Optimal Reinsurance Strategies in Regime-switching Jump Diffusion Models: Stochastic Differential Game Formulation and Numerical Methods

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Abstract

This work develops a stochastic differential game model between two insurance companies who adopt the optimal reinsurance strategies to reduce the risk. The surplus is modeled by a regime-switching jump diffusion process. A single payoff function is imposed, and one player devises an optimal strategy to maximize the expected payoff function, whereas the other player is trying to minimize the same quantity. Using dynamic programming principle, the upper and lower values of the game satisfy a coupled system of nonlinear integro-differential Hamilton-Jacobi-Issacs (HJI) equations. Moreover, the existence of the saddle point for this game problem is verified. Because of the jumps and regime-switching, closed-form solutions are virtually impossible to obtain. Our effort is devoted to designing numerical methods. We use Markov chain approximation techniques to construct a discrete-time controlled Markov chain to approximate the value functions and optimal controls. Convergence of the approximation algorithms is proved. Examples are presented to illustrate the applicability of the numerical methods.

Key Words. Stochastic differential game, reinsurance strategy, regime switching, Markov chain approximation.

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1 Introduction

To design optimal risk controls for an insurance corporation has drawn increasing attention since the introduction of the classical collective risk model in Lundberg (1903). Since then, many researchers have analyzed this problem under more realistic assumptions and extended its range of applicability. To protect insurance companies against the impact of claim volatilities, reinsurance is a standard tool with the goal of reducing and eliminating risk. The primary insurance carrier pays the reinsurance company a certain part of the premiums. In return, the reinsurance company is obliged to share the risk of large claims. Some recent work can be found in Asmusen et al. (2000); Choulli et al. (2001) and references therein. Proportional reinsurance is one of such reinsurance policies. Using such an approach, the reinsurance company covers a fixed percentage of losses and the premium of reinsurance is determined. Hald and Schmidli (2004) studied the optimal reinsurance policy to maximize the adjustment coefficient of the ruin probability with the variance premium principle. Lin and Yang (2011) treated application of the variance premium principle to optimal investment and reinsurance for the jump diffusion risk process. The insurance company is allowed to take proportional reinsurance with the variance premium principle and invests its surplus into a financial market consisting of a risky asset whose return follows a diffusion process. In this paper, we propose a model based on game theoretic framework. We consider only cheap reinsurance, where the safety loading for the reinsurer is the same as that for the cedent. The insurance company aims to find the optimal proportional reinsurance policy to optimize the expected exponential utility of the payoff function.

One of the new angles considered in this paper is that two insurance companies are considered. Each insurance company adopts a reinsurance strategy among a set of available options to optimize the corresponding payoff function. The payoff, however, depends also on the choices made by the other insurance company. This competition between these two insurance companies can be formulated as a stochastic differential game with two players. In previous literature, using limits of discrete strategies to define the upper and lower values of games can be traced back to Fleming (1961). Elliott and Kalton (1972) analyzed the existence of the values in differential games. For the study of stochastic differential games related to reinsurance and investment, we refer to Elliott and Siu (2011); Suijs et al. (1998); Zeng (2010).

It has been widely recognized, in particular, that traditional surplus models fail to capture discrete movements (such as random environment, market trends, interest rates, business cycles, etc.). To reflect the reality, one of the recent trends is to use regime-switching models, where continuous dynamics and discrete events coexist in the systems. Whilst traditional models rely on either ordinary or stochastic differential equations in the continuous setting alone, one of the advantages of regime-switching models is that the models contain discrete events, which describe the economical movements and impacts that cannot be modeled as either ordinary or stochastic differential equations. The switching process between regimes is modulated as a finite state Markov chain. Thus the formulation of regime-switching models is a more general and versatile framework to describe the complicated financial markets and their inherent uncertainty and randomness. In light of these developments, it is natural to consider stochastic differential games in which both the drift and the diffusion terms are modulated by another random process. Hamilton introduced a regime-switching time series model in Hamilton (1989). Recent work on risk models and related issues can be found in Yang and Yin (2004). In Jin et al. (2011), the optimal dividend strategy with restricted payment rate was studied for the regime-switching jump diffusion model. Sotomayor and Cadenillas (2011) obtained optimal dividend strategies under a regime-switching diffusion model. Concerning switching diffusion processes, a comprehensive study with “state-dependent” switching is in Yin and Zhu (2010).

Compared with Taksar and Zeng (2011), we analyze the competitions between two insurance companies in the Markovian regime-switching environment, each of which makes individual decisions on the reinsurance policies to reduce the exposure to risk. One company's decision is assumed to be completely observed by its opponent. That is, assume that, at each time t, each insurance company can observe the current surplus process of the system and get information about the reinsurance strategies taken by the other company. However, one cannot predict future actions of the competitor. We assume that player 1 (the leader) announces his strategy in advance, and then player 2 (the follower) makes his choice accordingly. In addition, one single payoff function is proposed. From the point of view of Player 1, the task is to devise a strategy to maximize the expected payoff function while the other company is trying to minimize the same payoff function.

We model the surplus process as a regime-switching jump diffusion process. Reinsurance strategies are introduced as control parameters for each player. The analysis of differential games relies heavily on concepts and techniques of optimal control theory. Equilibrium strategies are studied by solving for a system of HJI equations for the value functions of the various players, derived from the principle of dynamic programming. The formulation of our model is very general and versatile. Nevertheless, due to the inclusion of the random switching environment and jump processes, the system of HJI equations becomes more complicated and closed-form solutions are virtually impossible to obtain. A viable alternative is to employ numerical approximations. In this work, we adopt the Markov chain approximation methodology developed in Kushner and Dupuis (2001), in which numerical approximation algorithms for stochastic controls were developed. Inspired by and generalizing the work of Kushner (2002), Song et al. (2008) deals with numerical methods for stochastic differential games of regime-switching diffusions. In that paper, one needs to deal with a system of HJI equations. This paper further treat models with jumps. As a result, we have to deal with a system of integro-differential HJI equations. Under simple conditions, we prove the existence of the saddle point for this game problem. The convergence of the approximation sequence to the jump diffusion process and the convergence of the approximation to the upper and lower value of the game will be confirmed. In the actual computation, we simply use the well-known value iteration method for our approximation schemes.

The rest of the paper is organized as follows. A generalized formulation of stochastic differential game for optimal risk control and assumptions are presented in Section 2. The optimal proportional reinsurance strategies and variance premium principle are considered in our study. Section 3 deals with the numerical algorithm of Markov chain approximation method. The upper and lower values of the game are well approximated by the approximating Markov chain and the dynamic programming equation are presented. Section 4 deals with the convergence of the approximation scheme. The existence of the saddle points are provided. Numerical examples of two classes of premium principles are provided in Section 5 to illustrate the performance of the approximation method. Finally, some additional remarks are provided in Section 6.

2 Formulation and Preliminaries

Let us work with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where $\{\mathcal{F}_t\}$ (or simply $\mathcal{F}_t$) is a filtration satisfying the usual condition. That is, $\mathcal{F}_t$ is a family of σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$ and that $\mathcal{F}_0$ contains all null sets. Following the classical Cramér Lundberg process, we assume that $X^{(1)}(t)$, the surplus of insurance company 1 satisfies

$$X^{(1)}(t) = x^{(1)} + c^{(1)}t - Y^{(1)}(t), \quad t \ge 0, \tag{2.1}$$

where $x^{(1)}$ is the initial surplus, the constant $c^{(1)}$ is the rate of premium, and $Y^{(1)}(t) = \sum_{i=1}^{N(t)} A_i^{(1)}$ is a compound Poisson process with the claim size $A_i^{(1)}$ with $\{A_i^{(1)} : i > 1\}$ being a sequence of positive, independent and identically distributed random variables.

Let u_1 be an exogenous retention level, which is a control chosen by the insurance company representing the reinsurance policy. We allow the insurance companies to continuously reinsure a fraction of its claim with the retention level $u_1 \in [0, 1]$. By using the variance premium principle, the reinsurance premium rate at time t is

$$g(u_1) = (1 - u_1)\vartheta_1^{(1)} + \beta(1 - u_1)^2\vartheta_2^{(1)}, \tag{2.2}$$

where $\vartheta_1^{(1)} = E[A_i^{(1)}]$, $\vartheta_2^{(1)} = E[A_i^{(1)}]^2 + \mathrm{Var}[A_i^{(1)}]$ and $\beta > 0$ is the safety loading of the reinsurer. Hence, combing the reinsurance control strategies, the surplus process of the insurance company follows

$$\begin{cases} dX^{(1)}(t) = (c^{(1)} - g(u_1))dt - u_1 dY^{(1)}(t), \\ X^{(1)}(0) = x^{(1)}. \end{cases} \tag{2.3}$$

Analogously, the surplus of insurance company 2, the competitor of insurance company 1, follows

$$\begin{cases} dX^{(2)}(t) = (c^{(2)} - g(u_2))dt - u_2 dY^{(2)}(t), \\ X^{(2)}(0) = x^{(2)}, \end{cases} \tag{2.4}$$

where $x^{(2)}$ is the initial surplus, $c^{(2)}$ is the rate of premium, $Y^{(2)}(t) = \sum_{i=1}^{N(t)} A_i^{(2)}$ is a compound Poisson process with the claim size $A_i^{(2)}$, $\{A_i^{(2)} : i > 1\}$ is a sequence of positive, independent and identically distributed random variables representing the claim amount. u_2 represents the corresponding retention level of the company 2's reinsurance strategy, and $g(u_2)$ denotes the the reinsurance premium rate.

In this work, we will model the competition of two insurance companies with reinsurance schemes. The performance of the two companies is measured by the difference of their surpluses $X^{(1)} - X^{(2)}$. Without loss of generality, we assume $x^{(1)} > x^{(2)}$. The company with more surplus pursues increasing the difference of the surplus, while the company with less surplus will try to decrease the surplus difference and narrow the gap. Thus, the competition between the two companies formulates a game with two players, each of which can adjust its reinsurance strategies based on the competitor's scheme. Let $X(t) = X^{(1)} - X^{(2)}$. Hence, the difference of the two surpluses $X(t)$ is governed by the following dynamics

$$\begin{cases} dX(t) = (c^{(1)} - g(u_1) - c^{(2)} + g(u_2))dt - u_1 dY^{(1)}(t) + u_2 dY^{(2)}(t), \\ X(0) = x^{(1)} - x^{(2)}. \end{cases} \tag{2.5}$$

To delineate the random environment and other random factors, we use a continuous-time Markov chain $\alpha(t)$ taking values in the finite space $\mathcal{M} = \{1, \dots, m\}$. The market states are represented by the Markov chain $\alpha(t)$, and they undergo a Markov regime switching. Let the continuous-time Markov chain $\alpha(t)$ be generated by $Q = (q_{ij}) \in \mathbb{R}^{m \times m}$. That is,

$$\mathbb{P}\{\alpha(t+\delta) = j | \alpha(t) = i, \alpha(s), s \le t\} = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } j \neq i, \\ 1 + q_{ii}\delta + o(\delta), & \text{if } j = i, \end{cases} \tag{2.6}$$

where $q_{ij} \ge 0$ for $i, j = 1, 2, \dots, m$ with $j \neq i$ and $q_{ii} = -\sum_{j \neq i} q_{ij} < 0$ for each $i = 1, 2, \dots, m$.

Remark 2.1. There are different alternatives for the modeling framework. (i) We may use the semi-martingale representation of Elliott and Siu (2011) starting with the representation

$$\alpha(t) = \alpha(0) + \int_0^t Q' \alpha(u) du + M(t),$$

where Q' is the transpose of Q and $\{M(t) : t \in [0, T]\}$ is an $\mathbb{R}^m$-valued martingale; see also Elliott and Siu (2010), Siu (2012), Siu (2013), and many references therein. We can then proceed to develop approximation schemes. (ii) We could rewrite the system by representing the Markov chain using another Poisson process in the form

$$d\alpha(t) = \int_{\mathbb{R}} h(X(t^-), \alpha(t^-), z) N_1(dt, dz)$$

where $h(x, i, z) : \mathbb{R} \times \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is a bounded measurable function and $N_1(dt, dz)$ is a Poisson measure with intensity $dt \times m_1(dz)$, and $m_1(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. The Poisson measure $N_1(\cdot, \cdot)$ is independent of the Brownian motion $W(\cdot)$ and Poisson measure $N(\cdot, \cdot)$ for claims; see Boukas et al (1997) and Yin and Xi (2010). We then could proceed to develop numerical schemes to approximate the solution. Note that in the system, we have two Poisson processes, one of them is the jumps due to claims and the other is from the modulating Markov chain. We can prove the convergence of the numerical algorithm using the methods of Kushner and Dupuis (2001). Effectively, we deal with the convergence of controlled martingale problems and obtain the limit problem. We choose the current setting in the paper since this is most convenient for us to separate the effect due to the Poisson jump and the Markov chain.

The surplus process $X^{(k)}(t), k = 1, 2$ under consideration is a jump process with regime-switching. For each $i \in \mathcal{M}$, the premium rate is $c(i) > 0$. Let ζ_n be the inter-arrival time of the nth claim, $\nu_n = \sum_{j=1}^n \zeta_j$. In general we consider a Poisson measure in lieu of the traditionally used Poisson process. Suppose $\Theta \subset \mathbb{R}_+$ is a compact set and the function $q(i, \rho)$ is the magnitude of the claim sizes, where ρ has distribution $\Pi(\cdot)$.

$$N(t, H) = \text{ number of claims on } [0, t] \text{ with claim size taking values in } H \in \Theta. \tag{2.7}$$

counts the number of claims up to time t, which is a Poisson counting process. For $k = 1, 2$, let $Y^{(k)}(t)$ be jump processes representing claims with arrival rate λ for each company. The function $q(i, \rho_k)$ is assumed to be the magnitude of the claim sizes, where ρ_k has distribution $\Pi_k(\cdot)$. Note that our formulation is general, the claim sizes are assumed to depend on the switching regimes. At different regimes, the values of ρ_k could be much different, which takes into consideration of random environment. Then the Poisson measure $N(\cdot)$ has intensity $\lambda_k dt \times \Pi_k(d\rho_k)$ where

$\Pi_k(d\rho_k) = f(\rho_k)d\rho_k$. Assume that $q(i, \rho_k)$ is continuous for each ρ_k and each $i \in \mathcal{M}$. Then the surplus process in absence of investment is shown as a regime-switching jump process

$$\begin{aligned} d\widetilde{X}(t) &= \sum_{i\in\mathcal{M}} I_{\{\alpha(s)=i\}}[(c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2))dt - u_1 dY^{(1)}(t) + u_2 dY^{(2)}(t)] \\ &= \Big[c^{(1)}(\alpha(t)) - g(u_1) - c^{(2)}(\alpha(t)) + g(u_2)\Big]dt - u_1(t)\int_{\mathbb{R}_+} q(\alpha(t),\rho_1)N(dt,d\rho_1) \\ &\quad + u_2(t)\int_{\mathbb{R}_+} q(\alpha(t),\rho_2)N(dt,d\rho_2). \end{aligned} \tag{2.8}$$

We allow the surplus to be invested in an asset $Z(t)$ in the financial market with prices satisfying the Geometric Brownian Motion process

$$\frac{dZ(t)}{Z(t)} = \mu(\alpha(t))dt + \sigma(\alpha(t))dW(t), \tag{2.9}$$

where for each $i \in \mathcal{M}$, where $\mu(i)$ is the return rate of the asset and $\sigma(i)$ is the corresponding volatility and $W(t)$ is a standard Brownian motion. Because of the randomness of the environment, the yield rate of the asset can be effected by the market modes. $\alpha(t)$ represents the switching process of the markets modes, and the yield rate of the asset is driven by a finite state Markov chain. We are now working on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{W(s), \alpha(s), N(s) : 0 \le s \le t\}$.

The surplus process considering reinsurance control and investment satisfy the following s-tochastic differential equation

$$\begin{cases} dX(t) = \Big[\mu(\alpha(t))X(t) + c^{(1)}(\alpha(t)) - g(u_1) - c^{(2)}(\alpha(t)) + g(u_2)\Big]dt + \sigma(\alpha(t))X(t)dW(t) \\ \qquad\qquad -u_1(t)\int_{\mathbb{R}_+} q(\alpha(t),\rho_1)N(dt,d\rho_1) + u_2(t)\int_{\mathbb{R}_+} q(\alpha(t),\rho_2)N(dt,d\rho_2), \\ X(0) = x = x^{(1)} - x^{(2)}. \end{cases} \tag{2.10}$$

for all $t < \tau$, where $\tau = \inf\{t \ge 0 : X(t) \notin (a, b)\}$ represents the time of exiting the game, and a, b are constants satisfying $a < b$. We assume that game will stop if the absolute value of the difference of the surpluses are too large. Either in positive or negative direction, the large difference means that one company dominates the market and wins the game.

Denote by $r > 0$ the discount factor. Let U_1 and U_2 be the collection of all investment and reinsurance strategies respectively, which are assumed to be compact sets. Let $u = (u_1, u_2)$ and $u \in U = U_1 \times U_2$. For an arbitrary admissible control u, the expected discounted payoff is

$$J(x, i, u) = E_{x,i}\left[\int_0^\tau e^{-rt}\Big[f(X(t), \alpha(t), u(t))dt + \tilde{h}(X(\tau^-), \alpha(\tau^-))\Big]\right]. \tag{2.11}$$

The control $u = (u_1, u_2)$ is said to be *admissible* if u_1, u_2 satisfy

(i) $u_1(t)$, $u_2(t)$ are nonnegative for any $t \ge 0$,

(ii) $X(t) \in (a, b)$, for any $t \le \tau$,

(iii) both u_1, u_2 are adapted to $\mathcal{F}_t$ that contains at least $\sigma\{W(s), \alpha(s), N(s), 0 \le s \le t\}$,

(iv) $J(x, i, u) < \infty$ for any $(x, i) \in G \times \mathcal{M}$ and admissible pair $u = (u_1, u_2)$, where J is the functional defined in (2.11).

Suppose that U is the collection of possible retention levels $u(t)$. Throughout the paper, we assume that U is a given compact set, and that for each $i \in \mathcal{M}$ we assume the function $f(\cdot)$ is concave-convex with respect to (u_1, u_2). That is, $f(\cdot)$ is concave with respect to u_1 for every (x, i, u_2) and convex with respect to u_2 for every (x, i, u_1). $\tilde{h}(\cdot)$ is assumed to be a continuous function.

For $k = 1, 2$, let $\mathcal{B}(U_k \times [0, \infty))$ be the σ-algebra of Borel subsets of $U_k \times [0, \infty)$. An *admissible relaxed control* (or deterministic relaxed control) $m_k(\cdot)$ is a measure on $\mathcal{B}(U_k \times [0, \infty))$ such that $m_k(U_k \times [0, t]) = t$ for each $t \geq 0$. With the given probability space, we say that $m_k(\cdot)$ is an admissible relaxed (stochastic) control for $(W(\cdot), \alpha(\cdot))$ or $(m(\cdot), W(\cdot), \alpha(\cdot))$ is admissible, if $m_k(\cdot, \omega)$ is a deterministic relaxed control with probability one and if $m_k(A \times [0, t])$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U_k)$. There is a derivative $m_{t,k}(\cdot)$ such that $m_{t,k}(\cdot)$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U_k)$.

Given a relaxed control $m_k(\cdot)$ of $u_k(\cdot)$, we define the derivative $m_{t,k}(\cdot)$ such that

$$m_k(K) = \int_{U_k \times [0,\infty)} I_{\{(u_k,t) \in K\}} m_{t,k}(d\phi_k) dt \tag{2.12}$$

for all $K \in \mathcal{B}(U_k \times [0, \infty))$, and that for each t, $m_{t,k}(\cdot)$ is a measure on $\mathcal{B}(U_k)$ satisfying $m_{t,k}(U_k) = 1$. For example, we can define $m_{t,k}(\cdot)$ in any convenient way for $t = 0$ and as the left-hand derivative for $t > 0$,

$$m_{t,k}(A) = \lim_{\delta \to 0} \frac{m_k(A \times [t - \delta, t])}{\delta}, \ \forall A \in \mathcal{B}(U_k). \tag{2.13}$$

Note that $m_k(d\phi_k dt) = m_{t,k}(d\phi_k)dt$. It is natural to define the relaxed control representation $m_k(\cdot)$ of $u_k(\cdot)$ by

$$m^h_{t,k}(A) = I_{\{u_k(t) \in A\}}, \ \forall A \in \mathcal{B}(U_k). \tag{2.14}$$

Define the relaxed control $m(\cdot) = (m_1(\cdot) \times m_2(\cdot))$ with derivative $m_t(\cdot) = m_{t,1}(\cdot) \times m_{t,2}(\cdot)$. Thus $m(\cdot)$ is a measure on the Borel sets of $(U_1 \times U_2) \times [0, \infty)$.

Analogous to the work in Kushner (2002), we will proceed to define the upper values, lower values and saddle points. Let $\mathcal{U}_k = \{u_k : \text{admissible ordinary control w.r.t. } (W(\cdot), \alpha(\cdot), N(\cdot))\}$ For $\Delta > 0$, denoted by $\mathcal{U}_k(\Delta)$ the collection of the piecewise constant controls $u_k(\cdot) = (u_{1,k}(\cdot), u_{2,k}(\cdot))$ on the intervals $[n\Delta, (n + 1)\Delta), n = 0, 1, 2, \ldots$, where $u_k(n\Delta)$ is $\mathcal{F}_{n\Delta}$-measurable and $\mathcal{U}_k(\Delta) \subset \mathcal{U}_k$. Let A_1 be a Borel subset of U_1 and $\mathcal{R}_1(\Delta) \subset \mathcal{U}_1(\Delta)$ denote the set of piecewise constant controls represented by $F_{1,j}(A_1, \cdot), j = 0, 1, 2, \ldots$ of the conditional probability type

$$\begin{aligned} &P\{u_1(jn) \in A_1 | W(s), \alpha(s), N(s), u_2(s), s < j\Delta; u_1(n\Delta), n < j\} \\ &\quad = F_{1,j}(A_1; W(s), \alpha(s), N(s), u(s), s < j\Delta), \end{aligned} \tag{2.15}$$

where $F_{1,j}(A_1, \cdot)$ is a measurable function for $A_1 \in \mathcal{B}(U_1)$. We denote $u_1(u_2)$ to emphasize the dependence of u_1 on u_2 if the control rule is given by (2.15). Similarly, we define $\mathcal{R}_2(\Delta)$ and the associated control rule $u_2(u_1)$.

To proceed, we need more assumption.

(A1) Let $u(\cdot)$ be an admissible ordinary control with respect to $\omega(\cdot)$ and $\alpha(\cdot)$, and suppose that $u(\cdot)$ is piecewise constant and takes only a finite number of values. For each initial condition, there exists a solution to the relaxed equation where $m(\cdot)$ is the relaxed control representation of $u(\cdot)$. This solution is unique in the weak sense.

The uniqueness assumption implies that, for an initial condition (x,i), by the weak sense uniqueness, we mean that the probability law of the admissible process $(\alpha(\cdot), m(\cdot), W(\cdot), N(\cdot))$ determines the probability law of solution $(X(\cdot), \alpha(\cdot), m(\cdot), W(\cdot), N(\cdot))$ to the relaxed equation, irrespective of probability space. Thus we can always suppose that if m_1 is determined by a form in (2.15). Then the law of $(W(\cdot), \alpha(\cdot)N(\cdot), m_2(\cdot))$ is determined recursively by a conditional law

$$\begin{aligned}P\{W(t), \alpha(t), N(t), m_2(t), n\Delta \le t < (n+1)\Delta \in \cdot | W(s), \alpha(s), N(s), u_2(s), s < t;\\ m_1(s), s \le n\Delta\}.\end{aligned} \tag{2.16}$$

Define the upper values as

$$V^{+}(x,i) = \lim_{\Delta\to 0} \inf_{u_1\in\mathcal{R}_1(\Delta)} \sup_{u_2\in\mathcal{U}_2} J(x,i,u_1(u_2),u_2), \quad \text{for each } \ i\in\mathcal{M}. \tag{2.17}$$

Analogously, the lower value is defined as

$$V^{-}(x,i) = \lim_{\Delta\to 0} \sup_{u_2\in\mathcal{R}_2(\Delta)} \inf_{u_1\in\mathcal{U}_1} J(x,i,u_1,u_2(u_1)), \quad \text{for each } \ i\in\mathcal{M}. \tag{2.18}$$

If the upper values and lower values are equal, then we say there exists a saddle point

$$V(x,i) = V^{+}(x,i) = V^{-}(x,i), \quad \text{for each } \ i\in\mathcal{M}. \tag{2.19}$$

For an arbitrary $u\in U$, $i\in\mathcal{M}$, and $V(\cdot,i)\in C^2(\mathbb{R})$, define an operator $\mathcal{L}^u$ by

$$\begin{aligned}\mathcal{L}^u V(x,i) &= V_x(x,i)[\mu(i)X(t)dt + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)] + \frac{1}{2}\sigma(i)^2 x^2 V_{xx}(x,i)\\ &\quad -\lambda_1\int_0^x [V(x-u_1 q(i,\rho_1),i) - V(x,i)] f(\rho_1)d\rho_1\\ &\quad +\lambda_2\int_0^x [V(x+u_2 q(i,\rho_2),i) - V(x,i)] f(\rho_2)d\rho_2 + QV(x,\cdot)(i),\end{aligned} \tag{2.20}$$

where V_x and V_{xx} denote the first and second derivatives with respect to x, and

$$QV(x,\cdot)(i) = \sum_{j\neq i} q_{ij}(V(x,j) - V(x,i)).$$

Formally, we conclude that V satisfies the following coupled system of integro-differential HJI equations: for each $i\in\mathcal{M}$,

$$\begin{cases}\displaystyle\inf_{u_1\in U_1}\sup_{u_2\in U_2}(\mathcal{L}^u V(x,i) - rV(x,i) + f(x,i,u))\\ \displaystyle\quad = \sup_{u_2\in U_2}\inf_{u_1\in U_1}(\mathcal{L}^u V(x,i) - rV(x,i) + f(x,i,u)) = 0, \quad x\in(a,b)\\ V(x,i) = \tilde{h}(x,i), \quad x = a,b.\end{cases} \tag{2.21}$$

3 Numerical Algorithm

To design a numerical scheme to approximate the optimal strategy for the stochastic differential games with regime-switchings, we will construct discrete-time and finite-state controlled Markov chain that is locally consistent with (2.10); see also Jin et al. (2012). In this problem, the surplus process has two components such that one represents the diffusive behavior with Poisson jumps and the other describes the regimes. Hence, in order to adopt the classical Markov chain approximation methodology in Kushner and Dupuis (2001), our approximating Markov chain must have two components. One component delineates the diffusive behavior whereas the other keeps track of the regimes. Because of the jump diffusions and the regime switching, the notation of local consistency, the interpolation, and the use of the relaxed control etc. have to be carefully redefined. We begin by constructing a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion process with regime-switching in the absence of jumps, with the dynamic system

$$\begin{cases} dX(t) = \Big[\mu(\alpha(t))X(t) + c^{(1)}(\alpha(t)) - g(u_1) - c^{(2)}(\alpha(t)) + g(u_2)\Big]dt + \sigma(\alpha(t))X(t)dW(t) \\ X(0) = x = x^{(1)} - x^{(2)}. \end{cases} \tag{3.1}$$

Let $h > 0$ be a discretization stepsize and the boundary points a, b be integer multiples of h. Define $L_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \dots\}$ and $S_h = L_h \cap G$, where $G = [a, b]$ and $G^o = (a, b)$. Let $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ be a controlled discrete-time Markov chain on $S_h \times \mathcal{M}$ and denote by $p_D^h((x, i), (y, j)|u^h)$ the transition probability from a state (x, i) to another state (y, j) under the control u^h. p_D^h is so defined that the constructed Markov chain's evolution well approximates the local behavior of the controlled regime-switching diffusion (2.10).

We denote by $u_n^h = \{(u_{1,n}^h, u_{2,n}^h)\} \subset U$ the random variable that is the regular control action for the chain at time n. Let $\Delta t^h(\cdot, \cdot, \cdot, \cdot) > 0$ be the *interpolation interval* on $S_h \times \mathcal{M} \times U_1 \times U_2$. Assume $\inf_{x,i,u} \Delta t^h(x, i, u_1, u_2) > 0$ for each $h > 0$ and $\lim_{h\to 0} \sup_{x,i,u} \Delta t^h(x, i, u_1, u_2) \to 0$. Hence,

$$\begin{aligned} &P_{x,i,n}^{u,h}\{\alpha_{n+1}^h = j\} = q_{ij}(x)\Delta t^h(x, i, u) + o(\Delta t^h(x, i, u)), \text{ for } j \neq i, \\ &P_{x,i,n}^{u,h}\{\alpha_{n+1}^h = i\} = 1 + q_{ii}(x)\Delta t^h(x, i, u) + o(\Delta t^h(x, i, u)), \\ &\sup_{n,\omega\in\Omega} |\Delta \xi_n^h| \to 0 \ \text{ as } \ h \to 0. \end{aligned}$$

Use $E_{x,i,n}^{u,h}$, $\text{Var}_{x,i,n}^{u,h}$ and $P_{x,i,n}^{u,h}$ to denote the conditional expectation, variance, and marginal probability given $\{\xi_k^h, \alpha_k^h, u_k^h, I_k^h, k \le n, \xi_n^h = x, \alpha_n^h = i, u_n^h = u\}$, respectively. The sequence $\{(\xi_n^h, \alpha_n^h)\}$ is said to be *locally consistent*, if it satisfies

$$\begin{aligned} &E_{x,i,n}^{u,h}[\Delta \xi_n^h] = [\mu(i)X(t) + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]\Delta t^h(x, i, u) + o(\tilde{\Delta} t^h(x, i, u)), \\ &\text{Var}_{x,i,n}^{u,h}(\Delta \xi_n^h) = \sigma(i)^2 x^2 \Delta t^h(x, i, u) + o(\Delta t^h(x, i, u)). \end{aligned} \tag{3.2}$$

Let $u^h := \{u_n^h, n \ge 0\}$ be the sequence of control actions. The sequence u^h is said to be *admissible* if u_n^h is $\sigma\left\{(\xi_0^h, \alpha_0^h), \dots, (\xi_n^h, \alpha_n^h), u_0^h, \dots, u_{n-1}^h\right\}$-adapted and for any $E \in \mathcal{B}(S_h \times \mathcal{M})$, we have

$$P\left\{(\xi_{n+1}^h, \alpha_{n+1}^h) \in E \middle| \sigma\{(\xi_0^h, \alpha_0^h), \dots, (\xi_n^h, \alpha_n^h), u_0^h, \dots, u_n^h\}\right\} = p^h((\xi_n^h, \alpha_n^h), E|u_n^h).$$

The piecewise constant interpolations $(\xi^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$ are naturally defined as

$$\xi^h(t) = \xi_n^h, \ \alpha^h(t) = \alpha_n^h, \ u^h(t) = u_n^h \tag{3.3}$$

for $t \in [t_n^h, t_{n+1}^h)$, where $t_0^h := 0$, $t_n^h := \sum_{k=0}^{n-1} \Delta t^h(\xi_k^h, \alpha_k^h, u_k^h)$. Let $\eta_h := \inf\left\{n : \xi_n^h \in \partial G\right\}$. Then the first exit time of ξ^h from G is $\tau^h = t_{\eta_h}^h$. Let $(\xi_0^h, \alpha_0^h) = (x, i) \in S_h \times \mathcal{M}$ and u^h be an admissible control. The cost function for the controlled Markov chain is defined as

$$J^h(x, i, u^h) = E \sum_{k=1}^{\eta_h - 1} e^{-rt_k^h}[f(\xi_k^h, \alpha_k^h, u_k^h)\Delta t_k^h + \tilde{h}(\xi_k^h, \alpha_k^h)]. \tag{3.4}$$

Based on the approximating Markov chain constructed above, the piecewise constant interpolation is obtained and the appropriate interpolation interval level is chosen. The continuous-time interpolations $(\xi^h(\cdot), \alpha^h(\cdot))$, $u^h(\cdot)$, and $g^h(\cdot)$ are defined. In addition, let $\mathcal{U}_k^h(1)$ denote the collection of controls that the player k goes first, which are determined by a sequence of measurable functions $F_n^h(\cdot)$ such that

$$u_{k,n}^h = F_n^h(\xi_i^h, \alpha_i^h, i \le n; u_i^h, i \le n). \tag{3.5}$$

Using $\mathcal{U}_k^h(2)$ to denote the collection of the ordinary controls that the player k goes last, the strategy is defined by a sequence of measurable functions $\tilde{F}_n^h(\cdot)$

$$u_{k,n}^h = \tilde{F}_n^h(\xi_i^h, \alpha_i^h, i \le n; u_i^h, i \le n; u_{l,n}^h, l \neq k). \tag{3.6}$$

Define $\mathcal{D}_t^h$ as the smallest σ-algebra generated by $\{\xi^h(s), \alpha^h(s), u^h(s), s \le t\}$. In addition, $\mathcal{U}^h = \mathcal{U}_k^h(1) \times \mathcal{U}_l^h(2)$ defined by (3.5) and (3.6) is equivalent to the collection of all piecewise constant admissible controls with respect to $\mathcal{D}_t^h$.

We then can define the upper and lower values with $J^h(\cdot)$ and $\mathcal{U}^h$ as

$$V^{h,+}(x,i) = \inf_{u_1^h \in \mathcal{U}_1^h(1)} \sup_{u_2^h \in \mathcal{U}_2^h(2)} J^h(x, i, u_1^h, u_2^h), \forall i \in \mathcal{M}. \tag{3.7}$$

Analogously, the lower value is defined as

$$V^{h,-}(x,i) = \sup_{u_2^h \in \mathcal{U}_2^h(1)} \inf_{u_1^h \in \mathcal{U}_1^h(2)} J^h(x, i, u_1^h, u_2^h), \forall i \in \mathcal{M}. \tag{3.8}$$

If a saddle point exists,

$$V^h(x,i) = V^{h,+}(x,i) = V^{h,-}(x,i), \ \forall i \in \mathcal{M}. \tag{3.9}$$

Practically, we compute $V^h(x,i)$ by solving the corresponding dynamic programming equation using either value iteration or policy iteration. In fact, for $i \in \mathcal{M}$, we can use

$$\begin{aligned} V^{h,+}(x,i) &= \inf_{u_1 \in U_1} \{ \sup_{u_2 \in U_2} \sum_{(y,j)} e^{-r\Delta t^h(x,i,u)} p^h((x,i),(y,j)|u) V^h(y,j) + f(x,i,u)\Delta t^h(x,i,u) \}, \\ V^{h,-}(x,i) &= \sup_{u_2 \in U_2} \{ \inf_{u_1 \in U_1} \sum_{(y,j)} e^{-r\Delta t^h(x,i,u)} p^h((x,i),(y,j)|u) V^h(y,j) + f(x,i,u)\Delta t^h(x,i,u) \}. \end{aligned} \tag{3.10}$$

On the other hand, for $i \in \mathcal{M}$, the saddle point $V(x,i)$ satisfies the HJI equation with only diffusion and regime switching as

$$\begin{aligned} V_x(x,i)[\mu(i)X(t) + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)] + \frac{1}{2} V_{xx}(x,u,i)\sigma^2(i) \\ + \sum_j V(x,\cdot) q_{ij} - rV(x,i) = 0. \end{aligned} \tag{3.11}$$

Define the approximation to the first and the second derivatives of $V(\cdot, i)$ by finite difference method using stepsize $h > 0$ as:

$$\begin{aligned} &V(x,i) \to V^h(x,i), \\ &V_x(x,i) \to \frac{V^h(x+h,i) - V^h(x,i)}{h} \quad \text{for} \ \ [\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)] > 0, \\ &V_x(x,t) \to \frac{V^h(x,i) - V^h(x-h,i)}{h} \quad \text{for} \ \ [\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)] < 0, \\ &V_{xx}(x,t) \to \frac{V^h(x+h,i) - 2V^h(x,i) + V^h(x-h,i)}{h^2}. \end{aligned} \tag{3.12}$$

Together with the boundary conditions, it leads to

$$
\begin{aligned}
&V^h(x,i) = \tilde{h}(x,i), \ \text{for } x \in \partial S_h, \\
&\frac{V^h(x+h,i) - V^h(x,i)}{h}[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]^+ \\
&\quad - \frac{V^h(x,i) - V^h(x-h,i)}{h}[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]^- \\
&\quad + \frac{V^h(x+h,i) - 2V^h(x,i) + V^h(x-h,i)}{h^2} \cdot \frac{\sigma^2(i)}{2} \\
&\quad + \sum_j^m V^h(x,\cdot)q_{ij} - rV^h(x,i) = 0, \quad \forall x \in S_h^o, i \in \mathcal{M},
\end{aligned}
\tag{3.13}
$$

where $[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]^+$ and $[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]^-$ are the positive and negative parts of $\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)$, respectively. Simplifying (3.13) and comparing the result with (3.10), we have

$$
\begin{aligned}
&p_D^h((x,i),(x+h,i)|u) = \frac{\sigma^2(i)/2 + h[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]^+}{\widetilde{D} - rh^2}, \\
&p_D^h((x,i),(x-h,i)|u) = \frac{\sigma^2(i)/2 + h[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]^-}{\widetilde{D} - rh^2}, \\
&p_D^h((x,i),(x,j)|u) = \frac{h^2}{\widetilde{D} - rh^2} q_{ij}, \quad \text{for} \ \ j \neq i, \\
&p_D^h(\cdot) = 0, \quad \text{otherwise}, \\
&\Delta t^h(x,i,u) = \frac{h^2}{\widetilde{D}},
\end{aligned}
\tag{3.14}
$$

with $\widetilde{D} = \sigma^2(i) + h|\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)| + h^2(r - q_{ii})$ being well defined.

We need to approximate the Poisson jumps ensuring the local properties of claims for (2.10). Recall that ν_n denote the time of the nth claim and $\zeta_n = \nu_{n+1} - \nu_n$. To proceed, for each player, let $q(\cdot, \rho_n)$ be the claim intensity with a suitable function of $q(\cdot)$ and $\{\zeta_n, \rho_n, n < \infty\}$ be mutually independent random variables with ζ_n being exponentially distributed with mean $1/\lambda$, and let ρ_n have a distribution $\Pi(\cdot)$. Furthermore, let $\{\zeta_k, \rho_k, k \geq n\}$ be independent of $\{x(s), \alpha(s), s < \nu_n, \zeta_k, \rho_k, k < n\}$. Then the nth claim term is $q(x(\nu_n^-), \alpha(\nu_n), \rho_n)$, and the claim amount $Y(t)$ can be written as $Y(t) = \sum_{\nu_n \leq t} q(\alpha(\nu_n), \rho_n)$. Since ζ_n is exponentially distributed, we have

$$
P\{\text{claim occurs on } \ [t, t+\Delta) | x(s), \alpha(s), W(s), N(s,\cdot), s \leq t\} = \lambda\Delta + o(\Delta). \tag{3.15}
$$

By the independence and the definition of ρ_n, for any $H \in \mathcal{B}(\Theta)$, we have

$$
\begin{aligned}
&P\{x(t) - x(t^-) \in H | t = \nu_n \ \text{ for some } n; W(s), x(s), \alpha(s), N(s,\cdot), s < t; x(t^-) = x, \alpha(t) = \alpha\} \\
&\quad = \Pi(\rho : q(\alpha(t), \rho) \in H).
\end{aligned}
\tag{3.16}
$$

It is shown that the process (2.10) will behave as a regime-switching diffusion process in the consecutive claim intervals. According to the claim rate defined by (3.15), it follows from the conditional probability law (3.16) that the nth claim will be valued as $q(\alpha(\nu_n), \rho_n)$ given that the nth claim occurs at time ν_n.

In (2.10), the surplus process is determined by two jump terms, with the arriving rate λ_1 and λ_2, respectively. Denote by $R(t)$ the difference of the two jumps. That is,

$$R(t) = u_1(t) \int_{\mathbb{R}_+} q(\alpha(t), \rho_1) N(dt, d\rho_1) - u_2(t) \int_{\mathbb{R}_+} q(\alpha(t), \rho_2) N(dt, d\rho_2).$$

Since the difference of two Poisson processes is again a Poisson process, events in the new process $R(t)$ will occur according to a Poisson process with rate λ, with $\lambda = \lambda_1 + \lambda_2$, and each event, independently, will be from the first jump process with probability $\lambda_1/(\lambda_1 + \lambda_2)$, yielding the generic claim size

$$\tilde{A} = \begin{cases} u_1(t) A^{(1)}, & \text{with probability } \dfrac{\lambda_1}{\lambda_1 + \lambda_2}, \\ -u_2(t) A^{(2)}, & \text{with probability } \dfrac{\lambda_2}{\lambda_1 + \lambda_2}. \end{cases} \tag{3.17}$$

Suppose that the current state is $\xi_n^h = x$, $\alpha_n^h = i$, and control is $u_n^h = u$. The next interpolation interval $\Delta t^h(x, i, u)$ is determined by (3.14) and $q_h(i, \rho) \in S_h \subseteq \mathbb{R}_+$ such that $q_h(i, \rho)$ is the nearest value of $q(i, \rho)$ so that $\xi_{n+1}^h \in S_h$. Then $|q_h(i, \rho) - q(i, \rho)| \to 0$ as $h \to 0$, uniformly in x. To present the claim terms, we determine the next state $(\xi_{n+1}^h, \alpha_{n+1}^h)$ by noting:

1. No claims occur in $[t_n^h, t_{n+1}^h)$ with probability $1 - \lambda \Delta t^h(x, i, u) + o(\Delta t^h(x, i, u))$; we determine $(\xi_{n+1}^h, \alpha_{n+1}^h)$ by transition probability $p_D^h(\cdot)$ as in (3.14).

2. There is a claim in $[t_n^h, t_{n+1}^h)$ with probability $\lambda \Delta t^h(x, i, u) + o(\Delta t^h(x, i, u))$, we determine $(\xi_{n+1}^h, \alpha_{n+1}^h)$ by
$$\xi_{n+1}^h = \xi_n^h - q_h(i, \rho), \alpha_{n+1}^h = \alpha_n^h.$$

Note that as noted before, we need to carefully redefine the notion of local consistency for Markov chain approximation to jump diffusion processes with regime switchings.

Definition 3.1. A controlled Markov chain $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ is said to be locally consistent with (2.10), if $\Delta t^h(x, i, u) \to 0$ as $h \to 0$ uniformly in x, i, and u such that

1. there is a transition probability $p_D^h(\cdot)$ that is locally consistent with (3.1) in the sense that (3.2) holds.

2. there is a $\delta^h(x, i, u) = o(\Delta t^h(x, i, u))$ such that $\{p^h((x, i), (y, j))|u\}$, the one-step transition probability is given by
$$\begin{aligned} p^h(((x, i), (y, j))|u) = (1 - \lambda \Delta t^h(x, i, u) + \delta^h(x, i, u)) p_D^h((x, i), (y, j)) \\ + (\lambda \Delta t^h(x, i, u) + \delta^h(x, i, u)) \Pi\{\rho : q_h(i, \rho) = x - y\}. \end{aligned} \tag{3.18}$$

Furthermore, the system of dynamic programming equations is a modification of (3.10). That is

$$V^h(x,i)$$

$$= \begin{cases} \sup_{u_2\in U_2}\Big\{\inf_{u_1\in U_1}\Big[(1-\lambda\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\sum_{y,j}(p^h_D((x,i),(y,j))|u) \\ \quad\times V^h(y,j)+(\lambda_1\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x-u_1q_h(i,\rho_1),i)\Pi(d\rho_1) \\ \quad+(\lambda_2\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x+u_2q_h(i,\rho_2),i)\Pi(d\rho_2) \\ \quad+f(x,i,u)\Delta t^h(x,i,u)\Big]\Big\}, & \text{for } x\in S_h^o, \\ \tilde{h}(x,i), & \text{for } x=\partial S_h. \end{cases} \tag{3.19}$$

4 Convergence of Numerical Approximation

4.1 Interpolations of Approximation Sequences

In this subsection, we deal with the piecewise constant interpolation with appropriately chosen interpolation intervals based on the constructed Markov chain approximation. Recalling the (3.3) in last section, we use (ξ_n^h,α_n^h) to approximate the continuous-time process $(x(\cdot),\alpha(\cdot))$ and define the continuous-time interpolations $(\xi^h(\cdot),\alpha^h(\cdot))$ and $u^h(\cdot)$. We further define the first exit time of $\xi^h(\cdot)$ from S_h^o by $\tau_h=t^h_{\eta_h}$. Let the discrete times at which claims occur be denoted by ν_j^h, $j=1,2,\ldots$ Then we have

$$\xi^h_{\nu^h_{j-1}}-\xi^h_{\nu^h_j}=q_h(\alpha^h_{\nu^h_{j-1}},\rho).$$

Define $\mathcal{D}_n^h$ as the smallest σ-algebra of $\{\xi_k^h,\alpha_k^h,u_k^h,H_k^h,k\le n;\nu_k^h,\rho_{1,k}^h,\rho_{2,k}^h:\nu_k^h\le t_n\}$. Then τ_h is a $\mathcal{D}_n^h$-stopping time. Using the interpolation process, we can rewrite (3.4) as

$$J^h(x,i,u^h)=E_{x,i}\left[\int_0^{\tau_h}e^{-rt}\left[f(\xi^h(t),\alpha^h(t),u^h(t))dt+\tilde{h}(\xi^h(\tau_h^-),\alpha^h(\tau_h^-))\right]\right]. \tag{4.1}$$

Let $\xi_0^h=x$, $\alpha_0^h=\alpha$, and use E_n^h to denote the expectation conditioned on the information up to time n, that is, conditioned on $\mathcal{D}_n^h$. Note that $\mathcal{U}^h$ defined by (3.5) and (3.6) is equivalent to the collection of all piecewise constant admissible controls with respect to $\mathcal{D}_n^h$. Let H_n^h denote the event that $(\xi_{n+1}^h,\alpha_{n+1}^h)$ is determined by the case of "no claim occurs" and use T_n^h to denote the event of "one claim occurs". Let $I_{H_n^h}$ and $I_{T_n^h}$ be corresponding indicator functions, respectively. Then $I_{H_n^h}+I_{T_n^h}=1$. Thus, we can write

$$\begin{aligned}\xi_n &= x+\sum_{k=0}^{n-1}[\Delta\xi_k^h I_{H_k^h}+(\Delta\xi_k^h(1-I_{H_k^h}))] \\ &= x+\sum_{k=0}^{n-1}E_k^h\Delta\xi_k^h I_{H_k^h}+\sum_{k=0}^{n-1}(\Delta\xi_k^h-E_k^h\Delta\xi_k^h)I_{H_k^h}+\sum_{k=0}^{n-1}(\Delta\xi_k^h(1-I_{H_k^h})).\end{aligned} \tag{4.2}$$

Denote

$$\begin{aligned}M_n^h &= \sum_{k=0}^{n-1}(\Delta\xi_k^h-E_k^h\Delta\xi_k^h)I_{H_k^h}, \\ R_n^h &= -\sum_{k=0}^{n-1}(\Delta\xi_k^h(1-I_{H_k^h}))=\sum_{j=1,2}\sum_{k:\nu_k\le n}(-1)^{j+1}q(\alpha^h(\nu_k),\rho_{j,k})u_{j,k}^h(\nu_k),\end{aligned} \tag{4.3}$$

where M_n^h is a martingale with respect to $\mathcal{D}_n^h$. We attempt to represent $M^h(t)$ similar to the diffusion term in (2.10). Define $W^h(\cdot)$ as

$$W^h(t) = \sum_{k=0}^{n-1} (\Delta\xi_k^h - E_k^h \Delta\xi_k^h)/\sigma(\alpha_k^h)\xi_k^h, = \int_0^t \sigma^{-1}(\alpha^h(s))\xi^h(s)dM^h(s). \tag{4.4}$$

The local consistency leads to

$$\begin{aligned}\sum_{k=0}^{n-1} E_k^h \Delta\xi_k^h I_{H_k^h} &= \sum_{k=0}^{n-1}((\mu(\alpha_k^h)\xi_k^h + c^{(1)}(\alpha_k^h) - g(u_{1,k}^h) - c^{(2)}(\alpha_k^h) + g(u_{2,k}^h))\Delta t_k^h + o(\Delta t_k^h))I_{H_k^h} \\ &= \sum_{k=0}^{n-1}((\mu(\alpha_k^h)\xi_k^h + c^{(1)}(\alpha_k^h) - g(u_{1,k}^h) - c^{(2)}(\alpha_k^h) + g(u_{2,k}^h))\Delta t_k^h + o(\Delta t_k^h)) \\ &\quad -(\max_{k'\le n} \Delta t_{k'}^h)O(\sum_{k=0}^{n-1} I_{T_k^h}).\end{aligned} \tag{4.5}$$

Since $(\max_{k'\le n} \Delta t_{k'}^h)O(\sum_{k=0}^{n-1} I_{T_k^h}) \to 0$ in probability as $h \to 0$, the term involving $I_{H_k^h}$ can be dropped without affecting the limit in (4.5). Combining (4.5)-(4.4), we rewrite (4.2) by

$$\begin{aligned}\xi^h(t) &= x + \int_0^t (\mu(\alpha^h(s))\xi^h(s) + c^{(1)}(\alpha^h(s)) - g(u_1^h) - c^{(2)}(\alpha^h(s)) + g(u_2^h))ds \\ &\quad + \int_0^t \sigma(\alpha^h(s))\xi^h(s)dW^h(s) - R^h(t) + \varepsilon^h(t) \\ R^h(t) &= \sum_{j=1,2}\sum_{\nu_n\le t} (-1)^{j+1} q(\alpha^h(\nu_n), \rho_{j,n})u_{j,n}^h(\nu_n),\end{aligned} \tag{4.6}$$

where $\varepsilon^h(t)$ is a negligible error satisfying

$$\lim_{h\to\infty} \sup_{0\le t\le T} E|\varepsilon^h(t)| \to 0 \text{ for any } 0 < T < \infty. \tag{4.7}$$

We can also rewrite (4.2) as

$$\begin{aligned}X(t) = x + \int_0^t (\mu(\alpha(s))X(s) + c^{(1)}(\alpha(s)) - g(u_1) - c^{(2)}(\alpha(s)) + g(u_2))ds \\ + \int_0^t \sigma(\alpha(s))X(t)dW(s) - R(t),\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}R(t) &= \sum_{j=1,2}\sum_{\nu_n\le t} (-1)^{j+1} q(\alpha(\nu_n), \rho_{j,n})u_j(\nu_n) \\ &= u_1(t)\int_{\mathbb{R}_+} q(\alpha(t), \rho_1)N(dt, d\rho_1) - u_2(t)\int_{\mathbb{R}_+} q(\alpha(t), \rho_2)N(dt, d\rho_2).\end{aligned}$$

Let $\mathcal{F}_t^h$ be a filtration, which denotes the minimal σ-algebra that measures

$$\{\xi^h(s), \alpha^h(\cdot), m_s^h(\cdot), W^h(s), N^h(s), s \le t\}. \tag{4.9}$$

Use Γ^h to denote the set of admissible relaxed controls $m^h(\cdot)$ with respect to $(\alpha^h(\cdot), W^h(\cdot))$ such that $m_t^h(\cdot)$ is a fixed probability measure in the interval $[t_n^h, t_{n+1}^h)$ given $\mathcal{F}_t^h$. Then $\Gamma^h = \Gamma_1^h \times \Gamma_2^h$

is a larger control space containing $\mathcal{U}^h$. With the notation of relaxed control given above, we can write (4.6), (4.1), the upper and lower value function as

$$\begin{aligned}\xi^h(t) = x \ &+ \int_0^t \int_{\Gamma_1^h \times \Gamma_2^h} (\mu(\alpha^h(s))\xi^h(s) + c^{(1)}(\alpha^h(s)) - g(\phi_1) - c^{(2)}(\alpha^h(s)) + g(\phi_2)) \\ &\times m_s^h(d\phi_1 \times d\phi_2)ds + \int_0^t \sigma(\alpha^h(s))\xi^h(s)dW^h(s) \\ &- \int_{\Gamma_1^h} m_{s,1}^h(d\phi_1)ds \int_{\mathbb{R}_+} q(\alpha(t),\rho_1)N(dt,d\rho_1) \\ &+ \int_{\Gamma_2^h} m_{s,2}^h(d\phi_2)ds \int_{\mathbb{R}_+} q(\alpha(t),\rho_2)N(dt,d\rho_2) + \varepsilon^h(t),\end{aligned} \tag{4.10}$$

$$J^h(x,i,u^h) = J^h(x,i,m^h) = J^h(x,i,m_1^h,m_2^h) \tag{4.11}$$

and

$$\begin{aligned}V^{h,+}(x,i) &= \inf_{m_1^h \in \Gamma_1^h} \sup_{m_2^h \in \Gamma_2^h} J^h(x,i,m_1^h,m_2^h) \\ V^{h,-}(x,i) &= \sup_{m_2^h \in \Gamma_2^h} \inf_{m_1^h \in \Gamma_1^h} J^h(x,i,m_1^h,m_2^h)\end{aligned} \tag{4.12}$$

To proceed, we need one more assumption.

(A2) Let $\widehat{\tau}(\phi) = \infty$, if $\phi(t) \in G^o$, for all $t < \infty$, otherwise, define $\widehat{\tau}(\phi) = \inf\{t : \phi \notin G^o\}$. The function $\widehat{\tau}(\cdot)$ is continuous (as a map from $D[0,\infty)$, the space of functions that are right continuous and have left limits endowed with the Skorohod topology to the interval $[0,\infty]$ (the extended and compactified positive real numbers)) with probability one relative to the measure induced by any solution to (4.8) with initial condition (x,α).

4.2 Convergence of Approximating Markov Chains

Lemma 4.1. *Using the transition probabilities $\{p^h(\cdot)\}$ defined in* (3.2) *and* (3.18)*, the interpolated process of the constructed Markov chain $\{\alpha^h(\cdot)\}$ converges weakly to $\alpha(\cdot)$, the Markov chain with generator $Q = (q_{i\ell})$.*

Proof. The proof can be obtained similar to (Yin et al., 2003, Theorem 3.1). □

Theorem 4.2. *Let the approximating chain $\{\xi_n^h, \alpha_n^h, n < \infty\}$ constructed with transition probabilities defined in* (3.14) *be locally consistent with* (2.10)*, $m^h(\cdot)$ be the relaxed control representation of $\{u_n^h, n < \infty\}$, $(\xi^h(\cdot), \alpha^h(\cdot))$ be the continuous-time interpolation defined in* (3.14)*, and $\{\widetilde{\tau}_h\}$ be a sequence of $\mathcal{F}_t^h$-stopping times. Then $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), W^h(\cdot), N^h(\cdot), \widetilde{\tau}_h\}$ is tight.*

Proof. Using one point compactification, $\widetilde{\tau} \in [0,\infty]$. In view of Lemma 4.1, $\{\alpha^h(\cdot)\}$ is tight. Let $T < \infty$, and let $\widetilde{\nu}_h$ be an $\mathcal{F}_t$-stopping time that is not larger than T. Then for $\delta > 0$,

$$E_{\widetilde{\nu}_h}^{u^h}(W^h(\widetilde{\nu}_h + \delta) - W^h(\widetilde{\nu}_h))^2 = \delta + \widetilde{\varepsilon}_h, \tag{4.13}$$

where $\widetilde{\varepsilon}_h \to 0$ uniformly in $\widetilde{\nu}_h$. Taking $\limsup_{h\to 0}$ followed by $\lim_{\delta \to 0}$ yield the tightness of $\{W^h(\cdot)\}$. A similar argument yields the tightness of $M^h(\cdot)$. The sequence $\{m^h(\cdot)\}$ is tight s-ince its range space is compact. In view of (Kushner and Dupuis, 2001, Theorem 9.2.1), the sequence $\{N^h(\cdot)\}$ is tight because the mean number of claims on any bounded interval $[t, t+s]$ is bounded and $\lim_{\delta\to 0}\inf_{h,n} P\{\nu^h_{n+1} - \nu^h_n > \delta | \text{data up to } \nu^h_n\} = 1$. This also implies the tightness of $\{R^h(\cdot)\}$. These results and the boundedness of $c(\cdot)$ and $u(\cdot)$ implies the tightness of $\{\xi^h(\cdot)\}$. Thus, $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), W^h(\cdot), N^h(\cdot), \widetilde{\tau}_h\}$ is tight. □

To proceed, the convergence of the approximation sequence to the regime-switching jump diffusion process and the approximation to the upper and lower value of the game are confirmed. We will derive the following two theorems, whose proof are provided in the appendix.

Theorem 4.3. *Let $(\xi(\cdot), \alpha(\cdot), u(\cdot), W(\cdot), N(\cdot), \widetilde{\tau})$ be the limit of weakly convergent subsequence and $\mathcal{F}_t$ the σ-algebra generated by $\{x(s), \alpha(s), u(s), W(s), N(s), s \le t, \widetilde{\tau} I_{\{\widetilde{\tau}<t\}}\}$. Then $W(\cdot)$ and $N(\cdot)$ are a standard $\mathcal{F}_t$-Wiener process and Poisson measure, respectively, and $\widetilde{\tau}$ is an $\mathcal{F}_t$-stopping time and $u(\cdot)$ is an admissible control. Let the claim times and claim sizes of $N(\cdot)$ be denoted by ν_n, ρ_n. Then,* (4.8) *is satisfied.*

Theorem 4.4. *Assume* (A1) *and* (A2). *$V^{h,+}(x,i)$, $V^{h,-}(x,i)$, $V^+(x,i)$ and $V^-(x,i)$ are value functions defined in* (3.7), (3.8), (2.17) *and* (2.18), *respectively. Then we have*

$$\lim_{h\to 0} V^{h,-}(x,i) = V^-(x,i), \tag{4.14}$$

$$\lim_{h\to 0} V^{h,+}(x,i) = V^+(x,i). \tag{4.15}$$

4.3 Existence of Saddle Points

To begin with, we need construct a new local consistent Markov chain by using central finite difference scheme. The new constructed Markov chain has different transition probabilities from the setup in (3.14) and is only for analysis purpose. The transition probabilities follow

$$\begin{aligned}
&\tilde{p}^h((x,i),(x+h,i)|u) = \frac{\sigma^2(i)/2 + h[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]/2}{\overline{D} - rh^2},\\
&\tilde{p}^h((x,i),(x-h,i)|u) = \frac{\sigma^2(i)/2 - h[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]/2}{\overline{D} - rh^2},\\
&\tilde{p}^h((x,i),(x,j)|u) = \frac{h^2}{\overline{D} - rh^2} q_{ij}, \quad \text{for} \quad j \ne i,\\
&\tilde{p}^h(\cdot) = 0, \quad \text{otherwise},\\
&\Delta t^h(x,i,u) = \frac{h^2}{\overline{D}},
\end{aligned} \tag{4.16}$$

with $\overline{D} = \sigma^2(i) + h^2(r - q_{ii})$ being well defined. To ensure the feasibility of transition probabilities, we require

$$h \le \frac{\sigma^2(i)}{\sup_u[\mu(i)x + c^{(1)}(i) - g(u_1) - c^{(2)}(i) + g(u_2)]}, \ \forall i \in \mathcal{M}. \tag{4.17}$$

Next, we prove a result of minimax principle in the game, which can be easily obtained using the results in Sion (1958). The proof will be omitted. The following lemma provides the conditions under which interchanging inf and sup is available.

Lemma 4.5. *Let G_1 and G_2 be compact spaces. Assume that a continuous function $f(x, y) \in G_1 \times G_2$ satisfies the concave-convex condition, that is, $f(\cdot, y)$ is convex $\forall y \in G_2$ and $f(x, \cdot)$ is concave $\forall x \in G_1$. Then*

$$\inf_{x \in G_1} \sup_{y \in G_2} f(x, y) = \sup_{y \in G_2} \inf_{x \in G_1} f(x, y). \tag{4.18}$$

Theorem 4.6. *Assume* (A1) *and* (A2). *For $x \in S_h$, a Markov chain is defined in* (4.16). *If* (4.17) *is satisfied, then there exists a saddle point*

$$V^{h,+}(x, i) = V^{h,-}(x, i), \text{ for } i \in \mathcal{M}. \tag{4.19}$$

The proof of this theorem is given in the appendix.

Theorem 4.7. *Assume the conditions in Theorem* 4.6 *are satisfied, then the saddle point exists as*

$$V^{+}(x, i) = V^{-}(x, i), \text{ for } i \in \mathcal{M}. \tag{4.20}$$

Proof. Since for the approximating Markov chain defined in (4.16), we can achieve

$$\lim_{h \to 0} V^{h,-}(x, i) = V^{-}(x, i) \tag{4.21}$$

$$\lim_{h \to 0} V^{h,+}(x, i) = V^{+}(x, i) \tag{4.22}$$

by using similar techniques in Theorem 4.4. By virtue of Theorem 4.6, we obtain that there exists a saddle point that

$$V^{+}(x, i) = V^{-}(x, i), \text{ for } i \in \mathcal{M}. \tag{4.23}$$

□

5 Numerical Examples

This section is devoted to several examples. For simplicity, we consider the case that the discrete event has two states. That is, the continuous-time Markov chain has two states with given claim size distributions. In addition, we assume the claim size distributions are identical in each regime. By using value iteration methods, we numerically solve the optimal control problems.

We impose the payoff function as the probability of the surplus difference between two players reaches the upper barrier before it reaches the lower barrier. Thus is, in the game, player 1 wants to maximize the probability while Player 2 wants to minimize the same probability. Then the cost function follows

$$J(x, i, u) = \mathbb{P}\{X(\tau) = b | X(0) = x\}. \tag{5.1}$$

Thus, the saddle point V satisfies the following coupled system of integro-differential HJI equations: for each $i \in \mathcal{M}$,

$$\begin{cases} \inf\limits_{u_1 \in U_1} \sup\limits_{u_2 \in U_2} \mathcal{L}^u V(x, i) - rV(x, i) = \sup\limits_{u_2 \in U_2} \inf\limits_{u_1 \in U_1} \mathcal{L}^u V(x, i) - rV(x, i) = 0, \quad x \in (a, b) \\ V(x, a) = 0, \\ V(x, b) = 1. \end{cases} \tag{5.2}$$

Based on the algorithm constructed above, we carry out the computation by value iterations. For $n \in Z^+$ and $i \in \mathcal{M}$, define the vectors

$$\begin{aligned} V_n^h &= \{V_n^h(a,1), V_n^h(a+h,1), \ldots, V_n^h(b,1), \ldots V_n^h(a,n_0), V_n^h(a+h,n_0), \ldots, V_n^h(b,n_0)\} \\ V^h &= \{V_n(a,1), V_n(a+h,1), \ldots, V_n(b,1), \ldots V_n(a,n_0), V_n(a+h,n_0), \ldots, V_n(b,n_0)\}. \end{aligned}$$

Using the method of value iteration, we obtain $V_n^h \to V^h$ as $n \to \infty$.

1. Set $n = 0$. $\forall x \in S_h$ and $i \in \mathcal{M}$, we set the initial policy $(u_{1,0}^h(x,i), u_{2,0}^h(x,i)) = (0,0)$.
2. Find improved values $u_{n+1}^h(x,i) = (u_{1,n+1}^h(x,i), u_{2,n+1}^h(x,i))$ by (3.19) and record the corresponding lower values.

$$\begin{aligned} &V_{n+1}^{h,-}(x,i) \\ &= \max_{u_2 \in U_2} \Big\{ \min_{u_1 \in U_1} \Big[(1 - \lambda \Delta t^h(x,i,u) + \delta^h(x,i,u)) e^{-r\Delta t^h(x,i,u)} \sum_{y,j} (p_D^h((x,i),(y,j))|u) \\ &\quad \times V_n^{h,-}(y,j) + (\lambda_1 \Delta t^h(x,i,u) + \delta^h(x,i,u)) e^{-r\Delta t^h(x,i,u)} \int_0^x V_n^{h,-}(x - u_1 y_1, i)\Pi(dy_1) \\ &\quad + (\lambda_2 \Delta t^h(x,i,u) + \delta^h(x,i,u)) e^{-r\Delta t^h(x,i,u)} \int_0^x V_n^{h,-}(x + u_2 y_2, i)\Pi(dy_2) \\ &\quad + f(x,i,u)\Delta t^h(x,i,u) \Big] \Big\} \end{aligned}$$

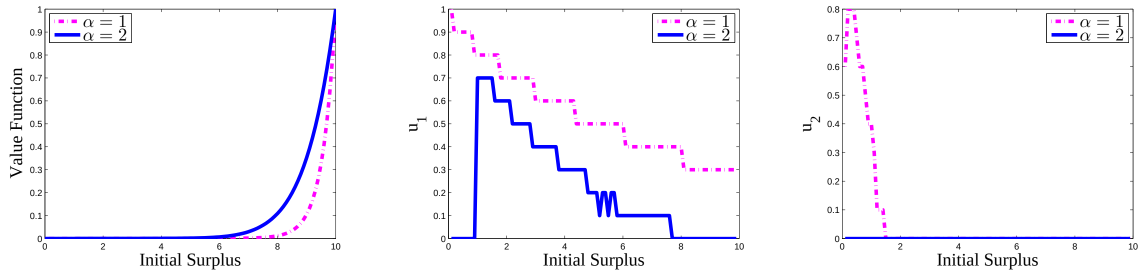
$$\begin{aligned} &u_{n+1}^h(x,i) \\ &= \operatorname*{argmax}_{u_2 \in U_2} \operatorname*{argmin}_{u_1 \in U_1} \Big[(1 - \lambda \Delta t^h(x,i,u) + \delta^h(x,i,u)) e^{-r\Delta t^h(x,i,u)} \sum_{y,j} (p_D^h((x,i),(y,j))|u) \\ &\quad \times V_n^{h,-}(y,j) + (\lambda_1 \Delta t^h(x,i,u) + \delta^h(x,i,u)) e^{-r\Delta t^h(x,i,u)} \int_0^x V_n^{h,-}(x - u_1 y_1, i)\Pi(dy_1) \\ &\quad + (\lambda_2 \Delta t^h(x,i,u) + \delta^h(x,i,u)) e^{-r\Delta t^h(x,i,u)} \int_0^x V_n^{h,-}(x + u_2 y_2, i)\Pi(dy_2) \\ &\quad + f(x,i,u)\Delta t^h(x,i,u) \Big] \end{aligned}$$

3. If $|V_{n+1}^h - V_n^h| >$ tolerance, then $n \to n+1$ and go to step 2; else the iteration stops.

The continuous-time Markov chain $\alpha(t)$ representing the discrete event state has the generator $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$, and takes values in $\mathcal{M} = \{1,2\}$. The claim severity distribution follows exponential distribution with density function $f(y) = 0.5e^{-0.5y}$. Furthermore, $\{\nu_{n+1} - \nu_n\}$ is a sequence of exponentially distributed random variables. Suppose the two companies has different claim densities. For company 1, the claim arrival rate $\lambda_1 = 4$, the premium rate depends on the discrete state with $c^{(1)}(1) = 0.05$ and $c^{(2)}(2) = 0.1$ in each regime. For company 2, the claim arrival rate $\lambda_1 = 6$, the premium rate depends on the discrete state with $c^{(2)}(1) = 0.02$ and $c^{(2)}(2) = 0.2$ in each regime. Corresponding to the different discrete states, the yield rate of the financial asset is $\mu(1) = 0.5$ and $\mu(2) = 1$, and the volatility of the financial market $\sigma(\alpha(t))$ is valued as $\sigma(1) = 0.1$ and $\sigma(2) = 1$. Let the discount rate $r = 0.05$, and the additional premium charged by the reinsurance company $\beta = 0.8$. We assume the game boundaries to be $a = 0$, and $b = 10$. That is, either the follower catches up the leader or the surplus difference is large enough, the game will be over. In the following examples, we will analyze the difference of two typical reinsurance premium rate principles.

Example 5.1. Variance Premium Principle. The premium rate principle is said to be Variance Premium Principle if

$$g(u_k) = (1 - u_k)\vartheta_1^{(k)} + \beta(1 - u_k)^2\vartheta_2^{(k)}, \text{ for } k = 1, 2.$$



5.1.1 Value function versus initial surplus

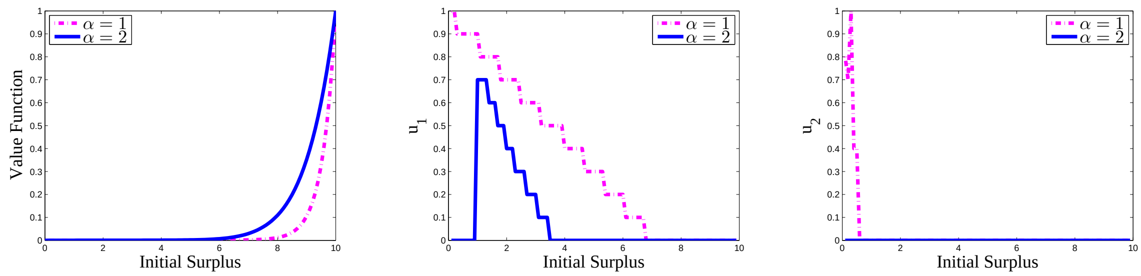
5.1.2 Reinsurance strategy of the game leader

5.1.3 Reinsurance strategy of the game follower

Figure 5.1: Variance Premium Principle

Example 5.2. Expectation Premium Principle. The premium rate priciple is said to be Expectation Premium Principle if

$$g(u_k) = (1 + \beta)(1 - u_k)\vartheta_1^{(k)}, \text{ for } k = 1, 2.$$



5.2.1 Value function versus initial surplus

5.2.2 Reinsurance strategy of the game leader

5.2.3 Reinsurance strategy of the game follower

Figure 5.2: Expectation Premium Principle

All the figures contain at least two lines since we consider the two-regime case. Figure 5.1.1 and Figure 5.2.1 show the probability of leader wins versus the surplus difference between the two companies under the variance premium principle and expectation premium principle. The curves in both cases are convex and increasing. We can also see that the leader will be more likely to win the game only if the surplus difference is beyond a relatively large level. This result may refer to the volatile return rate of the game follower, which has a higher expectation of the return rate than the game leader.

Figure 5.1.2 and Figure 5.2.2 represent the game leader's strategies in the two cases. We see that, in each regime, the retention level of the game leader is decreasing with respect to the surplus difference. With large surplus difference, which shows a big advantage to game follower, the game leader is less willing to be exposed to the risk and the retention goes to zero in the expectation premium principle case. With a relative small surplus advantage to the follower, the game leader does not need as low retention level on the reinsurance product as in the situation of larger surplus advantage. Figure 5.1.3 and Figure 5.2.3 represent the game follower's strategies in the two cases. In both premium principles, we see that the game follower is more likely to choose low retention level on reinsurance strategies, and does not want to be exposed to the risk of claim losses. It can be interpreted as the game follower is more relying on the reinsurance products to shorten the surplus difference with the leader.

Moreover, from Figures 5.1.2-5.1.3 and 5.2.2-5.2.3, we also find that the game follower has lower retention level than the game leader. That is, to achieve the goal, the company with less surplus pays more money on the premium of the reinsurance product to catch up with game leader and can be seen as risk averse. Particularly, in certain regimes, the game follower choose the retention level as 0 to completely remove the risk of large claim losses. Comparing with the variance premium principle and expectation premium principle, we see that the retention level is lower in the variance premium principle generally. Especially for the reinsurance strategies of the game follower, the maximal retention level is 0.8 whilst it can achieve 1 in the case of expectation premium principle. Furthermore, all of the four pictures show the effects of the regime-switchings. The strategy varies in different regimes due to the Markov switching.

From the numerical examples, we obtain dynamic strategies instead of static strategies for decision makers in the insurance industry driven by a dynamic business environment and increased competition between insurance companies. For diffusion models, optimal game strategies could be obtained as in Taksar and Zeng (2011). Nevertheless, jump diffusion models incorporated with market switching are more realistic. The solution of which is certainly more difficult than those considered in the existing literature. For this class of problems, designing numerical schemes for finding optimal strategies becomes a natural choice. Moreover, the discrete jumps add additional challenges in designing the numerical algorithm. Under the formulation of stochastic games, each company could adjust its optimal reinsurance strategies based on one's opponent. The continuously adjusted strategies are desirable to optimize the allocation of cash reserve for each company. These achieved optional strategies shown in the numerical examples will provide operation guidance for decision makers in the insurance and financial industries and reduce the risk of insurance and financial institutions incurring large losses through misinformed decision.

6 Concluding Remarks

We model and analyze the competitions between two insurance companies in the Markovian regime-switching environment. Each of the companies makes decisions on reinsurance policies to reduce the exposure to risk by adopting a reinsurance strategy among a set of available options to optimize the corresponding payoff function. The competition between these two insurance companies is formulated as a stochastic differential game with two players. A single payoff function is imposed, and one player devises an optimal strategy to maximize the expected payoff function while the other player is trying to minimize the same quantity. Using dynamic programming principle, we show that the upper and lower values of the game satisfy a coupled system of nonlinear integro-differential HJI equations. It is virtually impossible to obtain a closed-form solution. As a viable alternative, we use the Markov chain approximation method to obtain numerical solutions. The existence of the saddle point for this game problem is verified; the convergence of the approximation sequence to the jump diffusion process and the convergence of the approximation to the upper and lower value of the game are demonstrated. In future study, we can further analyze the stochastic differential games involving the investment strategies under regime-switching jump diffusion models. Considering the investment in different financial assets and reinsurance strategies, together with Markov regime-switching jump diffusions. The stochastic systems will be more realistic but more complicated.

A Appendix

A.1 Proof of Theorem 4.3

Proof. We shall show that $X(\cdot)$ is a solution of a stochastic differential equation with driving processes $\alpha(\cdot)$, $m(\cdot)$, $W(\cdot)$, and $N(\cdot)$. $\{\xi^h(\cdot),\alpha^h(\cdot),m^h(\cdot),W^h(\cdot),N^h(\cdot),\tilde{\tau}_h\}$ is tight, a weakly convergent subsequence can be extracted and indexed by h for notational simplicity. Denote by $(\xi(\cdot),\alpha(\cdot),m(\cdot),W(\cdot),N(\cdot),\tilde{\tau})$ the limit of the weakly convergent subsequence. By the Skorohod representation, we may assume that $\{\xi^h(\cdot),\alpha^h(\cdot),m^h(\cdot),W^h(\cdot),N^h(\cdot),\tilde{\tau}_h\}$ converges to $(\xi(\cdot),\alpha(\cdot),m(\cdot),$ $W(\cdot),N(\cdot),\tilde{\tau})$ w.p.1 and the convergence is uniform on compact set.

To proceed, we need to characterize $W(\cdot)$. Define

$$(l,m)_t=\int_0^t\int_U l(\zeta,s)m(d\phi ds). \tag{A.1}$$

for any real-valued and continuous functions $l(\cdot)$ on $U\times[0,\infty)$. Let $t>0$, $\delta>0$, p, q, $\{t_j:j\le p\}$, $l_i(\cdot)$ be given such that $t_j\le t\le t+\widetilde{t}$ for all $j\le p$, $\mathrm{P}(\tilde{\tau}_h=t_j)=0$, $l_i(\cdot)$ are continuous functions with compact support for $i\le q$. Let $\{\Gamma_i^\kappa,i\le\kappa\}$ be a sequence of nondecreasing partition of Γ such that $\Pi(\partial\Gamma_i^\kappa)=0$ for all i and all κ, where $\partial\Gamma_i^\kappa$ is the boundary of the set Γ_i^κ. As $\kappa\to\infty$, let the diameter of the sets Γ_i^κ go to zero. By (4.4), $W^h(\cdot)$ is an $\mathcal{F}_t-$martingale. Let $S(\cdot)$ be a real-valued and continuous function of its arguments with compact support. We have

$$ES(\xi^h(t_j),\alpha^h(t_j),W^h(t_j),(l_i,m^h)_{t_j},N(t_j,\Gamma_i^q),i\le q,j\le p,\tilde{\tau}I_{\{\tilde{\tau}\le t\}})[W^h(t+\widetilde{t})-W^h(t)]=0. \tag{A.2}$$

By using the Skorohod representation and the dominated convergence theorem, letting $h\to0$,

$$\begin{aligned}&ES(\xi^h(t_j),\alpha^h(t_j),W^h(t_j),(l_i,m^h)_{t_j},N(t_j,\Gamma_i^q),i\le q,j\le p,\tilde{\tau}I_{\{\tilde{\tau}\le t\}})[W^h(t+\widetilde{t})-W^h(t)]\\&\to ES(\xi^h(t_j),\alpha^h(t_j),W^h(t_j),(l_i,m^h)_{t_j},N(t_j,\Gamma_i^q),i\le q,j\le p,\tilde{\tau}I_{\{\tilde{\tau}\le t\}})[W(t+\widetilde{t})-W(t)].\end{aligned} \tag{A.3}$$

Thus, it follows that

$$ES(\xi^h(t_j),\alpha^h(t_j),W^h(t_j),(l_i,m^h)_{t_j},N(t_j,\Gamma_i^q),i\le q,j\le p,\tilde{\tau}I_{\{\tilde{\tau}\le t\}})[W(t+\widetilde{t})-W(t)]=0. \tag{A.4}$$

Since $W(\cdot)$ has continuous sample paths, (A.4) implies that $W(\cdot)$ is a continuous $\mathcal{F}_t$-martingale. Using the Skorohod representation and the dominated convergence theorem together with (4.13), we have

$$ES(\xi(t_j),\alpha(t_j),W(t_j),(l_i,m)_{t_j},i\le q,j\le p)[W^2(t+\delta)-W^2(t)-\delta]=0. \tag{A.5}$$

The quadratic variation of the martingale $W(t)$ is t, then $W(\cdot)$ is an $\mathcal{F}_t$-Wiener process.

To continue, we need to show that $N(\cdot)$ is an $\mathcal{F}_t-$Poisson measure. Let $\varphi(\cdot)$ be a continuous function on $\mathbb{R}_+$, and define the process

$$\Phi_N(t)=\int_0^t\int_{\mathbb{R}_+}\varphi(\rho)N(ds,d\rho).$$

Let $f(\cdot)$ be a continuous function with compact support, then

$$\begin{aligned}&ES(\xi^h(t_j),\alpha^h(t_j),W^h(t_j),(l_i,m^h)_{t_j},N(t_j,\Gamma_i^q),i\le q,j\le p,\tilde{\tau}I_{\{\tilde{\tau}\le t\}})\\&\times\Big[f(\Phi_N(t+\tilde{t}))-f(\Phi_N(t))-\lambda\int_t^{t+\tilde{t}}\int_{\mathbb{R}_+}[f(\Phi_N(s)+\theta(\rho))-f(\Phi_N(s))]\Pi(d\rho)ds\Big]=0.\end{aligned} \tag{A.6}$$

Equation (A.6) and the arbitrariness of $S(\cdot),p,q,t_j,\Gamma_j^q,f(\cdot)$ and $\theta(\cdot)$ imply that $N(\cdot)$ is an $\mathcal{F}_t-$Poisson measure.

For $\delta>0$, define the process $q(\cdot)$ by $q^{h,\delta}(t)=q^h(n\delta),t\in[n\delta,(n+1)\delta)$. Then, by the tightness of $\{\xi^h(\cdot),\alpha^h(\cdot)\}$, (4.10) can be rewritten as

$$\begin{aligned}\xi^h(t)=w&+\int_0^t\int_{\Gamma_1^h\times\Gamma_2^h}(\mu(\alpha^h(s))\xi^h(s)+c^{(1)}(\alpha^h(s))-g(\phi_1)-c^{(2)}(\alpha^h(s))+g(\phi_2))\\&\times m_s^h(d\phi_1\times d\phi_2)ds+\int_0^t\widehat{\sigma}(\alpha^{h,\delta}(s))u^{h,\delta}(s)dW^h(s)\\&-\int_{\Gamma_1^h}m_{s,1}^h(d\phi_1)ds\int_{\mathbb{R}_+}q(\alpha(t),\rho_1)N(dt,d\rho_1)\\&+\int_{\Gamma_2^h}m_{s,2}^h(d\phi_2)ds\int_{\mathbb{R}_+}q(\alpha(t),\rho_2)N(dt,d\rho_2)+\varepsilon^{h,\delta}(t),\end{aligned} \tag{A.7}$$

where

$$\lim_{\delta\to0}\limsup_{h\to0}E|\varepsilon^{h,\delta}(t)|=0. \tag{A.8}$$

Letting $h\to0$ and using the Skorohod representation, we obtain

$$\begin{aligned}E\Big|&\int_0^t\int_{\Gamma_1^h\times\Gamma_2^h}(\mu(\alpha^h(s))\xi^h(s)+c^{(1)}(\alpha^h(s))-g(\phi_1)-c^{(2)}(\alpha^h(s))+g(\phi_2))m_s^h(d\phi_1\times d\phi_2)ds\\&-\int_0^t\int_{\Gamma_1^h\times\Gamma_2^h}(\mu(\alpha(s))\xi(s)+c^{(1)}(\alpha(s))-g(\phi_1)-c^{(2)}(\alpha(s))+g(\phi_2))m_s^h(d\phi_1\times d\phi_2)ds\Big|=0\end{aligned} \tag{A.9}$$

uniformly in t with probability one. On the other hand, $\{m^h(\cdot)\}$ converges in the compact weak topology, that is, for any bounded and continuous function $l(\cdot)$ with compact support,

$$\int_0^\infty\int_{\Gamma_1^h\times\Gamma_2^h}l(\phi,s)m^h(d\phi ds)\to\int_0^\infty\int_{\Gamma_1^h\times\Gamma_2^h}l(\phi,s)m(d\phi ds). \tag{A.10}$$

Again, the Skorohod representation implies that as $h\to0$,

$$\begin{aligned}&\int_0^t\int_{\Gamma_1^h\times\Gamma_2^h}(\mu(\alpha(s))\xi(s)+c^{(1)}(\alpha(s))-g(\phi_1)-c^{(2)}(\alpha(s))+g(\phi_2))m_s^h(d\phi_1\times d\phi_2)ds\\&\to\int_0^t\int_{\Gamma_1^h\times\Gamma_2^h}(\mu(\alpha(s))\xi(s)+c^{(1)}(\alpha(s))-g(\phi_1)-c^{(2)}(\alpha(s))+g(\phi_2))m_s(d\phi_1\times d\phi_2)ds\end{aligned} \tag{A.11}$$

uniformly in t with probability one on any bounded interval.

Since $\xi^{h,\delta}(\cdot)$ and $\alpha^{h,\delta}(\cdot)$ are piecewise constant functions, we obtain

$$\begin{aligned}\int_0^t \widehat{\sigma}(\alpha^{h,\delta}(s))u^{h,\delta}(s)dW^h(s) &= \sum_{i=0}^{t/\delta} \widehat{\sigma}(\alpha^{h,\delta}(s))u^{h,\delta}(s)(W^h((i+1)\delta) - W^h(i\delta)) \\ &\to \int_0^t \widehat{\sigma}(\alpha^{\delta}(s))u^{\delta}(s)dW(s) \quad \text{as} \quad h \to 0\end{aligned} \tag{A.12}$$

with probability one. Combining (A.1)-(A.12), we have

$$X(t) = x + \int_0^t \int_U b(w(s), \alpha(s), u(s))m_s^h(d\phi)dt + \int_0^t \widehat{\sigma}(\xi^{\delta}(s), \alpha^{\delta}(s), u^{\delta}(s))dW(s) - R(t) + \varepsilon^{\delta}(t), \tag{A.13}$$

where $\lim_{\delta\to 0} E|\varepsilon^{\delta}(t)| = 0$. Finally, taking limits in the above as $\delta \to 0$, (4.8) is obtained. □

A.2 Proof of Theorem 4.4

Proof. We will only prove (4.14), and the proof of (4.15) is similar.

Suppose play 1 goes last. Given $\varepsilon > 0$, there exists a small $\Delta > 0$ and a ε-optimal maximizing rule $u_2^{\varepsilon,\Delta}(\cdot|m_1) \in \mathcal{L}_1(\Delta)$ such that $u_2^{\varepsilon,\Delta}(\cdot|m_1) \in \mathcal{L}_2(\Delta)$ follows the conditional probability law: for small $\zeta > 0$

$$\begin{aligned}&P\{u_2^{\varepsilon,\Delta}(n\Delta) = \phi_2 | u_2^{\varepsilon,\Delta}(n\Delta), j < n; W(s), \alpha(s), N(s), m_1(s), s < n\Delta\} \\ &= P\{u_2^{\varepsilon,\Delta}(n\Delta) = \phi_2 | u_2^{\varepsilon,\Delta}(n\Delta), j < n; W(j\zeta), \alpha(j\zeta), N(j\zeta), u_1^{\zeta}(j\zeta|m_1), j\zeta < n\Delta\} \\ &= \tilde{F}_{2,n}(\phi_2; u_2^{\varepsilon,\Delta}(n\Delta), j < n; W(j\zeta), \alpha(j\zeta), N(j\zeta), u_1^{\zeta}(j\zeta|m_1), j\zeta < n\Delta\},\end{aligned} \tag{A.14}$$

where $u_1^{\zeta}(\cdot|m_1)$ is player 1's strategy and it is constant on $[j\zeta, (j+1)\zeta)$. Recall that a control in $\mathcal{U}_2^h(1)$ is a constant on interval $[t_n^h, t_{n+1}^h)$ and depends on the passed information $u_1^h(s), s < t_n^h$. Then $u_2^{\varepsilon,\Delta}(\cdot|m_1) \in \mathcal{L}_2(\Delta)$ needs to be adapted for the constructed Markov chain (ξ_n^h, α_n^h). For small ζ, the adaptation of $u_2^{\varepsilon,\Delta}(\cdot|m_1)$, denoted by $u_2^{\varepsilon,h}(\cdot|m_1^h) \in \mathcal{U}_2^h(1)$, is represented by

$$\begin{aligned}&P\{u_2^{\varepsilon,h}(t_n^h) = \phi_2 | u_2^{\varepsilon,h}(t_j^h), j < n; W(s), \alpha(s), N(s), m_1(s), s < t_n^h\} \\ &= P\{u_2^{\varepsilon,h}(t_n^h) = \phi_2 | u_2^{\varepsilon,h}(t_j^h), j < n; W(j\zeta), \alpha(j\zeta), N(j\zeta), u_1^{\zeta}(j\zeta|m_1^h), j\zeta < t_n^h\} \\ &= \tilde{F}_{2,[t_n^h/\Delta]}(\phi_2; u_2^{\varepsilon,h}(t_j^h), j < n; W(j\zeta), \alpha(j\zeta), N(j\zeta), u_1^{\zeta}(j\zeta|m_1^h), j\zeta < t_n^h\}.\end{aligned} \tag{A.15}$$

Given $u_2^{\varepsilon,h}(\cdot|m_1^h)$ for play 2, player 1 selects a minimizing control. Let $u_1^{h,*}$ denote the player 1's minimizing choice in $\mathcal{U}_1^1(2)$ with the corresponding relaxed control representation $m_1^{h,*}$. Moreover, let $m^h = (m_1^h, m_2^h)$ be the relaxed control representation of $(u_1^{h,*}(\cdot), u_2^{\varepsilon,\Delta}(\cdot|u_1^{h,*}))$.

Note that the cost function $J^h(x, i, m^h)$ is given by (4.11). By virtue of Theorem 4.2, each sequence $\{\xi^h(\cdot), \alpha^h(\cdot), m^h(\cdot), W^h(\cdot), N^h(\cdot), \tau_h\}$ has a weakly convergent subsequence with the limit satisfying (4.8). Abusing notation, still index the convergent subsequence by h with the limit denoted by $(x(\cdot), \alpha(\cdot), m(\cdot), W(\cdot), N(\cdot), \widetilde{\tau})$. By assumption (A2), $\{\tau_h\}$ is uniformly integrable, it guarantees that the exit time of $x(\cdot)$ from G^o is $\widetilde{\tau} = \tau$. Using the Skorohod representation and the weak convergence, as $h \to 0$, we obtain

$$J^h(x, i, m^h) \to J(x, i, m) \quad \text{as} \quad h \to 0. \tag{A.16}$$

Furthermore, by virtue of the Skorohod representation,

$$\begin{aligned}&\tilde{F}_{2,n}(\phi_2; u_2^{\varepsilon,h}(j\Delta), j < n; W(j\zeta), \alpha(j\zeta), N(j\zeta), u_1^{\zeta}(j\zeta|m_1^{h,*}), j\zeta < n\Delta\},\\ &\to \tilde{F}_{2,n}(\phi_2; u_2^{\varepsilon,\Delta}(j\Delta), j < n; W(j\zeta), \alpha(j\zeta), N(j\zeta), u_1^{\zeta}(j\zeta|m_1), j\zeta < n\Delta\},\end{aligned} \tag{A.17}$$

with probability one as $h \to 0$. Thus,

$$(m_1^{h,*}(\cdot), u_2^{\varepsilon,h}(\cdot|m_1^{h,*})) \Rightarrow (m_1(\cdot), u_2^{\varepsilon,\Delta}(\cdot|m_1)) \text{ as } h \to 0.$$

Then we have

$$J^h(x, i, u_1^{h,*}(\cdot), u_2^{\varepsilon,h}(\cdot|u_1^{h,*})) \to J(x, i, m_1(\cdot), u_2^{\varepsilon,\Delta}(\cdot|m_1)) \quad \text{as} \quad h \to 0. \tag{A.18}$$

Therefore,

$$\begin{aligned}V^{h,-}(x,i) &= \sup_{u_2^h \in \mathcal{U}_2^h(1)} \inf_{u_1^h \in \mathcal{U}_1^h(2)} J^h(x, i, u_1^h, u_2^h)\\ &\ge \inf_{u_1^h \in \mathcal{U}_1^h(2)} J^h(x, i, u_1^h, u_1^h, u_2^{\varepsilon,h}(\cdot|u_1^h))\\ &= J^h(x, i, u_1^{h,*}, u_2^{\varepsilon,h}(\cdot|u_1^{h,*}))\\ &\to J(x, i, m_1(\cdot), u_2^{\varepsilon,\Delta}(\cdot|m_1))\end{aligned} \tag{A.19}$$

as $h \to 0$. On the other hand, for small Δ and largeΔ/ζ, we have

$$\inf_{m_1 \in \mathcal{U}_1} J(x, i, m_1(\cdot), u_2^{\varepsilon,\Delta}(\cdot|m_1)) \ge V^-(x,i) - \varepsilon. \tag{A.20}$$

Combining (A.19) and (A.20), we obtain

$$\liminf_{h\to 0} V^{h,-}(x,i) \ge V^-(x,i) - \varepsilon. \tag{A.21}$$

We proceed to prove the reverse inequality. We claim that

$$\limsup_{h} V^{h,-}(x,i) \le V^-(x,i). \tag{A.22}$$

For $\Delta > 0$ and small $h > 0$, there exists an $\varepsilon^{h,\Delta}$-optimal maximizing control $\overline{u}_2^{h,\Delta}(\cdot|m_1^h) \in \mathcal{U}_2^h(1)$ that takes only finite many values, that $\overline{u}_2^{h,\Delta}(\cdot|m_1^h) \in \mathcal{U}_2^h(1)$ is a constant on $[k\Delta, k\Delta + \Delta)$. Let $\overline{x}^h(\cdot)$ and $\overline{\tau}^h$ be the associated solution and stopping time. With the corresponding relaxed control representation $\overline{m}_2^{h,\Delta}(\cdot|m_1^h)$, we have

$$\inf_{m_1^h \in \Gamma_1^h} J^h(x, i, m_1^h, \overline{m}_2^{h,\Delta}(\cdot|m_1^h)) \ge V^{h,-}(x,i) - \varepsilon^{h,\Delta}, \tag{A.23}$$

where $\lim_{\Delta\to 0} \limsup_{h\to 0} \varepsilon^{h,\Delta} = 0$. Given $\overline{m}_2^{h,\Delta}(\cdot|m_1^h) \in \Gamma_2^h(1)$, there exists a minimizing control $\overline{m}_1^{h,\Delta} \in \Gamma_1^h(2)$. Then if $(\overline{m}_1^{h,\Delta}(\cdot), \overline{m}_2^{h,\Delta}(\cdot), \alpha(\cdot), W(\cdot), N(\cdot))$ converges weakly to $(\overline{m}_1^{\Delta}(\cdot), \overline{m}_2^{\Delta}(\cdot), \alpha(\cdot), W(\cdot), N(\cdot))$, we also have $(\overline{x}^h(\cdot), \overline{m}_1^{h,\Delta}(\cdot), \overline{m}_2^{h,\Delta}(\cdot), \alpha(\cdot), W(\cdot), N(\cdot), \overline{\tau}^h)$ converges weakly to $(x(\cdot), \overline{m}_1^{\Delta}(\cdot), \overline{m}_2^{\Delta}(\cdot), \alpha(\cdot), W(\cdot), N(\cdot), \overline{\tau})$, where $\overline{m}_1^{\Delta} \in \Gamma_1$ and $\overline{m}_2^{\Delta} \in \mathcal{L}_2(\Delta)$. Hence, as $h \to 0$,

$$J^h(x, i, m_1^{h,\Delta}, \overline{m}_2^{h,\Delta}(\cdot|m_1^{h,\Delta})) \to J^h(x, i, m_1^{\Delta}, \overline{m}_2^{\Delta}(\cdot|m_1^{\Delta})). \tag{A.24}$$

Moreover, $\overline{m}_1^{\Delta}$ is an optimal minimizing control in $\mathcal{U}_1$, given $\overline{m}_2^{\Delta} \in \mathcal{L}_2(\Delta)$. That is,

$$\inf_{m_1 \in \Gamma_1} J^h(x, i, m_1, \overline{m}_2^{\Delta}(\cdot|m_1)) = J(x, i, m_1^{\Delta}, \overline{m}_2^{\Delta}(\cdot|m_1^{\Delta})). \tag{A.25}$$

Therefore, combining (A.23) - (A.25), we have

$$\begin{aligned} V^{h,-}(x,i) &= \sup_{m_2^h\in\Gamma_2^h}\inf_{m_1^h\in\Gamma_1^h} J^h(x,i,m_1^h,m_2^h) \\ &\le \inf_{m_1^h\in\Gamma_1^h} J^h(x,i,m_1^h,\overline{m}_2^{h,\Delta}(\cdot|m_1^h))+\varepsilon^{h,\Delta} \\ &= J^h(x,i,m_1^{h,\Delta},\overline{m}_2^{h,\Delta}(\cdot|m_1^{h,\Delta}))+\varepsilon^{h,\Delta} \\ &\to J(x,i,m_1^{\Delta},\overline{m}_2^{\Delta}(\cdot|m_1^{\Delta}))-\varepsilon^{\Delta} \end{aligned} \tag{A.26}$$

as $h\to 0$. Moreover,

$$\begin{aligned} V^{h,-}(x,i) &\le J(x,i,m_1^{\Delta},\overline{m}_2^{\Delta}(\cdot|m_1^{\Delta}))-\varepsilon^{\Delta} \\ &= \inf_{m_1\in\Gamma_1} J^h(x,i,m_1,\overline{m}_2^{\Delta}(\cdot|m_1))-\varepsilon^{\Delta} \\ &\le \sup_{m_2\in\mathcal{L}_2(\Delta)}\inf_{m_1\in\Gamma_1} J^h(x,i,m_1,\overline{m}_2(\cdot|m_1))-\varepsilon^{\Delta} \\ &\to V^-(x,i) \end{aligned} \tag{A.27}$$

as $\Delta\to 0$. This implies

$$\limsup_h V^{h,-}(x,i)\le V^-(x,i). \tag{A.28}$$

Combing (A.21) and (A.28), (4.14) is obtained. □

A.3 Proof of Theorem 4.6

Proof. Recall (3.19), the upper and lower values can be written as

$$\begin{aligned} &V^{h,+}(x,i) \\ &= \inf_{u_1\in U_1}\Big\{\sup_{u_2\in U_2}[(1-\lambda\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\sum_{y,j}(\tilde{p}^h((x,i),(y,j))|u)V^{h,+}(y,j) \\ &\quad +(\lambda_1\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x-u_1q_h(i,\rho_1),i)\Pi(d\rho_1) \\ &\quad +(\lambda_2\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x+u_2q_h(i,\rho_2),i)\Pi(d\rho_2) \\ &\quad +f(x,i,u)\Delta t^h(x,i,u)]\Big\}, \end{aligned} \tag{A.29}$$

and

$$\begin{aligned} &V^{h,-}(x,i) \\ &= \sup_{u_2\in U_2}\Big\{\inf_{u_1\in U_1}[(1-\lambda\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\sum_{y,j}(\tilde{p}^h((x,i),(y,j))|u)V^{h,-}(y,j) \\ &\quad +(\lambda_1\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x-u_1q_h(i,\rho_1),i)\Pi(d\rho_1) \\ &\quad +(\lambda_2\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x+u_2q_h(i,\rho_2),i)\Pi(d\rho_2) \\ &\quad +f(x,i,u)\Delta t^h(x,i,u)]\Big\}. \end{aligned} \tag{A.30}$$

Define two functions $\psi^{h,+}(x,i,u_1,u_2)$ and $\psi^{h,-}(x,i,u_1,u_2)$ such that

$$\begin{aligned}\psi^{h,+}(x,i,u_1,u_2) &= (1-\lambda\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\sum_{y,j}(\tilde{p}^h((x,i),(y,j))|u)V^{h,+}(y,j) \\ &\quad +(\lambda_1\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x-u_1q_h(i,\rho_1),i)\Pi(d\rho_1) \\ &\quad +(\lambda_2\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x+u_2q_h(i,\rho_2),i)\Pi(d\rho_2) \\ &\quad +f(x,i,u)\Delta t^h(x,i,u),\end{aligned} \tag{A.31}$$

and

$$\begin{aligned}\psi^{h,-}(x,i,u_1,u_2) &= (1-\lambda\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\sum_{y,j}(\tilde{p}^h((x,i),(y,j))|u)V^{h,-}(y,j) \\ &\quad +(\lambda_1\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x-u_1q_h(i,\rho_1),i)\Pi(d\rho_1) \\ &\quad +(\lambda_2\Delta t^h(x,i,u)+\delta^h(x,i,u))e^{-r\Delta t^h(x,i,u)}\int_0^x V^h(x+u_2q_h(i,\rho_2),i)\Pi(d\rho_2) \\ &\quad +f(x,i,u)\Delta t^h(x,i,u).\end{aligned} \tag{A.32}$$

Then (A.29) and (A.30) can be rewritten as

$$V^{h,+}(x,i) = \inf_{u_1\in U_1}\sup_{u_2\in U_2}\psi^{h,+}(x,i,u_1,u_2), \tag{A.33}$$

and

$$V^{h,-}(x,i) = \sup_{u_2\in U_2}\inf_{u_1\in U_1}\psi^{h,-}(x,i,u_1,u_2). \tag{A.34}$$

From (4.16), we see that $(\tilde{p}^h((x,i),(y,j))|u)$ is separable in u_1 and u_2. Moreover, $f(x,i,u)$ satisfies the concave-convex condition. Hence, by virtue of Lemma 4.5, inf and sup can be interchanged in (A.33) and (A.34). That is,

$$\begin{aligned}V^{h,+}(x,i) &= \inf_{u_1\in U_1}\sup_{u_2\in U_2}\psi^{h,+}(x,i,u_1,u_2) = \sup_{u_2\in U_2}\inf_{u_1\in U_1}\psi^{h,+}(x,i,u_1,u_2), \\ V^{h,-}(x,i) &= \sup_{u_2\in U_2}\inf_{u_1\in U_1}\psi^{h,-}(x,i,u_1,u_2) = \inf_{u_1\in U_1}\sup_{u_2\in U_2}\psi^{h,-}(x,i,u_1,u_2).\end{aligned} \tag{A.35}$$

To proceed, we claim that

$$V^{h,+}(x,i) = \inf_{u_1\in U_1}\sup_{u_2\in U_2}\psi^{h,+}(x,i,u_1,u_2) \geq \sup_{u_2\in U_2}\inf_{u_1\in U_1}\psi^{h,-}(x,i,u_1,u_2) = V^{h,-}(x,i). \tag{A.36}$$

Without loss of generality, we let player 2 go first. Let u_2 be the strategy of player 2 , denote by $\widehat{u}_1(u_2)$ the best reply of player 1. Since $V^-(x,i) = \sup_{u_2\in U_2}\psi(x,i,u_1(u_2),u_2)$, for all $\varepsilon > 0$, given u_2, there exists $u_{2,\varepsilon}$ such that $\psi(x,i,u_1(u_{2,\varepsilon}),u_{2,\varepsilon}) \geq V^-(x,i)-\varepsilon$. Hence we have

$$\begin{aligned}V^+(x,i) &= \inf_{u_1\in U_1}\sup_{u_2\in U_2}\psi^{h,+}(x,i,u_1,u_2) \geq \inf_{u_1\in U_1}\psi^{h,+}(x,i,u_1,u_2) = \psi^{h,+}(x,i,u_1(u_{2,\varepsilon}),u_2) \\ &\geq V^-(x,i)-\varepsilon.\end{aligned}$$

Because of the arbitrary of ε, (A.36) is satisfied.

To complete the proof, let $\beta \geq 0$ and

$$\beta = \sup_{(x,i)\in(S_h,\mathcal{M})} (V^{h,+}(x,i) - V^{h,-}(x,i)). \tag{A.37}$$

Then $\forall(x,i) \in (S_h \times \mathcal{M})$, $V^{h,+}(x,i) - V^{h,-}(x,i) \leq \beta$, and there exists $(x_0, i_0) \in (S_h \times \mathcal{M})$ such that $V^{h,+}(x_0,i_0) - V^{h,-}(x_0,i_0) = \beta$. Therefore, for any $(x_0,i_0) \in (S_h \times \mathcal{M})$,

$$\begin{aligned}
&V^{h,-}(x_0,i_0)\\
&= \sup_{u_2\in U_2} \inf_{u_1\in U_1} \psi^{h,-}(x_0,i_0,u_1,u_2)\\
&= \inf_{u_1\in U_1} \sup_{u_2\in U_2} \psi^{h,-}(x_0,i_0,u_1,u_2)\\
&= \inf_{u_1\in U_1} \sup_{u_2\in U_2} (1-\lambda\Delta t^h(x_0,i_0,u) + \delta^h(x_0,i_0,u))e^{-r\Delta t^h(x_0,i_0)} \sum_{y,j} (\tilde{p}^h((x_0,i_0),(y,j))|u)\\
&\quad \times V^{h,-}(y,j) + (\lambda_1 \Delta t^h(x_0,i_0) + \delta^h(x_0,i_0))e^{-r\Delta t^h(x_0,i_0)} \int_0^{x_0} V^{h,-}(x_0 - q_h(i_0,\rho_1),i_0)\Pi(d\rho_1)\\
&\quad +(\lambda_2 \Delta t^h(x_0,i_0) + \delta^h(x_0,i_0))e^{-r\Delta t^h(x_0,i_0)} \int_0^{x_0} V^{h,-}(x_0 + q_h(i_0,\rho_2),i_0)\Pi(d\rho_2)\\
&\quad +f(x_0,i_0,u)\Delta t^h(x_0,i_0,u)\\
&\geq \inf_{u_1\in U_1} \sup_{u_2\in U_2} (1-\lambda\Delta t^h(x_0,i_0,u) + \delta^h(x_0,i_0,u))e^{-r\Delta t^h(x_0,i_0)} \sum_{y,j} (\tilde{p}^h((x_0,i_0),(y,j))|u)\\
&\quad \times (V^{h,+}(y,j) - \beta) + (\lambda_1 \Delta t^h(x_0,i_0) + \delta^h(x_0,i_0))e^{-r\Delta t^h(x_0,i_0)} \int_0^{x_0} V^{h,-}(x_0 - q_h(i_0,\rho_1),i_0)\\
&\quad \times \Pi(d\rho_1) + (\lambda_2 \Delta t^h(x_0,i_0) + \delta^h(x_0,i_0))e^{-r\Delta t^h(x_0,i_0)} \int_0^{x_0} V^{h,-}(x_0 + q_h(i_0,\rho_2),i_0)\Pi(d\rho_2)\\
&\quad +f(x_0,i_0,u)\Delta t^h(x_0,i_0,u)\\
&= \inf_{u_1\in U_1} \sup_{u_2\in U_2} \{\psi^{h,+}(x_0,i_0,u_1,u_2) - e^{-r\Delta t^h(x_0,i_0)}\beta\}\\
&= V^{h,+}(x_0,i_0) - e^{-r\Delta t^h(x_0,i_0)}\beta\\
&\geq V^{h,+}(x_0,i_0) - \beta\\
&= V^{h,-}(x_0,i_0)
\end{aligned}$$

In view of $r > 0$, we obtain $\beta = 0$. Hence the saddle point exists. □

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