,\delta}(u; z). \tag{4.4}$$

Let $t = \beta(u + c/\delta)$ and $Q_{\alpha,\delta}(t; z) = e^t \Phi_{\alpha,\delta}(t/\beta - c/\delta; z)$, and let $\gamma_1 = \lambda/\delta$, $\gamma_2 = \alpha/\delta$, and $\gamma_3 = (\lambda + \alpha)/\delta$ for notation convenience. From (4.4), we can get that $Q_{\alpha,\delta}(t)$ satisfies

$$tQ''_{\alpha,\delta}(t; z) + (1 - \gamma_3 - t)\, Q'_{\alpha,\delta}(t; z) - (1 - \gamma_1)\, Q_{\alpha,\delta}(t; z) = 0\,,$$

which is a standard Kummer's equation. The solution to Kummer's equation can be expressed in term of the confluent hypergeometric functions; see, for example, Abramowitz and Stegun (1970). In fact,

$$Q_{\alpha,\delta}(t; z) = t^{\gamma_3} \left[C_1(z)M(1 + \gamma_2, 1 + \gamma_3, t) + C_2(z)U(1 + \gamma_2, 1 + \gamma_3, t)\right]\,, \tag{4.5}$$

where $C_1(z)$ and $C_2(z)$ are two constants (or functions of parameter z) to be determined, and $M(a, b; t)$ and $U(a, b; t)$ are confluent hypergeometric functions of the first and second kind, respectively.

Recall that $\Phi_{\alpha,\delta}(u; z) = e^{-\beta(u+c/\delta)} Q_{\alpha,\delta}(\beta(u+c/\delta); z)$. Then by (4.5) the solution to (4.4) can be expressed as

$$\begin{aligned}\Phi_{\alpha,\delta}(u; z) = &\left(u + \frac{c}{\delta}\right)^{\gamma_3} e^{-\beta\left(u+\frac{c}{\delta}\right)} \left[C_1^*(z)M\left(1 + \gamma_2, 1 + \gamma_3, \beta\left(u + \frac{c}{\delta}\right)\right)\right.\\ &\left.+C_2^*(z)U\left(1 + \gamma_2, 1 + \gamma_3, \beta\left(u + \frac{c}{\delta}\right)\right)\right]\,,\end{aligned} \tag{4.6}$$

where $C_i^*(z) = \beta^{\gamma_3} C_i(z)$, for $i = 1, 2$.

For $\alpha > 0$, by properties in Abramowitz and Stegun (1970, page 504, 13.1.4. and 13.1.8), we observe, as $u \to \infty$, that

$$\begin{aligned}&\left(u + \frac{c}{\delta}\right)^{\gamma_3} e^{-\beta\left(u+\frac{c}{\delta}\right)} M\left(1 + \gamma_2, 1 + \gamma_3, \beta\left(u + \frac{c}{\delta}\right)\right)\\ &\qquad \sim \left(u + \frac{c}{\delta}\right)^{\gamma_2} \left[1 + O\left(\left|u + \frac{c}{\delta}\right|^{-1}\right)\right] \to \infty\\ &\left(u + \frac{c}{\delta}\right)^{\gamma_3} e^{-\beta\left(u+\frac{c}{\delta}\right)} U\left(1 + \gamma_2, 1 + \gamma_3, \beta\left(u + \frac{c}{\delta}\right)\right)\\ &\qquad \sim \left(u + \frac{c}{\delta}\right)^{\gamma_1 - 1} e^{-\beta\left(u+\frac{c}{\delta}\right)} \left[1 + O\left(\left|u + \frac{c}{\delta}\right|^{-1}\right)\right] \to 0\,,\end{aligned}$$

implying immediately that $C_1^*(z) = 0$ by the boundary condition $\Phi_{\alpha,\delta}(\infty; z) = 0$. Furthermore, letting $u = z$ in (4.6), and using the expression of $\Phi_{\alpha,\delta}(z; z)$ from (3.12), we find the expression of $C_2^*(z)$, with $z_1 = z + c/\delta$, as

$$C_2^*(z) = \frac{z_1^{-\gamma_3} e^{\beta z_1} \Phi_{\alpha,\delta}(z; z)}{U\left(1 + \gamma_2, 1 + \gamma_3, \beta z_1\right)} = \frac{\gamma_1 \tilde{w}(\beta) z_1^{-\gamma_3 - 1} e^{\beta z_1}}{U\left(1 + \gamma_2, 2 + \gamma_3, \beta z_1\right)}\,.$$

Finally, we obtain from (4.6) that

$$\Phi_{\alpha,\delta}(u;z) = \frac{\gamma_1 \tilde{w}(\beta)}{z+\frac{c}{\delta}} \left(\frac{u+\frac{c}{\delta}}{z+\frac{c}{\delta}}\right)^{\gamma_3} e^{-\beta(u-z)} \frac{U\left(1+\gamma_2, 1+\gamma_3, \beta\left(u+\frac{c}{\delta}\right)\right)}{U\left(1+\gamma_2, 2+\gamma_3, \beta\left(z+\frac{c}{\delta}\right)\right)}, \tag{4.7}$$

which is an explicit expression of $\Phi_{\alpha,\delta}(u;z)$ when the claim amounts are exponentially distributed and the penalty function $w(x,y) = w(y)$.

In the following, we consider some special cases.

1. When $\alpha = 0$, we have $\gamma_1 = \gamma_3 = \lambda/\delta$ and $\gamma_2 = 0$. Then (4.7) simplifies to

$$\Phi_{0,\delta}(u;z) = \frac{\gamma_1 \tilde{w}(\beta)}{z+\frac{c}{\delta}} \left(\frac{u+\frac{c}{\delta}}{z+\frac{c}{\delta}}\right)^{\gamma_1} e^{-\beta(u-z)} \frac{U\left(1, 1+\gamma_1, \beta\left(u+\frac{c}{\delta}\right)\right)}{U\left(1, 2+\gamma_1, \beta\left(z+\frac{c}{\delta}\right)\right)}. \tag{4.8}$$

   Using the relationship $U(a,b,z) = x^{1-b}U(1+a-b, 2-b, x)$ and the relationship with the incomplete Gamma function

$$\Gamma(a,x) = e^{-x}U(1-a, 1-a, x) = x^a e^{-x} U(1, 1+a, x),$$

   $\Phi_{0,\delta}(u;z)$ in (4.8) can be further simplified to

$$\Phi_{0,\delta}(u;z) = \gamma_1 \beta \tilde{w}(\beta) \frac{\Gamma\left(\gamma_1, \beta\left(u+\frac{c}{\delta}\right)\right)}{\Gamma\left(1+\gamma_1, \beta\left(z+\frac{c}{\delta}\right)\right)} = \frac{\lambda\beta}{\delta}\tilde{w}(\beta) \frac{\Gamma\left(\frac{\lambda}{\delta}, \beta\left(u+\frac{c}{\delta}\right)\right)}{\Gamma\left(1+\frac{\lambda}{\delta}, \beta\left(z+\frac{c}{\delta}\right)\right)}. \tag{4.9}$$

2. With $\alpha = 0$ and $w(x,y) = 1$, $\Phi_{0,\delta}(u;z)$ reduces to $\psi_z(u)$ and hence (4.9) gives

$$\psi_z(u) = \frac{\lambda}{\delta} \frac{\Gamma\left(\frac{\lambda}{\delta}, \beta\left(u+\frac{c}{\delta}\right)\right)}{\Gamma\left(1+\frac{\lambda}{\delta}, \beta\left(z+\frac{c}{\delta}\right)\right)}, \qquad u \geq 0. \tag{4.10}$$

3. When $\alpha = 0$, $w(y) = 1$, and $z = 0$, $\Phi_{0,\delta}(u;0)$ is the ruin probability in the classical risk model with constant force of interest. Using relationship $\Gamma(a+1,x) = a\Gamma(a,x) + x^a e^{-x}$, (4.9) is equivalent to

$$\Phi_{0,\delta}(u;0) = \frac{\Gamma\left(\frac{\lambda}{\delta}, \beta\left(u+\frac{c}{\delta}\right)\right)}{\Gamma\left(1+\frac{\lambda}{\delta}, \frac{c\beta}{\delta}\right) + \frac{\delta}{\lambda}\left(\frac{c\beta}{\delta}\right)^{\frac{\lambda}{\delta}} e^{-\frac{c\beta}{\delta}}}, \qquad u \geq 0,$$

   which is given in Sundt and Teugels (1995).

4. When $\alpha = 0$, $w(y) = 1$, and $z \downarrow -c/\delta$, $\Phi_{0,\delta}(u;-c/\delta)$ is the absolute ruin probability in the classical risk model with constant force of interest. From (4.7) we obtain

$$\Phi_{0,\delta}\left(u; -\frac{c}{\delta}\right) = \frac{\lambda}{\delta} \frac{\Gamma\left(\frac{\lambda}{\delta}, \beta\left(u+\frac{c}{\delta}\right)\right)}{\Gamma\left(1+\frac{\lambda}{\delta}\right)}, \qquad u \geq 0.$$

5 The probability of hitting an upper level without dropping below a lower level

For $-\frac{c}{\delta} \le z < u < b$, define

$$T_u^b = \inf\{t \ge 0 : U_\delta(t) = b\}$$

to be the first time that the surplus process reaches b from u and define

$$\chi_\delta(u; b, z) = \mathbb{P}(T_u^b < T_{u,z})$$

to be the probability that the surplus hits an upper level b from level u without having dropped below a lower level z. Then $\xi_\delta(u; b, z) = \mathbb{P}(T_{u,z} < T_u^b) = 1 - \chi_\delta(u; b, z)$ is the probability that the surplus process drops below z from $u\,(u > z)$ without reaching level b. Apparently, we have $\chi_\delta(b; b, z) = 1$.

If the surplus never drop bellow z from the initial level u, the surplus process must pass through the level $b(> u)$ at some time point due to the positive loading condition, and the surplus process never drops below z from new level b. Therefore, $1 - \psi_z(u) = \chi_\delta(u, b, z)[1 - \psi_z(b)]$, or,

$$\chi_\delta(u, b, z) = \frac{1 - \psi_z(u)}{1 - \psi_z(b)}\,, \qquad -\frac{c}{\delta} \le z < u < b\,. \tag{5.1}$$

Similarly, if the surplus drops below z from the level u, then either the surplus process does or does not reach b before it drops below z. We thus have

$$\psi_z(u) = \xi_\delta(u, b, z) + \chi_\delta(u, b, z)\psi_z(b)\,,$$

implying

$$\xi_\delta(u, b, z) = \frac{\psi_z(u) - \psi_z(b)}{1 - \psi_z(b)}\,, \qquad -\frac{c}{\delta} \le z < u < b\,. \tag{5.2}$$

In particular, for the classical risk model, i.e., $\delta = 0$, we have

$$\begin{aligned} \chi_0(u, b, z) &= \frac{1 - \psi_0(u - z)}{1 - \psi_0(b - z)}\,, \\ \xi_0(u, b, z) &= \frac{\psi_0(u - z) - \psi_0(b - z)}{1 - \psi_0(b - z)}\,, \qquad z < u < b\,, \end{aligned}$$

which can be found in Dickson and Gray (1984). Furthermore, we have the following special cases.

1. When $z = 0$, $\chi_\delta(u; b, 0) = \frac{1-\psi_0(u)}{1-\psi_0(b)}$, for $0 \le u < b$, is the probability that the surplus hits level b without having ruin occurred. For simplicity, we denote this particular probability as $\chi_\delta(u, b)$, and let $\xi_\delta(u, b) = 1 - \chi_\delta(u, b)$. Then $\xi_\delta(u, b)$ is the probability that ruin occurs from initial surplus u without the surplus ever reaching level $b > u$. A mathematical definition is

$$\xi_\delta(u, b) = \mathbb{P}\left(\sup_{0 \le t \le T} U_\delta(t) < b, T_{u,0} < \infty \middle| U_\delta(0) = u \right).$$

   Here $\xi_\delta(u, b)$ is also called the distribution of the maximum surplus prior to ruin if ruin occurs.

2. When $b = \infty$, $\chi_\delta(u; \infty, z) = 1 - \psi_z(u)$ is the probability that surplus never drop below level $z (< u)$. In particular, $\chi_\delta(u; \infty, 0) = 1 - \psi_0(u)$, the non-ruin probability in the classical compound Poisson risk model with a constant force of interest, which has been studied in Sundt and Teugels (1995).

3. When $b = 0$, $\chi_\delta(u; 0, z) = \frac{1-\psi_z(u)}{1-\psi_z(0)}$, for $-\frac{c}{\delta} \le z < u < 0$, is the probability that the surplus will get recovered from a negative level u (if ruin has occurred with a deficit $-u$) without dropping below a lower level z. Further, when $z = -c/\delta$, $\chi_\delta(u; 0, -c/\delta)$, for $u < 0$, is the probability that the surplus will get recovered from a negative level u (if ruin has occurred with a deficit $-u$) without absolute ruin occurring.

4. When $-c/\delta < u < \infty$, $\xi_\delta(u, \infty, -c/\delta) = \psi_{-c/\delta}(u)$ is the probability that the surplus will ever drop below $-c/\delta$, i.e., the probability of absolute ruin.

6 The distribution of the maximum severity of ruin

In this section, we allow the surplus process to continue if ruin occurs but the deficit at ruin, $|U_\delta(T_{u,0})|$, is less than c/δ so that the surplus can return to zero; if ruin occurs but the deficit at ruin is more than c/δ, then the surplus process stops as it is impossible for the surplus to return to zero again. We consider the insurer's maximum severity of ruin from the time of ruin until the time that the surplus returns to level 0 (time of recovery after ruin).

We define $\bar{T}_0$ to be the time of recovery after ruin, i.e.,

$$\bar{T}_0 = \inf\{t : t > T_{u,0}, U_\delta(t) \ge 0\},$$

and $\bar{T}_0 = \infty$ if the deficit at ruin is more than c/δ. Further define

$$M_u = \sup\{|U_\delta(t)|, T_{u,0} \le t < \bar{T}_0\}$$

to be the maximum severity of ruin. Let

$$F(z;u) = \mathbb{P}(M_u \le z \big| T_{u,0} < \infty)\,, \qquad z \ge 0\,,$$

denote the distribution function of the maximum severity of ruin given that ruin occurs.

To evaluate $F(u;z)$, we define for $-c/\delta \le z < u < b$,

$$H_{z,b}(u,y) = \mathbb{P}\big(z - U_\delta(T_{u,z}) \le y, T_{u,z} < \infty, \max_{0 \le t \le T_{u,z}} U_\delta(t) < b \,\big|\, U_\delta(0) = u\big)$$

to be the distribution of the amount by which the surplus falls below z if the surplus drops below z without ever reaching level b. By considering whether or not the surplus process attains level b prior to the stopping time $T_{u,z}$, we have the following formula for the distribution function $G_z(u,y)$ defined in (1.3):

$$\begin{aligned} G_z(u,y) &= H_{z,b}(u,y) + \xi_\delta(u,b,z)G_z(b,y), \\ &= H_{z,b}(u,y) + \frac{1-\psi_z(u)}{1-\psi_z(b)}G_z(b,y)\,, \qquad -\frac{c}{\delta} \le z < u\,,\ y \ge 0\,. \end{aligned} \quad (6.1)$$

$F(u;z)$ is given in the following theorem.

Theorem: The distribution function of the maximum severity of ruin, $F(z;u)$, is given as follows.

1. If $0 \le z \le c/\delta$, we have

$$F(z;u) = \frac{\psi_0(u) - \psi_{-z}(u)}{\psi_0(u)[1-\psi_{-z}(0)]}\,, \qquad u \ge 0\,. \quad (6.2)$$

2. If $z > c/\delta$, then

$$F(z;u) = F\left(\frac{c}{\delta};u\right)\left[1 - G_{-\frac{c}{\delta}}\left(0, z - \frac{c}{\delta}\right)\right] + \frac{G_{-\frac{c}{\delta}}\left(u, z - \frac{c}{\delta}\right)}{\psi_0(u)}\,, \qquad u \ge 0\,. \quad (6.3)$$

Proof: We prove this theorem under the two cases.

1. If $0 \le z \le c/\delta$, the maximum severity of ruin will be no more than z if ruin occurs with a deficit $y \le z$ and the surplus reaches 0 from $-y$ without falling below $-z$. The probability of latter event is

$$\xi_\delta(-y,0,-z) = \frac{1-\psi_{-z}(-y)}{1-\psi_{-z}(0)}\,.$$

Thus

$$F(z;u) = \int_0^z \frac{g_0(u,y)}{\psi_0(u)} \xi_\delta(-y,0,-z)dy \tag{6.4}$$
$$= \frac{\int_0^z g_0(u,y)dy - \int_0^z g_0(u,y)\psi_{-z}(-y)dy}{\psi_0(u)[1-\psi_{-z}(0)]} .$$

The integral in the numerator can be evaluated by the expression

$$\psi_{-z}(u) = \int_0^z g_0(u,y)\psi_{-z}(-y)dy + \int_z^\infty g_0(u,y)dy . \tag{6.5}$$

Then equation (6.5) together with formula

$$\psi_0(u) = \int_0^\infty g_0(u,y)dy$$

give (6.2).

2. If $z > c/\delta$, then

$$F(z;u) = F\left(\frac{c}{\delta};u\right) + \mathbb{P}\left(\frac{c}{\delta} < M_u \le z \middle| T_{u,0} < \infty\right) . \tag{6.6}$$

It follows from probability reasonings and formulae (6.4) and (6.1) that

$$\begin{aligned}
&\mathbb{P}\left(\frac{c}{\delta} < M_u \le z \middle| T_{u,0} < \infty\right) \\
&= \frac{G_0(u,z) - G_0\left(u,\frac{c}{\delta}\right)}{\psi_0(u)} + \frac{1}{\psi_0(u)} \int_0^{\frac{c}{\delta}} g_0(u,y) H_{-\frac{c}{\delta},0}\left(-y, z-\frac{c}{\delta}\right) dy \\
&= \frac{G_0(u,z) - G_0\left(u,\frac{c}{\delta}\right)}{\psi_0(u)} + \frac{1}{\psi_0(u)} \int_0^{\frac{c}{\delta}} g_0(u,y) G_{-\frac{c}{\delta}}\left(-y, z-\frac{c}{\delta}\right) dy \\
&\quad - \frac{G_{-\frac{c}{\delta}}\left(0, z-\frac{c}{\delta}\right)}{\psi_0(u)} \int_0^{\frac{c}{\delta}} g_0(u,y) \xi_\delta\left(-y, 0, -\frac{c}{\delta}\right) dy \\
&= \frac{G_{-\frac{c}{\delta}}\left(u, z-\frac{c}{\delta}\right)}{\psi_0(u)} - F\left(\frac{c}{\delta};u\right) G_{-\frac{c}{\delta}}\left(0, z-\frac{c}{\delta}\right) .
\end{aligned} \tag{6.7}$$

Finally, formulae (6.6) and (6.7) give (6.3). This completes the proof. □

Remarks:

1. When $\delta = 0$, the model simplifies to the classical compound Poisson risk model, $\psi_z(u) = \psi(u-z)$ with $\psi(u)$ being the probability of ruin for the classical risk model, and (6.2) simplifies to

$$F(z;u) = \frac{\psi(u) - \psi(u+z)}{\psi(u)[1-\psi(z)]} , \qquad u, z \ge 0 ,$$

which is the formula given in Picard (1994).

2. The probability that the maximum deficit occurs at ruin is given by

$$\begin{aligned}
&\mathbb{P}\left(M_u = |U_\delta(T_{u,0})| \;\middle|\; T_{u,0} < \infty\right) \\
&= \int_{\frac{c}{\delta}}^{\infty} \frac{g_0(u,y)}{\psi_0(u)} dy + \int_0^{\frac{c}{\delta}} \frac{g_0(u,y)}{\psi_0(u)} \xi_\delta(-y,0,-y) dy \\
&= \frac{\psi_0(u) - G_0\left(u, \frac{c}{\delta}\right)}{\psi_0(u)} + \int_0^{\frac{c}{\delta}} \frac{g_0(u,y)}{\psi_0(u)} \frac{1-\psi_{-y}(-y)}{1-\psi_{-y}(0)} dy\,.
\end{aligned}$$

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References

[1] Abramowitz, M. and Stegun, I.A. (1970). *Handbook of Mathematical Functions with Formulas, graphs, and Mathematical Tables*. Dover Publications, Inc., New York.

[2] Cai, J. (2007). On the time value of absolute ruin with debit interest. *Advances in Applied Probability*, **39**, 343-359.

[3] Cai, J. and Dickson, D.C.M. (2002). On the expected discounted penalty function at ruin of a surplus process with interest. *Insurance: Mathematics and Economics*, **30**, 389-404.

[4] Cai, J., Feng, R. and Willmot, G.E. (2009). On the expectation of total discounted operating costs up to default and its applications. *Advances in Applied Probability*, **41**(2), 495-522.

[5] Dassios, A. and Embrechts, P. (1989). Martingales and insurance risk. *Commun. Statist. Stoch. Models* **5**, 181–217.

[6] Embrechts, P. and Schmidli, H. (1994). Ruin estimation for a general insurance risk model. *Advances in Applied Probability*, **26**, 404-422.

[7] Dickson, D.C.M. and Egídio dos Reis, A.D. (1997). The effect of interest on negative surplus. *Insurance: Mathematics and Economics*, **21**, 1-16.

[8] Dickson, D.C.M. and Gray, J. (1984). Approximations to ruin probability in the presence of an upper absorbing barrier. *Scandinavian Actuarial Journal*, 105-115.

[9] Garrido, J. (1989). Stochastic differential equations for compounded risk reserves. *Insurance: Mathematics and Economics*, **8**, 165-173.

[10] Gerber, H.U. (1971). Der Einfluss von Zins auf die Ruinwahrscheinlichkeit. *Bulletin of the Swiss Association of Actuaries*, 63-70.

[11] Gerber, H.U. and Shiu, E.S.W. (1998). On the time value of ruin. *North American Actuarial Journal*, **2**, 48-78.

[12] Gerber, H.U. and Yang, H. (2007). Absolute ruin probabilities in a jump diffusion risk model with investment. *North American Actuarial Journal*, **11**(3), 159-169.

[13] Liu, L. and Mao, S. (2006). The risk model of the expected discounted penalty function with constant interest force. *Acta Mathematica Scientia*, **26B**(3), 509-518.

[14] Mitric, I.R., Badescu, A. and Stanford, D.A. (2012). On the absolute ruin problem in a Sparre Andersen risk model with constant interest. *Insurance: Mathematics and Economics*, **50**, 167-178.

[15] Picard, P. (1994). On some measures of the severity of ruin in the classical Poisson model. *Insurance: Mathematics and Economics*, **14**, 107-115.

[16] Sundt, B. and Teugels, J.L. (1995). Ruin estimates under interest force. *Insurance: Mathematics and Economics*, **16**, 7-22.

[17] Wang, C., Yin, C. and Li, E. (2010). On the classical risk model with credit and debit interests under absolute ruin. *Statistics and Probability Letters*, **80**, 427-426.

[18] Yang, H., Zhang, Z. and Lan, C. (2008). On the time value of absolute ruin for a multi-layer compound Poisson model under interest force. *Statistics and Probability Letters*, **78**, 1835-1845.

[19] Yuen, K.C., Wang, G. and Li, W.K. (2007). The Gerber-Shiu expected discounted penalty function for risk process with interest and a constant dividend barrier. *Insurance: Mathematics and Economics*, **40**, 104-112.