Excessive Dynamic Trading: Propagation of Belief Shocks in Small Markets*

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Abstract

Can belief shocks make trading excessive ? We present a dynamic inventory management model in which belief shocks gradually propagate across traders, leading to the inflated trading activity which reduces traders' welfare. Trading can be socially beneficial because smoothing heterogeneous asset positions saves inventory costs. Without belief shocks, traders focus on the socially beneficial trading and the dispersion of the asset positions decreases monotonically. We show that one-shot belief shocks induce a speculative trading, which aggregates information but slows down the convergence of the asset positions. When traders' beliefs change quickly, the dispersion of the asset positions goes up, creating a cyclical pattern in volume. We also show that the high frequency trading amplifies the impact of belief shocks by making the speculation less costly, and therefore steering traders away from the socially beneficial trading motive.

Keywords: Asymmetric information, High-frequency trading, Information aggregation, Volume, Welfare.

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1 Introduction

Many, both in academia and practice, argue that the trading activity in financial markets appears excessive because it is difficult to rationalize the level and rich dynamic patterns of asset trading on the basis of traders' fundamental needs. One explanation for the excessiveness is that a large amount of trading may be generated by belief shocks. Indeed, the high correlation between volume and information flows has been well documented.[1] Two questions arise: (i) how do belief shocks affect the dynamics of trading? and (ii) does the trading generated by belief shocks reduce traders' welfare? Regarding the first question, it is important to identify an *internal* and *rational* economic mechanism because an arbitrary sequence of news and/or arbitrary reactions to news can explain almost anything. Specifically we ask: can a *one-shot* arrival of belief shocks followed by a *rational* learning process generate rich patterns of dynamic trading? We present a model in which belief shocks gradually propagate across traders, leading to the inflated and fluctuating trading activity. Based on this model, our answer to the second question is yes: the additional trading induced by belief shocks wastes resources – it is *inefficiently* excessive.

We present a finite-horizon inventory management model with information frictions. There is a single risky asset and each trader is hit by an idiosyncratic endowment shock at the beginning. Due to convex inventory costs, the higher cross-sectional dispersion of the asset positions wastes more resources. Hence trading that reduces (increases) the dispersion improves (hurts) traders' welfare. We assume that the large market where everyone can trade at once does not exist and that the trading process is local and gradual: in each period there are many trading venues, and at each venue, a small number of traders are randomly drawn from the population and trade within the group. After each round of trading, traders update their beliefs about the distribution of the asset positions across all traders, knowing that everyone is going through the same trading process. No pair of traders meets twice. The model is set up such that without belief shocks (i.e., all traders have the same belief

[1] See for example Chae 2005, Graham 2006, Engelberg and Parsons 2011, Peress 2014.

about the asset value) the cross-sectional dispersion of the asset positions decreases in each period. As the allocation approaches the first best of zero dispersion, volume also decreases. We use this dynamic path as a benchmark (**Lemma 1**).

To the benchmark we add belief shocks. Before the trading process starts, each trader observes a noisy signal about the asset value. The belief shocks are idiosyncratic, but they affect the equilibrium because the Law of Large Numbers does not apply in small markets. In each trading round, prices that clear a local market aggregate the information within each market. Because the dispersion of asset positions acts as an endogenous noise in the dynamic learning process, the extent of the information aggregation depends on how much traders speculate on their signals relative to trading motivated by the inventory management. Thus, not only the asset positions but also beliefs about the asset value endogenously change over time. In turn, changes in the distribution of the asset positions and beliefs affect traders' strategies through rational expectations.

We show that belief shocks and the subsequent learning process significantly alter the dynamics of trading. First, one-shot belief shocks persistently slow down the convergence of the asset positions relative to the benchmark, and the faster learning is associated with the slower convergence of the asset positions (**Proposition 2**). Second, when traders speculate aggressively and hence their beliefs change sufficiently quickly, the dispersion of their asset positions goes up. As a result, the volume can temporarily go up, creating a cyclical pattern (**Proposition 3**). **Figure 1** shows a time-series of the volume. In the benchmark, the volume monotonically approaches zero (the red dotted line labeled as "w/o belief shocks"). With belief shocks, the volume is persistently higher and temporarily goes up without any external shocks (the blue solid line). Thus, the model can rationalize an inflated and non-monotonic trading dynamics by one-shot belief shocks.

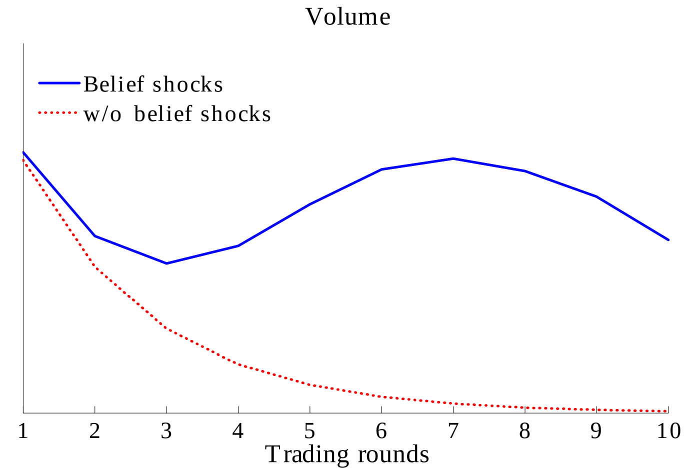

Figure 1. The time series of the volume.
Notes. The number of traders in each local venue is $n + 1 = 4$.
Other parameter values are given in the Appendix.

Our analysis sheds a new light on the role of belief shocks in financial markets. In many finance models, idiosyncratic shocks do not affect the equilibrium. This largely reflects the view that idiosyncratic shocks are eliminated by the Law of Large Numbers in a large economy.[2] We offer an alternative view: the trading activity is local in that the number of traders *directly interacting at a time* is negligible compared to the total number of traders. Because idiosyncratic shocks are averaged only within a small group, they propagate across traders over time. Our focus on idiosyncratic shocks is the extreme opposite of the standard approach: while the standard approach purges idiosyncratic shocks in a single large market, we introduce many small markets to study a dynamic propagation of idiosyncratic shocks, but purge aggregate shocks from the model.

The inefficiency in our model is caused by the rationality: if all traders (irrationally) ignored the signals, trading would not be excessive (i.e., the lower line in **Figure 1**). The model highlights the negative externality of idiosyncratic belief shocks in small markets. Each trader is forward-looking and optimizes his dynamic trading strategy by using a noisy but informative signal in a rational (Bayesian) way. While traders understand that the distribution of the asset positions becomes more dispersed and converges more slowly because

[2] Hellwig (1980) is an example where idiosyncratic shocks do not affect the equilibrium.

they speculate on noisy signals, they fail to internalize welfare costs of such speculation.

The model identifies a dynamic leverage mechanism that amplifies the impact of belief shocks. For given belief shocks, the more trading rounds are anticipated, the more amplified is the non-monotonicity of the dispersion and volume. With more future trading rounds, traders put a larger weight on future payoffs than on the current payoff. Future payoffs depend on future prices, which are *more correlated* with the asset value than current prices are, because of the non-stationary learning process. Therefore, as the importance of future trading rounds increases, the signals are given more leverage for forecasting future prices. Accordingly, traders speculate more relative to the inventory management in early rounds. As a result, traders will have relatively more dispersed asset positions and more symmetric beliefs in later rounds. Because there are less room for the speculation and more gains from the inventory management, the convergence of the asset positions becomes faster. Thus, the slower convergence/faster learning in early rounds and the subsequent faster convergence/slower learning are a rational response to more trading opportunities. The aggressive speculation in early rounds arise precisely because traders rationally anticipate that they will have trading opportunities when beliefs become more symmetric after the speculation.

Our model shows that when markets are segmented, giving more trading opportunities to traders can generate more volume by endogenously changing traders' objectives. While we do not explicitly model an intermediation sector, the result is suggestive for the industrial organization of financial markets: intermediaries who profit from the volume may find such a market structure attractive at the expense of traders.

To characterize an equilibrium in a decentralized trading model with heterogeneous information, we need to keep track of a joint distribution of asset positions and beliefs. This is a technically hard and intricate problem. We use two assumptions to overcome the issue of the dimensionality: (i) there is a known bound on the number of trading rounds, (ii) all the random variables are normally distributed. This allows us to describe the dynamics of the joint distribution as a solution to a simple fixed point problem.

Related literature. Our model captures a potential implication of the high-frequency trading: giving more trading opportunities may endogenously distort traders' objectives away from a socially beneficial one. The literature on the high-frequency trading mostly focused on the speed *difference* among traders (Biais et al 2014, Ait-Sahalia and Saglam 2014), and abstracted from the dynamic learning. In these models, the welfare implication requires taking a stand on how to weight different classes of traders because fast traders typically win at the expense of slow traders. Our work complements the literature by studying the welfare implication of giving *all* traders more opportunities to trade. In our ex ante symmetric environment, the welfare implication is unambiguous: when trading becomes excessive, more inventory costs are wasted and all traders are worse off.

Many market-microstructure models directly assume a volume process (e.g. noise traders in Kyle 1985). This left us in a situation with "trading volume whose size, function, and operation we do not understand" (Cochrane 2013).[3] In our model, trading volume is not a side show, but essential in improving welfare. We show that idiosyncratic belief shocks in small markets can prevent the volume from performing its socially beneficial function. To our knowledge, our work is the first normative assessment of the volume in the presence of non-stationary learning. Empirically, Bessembinder et al (1996) document an inverted-U pattern in volume *across days* in a week, while Foster and Viswanathan (1993) find the U-shaped *intraday* pattern in volume. Bernhardt and Miao (2004) present a model which can explain the U-shaped pattern, but they point out that a sequential arrival of information is necessary. Our model can explain the non-monotonic patterns without relying on the sequential exogenous shocks.

This paper also contributes to the literature on the dynamic information aggregation.[4]

[3] The full sentence from which the quote was taken is "But as with active management, perhaps we should work just a little harder before dismissing the hundreds of years of trading activity, and the entire existence of the New York Stock Exchange, Chicago Mercantile Exchange, and other markets, as monuments to human folly, or before advocating regulations such as transactions taxes —the perennial favorite answer in search of a question—to reduce trading volume whose size, function, and operation we do not understand." (page 44, Cochrane 2013).

[4] See, for example, Amador and Weill (2012), Duffie et al (2009, 2013), Iyer et al (2011), Ostrovsky (2012).

The most closely related work is Golosov et al (2013). They study a model of dynamic asset trading with asymmetric information, in which traders have two motives for trading as in our model. Their model has an infinite time horizon, and they focus on long-run consequences of the one-sided learning. Our model has a finite time horizon and everyone is learning. This allows us to study short-run consequences of the dynamic information aggregation process.

The rest of this paper is organized as follows. Section 2 describes a model environment, a trading rule, and a solution concept. We also set up a benchmark without belief shocks (Lemma 1). Section 3 presents a two-period example to illustrate the key mechanism (Lemma 2, 3). Section 4 contains main results (Proposition 0, 1, 2, 3). Section 5 concludes. The Appendix contains all proofs.

2 Model

We describe a model environment, a trading rule, and a dynamic solution concept in the next three subsections. The last subsection presents a benchmark without belief shocks.

2.1 Environment

The economy has two assets and a continuum of risk neutral traders. A risky asset (henceforth, the asset) is perfectly divisible and has uncertain payoff. There is a convex cost of holding the asset. This creates gains from smoothing the asset positions by using a non-risky asset (henceforth, the money) as a means of exchange. Each trader starts with a different amount of the asset and an independent noisy signal about its value. Thus, each trader has two pieces of private information: the signal and the position of the asset.

The trading is locally intermediated in the following sense. There is a continuum of locations. In each location, $n+1$ traders are randomly drawn from the population distribution at each time. A finite number $n \geq 2$ is fixed.[5] Each trader submits his demand for the asset,

[5] We focus on $n \geq 2$ so that smoothing can occur in the benchmark case.

which is explicitly conditioned on the price he must pay (the amount of the money exchanged for one unit of the asset). In each location, given submitted orders, *a price for each trader* $\{p_i\}_{i=1}^{n+1}$ is determined such that a market is "locally cleared". In each period, the game ends with some probability. When the game does not end, each trader is matched with a new n traders randomly drawn from the population. Following the literature on dynamic random matching, we assume that no pair of traders meet twice. As the trading process continues, the cross-sectional joint distribution of the asset positions and traders' beliefs about the asset value endogenously change. Each trader rationally forms an expectation about how the distribution evolves over time, and knows that he or she will trade with a random sample of n traders whose types are drawn from the rationally anticipated distribution.

We index a unit measure of traders by $i \in I$. The asset has the uncertain unit payoff v in units of the money. The unit payoff v is realized at time $t = T + 1$ and not known to anyone until then. Traders have a common prior that v is drawn from a normal distribution with mean zero and variance τ_v^{-1}. Each trader has two types of private information: (i) endowment of the asset x_{i0}, and (ii) a noisy signal s_{i0} about v. Each trader's endowment is a realization of an independent normal random variable with mean zero and variance τ_x^{-1}. The signal takes the form

$$s_{i0} = v + \varepsilon_i,$$

where ε_i is unobserved noise in the signal, and follows a normal distribution with mean zero and variance τ_ε^{-1}. To summarize, random variables $v, \{x_{i0}, \varepsilon_i\}_{i \in I}$ are normally and independently distributed with zero means, and variances

$$Var\left[v\right] = \tau_v^{-1},\ Var\left[x_{i0}\right] = \tau_x^{-1},\ Var\left[\varepsilon_i\right] = \tau_\varepsilon^{-1}.$$

Let b_{i0} be trader i's initial money position. We assume that the net return on the money is zero. Given an initial position (x_{i0}, b_{i0}) of the asset and the money, the payoff from adding

q_i units of the asset and r_i units of the money is

$$\pi_i (q_i, r_i; x_{i0}, b_{i0}) = v (q_i + x_{i0}) + r_i + b_{i0}.$$

We call (q_i, r_i) *a trade* for trader i. Traders face inventory costs[6] of the form:

$$C_i (q_i + x_{i0}) = \frac{1}{2} (q_i + x_{i0})^2 .$$

Thus, the expected utility from a trade (q_i, r_i) is

$$u_i (\pi_i) = E_i[\pi_i] - \frac{1}{2} (q_i + x_{i0})^2 , \tag{1}$$

where $E_i[\cdot]$ denotes trader i's conditional expectation.[7][8] The second term in (1) combined with the exogenous initial dispersion of asset positions creates gains from trade.

Welfare measure. We study the efficiency of the asset allocation, i.e., $E_i[v (q_i + x_{i0})] - \frac{1}{2} (q_i + x_{i0})^2$.[9] As traders are ex ante symmetric, we use the following welfare measure

$$W \equiv E [v (q_i + x_{i0})] - \frac{1}{2} E \left[(q_i + x_{i0})^2\right] .$$

Recalling x_{i0} has a distribution $N (0, \tau_x^{-1})$ and is independent of v, the first best allocation is the perfect smoothing of the asset positions, which yields $W = 0$. On the other hand, no trade yields $W = -\frac{1}{2}\tau_x^{-1}$. Because belief shocks are unbiased, the associated dispersion in beliefs $E_i[v]$ does not directly affect the welfare. However, it will change the distribution of the asset positions when traders act on belief shocks.

[6] For the importance of the inventory management in financial markets, see Comerton-Forde et al (2010).

[7] We can allow for $v = \sqrt{\rho} v_A + \sqrt{1-\rho} v_B$, where v_A and v_B are independent draws from $N\left(0, \tau_v^{-1}\right)$ and $s_{i0} = v_A + \varepsilon_i$, and also $u_i (\pi_i) = E_i[\pi_i] - \frac{\kappa}{2} (q_i + x_{i0})^2$. It turns out that comparative statics depends on $\frac{\sqrt{\rho}}{\kappa} \frac{\tau_x}{\tau_\varepsilon}$ and $\frac{\tau_v}{\tau_\varepsilon}$. Therefore, we normalize $\rho = \kappa = 1$ without loss of generality.

[8] Our model environment is similar to Vives (2011) and Rostek and Weretka (2012), but it is simplified to keep the dynamic analysis tractable.

[9] From (1), utility is transferable via money. Because we do not impose any restriction on the money position, the money allocation is not important for our analysis.

2.2 Trading rule

We study a dynamic variation of the order-submission game in Kyle (1989).[10] To keep the dynamic analysis tractable, we modify a trading rule to induce a price-taking behavior. Trader i's money trade is determined by

$$r_i = -q_i p_i,$$

where p_i is the unit price of the asset charged for trader i. The individual price p_i is determined by the following local pricing rule

$$\sum_{j \neq i} q_j(p_i) = 0, \tag{2}$$

where $\{q_i(\cdot)\}_{i=1}^{n+1}$ is price-contingent orders submitted by traders. Hence, trader i's unit price p_i is determined independent of his order $q_i(\cdot)$, while his asset trade is determined by $q_i(p_i)$. This makes traders a price-taker, but allows them to internalize informational contents of other n traders' orders $\{q_j(\cdot)\}_{j \neq i}$ by best-responding to every possible realization of p_i. Due to the ex ante symmetry among traders, prices $\{p_i\}_{i=1}^{n+1}$ determined by (2) satisfy the local market clearing $\sum_{i=1}^{n+1} q_i(p_i) = 0$ in equilibrium.

Remark 1. In practice, a market maker can offer different prices for different traders to "locally clear" a market he makes, while allowing traders to adjust the quantity traded at the offered prices. In the model, this is achieved by orders conditioned on prices. Our trading rule also captures the idea that market makers facilitate the communication among traders. One interpretation of the pricing rule (2) is that, the market maker offers trader i *a hypothetical price at which the other n traders would trade without i*. Such a price provides a useful information to trader i, which he internalizes in his order. Because every trader does the same reasoning, the market maker facilitates the communication in equilibrium.

[10] Bernhardt, Seiler and Taub (2010) and Rostek and Weretka (2011) also present a dynamic analysis of order-submission markets. However, neither studies local trading.

Remark 2. In the Appendix we consider a generalized payment with a transaction cost $r_i = -\left(q_i + c\left|q_i\right|\right)p_i$, $c \geq 0$, and show that $c > 0$ is necessary to make the total payment by $n+1$ traders non-negative. Because this form of transaction costs does not affect the asset allocation in the linear equilibrium we study, we assume $c = 0$ throughout our analysis.

2.3 Dynamic equilibrium

We denote each trader's asset trade at time t by q_{it} and the associated money trade by r_{it}. Given v, the gross payoff at the end of time t is

$$\pi_{it} = v\left(\sum_{s=1}^{t} q_{is} + x_{i0}\right) + \sum_{s=1}^{t} r_{is} + b_{i0}.$$

If the game ends after trading at time t, the trader i's expected utility is

$$U_{it} = E\left[v|\mathcal{F}_{it}\right]\left(\sum_{s=1}^{t} q_{is} + x_{i0}\right) + \sum_{s=1}^{t} r_{is} + b_{i0} - \frac{1}{2}\left(\sum_{s=1}^{t} q_{is} + x_{i0}\right)^2. \tag{3}$$

The information set $\mathcal{F}_{it}$ starts with $\mathcal{F}_{i0} = \{s_{i0}, x_{i0}\}$, and expands over time reflecting the new information that trader i learns over time. At the end of each trading round $t \leq T-1$, there is positive probability $1-\gamma \in (0,1)$ that the game ends and no more trading occurs. Therefore, the expected lifetime utility evaluated at time t is

$$\begin{aligned} V_{it} &\equiv \frac{1-\gamma}{1-\gamma^{T-t+1}} E\left[\sum_{s=t}^{T} \gamma^{s-t} U_{is}|\mathcal{F}_{it}\right] \\ &= \begin{cases} (1-\gamma)U_{it} + \gamma E[V_{it+1}|\mathcal{F}_{it}] & \text{for } t \leq T-1 \\ U_{iT} & \text{for } t = T \end{cases}. \end{aligned} \tag{4}$$

The parameter γ is a probability weight put on future trading rounds, which measures the expected frequency of trading.[11] We assume $\gamma \leq \frac{1}{2}$, i.e., traders weakly discount the future

[11] An alternative interpretation is that $1-\gamma$ is a probability of an aggregate event that forces traders to be stuck with their positions.

trading rounds relative to the current trading round.

At each time $t = 1, .., T$, given that the game has not ended, each trader is matched with a finite number $n \in \mathbb{N}$ of other traders randomly drawn from the population. In each local market, each trader submits his order $q_{it}(\cdot; \mathcal{F}_{it})$ to a market maker, which is explicitly conditioned on the unit price p_{it} he must pay. Given submitted $n+1$ orders $\{q_{it}(\cdot; \mathcal{F}_{it})\}_{i=1}^{n+1}$, the market maker determines $\{p_{it}\}_{i=1,..,n+1}$ by

$$\sum_{j \neq i} q_{jt}(p_{it}; \mathcal{F}_{jt}) = 0. \tag{5}$$

The asset is traded according to $\{q_{it}(p_{it}; \mathcal{F}_{it})\}_{i=1}^{n+1}$. The monetary payment is $r_{it} = -q_{it}(p_{it}; \mathcal{F}_{it})p_{it}$. In the next period, each trader is matched with new traders.

Definition *A dynamic equilibrium is a collection* $\{q_{it}(\cdot; \mathcal{F}_{it}),\, p_{it},\, \mathcal{F}_{it}\}_{i \in I, t=1,..,T}$ *which satisfy for* $t = 1, .., T$,

(i) For all $i \in I$, $q_{it}(\cdot; \mathcal{F}_{it}) = \arg\max_{q_{it}} V_{it}$, where $\mathcal{F}_{it} = \mathcal{F}_{it-1} \cup \{p_{it}, x_{it-1}\}$, $\mathcal{F}_{i0} = \{s_{i0}, x_{i0}\}$.

(ii) For each local trading, (5) determines p_{it} for $i = 1, .., n+1$.

(iii) Each trader forms a Bayesian belief about the distributions of v and $\{p_{it}, x_{it-1}\}_{i \in I, t=1,..,T}$ consistent with the other equilibrium variables.

At time $t \geq 1$, each trader's information set $\mathcal{F}_{it}$ contains the initial private information (s_{i0}, x_{i0}) and endogenous information obtained in the previous $t-1$ trading rounds. In the definition above, $\mathcal{F}_{it}$ contains the price he pays at time t (i.e., $p_{it} \in \mathcal{F}_{it}$) even though each trader does not know the realization of p_{it}. This is because each trader can choose his best response for each realization of p_{it} by making the order conditioned on p_{it}.

2.4 A benchmark without belief shocks

We study a case $\tau_\varepsilon = 0$, where the information about v symmetric, i.e., $E[v|\mathcal{F}_{it}] = 0$ for all i and t. The first best allocation (without cross-sectional restrictions on trading) is that everyone holds the population average asset position $E[x_{i0}] = 0$. We show that without belief shocks the equilibrium allocation monotonically approaches the first best allocation. We the local average by $\overline{x}_t \equiv \frac{1}{n+1}\sum_{i=1}^{n+1} x_{it}$ etc, and the local average except trader i by $\overline{x}_{it} \equiv \frac{1}{n}\sum_{j\neq i} x_{jt}$ etc. Also, $\frac{1}{\tau_{xt}} \equiv Var[x_{it}]$ is the cross-sectional variance of the asset positions.

Lemma 1 (benchmark dynamics)

(a) *Equilibrium trade is* $q_{it} = \overline{x}_{it-1} - x_{it-1}$.

(b) *The dispersion of the asset positions changes according to*

$$\frac{1}{\tau_{xt}} = \frac{1}{n}\frac{1}{\tau_{xt-1}}, \tau_{x0} = \tau_x,$$

and the social welfare at the end of period t *is* $W_t = -\frac{1}{2}\frac{1}{\tau_{xt}} = -\frac{1}{2}\left(\frac{1}{n}\right)^t \frac{1}{\tau_x}$.

In each round t, trader i is matched with n new traders. By the end of time t, trader i has traded with nt other traders, but he indirectly traded with n^t traders because sets of trading counterparties do not overlap. Therefore, the asset positions converge at the rate n^{-t}. Thus, as long as $n \geq 2$, the allocation approaches the first best at the rate $\frac{1}{n}$.[12] The next section studies how the belief shocks at $t = 0$ alter this dynamics.

3 Two-period example

This section presents a two-period example to illustrate a key economic force.

Incorporation of belief shocks into allocations. First, we consider a one-shot trading ($T = 1$). Following the microstructure literature, we use a guess-and-verify method to characterize a linear equilibrium. Conjecture the order of the form:

$$q_i(p; s_{i0}, x_{i0}) = \beta^s s_{i0} - \beta^x x_{i0} - \beta^p p, \tag{6}$$

[12] When trading is bilateral ($n = 1$), this convergence does not occur under our trading rule, because two traders switch their positions: $\overline{x}_{it-1} = x_{jt-1}$.

where $(\beta^s, \beta^x, \beta^p)$ are determined in equilibrium. From (2), local prices are given by

$$p_i = \frac{\beta^s}{\beta^p}\overline{s}_{i0} - \frac{\beta^x}{\beta^p}\overline{x}_{i0}.$$

By substituting this into (6), the equilibrium trade is[13]

$$q_i = \beta^s \left(s_{i0} - \overline{s}_{i0}\right) - \beta^x \left(x_{i0} - \overline{x}_{i0}\right).$$

The first term is the trade generated by belief shocks $s_{i0} - \overline{s}_{i0} = \varepsilon_{i0} - \overline{\varepsilon}_{i0}$, while the second term is the trade that smooths the asset positions. These reflect two motives for trading: speculation and the inventory management. The relative importance of the two motives can be measured by $\frac{\beta^x}{\beta^s}$, which is endogenous.

Local prices are new signals about the asset value because

$$\frac{\beta^p}{\beta^s}p_i = \frac{1}{n}\sum_{j\neq i}\left(s_{j0} - \frac{\beta^x}{\beta^s}x_{j0}\right) = v + \frac{1}{n}\sum_{j\neq i}\left(\varepsilon_{j0} - \frac{\beta^x}{\beta^s}x_{j0}\right). \tag{7}$$

Local prices (7) are noisy for two reasons. First, idiosyncratic belief shocks $\{\varepsilon_{j0}\}_{j\neq i}$ are averaged, but do not disappear because n is small. Second, $\frac{1}{n}\sum_{j\neq i} x_{j0}$ is also stochastic from trader i's point of view. By the Bayes rule, the informational content of (7) is

$$\left(Var\left[\frac{\beta^p}{\beta^s}p_i|v\right]\right)^{-1} = n\tau_\varepsilon\left\{1 + \left(\frac{\beta^x}{\beta^s}\right)^2\frac{\tau_\varepsilon}{\tau_x}\right\}^{-1}.$$

To measure the share of the information revealed by p_i relative to n signals, we define

$$\varphi \equiv \left\{1 + \left(\frac{\beta^x}{\beta^s}\right)^2\frac{\tau_\varepsilon}{\tau_x}\right\}^{-1} \in (0,1).$$

Note that φ depends on the equilibrium trading behavior through $\frac{\beta^x}{\beta^s}$.

[13] We used q_i for $q_i\left(p_i; s_{i0}, x_{i0}\right)$ to suppress the notation.

Lemma 2 (one-shot trade with belief shocks)

(a) *Equilibrium trade is*

$$q_i = \sqrt{\frac{\varphi}{1-\varphi}\frac{\tau_\varepsilon}{\tau_x}}\left(s_{i0} - \overline{s}_{i0}\right) + \overline{x}_{i0} - x_{i0}, \tag{8}$$

where $\varphi \in (0,1)$ *is characterized in the proof.*

(b) *The dispersion of the asset positions changes according to*

$$\frac{1}{\tau_{x1}} = \frac{1+n\varphi}{n-n\varphi}\frac{1}{\tau_x},$$

and the social welfare at the end of period 1 is $W_1 = -\frac{1}{2}\frac{1+n\varphi}{n-n\varphi}\frac{1}{\tau_x}$.

(c) *The dispersion goes up when traders' beliefs change sufficiently, i.e.,*

$$Var\left[x_{i0}\right] < Var\left[x_{i1}\right] \Leftrightarrow \frac{n-1}{2} < n\varphi. \tag{9}$$

For any $n \geq 2$, $\tau_v \in [0,\infty)$, *and* $\tau_\varepsilon \in (0,\infty)$, *this occurs if* τ_x *is sufficiently large.*

Lemma 2(a) shows that traders with high (low) realizations of belief shocks buy more (less) of the asset relative to the benchmark trade. This speculative trading has the following properties: (i) it has mean zero, (ii) its variance positively depends on the informativeness of the prices φ, (iii) it is orthogonal to $\overline{x}_{i0} - x_{i0}$. Thus, the speculation increases the dispersion of the asset positions without changing its mean. These properties also hold with an arbitrary number of periods. Note that $\frac{1}{n} < \frac{1+n\varphi}{n-n\varphi}$ for $\varphi > 0$. Thus, **Lemma 2(b)** shows the slower convergence of the asset positions than the benchmark. The larger belief change is associated with the slower convergence, as a larger φ in (8) implies a larger impact of belief shocks. This reduces welfare because more inventory costs are wasted. **Lemma 2(c)** provides a condition for the dispersion to increase. Because $n\varphi$ measures the total amount of information each trader learns from prices, the condition (9) means that the change in traders' beliefs is large relative to the amount of inventory smoothing. Any parameter change that makes φ sufficiently large can satisfy (9). For example, the dispersion does *not* increase in any of the following cases: $\tau_\varepsilon \to 0$ or $\tau_\varepsilon \to \infty$ or $\tau_v \to \infty$ or $\tau_x \to 0$.[14] Intuitively, more symmetric information ($\tau_\varepsilon \to 0$ or $\tau_\varepsilon \to \infty$ or $\tau_v \to \infty$) or higher gains from the inventory smoothing ($\tau_x \to 0$) imply less speculation in equilibrium. As a result, the impact of belief shocks is not strong enough to overturn the effect of the inventory management.

Dynamic amplification. Next, we introduce the second round. After the first round, traders have asset positions $\{x_{i1}\}$ and two signals $\{s_{i0}, s_{i1}\}$, where $s_{i1} = \frac{\beta^p}{\beta^s} p_{i1}$ as in (7). Prices in the second round will aggregate information contained in $\{s_{i0}, s_{i1}\}$. There are two connections between the two rounds: (i) the sharing of information in the second round is affected by the dispersion of the asset positions and beliefs achieved in the first round, (ii) trading in the first round is affected by the anticipated informativeness of prices in the second round. The first connection should be straightforward, while the second connection may be less obvious. We elaborate more on the latter.

We construct a linear equilibrium of the form:

$$\begin{aligned} q_{i1}(p; s_{i0}, x_{i0}) &= \beta_1^s s_{i0} - \beta_1^x x_{i0} - \beta_1^p p, \\ q_{i2}(p; s_{i0}, s_{i1}, x_{i1}) &= \beta_{2,0}^s s_{i0} - \beta_{2,1}^s s_{i1} - \beta_2^x x_{i1} - \beta_2^p p, \end{aligned}$$

where coefficients $\left\{\beta_1^s, \beta_1^x, \beta_1^p, \beta_{2,0}^s, \beta_{2,1}^s, \beta_2^x, \beta_2^p\right\}$ depend only on primitive parameters. In the first round, anticipating the second round with probability γ, trader i solves

$$\begin{aligned} \max_{q_{i1}} \Bigg[(1-\gamma) \left\{ E_{i1}[v](q_{i1} + x_{i0}) - \frac{1}{2}(q_{i1} + x_{i0})^2 - p_{i1} q_{i1} \right\} \qquad (11) \\ + \gamma E_{i1} \left[E_{i2}[v] x_{i2} - \frac{1}{2} x_{i2}^2 - (p_{i1} q_{i1} + p_{i2}^* q_{i2}^*) \right] \Bigg], \end{aligned}$$

where $E_{i2}[v]$, p_{i2}^*, q_{i2}^*, $x_{i2} = q_{i2}^* + x_{i1}$ are determined in the second round. The first line in

[14]For all cases, φ approaches zero. In the Appendix, we show that φ decreases in τ_v, increases in τ_x, and non-monotonic in τ_ε. We discuss the effect of changing n in Section 4.

(11) is the expected utility for the case the game ends after the first round, while the second line corresponds to the case where there are two rounds. In the second round trader i solves

$$\max_{q_{i2}} \left\{ E_{i2}[v](q_{i2}+x_{i1}) - \frac{1}{2}(q_{i2}+x_{i1})^2 - (p_{i1}q_{i1}+p_{i2}q_{i2}) \right\}. \tag{12}$$

In the second round equilibrium,

$$\begin{aligned} p_{i2}^* &= \frac{\beta_{2,0}^s}{\beta_2^p}\overline{s}_{i0} + \frac{\beta_{2,1}^s}{\beta_2^p}\overline{s}_{i1} - \frac{\beta_2^x}{\beta_2^p}\overline{x}_{i1}, \\ q_{i2}^* &= \beta_{2,0}^s(s_{i0}-\overline{s}_{i0}) + \beta_{2,1}^s(s_{i1}-\overline{s}_{i1}) - \beta_2^x(x_{i1}-\overline{x}_{i1}), \\ x_{i2} &= \beta_{2,0}^s(s_{i0}-\overline{s}_{i0}) + \beta_{2,1}^s(s_{i1}-\overline{s}_{i1}) + \beta_2^x\overline{x}_{i1} + (1-\beta_2^x)x_{i1}. \end{aligned} \tag{13}$$

These equilibrium objects appear in (11) if and only if $\gamma > 0$. First, p_{i2}^* does not depend on q_{i1}. Second, taking the first order condition of (12), it is easy to verify $\beta_2^x = 1$. Hence, x_{i2} does not depend on q_{i1} either. Finally, q_{i2}^* depends on q_{i1} through $x_{i1} = q_{i1} + x_{i0}$. Dropping irrelevant terms, (11) can be written as

$$\max_{q_{i1}} \left[\{(1-\gamma)E_{i1}[v] + \gamma E_{i1}[p_{i2}^*] - p_{i1}\} q_{i1} - \frac{1-\gamma}{2}(q_{i1}+x_{i0})^2 \right].$$

The expected gross payoff from the asset consists of the asset's fundamental value $E_{i1}[v]$ and its *expected resale price* $E_{i1}[p_{i2}^*]$. The latter depends on the second round trading environment. Importantly, from (13), p_{i2}^* is correlated with v because of the speculation in the second round. Formally, we show that

$$E_{i1}[p_{i2}^*] = \frac{\tau_\varepsilon(1+n\varphi_1+n^2\varphi_2)}{\tau_v + \tau_\varepsilon(1+n\varphi_1+n^2\varphi_2)} E_{i1}[v],$$

where φ_1 and φ_2 are defined by

$$\begin{aligned}\left(Var\left[\frac{\beta_1^p}{\beta_1^s}p_{i1}|v\right]\right)^{-1} &= n\tau_\varepsilon\varphi_1, \\ \left(Var\left[\frac{\beta_2^p}{\beta_{2,0}^s+\beta_{2,1}^s}p_{i2}|v\right]\right)^{-1} &= n^2\tau_\varepsilon\varphi_2.\end{aligned}$$

Note that φ_2 measures the fraction of information revealed by p_{i2} relative to n^2 of original signals (i.e. $\{s_{j0}\}$ for all j with whom i has directly and indirectly interacted). All in all, the first round problem is

$$\max_{q_{i1}}\left[\left\{\left(1-\gamma+\gamma\frac{\tau_\varepsilon\left(1+n\varphi_1+n^2\varphi_2\right)}{\tau_v+\tau_\varepsilon\left(1+n\varphi_1+n^2\varphi_2\right)}\right)E_{i1}[v]-p_{i1}\right\}q_{i1}-\frac{1-\gamma}{2}\left(q_{i1}+x_{i0}\right)^2\right]. \quad (14)$$

The amount of speculation increases if the weight put on $E_{i1}[v]$ relative to the weight on $q_{i1}+x_{i0}$ increases. From the first order condition of (14), this relative weight is

$$1+\frac{\gamma}{1-\gamma}\frac{\tau_\varepsilon\left(1+n\varphi_1+n^2\varphi_2\right)}{\tau_v+\tau_\varepsilon\left(1+n\varphi_1+n^2\varphi_2\right)}\in\left[1,\frac{1}{1-\gamma}\right]. \quad (15)$$

There are two forces in (15). First, an increase in γ shifts traders' objective from the inventory management to the speculation. Second, the more *anticipated* learning in the second round (i.e. a larger φ_2) encourages more speculation in the first round. The two forces complement each other, leading to higher φ_1 (relative to φ in **Lemma 2**). In a dynamic environment, forward-looking traders rationally anticipate that future prices are more correlated with v. Because the future prices enter the objective positively through the resale possibility, as γ increases, the current signal is given a higher leverage in forecasting v.

Lemma 3 (dynamic amplification)

In anticipation of the second round trading (i.e. $\gamma > 0$), equilibrium trade in the first round incorporates more belief shocks. Specifically, $q_{i1} = \sqrt{\frac{\varphi_1}{1-\varphi_1}\frac{\tau_\varepsilon}{\tau_x}}\left(s_{i0}-\overline{s}_{i0}\right)+\overline{x}_{i0}-x_{i0}$, where φ_1 is greater than φ for the one-shot trading.

Because φ_1 and φ_2 are mutually dependent, they must be jointly determined. For a general case with T rounds, we solve a fixed point problem in $(\varphi_1, .., \varphi_T)$.

4 Dynamic equilibrium with belief shocks

We first sketch the construction of a dynamic equilibrium. Readers who are not interested in technical details may skip to subsection 4.2 where main results are presented.

4.1 Equilibrium construction

We construct a dynamic equilibrium with the following properties:

Property I: At the beginning of period t, the asset positions, x_{it-1}, has a cross-sectional distribution $N\left(0, \tau_{xt-1}^{-1}\right)$.

Property II: At the beginning of period t, trader i has t signals $\{s_{ik}\}_{k=0}^{t-1}$, where $s_{ik} = v + \varepsilon_{ik}$, $k = 0, .., t-1$, and the noise ε_{ik} has the distribution $N\left(0, \frac{1}{\tau_\varepsilon n^k \varphi_k}\right)$ independent across traders.

Property III: For time t trading, trader i submits an order

$$q_{it}(p; \mathcal{F}_{it}) = \sum_{k=0}^{t-1} \beta_{t,k}^s s_{ik} - \beta_t^x x_{it-1} - \beta_t^p p. \tag{16}$$

We show below that signals $\{s_{ik}\}_{k=1}^{t-1}$ except the initial signal s_{i0} are endogenously generated by equilibrium prices. Also, $\{\tau_{xt}, \varphi_t\}_{t=1}^{T}$ in Property I and II is endogenously determined by the orders in Property III, while the orders are chosen by traders who rationally anticipate $\{\tau_{xt}, \varphi_t\}_{t=1}^{T}$. We characterize this dynamic equilibrium in two steps. First, we establish Property I and II taking Property III as given. Second, we characterize $\left\{\{\beta_{t,k}^s\}_{k=0}^{t-1}, \beta_t^x, \beta_t^p\right\}_{t=1}^{T}$ by the guess-and-verify method combined with a backward induction.

Step 1: Allocation and beliefs. Property I and II are satisfied at $t = 1$ with

$\varphi_0 \equiv 1$. Given the conjecture (16), equilibrium prices and quantities at time t must satisfy

$$\beta_t^p p_{it} = \sum_{k=0}^{t-1} \beta_{t,k}^s \overline{s}_{ik} - \beta_t^x \overline{x}_{it-1}, \tag{17}$$

$$q_{it} = \sum_{k=0}^{t-1} \beta_{t,k}^s \left(s_{ik} - \overline{s}_{ik}\right) - \beta_t^x \left(x_{it-1} - \overline{x}_{it-1}\right). \tag{18}$$

Hence, x_{it-1} and $x_{it} = q_{it} + x_{it-1}$ are related by the following condition:

$$x_{it} = \sum_{k=0}^{t-1} \beta_{t,k}^s \left(s_{ik} - \overline{s}_{ik}\right) + \beta_t^x \overline{x}_{it-1} + \left(1 - \beta_t^x\right) x_{it-1}. \tag{19}$$

Suppose that Property I and II hold at time t. Given that x_{it-1} follows $N\left(0, \tau_{xt-1}^{-1}\right)$ and ε_{ik} follows $N\left(0, \frac{1}{\tau_\varepsilon n^k \varphi_k}\right)$ independent across i for all $k \leq t-1$, (19) implies $E\left[x_{it}\right] = 0$ and

$$V\left[x_{it}\right] = \frac{1}{\tau_\varepsilon} \frac{1}{n^t \varphi_t} \left(\sum_{k=0}^{t-1} \beta_{t,k}^s\right)^2 + \sum_{k=0}^{t-1} \frac{\left(\beta_{t,k}^s\right)^2}{\tau_\varepsilon n^k \varphi_k} + \left(1 - \beta_t^x\right)^2 \frac{1}{\tau_{xt-1}}.$$

Therefore,

$$\frac{\tau_\varepsilon}{\tau_{xt}} = \left(\sum_{k=0}^{t-1} \beta_{t,k}^s\right)^2 \left(\frac{1}{n^t \varphi_t} + \sum_{k=0}^{t-1} \frac{\left(\widetilde{\beta}_{t,k}^s\right)^2}{n^k \varphi_k}\right) + \left(1 - \beta_t^x\right)^2 \frac{\tau_\varepsilon}{\tau_{xt-1}}. \tag{20}$$

Thus, Property I holds at time $t+1$ with τ_{xt} determined by (20).

After time t trading, the most each trader can hope to learn is the information contained in n^t signals. Thus, φ_k in Property II measures a fraction of information each trader learns at time k relative to $\tau_\varepsilon n^k$. For each $t = 1, .., T$, signals $\{s_{ik}\}_{k=1}^{t-1}$ constructed from the equilibrium prices must satisfy Property II. From (17), information learned from p_{it} is

$$s_{it} \equiv \frac{\beta_t^p}{\sum_{k=0}^{t-1} \beta_{t,k}^s} p_{it} = \frac{\frac{1}{n} \sum_{j \neq i} \sum_{k=0}^{t-1} \beta_{t,k}^s s_{jk}}{\sum_{k=0}^{t-1} \beta_{t,k}^s} - \frac{\beta_t^x}{\sum_{k=0}^{t-1} \beta_{t,k}^s} \overline{x}_{it-1}.$$

Using $\widetilde{\beta}^s_{t,k} \equiv \frac{\beta^s_{t,k}}{\sum_{k=0}^{t-1}\beta^s_{t,k}}$, $\widetilde{\beta}^x_t \equiv \frac{\beta^x_t}{\sum_{k=0}^{t-1}\beta^s_{t,k}}$ and $\widetilde{\beta}^p_t \equiv \frac{\beta^p_t}{\sum_{k=0}^{t-1}\beta^s_{t,k}}$, this can be written as

$$\begin{aligned} s_{it} &\equiv \widetilde{\beta}^p_t p_{it} \\ &= \sum_{k=0}^{t-1}\widetilde{\beta}^s_{t,k}\overline{s}_{ik} - \widetilde{\beta}^x_t \overline{x}_{it-1} = v + \sum_{k=0}^{t-1}\widetilde{\beta}^s_{t,k}\overline{\varepsilon}_{ik} - \widetilde{\beta}^x_t \overline{x}_{it-1}. \end{aligned} \tag{21}$$

Suppose Property II holds in period t. The signal s_{it} is independent across i and also independent from $\{s_{ik}\}_{k=0}^{t-1}$ conditional on v, because no pair of traders meets twice and does not share the previous history. Therefore,

$$(Var[s_{it}|v])^{-1} = \left(\sum_{k=0}^{t-1}\left(\widetilde{\beta}^s_{t,k}\right)^2 \frac{1}{n}\frac{1}{\tau_\varepsilon n^k \varphi_k} + \left(\widetilde{\beta}^x_t\right)^2 \frac{1}{n}\frac{1}{\tau_{xt-1}}\right)^{-1}. \tag{22}$$

We define φ_t by equating (22) with $\tau_\varepsilon n^t \varphi_t$, i.e.,

$$\frac{1}{n^{t-1}\varphi_t} = \sum_{k=0}^{t-1}\frac{\left(\widetilde{\beta}^s_{t,k}\right)^2}{n^k\varphi_k} + \left(\widetilde{\beta}^x_t\right)^2 \frac{\tau_\varepsilon}{\tau_{xt-1}} \tag{23}$$

Thus, Property II holds in period $t+1$ with s_{it} defined by (21) and φ_t defined by (23). By the end of period $t-1$, trader i has signals $\{s_{ik}\}_{k=1}^{t-1}$, which are informationally equivalent to his price history $\{p_{ik}\}_{k=1}^{t-1}$.

Two dynamic equations (20) and (23) jointly describe dynamics of beliefs and the allocation given the trading behavior $\left\{\{\beta^s_{t,k}\}_{k=0}^{t-1}, \beta^x_t, \beta^p_t\right\}_{t=1}^{T}$. It should be clear from the derivation of these two equations that Properties I and II hold for time t if they hold for time up to $t-1$. Hence, Properties I and II were verified given Property III.

Step 2: Strategies. We verify Property III by showing that trader i's optimal order takes the conjectured form (16) given that all the others use the same form. Each trader i's belief about v at time t is summarized by its conditional mean and variance, and characterized by the Bayes rule. We measure *the cumulative amount of information* held by each trader at the end of period t by

$$\chi_t \equiv \sum_{k=0}^{t} n^k \varphi_k,\ \chi_0 = \varphi_0 \equiv 1,$$

which is bounded below by one and increases in t. The conditional expectation of v is

$$\begin{aligned} E_{it}[v] &= \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon \chi_t} \sum_{k=0}^{t} n^k \varphi_k s_{ik} \\ &= \frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon \chi_t} \left(\sum_{k=0}^{t-1} n^k \varphi_k s_{ik} + n^t \varphi_t \widetilde{\beta}_t^p p_{it} \right). \end{aligned} \tag{24}$$

The expression (24) is linear in $\{s_{ik}\}_{k=0}^{t-1}$ and p_{it}. Therefore, the optimal order takes the linear form of Property III. We relegate the rest of the characterization to the Appendix.

4.2 Main results

To capture the non-stationarity induced by learning, we define *a dynamic leverage factor* $\{\Gamma_{T-t}\}_{t=1}^{T} = \{\Gamma_{T-1}, \Gamma_{T-2}, .., \Gamma_1, \Gamma_0\}$ as follows:

$$\Gamma_{T-t} \equiv 1 + \frac{\gamma}{1 - \gamma \mathbf{1}\{t = T-1\}} \frac{\tau_\varepsilon \chi_{t+1}}{\tau_v + \tau_\varepsilon \chi_{t+1}} \Gamma_{T-(t+1)} \text{ for } t \leq T-1 \text{ and } \Gamma_0 \equiv 1. \tag{25}$$

The subscript $T-t$ denotes the maximum number of remaining rounds at time t. The leverage factor (25) generalizes (15).[15] It measures the time-varying importance of future prices for each round t by taking into account the probability $\gamma \in [0, 1]$ that the game continues and the information accumulation $\{\chi_t\}_{t=1}^{T}$. First, suppose $\gamma = 0$. In this case, traders do not internalize the inter-temporal implication of the information aggregation and $\Gamma_{T-t} = 1$ for all t. In other words, all rounds are the same as far as the relevance of future prices is concerned. With $\gamma > 0$, traders internalize the fact that prices become more informative over time relative to prior belief τ_v. This is captured by $\frac{\tau_\varepsilon \chi_{t+1}}{\tau_v + \tau_\varepsilon \chi_{t+1}}$, which increases over time

[15] To see this, set $T - t = 2 - 1 = 1$.

given $\tau_v > 0$.[16] Thus, a higher γ raises Γ_{T-t} uniformly for all $t < T$. Second, for a fixed $\gamma > 0$, $\frac{\tau_\varepsilon \chi_{t+1}}{\tau_v + \tau_\varepsilon \chi_{t+1}} \in [0,1]$ has a time-varying impact on $\{\Gamma_{T-t}\}_{t=1}^{T}$. If $\tau_\varepsilon = 0$ (i.e., no belief shocks), then $\frac{\tau_\varepsilon \chi_{t+1}}{\tau_v + \tau_\varepsilon \chi_{t+1}} = 0$ for all t and $\{\Gamma_{T-t}\}_{t=1}^{T}$ has the minimum path $\Gamma_{T-t} = 1$ for all t. If $\frac{\tau_\varepsilon \chi_{t+1}}{\tau_v + \tau_\varepsilon \chi_{t+1}} = 1$ for all t, then $\{\Gamma_{T-t}\}_{t=1}^{T}$ has the maximum path $\Gamma_{T-t} = \frac{1}{1-\gamma}$ for all $t < T$. More generally, $\frac{\tau_\varepsilon \chi_{t+1}}{\tau_v + \tau_\varepsilon \chi_{t+1}} \in (0,1)$ increases over time, and so is $\Gamma_{T-t} \in \left[1, \frac{1}{1-\gamma}\right]$.[17]

The Appendix shows that in round t, the weight put on s_{ik}, $k < t$, is

$$\beta_{t,k}^{s} = \Gamma_{T-t} \frac{\tau_\varepsilon n^k \varphi_k}{\tau_v + \tau_\varepsilon \chi_t}. \tag{26}$$

This is the dynamic leverage factor Γ_{T-t} times the standard Bayesian formula for $E\left[v|s_{ik}\right]$. Because $\beta_t^x = 1$ for all t, the weights (26) determine the amount of information that goes into current prices. To describe equilibrium dynamics, we define the change in (26) by

$$\text{For } t \geq 2,\ \beta_{t/t-1}^{s} \equiv \frac{\beta_{t,k}^{s}}{\beta_{t-1,k}^{s}} = \frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t}. \tag{27}$$

If the dynamic leverage factors are constant, (27) is smaller than one because $\chi_{t-1} < \chi_t$.

Proposition 0 (equilibrium with belief shocks)

For $t \in \{1, .., T\}$, equilibrium trade is

$$q_{it} = \frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \sum_{k=0}^{t-1} n^k \varphi_k \left(s_{ik} - \overline{s}_{ik}\right) + \overline{x}_{it-1} - x_{it-1}, \tag{28}$$

where $\{\varphi_1, .., \varphi_T\}$ solves

$$\frac{\varphi_1}{1-\varphi_1} = \frac{\tau_x}{\tau_\varepsilon} \left(\frac{\Gamma_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_1} \right)^2, \tag{29}$$

$$\varphi_t = \left(1 + \frac{n^{t-1}}{\chi_{t-2}} \left(\varphi_{t-1} - \varphi_t\right)\right) \left(\beta_{t/t-1}^{s}\right)^2 \varphi_{t-1} \text{ for } t \geq 2. \tag{30}$$

[16] With a diffused prior $\tau_v = 0$, in each period beliefs are completely replaced by $\tau_\varepsilon \chi_t$. In this case, the non-stationary of the *relative* importance of prices as an endogenous information is lost.

[17] Except in the final round $\Gamma_1 \geq \Gamma_0$. See the Appendix for more details.

From (28), the larger dynamic leverage factor Γ_{T-t} makes time t trading more speculation-driven, and hence more susceptible to belief shocks. The larger dynamic leverage factor also implies that prices are more informative through (29) and (30). Note that (29) and (30) jointly define a continuous mapping from $\mathbb{R}^T$ into itself, because $\{\Gamma_{T-t}\}_{t=1}^{T-1}$ are continuous in $\{\varphi_t\}_{t=1}^{T}$ and bounded in $\left[1, \frac{1}{1-\gamma}\right]$. For any T, a fixed point exists since $\varphi_t \in [0, \overline{\varphi}_t]$ is bounded.[18] This characterizes the equilibrium value of $\{\varphi_t\}_{t=1}^{T}$.

Learning speed. From (30), the speed of the information aggregation is governed by $\beta^s_{t/t-1}$ defined in (27), which measures how much traders act on the signals relative to the previous round. The following Proposition characterizes the dynamic behavior of φ_t.

Proposition 1 (learning dynamics)

(a) *If* $\gamma\tau_v = 0$, *then* $\varphi_1 > \varphi_2 > .. > \varphi_T$.

(b) *If* $\gamma\tau_v > 0$, *then* $\varphi_{T-1} > \varphi_T$ *and for each* $t = 2, .., T-1$, *only one of the following three cases is possible*:

$$
\begin{aligned}
\text{(i) } 1 &< \frac{\varphi_t}{\varphi_{t-1}} < \beta^s_{t/t-1}, \\
\text{(ii) } 1 &> \frac{\varphi_t}{\varphi_{t-1}} > \beta^s_{t/t-1}, \\
\text{(iii) } 1 &= \frac{\varphi_t}{\varphi_{t-1}} = \beta^s_{t/t-1}.
\end{aligned}
$$

Proposition 1 shows that φ_t can go up or down but the change is bounded by the value of $\beta^s_{t/t-1}$. In particular, it increases if and only if the speculation intensifies, i.e., $1 < \beta^s_{t/t-1}$. To see when this can happen, recall (27). First, an increase in the stock of information χ_t works against the speculation, captured by the term $\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t} < 1$. To overturn this force and let traders act more aggressively on the signals than in the previous round, $\frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} > 1$, and hence $\gamma\tau_v > 0$ is necessary. When $\frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}}$ is sufficiently large, the additional benefit of the

[18] The upper bound $\overline{\varphi}_t$ is given by the following recursive relationship: $\overline{\varphi}_{t+1} = \frac{1}{n^t}\sum_{k=0}^{t} n^k \overline{\varphi}_k$ with $\overline{\varphi}_0 \equiv 1$.

signals for forecasting future prices leads to more speculation and faster learning. **Figure 2** below illustrates the case where φ_t can go up only for the sufficiently high value of γ.

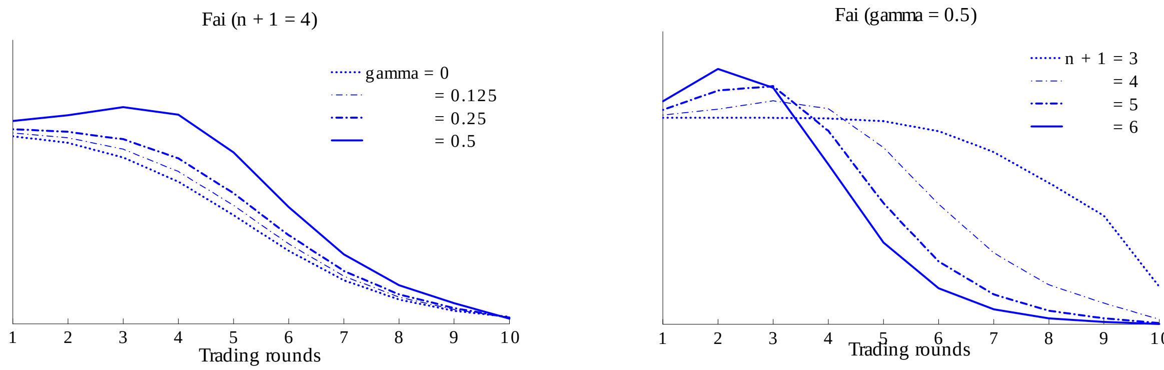

Figure 2. The dynamics of φ_t.
Note. The left panel shows the dynamics for different values of γ.
The right panel shows it for different values of n.

The left panel shows the impact of changes in γ for a fixed n. Letting traders anticipate more trading rounds has a positive effect on the information aggregation. This effect is uniform over time. The right panel shows the impact of changes in n for $\gamma = 0.5$. The larger n favors the earlier round of information aggregation.[19]

Dispersion, welfare, and volume. We measure the rate of changes in traders' beliefs by $\frac{n^t\varphi_t}{\chi_{t-1}}$, i.e., the flow amount of information $n^t\varphi_t$ divided by the stock χ_{t-1}.

Proposition 2 (dispersion and welfare)

The dispersion of the asset positions changes according to

$$\frac{1}{\tau_{xt}} = \frac{1 + \frac{n^t\varphi_t}{\chi_{t-1}}}{n - \frac{n^t\varphi_t}{\chi_{t-1}}} \frac{1}{\tau_{xt-1}}, \tag{31}$$

and the social welfare at the end of period t is $W_t = -\frac{1}{2}\left(\prod_{s=1}^{t} \frac{1+\frac{n^s\varphi_s}{\chi_{s-1}}}{n-\frac{n^s\varphi_s}{\chi_{s-1}}}\right)\frac{1}{\tau_x}$.

[19] With sufficiently large n, φ_t decreases over time. When traders anticipate many trading rounds to come (i.e., high γ), relaxing the local trading friction shifts the information aggregation to the earlier rounds.

Proposition 2 shows that the convergence rate of the asset positions between two rounds is bounded below by the benchmark rate $\frac{1}{n}$, and it increases in the rate of changes in traders beliefs $\frac{n^t \varphi_t}{\chi_{t-1}}$. Therefore, one-shot belief shocks persistently slow down the convergence, and more quickly changing beliefs lead to the slower convergence. As the belief change becomes negligible ($\frac{n^t \varphi_t}{\chi_{t-1}} \to 0$), the convergence rate approaches $\frac{1}{n}$. From (29) and (30), for a given n, setting $\tau_\varepsilon \to 0$ (given $\tau_v > 0$) or $\tau_\varepsilon \to \infty$ or $\tau_v \to \infty$ or $\tau_x \to 0$ leads to $\varphi_t \to 0$ for all t and the benchmark is obtained. As we discussed in Section 3, these are the cases where the speculation becomes negligible.

Finally, we study the volume. From **Proposition 0**, the equilibrium trade is

$$q_{it} = \frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \sum_{k=0}^{t-1} n^k \varphi_k \left(\varepsilon_{ik} - \overline{\varepsilon}_{ik}\right) - \left(x_{it-1} - \overline{x}_{it-1}\right). \tag{32}$$

We use $|q_{it}|$ to denote the absolute value of (32), and measure the volume by $E\left[|q_{it}|\right]$.[20]

Proposition 3 (dispersion and volume)

When the rate of changes in beliefs is sufficiently high, the dispersion and volume go up:

$$Var\left[x_{it-1}\right] < Var\left[x_{it}\right] \Leftrightarrow \frac{n-1}{2} < \frac{n^t \varphi_t}{\chi_{t-1}}, \tag{33}$$

$$E\left[|q_{it-1}|\right] < E\left[|q_{it}|\right] \Leftrightarrow n - 1 < \frac{n^{t-1} \varphi_{t-1}}{\chi_{t-2}} + \frac{n^t \varphi_t}{\chi_{t-1}}. \tag{34}$$

Proposition 3 shows that the dispersion and volume increase if and only if traders' beliefs change sufficiently quickly relative to n. For the dispersion to go up, we need two things: (i) a sufficient amount of belief shocks goes into trading, (ii) the inventory management is not too effective. The former requires that traders' beliefs change quickly consistent with the aggressive speculation, while the latter requires markets be small. The volume goes up if

[20] Strictly speaking this should be divided by 2 because on average a half of trade is a buy order and the other half is a sell order.

and only if the *average* rate of changes in beliefs for two rounds is sufficiently high. In fact, the condition (34) is a two-period average of the condition (33). Thus, if the dispersion goes up for two rounds, then the volume also goes up, but the converse is not necessarily true.

Discussion. Our model predicts that, given the cross-sectional trade restriction, an asset for which traders acquire noisy information is more likely to have excessive trading compared with an asset not subject to such belief shocks. It is well known that "glamour" stocks with high M/B ratios tend to have higher volume.[21] According to our model, if noisy private information tends to be generated more for the glamour stocks[22], then high volume can be generated by rational traders acting on idiosyncratic belief shocks.

4.3 Other results

Lemma 4 (volume dynamics)

Trading volume changes according to:

$$E\left[|q_{it}|\right] = \sqrt{\frac{2\left(n+1\right)}{\pi} Var\left[x_{it-1}\right]\left(n - \frac{n^t \varphi_t}{\chi_{t-1}}\right)^{-1}}, \tag{35}$$

$$\frac{E\left[|q_{it}|\right]}{E\left[|q_{it-1}|\right]} = \sqrt{\frac{Var\left[x_{it-1}\right]}{Var\left[x_{it-2}\right]} \frac{n - \frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}{n - \frac{n^t \varphi_t}{\chi_{t-1}}}}. \tag{36}$$

Lemma 4 decomposes the volume into the dispersion and the rate of changes in beliefs. In the benchmark, (35) becomes $E\left[|q_{it}|\right] = \sqrt{\frac{2}{\pi}\frac{n+1}{n} Var\left[x_{it-1}\right]}$, i.e., the volume and the dispersion is tightly connected. Since the dispersion $Var\left[x_{it-1}\right]$ monotonically decreases in this case, i.e., $\frac{Var[x_{it-1}]}{Var[x_{it-2}]} = \frac{1}{n}$, so does the volume. The extent in which belief shocks disturb the asset positions is captured by the rate of changes in beliefs (i.e. $\frac{n^t \varphi_t}{\chi_{t-1}} > 0$). Therefore, there are two forces that govern the dynamics of the volume. First, as **Proposition 2** showed, the

[21] See Hong and Stein (2007), Figure 2 on page 114.

[22] Or more generally for any stocks which attract attentions. Consider for example internet stocks during the dot-com bubble period.

dispersion itself can go up, which tends to raise the volume through the inventory smoothing motive. Second, there is a gap between the volume ratio and the variance ratio, as shown in (36). When the rate of belief changes goes up, i.e., $\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}} < \frac{n^t\varphi_t}{\chi_{t-1}}$, this gap takes the value greater than one. In other words, when the learning accelerates, the change in the volume relative to the change in the dispersion goes up.

Lemma 5 (market size)

For sufficiently large n, the dispersion and volume monotonically decrease over time.

The Appendix shows that as $n \to \infty$, $\frac{n^t\varphi_t}{\chi_{t-1}}$ increases in n at the rate slower than n. Hence, the difference between the convergence rate of (31) and that of the benchmark is small for sufficiently large n. Of course, with the maintained anonymity assumption, we cannot take the limit $n \to \infty$ literally. Still, it is reassuring that this "large market limit" restores the benchmark dynamics, suggesting that the cross-sectional trading restriction is the key driver of the main results. **Figure 3** below shows how the volume dynamics changes as n increases.

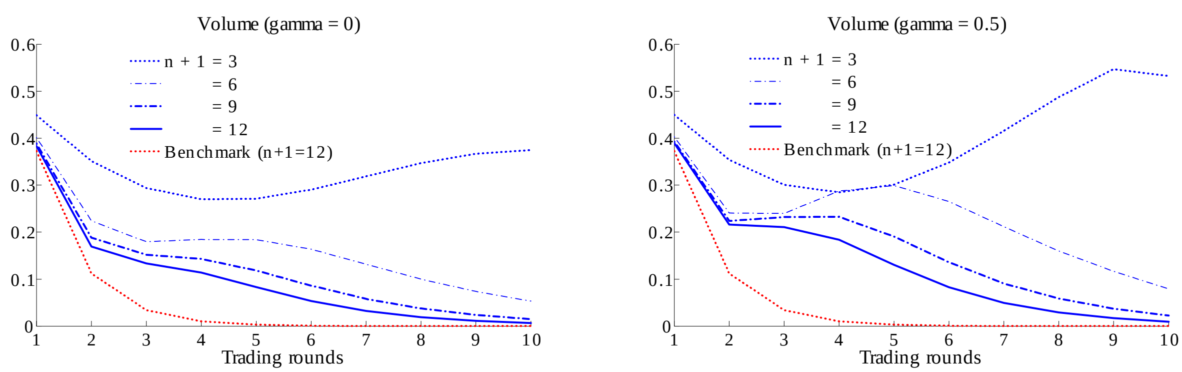

Figure 3. The volume for different n.
Note. The left panel uses $\gamma = 0$. The right panel uses $\gamma = 0.5$.

5 Conclusion

We studied dynamic asset trading with two frictions: 1) trading is locally intermediated, and 2) traders are hit by one time belief shocks that generates learning. Without belief shocks, traders focus on the socially beneficial trading and the allocation approaches the efficient allocation over time. When belief shocks induce the speculation, the convergence becomes slower, and the allocation may move away from the efficient allocation. When traders anticipate many trading rounds, traders' motives shift away from the socially beneficial inventory management to the speculation that has no social value. A highly non-monotonic pattern in trading activity can arise from the non-stationarity of the endogenous learning process.

In this paper, we focused on the propagation of belief shocks, and did not address the strategic issue that are likely to be present in small markets. We leave the analysis of the interaction between these two aspects of the local trading for a future work.

6 Appendix

6.1 Numerical solutions

Unless noted otherwise, we solved the model for $T = 10$. Other model parameters are set as follows.

$$\begin{array}{lll} \tau_\varepsilon = 0.05, & \tau_x = 5, & n \in \{2, 3, 4, 5\} \\ \tau_v = 1, & \gamma \in \{0, 0.125, 0.25, 0.5\} & \end{array}$$

6.2 Proof

Proof of Lemma 1.

This is a special case of **Proposition 0**. ■

Proof of Lemma 2.

We consider a generalized payment:

$$r_i = -\left(q_i + c\,|q_i|\right) p_i,\ c \geq 0,$$

where p_i is the unit price of the asset charged for trader i and the term $c\,|q_i|\,p_i$ represents a transaction cost that depends on the dollar volume.

From the conjecture $q_i(p) = \beta^s s_{i0} - \beta^x x_{i0} - \beta^p p$,

$$p_i = \frac{\beta^s}{\beta^p}\overline{s}_{i0} - \frac{\beta^x}{\beta^p}\overline{x}_{i0},$$

where $\overline{s}_{i0} \equiv \frac{1}{n}\sum_{j\neq i} s_{j0}$ and $\overline{x}_{i0} \equiv \frac{1}{n}\sum_{j\neq i} x_{j0}$. The net trade at this price is

$$q_i(p_i) = \beta^s(s_{i0} - \overline{s}_{i0}) - \beta^x(x_{i0} - \overline{x}_{i0}).$$

Note that β^p affects p_i but not $q_i(p_i)$.

This price is informationally equivalent to

$$\begin{aligned} h_i &\equiv \frac{\beta^p}{\beta^s} p_i \\ &= \overline{s}_{i0} - \frac{\beta^x}{\beta^s}\overline{x}_{i0} = v_A + \overline{\varepsilon}_{i0} - \frac{\beta^x}{\beta^s}\overline{x}_{i0}, \end{aligned}$$

which has

$$Var[h_i|v] = \frac{1}{n\tau_\varepsilon}\left\{1 + \left(\frac{\beta^x}{\beta^s}\right)^2 \frac{\tau_\varepsilon}{\tau_x}\right\}^{-1}.$$

Define

$$\varphi \equiv \left\{1 + \left(\frac{\beta^x}{\beta^s}\right)^2 \frac{\tau_\varepsilon}{\tau_x}\right\}^{-1}$$

to write

$$\begin{aligned} E_i[v] &= \frac{\tau_\varepsilon s_{i0} + \tau_\varepsilon n\varphi h_i}{\tau_v + \tau_\varepsilon(1 + n\varphi)} \\ &= \frac{1}{\frac{\tau_v}{\tau_\varepsilon} + 1 + n\varphi}\left(s_{i0} + n\varphi\frac{\beta^p}{\beta^s}p_i\right). \end{aligned}$$

Trader i's problem is

$$\max_{q_i}\left\{E_i[v](q_i + x_{i0}) - \frac{1}{2}(q_i + x_{i0})^2 - (q_i + c|q_i|)p_i\right\}.$$

First consider the realization of p_i for which $q_i \neq 0$. The first order condition is

$$E_i[v] - p_i(1 + c) = q_i + x_{i0}.$$

The second order condition $-\kappa < 0$ is satisfied. The optimal order, where p_i is replaced with p to emphasize that it is to be determined, is

$$\begin{aligned} q_i(p) &= E_i[v] - p(1 + c) - x_{i0} \\ &= \frac{1}{\frac{\tau_v}{\tau_\varepsilon} + 1 + n\varphi}\left(s_{i0} + n\varphi\frac{\beta^p}{\beta^s}p\right) - p(1 + c) - x_{i0}. \end{aligned}$$

This verifies the conjectured linear form and

$$\begin{aligned}
\beta^s &= \frac{1}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}, \\
\beta^x &= 1, \\
\beta^p &= 1+c-\frac{n\varphi}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}\frac{\beta^p}{\beta^s}.
\end{aligned}$$

By taking the ratio $\frac{\beta^p}{\beta^s}$,

$$\begin{aligned}
\frac{\beta^p}{\beta^s} &= \frac{1+c-\frac{n\varphi}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}\frac{\beta^p}{\beta^s}}{\frac{1}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}} \Leftrightarrow \frac{\beta^p}{\beta^s}\frac{1}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}(1+n\varphi) = 1+c \\
\therefore \quad \beta^p &= \beta^s\frac{\left(\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi\right)(1+c)}{1+n\varphi} = \frac{1+c}{1+n\varphi}.
\end{aligned}$$

The optimal order is

$$q_i(p) = \frac{1}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}s_{i0} - x_{i0} - \frac{1+c}{1+n\varphi}p.$$

The price for i is

$$\begin{aligned}
p_i &= \frac{\beta^s}{\beta^p}\overline{s}_{i0} - \frac{\beta^x}{\beta^p}\overline{x}_{i0} \\
&= \frac{1+n\varphi}{1+c}\left\{\frac{\sqrt{\rho}}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}\overline{s}_{i0} - \kappa\overline{x}_{i0}\right\}
\end{aligned}$$

and the trade for i is

$$q_i(p_i) = \frac{1}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}(s_{i0}-\overline{s}_{i0}) - (x_{i0}-\overline{x}_{i0}). \tag{37}$$

Because only β^p depends on c, neither trade $q_i(p_i)$ nor allocation $q_i(p_i)+x_{i0}$ depends on c. Also, when traders use this order in equilibrium, there is a unique value of p_i that makes $q_i(p_i)=0$. Because this is a zero probability event, using this order does not violate the optimality condition.

By taking the ratio $\frac{\beta^x}{\beta^s}$,

$$\begin{aligned}
\frac{1}{\varphi} &= \left\{1+\left(\frac{\beta^x}{\beta^s}\right)^2\frac{\tau_\varepsilon}{\tau_x}\right\} \\
&= 1+\left(\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi\right)^2\frac{\tau_\varepsilon}{\tau_x}.
\end{aligned}$$

$$\therefore \frac{1-\varphi}{\varphi} = \left(\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi\right)^2 \frac{\tau_\varepsilon}{\tau_x} \Leftrightarrow \frac{\varphi}{1-\varphi} = \frac{\tau_x}{\tau_\varepsilon}\frac{1}{\left(\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi\right)^2}. \tag{38}$$

This has a unique solution $\varphi \in (0,1)$. Using this, (37) can be written as

$$q_i(p_i) = \sqrt{\frac{\varphi}{1-\varphi}\frac{\tau_\varepsilon}{\tau_x}}(s_{i0}-\overline{s}_{i0}) - (x_{i0}-\overline{x}_{i0}).$$

Finally, from $x_{i1} = q_i(p_i) + x_{i0} = \sqrt{\frac{\varphi}{1-\varphi}\frac{\tau_\varepsilon}{\tau_x}}(s_{i0}-\overline{s}_{i0}) + \overline{x}_{i0}$,

$$\begin{aligned}
\frac{1}{\tau_{x1}} &= \frac{n+1}{n}\frac{\varphi}{1-\varphi}\frac{\tau_\varepsilon}{\tau_x}\frac{1}{\tau_\varepsilon} + \frac{1}{n}\frac{1}{\tau_x} \\
&= \frac{1}{n}\left(\frac{\varphi}{1-\varphi}(n+1)+1\right)\frac{1}{\tau_x} \\
&= \frac{1+n\varphi}{n-n\varphi}\frac{1}{\tau_x} \qquad \blacksquare.
\end{aligned}$$

Other results related to Lemma 2.
(a) $c = 0$ implies that the total payment by traders is non-positive.
(b) With the adjustment to mean parameters such that $E[p_i] > 0$, there is $c > 0$ that makes the total payment non-negative in expectation.

Proof.
(a) Define

$$\begin{aligned}
X_i &\equiv \frac{s_{i0}}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi} - x_{i0}, \\
\overline{X}_i &\equiv \frac{1}{n}\sum_{j\neq i} X_j.
\end{aligned}$$

Using these,

$$\begin{aligned}
p_i &= \frac{1+n\varphi}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}\overline{s}_{i0} - (1+n\varphi)\overline{x}_{i0} \\
&= (1+n\varphi)\overline{X}_i,
\end{aligned}$$

$$\begin{aligned}
q_i(p_i) &= \frac{1}{\frac{\tau_v}{\tau_\varepsilon}+1+n\varphi}(s_{i0}-\overline{s}_{i0}) - (x_{i0}-\overline{x}_{i0}) \\
&= X_i - \overline{X}_i.
\end{aligned}$$

Market clears because

$$\begin{aligned}
\sum_i \overline{X}_i &= \sum_i \frac{1}{n} \sum_{j \neq i} X_j \\
&= \frac{1}{n} \sum_i \left((n+1) \overline{X} - X_i \right) \\
&= \frac{(n+1)^2}{n} \overline{X} - \frac{n+1}{n} \overline{X} \\
&= (n+1) \overline{X} \\
&= \sum_i X_i .
\end{aligned}$$

The payment of trader i is

$$p_i \left\{ q_i \left(p_i \right) + c \left| q_i \left(p_i \right) \right| \right\} = (1 + n\varphi) \overline{X}_i \left\{ X_i - \overline{X}_i + c \left| X_i - \overline{X}_i \right| \right\} .$$

First,

$$\begin{aligned}
\sum_i X_i \overline{X}_i &= \sum_i X_i \left(\frac{1}{n} \sum_{j \neq i} X_j \right) \\
&= \frac{1}{n} \sum_i X_i \left\{ (n+1) \overline{X} - X_i \right\} \\
&= \frac{1}{n} \left\{ (n+1)^2 \overline{X}^2 - \sum_i X_i^2 \right\} .
\end{aligned}$$

Next,

$$\begin{aligned}
\sum_i \overline{X}_i^2 &= \sum_i \left(\frac{1}{n} \sum_{j \neq i} X_j \right)^2 \\
&= \frac{1}{n^2} \sum_i \left\{ (n+1) \overline{X} - X_i \right\}^2 \\
&= \frac{1}{n^2} \sum_i \left\{ (n+1)^2 \overline{X}^2 - 2 (n+1) \overline{X} X_i + X_i^2 \right\} \\
&= \frac{1}{n^2} \left\{ (n+1)^3 \overline{X}^2 - 2 (n+1)^2 \overline{X}^2 + \sum_i X_i^2 \right\} \\
&= \frac{1}{n^2} \left\{ (n+1)^2 (n-1) \overline{X}^2 + \sum_i X_i^2 \right\} .
\end{aligned}$$

Combining these two,

$$\begin{aligned}\sum_i \left(X_i - \overline{X}_i\right)\overline{X}_i &= \sum_i X_i\overline{X}_i - \sum_i \overline{X}_i^2 \\ &= \frac{(n+1)^2}{n}\left(1 - \frac{n-1}{n}\right)\overline{X}^2 - \left(\frac{1}{n} + \frac{1}{n^2}\right)\sum_i X_i^2 \\ &= \left(\frac{n+1}{n}\right)^2\left(\overline{X}^2 - \frac{1}{n+1}\sum_i X_i^2\right) \\ &= -\left(\frac{n+1}{n}\right)^2\frac{1}{n+1}\sum_i \left(X_i - \overline{X}\right)^2 \leq 0.\end{aligned}$$

Therefore, the total payment by $n+1$ traders when $c = 0$ is non-positive.

(b) With $c \geq 1$, $X_i - \overline{X}_i + c\left|X_i - \overline{X}_i\right| \geq 0$ for all i. Hence, as long as $E\left[\sum_i \overline{X}_i\right] = \sum_i E\left[\frac{p_i}{1+n\varphi}\right] > 0$, the expected value of total payments is non-negative. ■

Proof of Lemma 3.

We prove the result for φ_1 with a general T period case. From (28) and (29) in **Proposition 0**,

$$\begin{aligned}q_{i1} &= \frac{\Gamma_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_1}\left(s_{i0} - \overline{s}_{i0}\right) + \overline{x}_{i0} - x_{i0} \\ &= \sqrt{\frac{\varphi_1}{1-\varphi_1}\frac{\tau_\varepsilon}{\tau_x}}\left(s_{i0} - \overline{s}_{i0}\right) + \overline{x}_{i0} - x_{i0}.\end{aligned}$$

Comparing (29) with (38), $\varphi_1 > \varphi$ because $\Gamma_{T-1} = 1 + \frac{\gamma}{1-\gamma}\frac{\tau_\varepsilon\chi_2}{\tau_v+\tau_\varepsilon\chi_2}\Gamma_{T-2} > 1$ for $\gamma > 0$. ■

Proof of Proposition 0.

First, we study the final period $t = T$ and one period before the final period, $t = T-1$.

Final period $t = T$. Because there is no more trading after period T, the optimal order takes the same form as in the static case:

$$\begin{aligned}q_{iT}(p_{iT}; \mathcal{F}_{iT}) &= E_{iT}[v] - p_{iT} - x_{iT-1} \\ &= \frac{\sum_{k=0}^{T-1} n^k\varphi_k s_{ik}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} - x_{iT-1} - \left(1 - \frac{n^T\varphi_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}\widetilde{\beta}_T^p\right)p_{iT}.\end{aligned}$$

By equating coefficients with those in (16) for $t = T$,

$$\begin{aligned}
\beta^s_{T,k} &= \frac{n^k \varphi_k}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}, \; k = 0, .., T-1, \\
\beta^x_T &= 1, \\
\beta^p_T &= 1 - \frac{n^T \varphi_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \widetilde{\beta}^p_T.
\end{aligned}$$

From the expression of $\beta^s_{T,k}$, $\sum\limits_{k=0}^{T-1} \beta^s_{T,k} = \frac{\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}$. Hence, by normalizing by $\sum\limits_{k=0}^{T-1} \beta^s_{T,k}$,

$$\begin{aligned}
\widetilde{\beta}^s_{T,k} &\equiv \frac{\beta^s_{T,k}}{\sum\limits_{k=0}^{T-1} \beta^s_{T,k}} = \frac{n^k \varphi_k}{\chi_{T-1}}, \; k = 0, .., T-1, \\
\widetilde{\beta}^x_T &\equiv \frac{\beta^x_T}{\sum\limits_{k=0}^{T-1} \beta^s_{T,k}} = \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_{T-1}}, \\
\widetilde{\beta}^p_T &\equiv \frac{\beta^p_T}{\sum\limits_{k=0}^{T-1} \beta^s_{T,k}} = \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_{T-1}} \left(1 - \frac{\sqrt{\rho} n^T \varphi_T \widetilde{\beta}^p_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \right) \\
&= \frac{1}{\chi_{T-1}} \left(\frac{\tau_v}{\tau_\varepsilon} + \chi_T - n^T \varphi_T \widetilde{\beta}^p_T \right).
\end{aligned}$$

The last condition can be solved for $\widetilde{\beta}^p_T$:

$$\widetilde{\beta}^p_T = \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_{T-1} + n^T \varphi_T} = \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_T}.$$

Hence,

$$\beta^p_T = 1 - \frac{n^T \varphi_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_T} = \frac{\chi_{T-1}}{\chi_T}.$$

For $t = T$, this verifies Property III and

$$q_{iT}(p; \mathcal{F}_{iT}) = \frac{1}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \sum_{k=0}^{T-1} n^k \varphi_k s_{ik} - x_{iT-1} - \frac{\chi_{T-1}}{\chi_T} p,$$

$$\begin{aligned}
p_{iT} &= \sum_{k=0}^{T-1} \frac{\beta^s_{T,k}}{\beta^p_T} \overline{s}_{ik} - \frac{\beta^x_T}{\beta^p_T} \overline{x}_{iT-1} \qquad (39) \\
&= \frac{\chi_T}{\chi_{T-1}} \left(\frac{\tau_\varepsilon}{\tau_v + \tau_\varepsilon \chi_T} \sum_{k=0}^{T-1} n^k \varphi_k \overline{s}_{ik} - \overline{x}_{iT-1} \right),
\end{aligned}$$

$$q_{iT} = \frac{1}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \sum_{k=0}^{T-1} n^k \varphi_k \left(s_{ik} - \overline{s}_{ik}\right) - \left(x_{iT-1} - \overline{x}_{iT-1}\right), \tag{40}$$

$$x_{iT} = \frac{1}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \sum_{k=0}^{T-1} n^k \varphi_k \left(s_{ik} - \overline{s}_{ik}\right) + \overline{x}_{iT-1}. \tag{41}$$

Substituting derived coefficients into (23) gives

$$\begin{aligned}
\frac{1}{n^{T-1}\varphi_T} &= \sum_{k=0}^{T-1} \frac{\left(\frac{n^k \varphi_k}{\chi_{T-1}}\right)^2}{n^k \varphi_k} + \left(\frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_{T-1}}\right)^2 \frac{\tau_\varepsilon}{\tau_{xT-1}} \\
&= \frac{1}{\chi_{T-1}} + \frac{\tau_\varepsilon}{\tau_{xT-1}} \left(\frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_{T-1}}\right)^2.
\end{aligned}$$

This can be seen as an equation in φ_T given $(\varphi_1, .., \varphi_{T-1})$ and τ_{xT-1}, i.e.,

$$\frac{n^{T-1}\varphi_T}{\chi_{T-1}} - 1 + \frac{\tau_\varepsilon}{\tau_{xT-1}} \left(\frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_{T-1}}\right)^2 n^{T-1}\varphi_T = 0.$$

$$\Leftrightarrow \frac{\tau_\varepsilon}{\tau_{xT-1}} \left(\frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}{\chi_{T-1}}\right)^2 n^{T-1}\varphi_T = 1 - \frac{n^{T-1}}{\chi_{T-1}}\varphi_T.$$

Because the right hand side must be positive for the solution to exist,

$$\frac{n^{T-1}\varphi_T}{1 - \frac{n^{T-1}}{\chi_{T-1}}\varphi_T} = \frac{\tau_{xT-1}}{\tau_\varepsilon} \left(\frac{\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}\right)^2. \tag{42}$$

For a fixed $(\varphi_1, .., \varphi_{T-1})$ and τ_{xT-1}, the left hand side of (42) is increasing in φ_T and continuously changes from zero to positive infinity for $\varphi_T \in \left[0, \frac{\chi_{T-1}}{n^{T-1}}\right)$ and the right hand side is decreasing in φ_T. Hence, there is a unique φ_T that solves (42) for any given $(\varphi_1, .., \varphi_{T-1})$ and τ_{xT-1}.

Next, substituting derived coefficients into (20) gives

$$\begin{aligned}
\frac{\tau_\varepsilon}{\tau_{xT}} &= \left(\frac{\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}\right)^2 \left(\frac{1}{n^T \varphi_T} + \sum_{k=0}^{T-1} \frac{\left(\frac{n^k \varphi_k}{\chi_{T-1}}\right)^2}{n^k \varphi_k}\right) \\
&= \left(\frac{\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}\right)^2 \left(\frac{1}{n^T \varphi_T} + \frac{1}{\chi_{T-1}}\right) \\
&= \left(\frac{\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T}\right)^2 \frac{\chi_T}{n^T \varphi_T \chi_{T-1}} \\
&= \frac{1}{n^T \varphi_T} \frac{\chi_{T-1}\chi_T}{\left(\frac{\tau_v}{\tau_\varepsilon} + \chi_T\right)^2}.
\end{aligned}$$

Because $\beta_T^x = 1$ under the current trading rule, whatever position traders have at the beginning of period T is traded away and x_{iT-1} does not directly affect x_{iT}. Hence $\frac{\tau_\varepsilon}{\tau_{xT-1}}$ does not explicitly show up in the expression of $\frac{\tau_\varepsilon}{\tau_{xT}}$ above. However, (42) shows that φ_T is decreasing in $\frac{\tau_\varepsilon}{\tau_{xT-1}}$. Hence, the distribution of the asset positions in the previous period affects the distribution in the following period through the learning channel.

Period $t = T-1$. Trader i solves

$$\max_{q_{iT-1}} \left[(1-\gamma) \left\{ E_{iT-1}[v] (q_{iT-1} + x_{iT-2}) - \frac{1}{2} (q_{iT-1} + x_{iT-2})^2 - p_{iT-1} q_{iT-1} \right\} \right.$$
$$\left. + \gamma E_{iT-1} \left[E_{iT}[v] x_{iT} - \frac{1}{2} x_{iT}^2 - (p_{iT-1} q_{iT-1} + p_{iT} q_{iT}) \right] \right],$$

where p_{iT}, q_{iT}, x_{iT} are given by (39), (40), (41). The first line in the expression above is the expected utility for the case the game ends after trading in period $T-1$, while the second line corresponds to the case where there is another trading round. Dropping irrelevant terms,

$$\max_{q_{iT-1}} \left[(1-\gamma) \left\{ E_{iT-1}[v] q_{iT-1} - \frac{1}{2} (q_{iT-1} + x_{iT-2})^2 \right\} - p_{iT-1} q_{iT-1} + \gamma E_{iT-1}[p_{iT}^*] q_{iT-1} \right].$$

The last term shows up because the additional unit of the risky asset today would save the purchase tomorrow if and only if the game continues. Hence, the optimal order at $T-1$ is

$$q_{iT-1}(p; \mathcal{F}_{iT-1}) = \frac{(1-\gamma) E_{iT-1}[v] + \gamma E_{iT-1}[p_{iT}^*] - p}{1-\gamma} - x_{iT-2}.$$

From (39),

$$E_{iT-1}[p_{iT}] = \sum_{k=0}^{T-1} \frac{\beta_{T,k}^s}{\beta_T^p} E_{iT-1}[\overline{s}_{k,-i}] = \frac{\chi_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} E_{iT-1}[v].$$

Hence,

$$\begin{aligned} q_{iT-1}(p; \mathcal{F}_{iT-1}) &= \frac{1}{1-\gamma} \left\{ \left(1 - \gamma + \gamma \frac{\chi_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \right) E_{iT-1}[v] - p \right\} - x_{iT-2} \\ &= \left(1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \right) E_{iT-1}[v] - \frac{p}{1-\gamma} - x_{iT-2}. \end{aligned}$$

We define

$$\Gamma_1 \equiv 1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\frac{\tau_v}{\tau_\varepsilon} + \chi_T} \in \left[1, \frac{1}{1-\gamma} \right].$$

Finally, recall from (24) that

$$E_{iT-1}[v] = \frac{1}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}} \left(\sum_{k=0}^{T-2} n^k \varphi_k s_{ik} + n^{T-1} \varphi_{T-1} \widetilde{\beta}_{T-1}^p p \right).$$

Therefore,

$$q_{iT-1}(p;\mathcal{F}_{iT-1}) = \Gamma_1 \frac{\sum_{k=0}^{T-2} n^k \varphi_k s_{ik}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}} - x_{iT-2} - \left\{ \frac{1}{1-\gamma} - \Gamma_1 \frac{n^{T-1}\varphi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}} \widetilde{\beta}^p_{T-1} \right\} p.$$

By equating coefficients with those in (16) for $t = T-1$,

$$\begin{aligned}
\beta^s_{T-1,k} &= \Gamma_1 \frac{n^k \varphi_k}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}},\ k = 0, .., T-2, \\
\beta^x_{T-1} &= 1, \\
\beta^p_{T-1} &= \frac{1}{1-\gamma} - \Gamma_1 \frac{n^{T-1}\varphi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}} \frac{\beta^p_{T-1}}{\sum_{k=0}^{T-2} \beta^s_{T-1k}}.
\end{aligned}$$

From the expression of $\beta^s_{T-1,k}$, $\sum_{k=0}^{T-2} \beta^s_{T-1,k} = \Gamma_1 \frac{\chi_{T-2}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}}$. By normalizing this,

$$\begin{aligned}
\widetilde{\beta}^s_{T-1k} &= \frac{n^k \varphi_k}{\chi_{T-2}},\ k = 0, .., T-2, \\
\widetilde{\beta}^x_{T-1} &= \frac{1}{\Gamma_1} \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}}{\chi_{T-2}}, \\
\beta^p_{T-1} &= \frac{1}{1-\gamma} - \frac{n^{T-1}\varphi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}} \left(\frac{\chi_{T-2}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}} \right)^{-1} \beta^p_{T-1} \\
&= \frac{1}{1-\gamma} - \frac{n^{T-1}\varphi_{T-1}}{\chi_{T-2}} \beta^p_{T-1}.
\end{aligned}$$

The last condition can be solved for β^p_{T-1}:

$$\begin{aligned}
\beta^p_{T-1} &= \frac{1}{1-\gamma} \left(1 + \frac{n^{T-1}\varphi_{T-1}}{\chi_{T-2}} \right)^{-1} \\
&= \frac{1}{1-\gamma} \frac{\chi_{T-2}}{\chi_{T-1}}.
\end{aligned}$$

Hence,

$$q_{iT-1}(p;\mathcal{F}_{iT-1}) = \Gamma_1 \frac{\sum_{k=0}^{T-2} n^k \varphi_k s_{ik}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{T-1}} - x_{iT-2} - \frac{1}{1-\gamma} \frac{\chi_{T-2}}{\chi_{T-1}} p.$$

Proceeding similarly as before,

$$\begin{aligned}\frac{1}{n^{T-2}\varphi_{T-1}} &= \sum_{k=0}^{T-2}\frac{\left(\frac{n^k\varphi_k}{\chi_{T-2}}\right)^2}{n^k\varphi_k} + \left(\frac{1}{\Gamma_1}\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}{\chi_{T-2}}\right)^2\frac{\tau_\varepsilon}{\tau_{xT-2}} \\ &= \frac{1}{\chi_{T-2}} + \frac{\tau_\varepsilon}{\tau_{xT-2}}\left(\frac{1}{\Gamma_1}\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}{\chi_{T-2}}\right)^2.\end{aligned}$$

This can be seen as an equation in φ_{T-1} given $(\varphi_1, .., \varphi_{T-2}, \varphi_T)$ and τ_{xT-2}:

$$\frac{n^{T-2}\varphi_{T-1}}{\chi_{T-2}} - 1 + \frac{\tau_\varepsilon}{\tau_{xT-2}}\left(\frac{1}{\Gamma_1}\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}{\chi_{T-2}}\right)^2 n^{T-2}\varphi_{T-1} = 0.$$

$$\Leftrightarrow \frac{\tau_\varepsilon}{\tau_{xT-2}}\left(\frac{1}{\Gamma_1}\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}{\chi_{T-2}}\right)^2 n^{T-2}\varphi_{T-1} = 1 - \frac{n^{T-2}}{\chi_{T-2}}\varphi_{T-1}.$$

Because the right hand side must be positive for the solution to exist,

$$\frac{n^{T-2}\varphi_{T-1}}{1-\frac{n^{T-2}}{\chi_{T-2}}\varphi_{T-1}} = \frac{\tau_{xT-2}}{\tau_\varepsilon}\left(\Gamma_1\frac{\chi_{T-2}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}\right)^2. \tag{43}$$

Given $(\varphi_1, .., \varphi_{T-2})$, the left hand side of (43) is increasing in φ_{T-1} and continuously changes from zero to positive infinity for $\varphi_{T-1} \in \left[0, \frac{\chi_{T-2}}{n^{T-2}}\right)$. On the other hand, the right hand side is decreasing in φ_{T-1} if $\gamma \leq \frac{1}{2}$.[23] We focus on the case where there is a unique φ_{T-1} that solves (43) for any given $(\varphi_1, .., \varphi_{T-2}, \varphi_T)$ and τ_{xT-2}.

Also, substituting derived coefficients into (20) gives

$$\begin{aligned}\frac{\tau_\varepsilon}{\tau_{xT-1}} &= \left(\Gamma_1\frac{\chi_{T-2}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}\right)^2\left(\frac{1}{n^{T-1}\varphi_{T-1}} + \sum_{k=0}^{T-2}\frac{\left(\frac{n^k\varphi_k}{\chi_{T-2}}\right)^2}{n^k\varphi_k}\right) \\ &= \left(\Gamma_1\frac{\chi_{T-2}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}\right)^2\left(\frac{1}{n^{T-1}\varphi_{T-1}} + \frac{1}{\chi_{T-2}}\right) \\ &= \left(\Gamma_1\frac{\chi_{T-2}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}\right)^2\frac{\chi_{T-1}}{n^{T-1}\varphi_{T-1}\chi_{T-2}} \\ &= \frac{\chi_{T-2}\chi_{T-1}}{n^{T-1}\varphi_{T-1}}\left(\frac{\Gamma_1}{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}\right)^2.\end{aligned}$$

[23] Γ_1 as a function of $\varphi_1, .., \varphi_T$ is increasing in each argument if $\gamma > 0$, but $\frac{\Gamma_1}{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}} = \frac{\frac{\tau_v}{\tau_\varepsilon}+\frac{1}{1-\gamma}\chi_T}{\left(\frac{\tau_v}{\tau_\varepsilon}+\chi_T\right)\left(\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}\right)}$ is decreasing in φ_{T-1} if $\gamma \leq \frac{1}{2}$.

Finally, this last expression can be substituted into (42) to obtain

$$\frac{n^{T-1}\varphi_T}{1-\frac{n^{T-1}}{\chi_{T-1}}\varphi_T} = \frac{n^{T-1}\varphi_{T-1}}{\chi_{T-2}\chi_{T-1}}\left(\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}{\Gamma_1}\right)^2\left(\frac{\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_T}\right)^2$$

$$\Leftrightarrow \varphi_T = \left(1+\frac{n^{T-1}}{\chi_{T-2}}\left(\varphi_{T-1}-\varphi_T\right)\right)\left(\frac{1}{\Gamma_1}\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_T}\right)^2\varphi_{T-1}.$$

The characterization for other periods is by a straightforward induction. ■

Proof of Proposition 1.

(a) If $\gamma = 0$, $\Gamma_{T-t} = 1$ for all $t = 1,..,T$. If $\tau_v = 0$, $\Gamma_{T-t} = \frac{1}{1-\gamma}$ for all $t = 1,..,T-1$. For both cases,

$$\beta^s_{t/t-1} = \frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}}\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t} = \frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t} < 1$$

and for $t = 2,..,T$, φ_t solves

$$\frac{\varphi_t}{\varphi_{t-1}} = \left(1+\frac{n^{t-1}}{\chi_{t-2}}\left(\varphi_{t-1}-\varphi_t\right)\right)\left(\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\right)^2.$$

Clearly, φ_t must decrease over time.

(b) From

$$\frac{\varphi_t}{\varphi_{t-1}} = \left(1+\frac{n^{t-1}}{\chi_{t-2}}\left(\varphi_{t-1}-\varphi_t\right)\right)\left(\beta^s_{t/t-1}\right)^2,$$

the three cases (i)-(iii) follow. Case (i) occurs if and only if $\beta^s_{t/t-1} = \frac{\beta^s_{t,k}}{\beta^s_{t-1,k}} > 1$. Because $\beta^s_{T/T-1} = \frac{\Gamma_0}{\Gamma_1}\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{T-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_T} < 1$, case (ii) always applies for the last period and $\frac{\varphi_T}{\varphi_{T-1}} < 1$. ■

Other results related to Γ_{T-t}.

For $\tau_\varepsilon \in (0,\infty)$, $\Gamma_{T-t} \leq \Gamma_{T-(t-1)}$ for $t = 1,..,T-1$, with strict inequality if $\gamma\tau_v > 0$.

Proof.

If $\gamma\tau_v = 0$, the result holds with equality. Otherwise, consider

$$\begin{aligned}
\Gamma_0 &= 1,\\
\Gamma_1 &= 1+\frac{\gamma}{1-\gamma}\frac{\tau_\varepsilon\chi_T}{\tau_v+\tau_\varepsilon\chi_T}\Gamma_0,\\
\Gamma_2 &= 1+\gamma\frac{\tau_\varepsilon\chi_{T-1}}{\tau_v+\tau_\varepsilon\chi_{T-1}}\Gamma_1,\\
\Gamma_3 &= 1+\gamma\frac{\tau_\varepsilon\chi_{T-2}}{\tau_v+\tau_\varepsilon\chi_{T-2}}\Gamma_2,\\
&\vdots
\end{aligned}$$

Comparing Γ_3 and Γ_2, that $\Gamma_2 < \Gamma_1$ implies $\Gamma_3 < \Gamma_2$, because $\frac{\tau_\varepsilon \chi_{T-2}}{\tau_v + \tau_\varepsilon \chi_{T-2}} < \frac{\tau_\varepsilon \chi_{T-1}}{\tau_v + \tau_\varepsilon \chi_{T-1}}$. By an induction, it suffices to show $\Gamma_2 < \Gamma_1$. This is true because $\Gamma_2 < \Gamma_1$ if and only if

$$\frac{\chi_{T-1}}{\tau_v + \tau_\varepsilon \chi_{T-1}} \Gamma_1 < \frac{1}{1-\gamma} \frac{\chi_T}{\tau_v + \tau_\varepsilon \chi_T} \Leftrightarrow (1-\gamma)\Gamma_1 < \frac{\chi_T}{\chi_{T-1}} \frac{\tau_v + \tau_\varepsilon \chi_{T-1}}{\tau_v + \tau_\varepsilon \chi_T}$$
$$\Leftrightarrow \quad 1 - \gamma + \gamma \frac{\tau_\varepsilon \chi_T}{\tau_v + \tau_\varepsilon \chi_T} < \frac{\chi_T}{\chi_{T-1}} \frac{\tau_v + \tau_\varepsilon \chi_{T-1}}{\tau_v + \tau_\varepsilon \chi_T}.$$

Note that the left hand side is smaller than 1 for $\gamma \tau_v > 0$ and the right hand side is greater than 1 for $\tau_v > 0$. ■

Proof of Proposition 2.
From **Proposition 0**,

$$\sum_{k=0}^{t-1} \beta_{t,k}^s = \frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \chi_{t-1}, \; \widetilde{\beta}_{t,k}^s = \frac{n^k \varphi_k}{\chi_{t-1}} \text{ and } \beta_t^x = 1.$$

Using these in (20),

$$\begin{aligned} \frac{\tau_\varepsilon}{\tau_{xt}} &= \left(\frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \chi_{t-1} \right)^2 \left(\frac{1}{n^t \varphi_t} + \sum_{k=0}^{t-1} \frac{\left(\frac{n^k \varphi_k}{\chi_{t-1}} \right)^2}{n^k \varphi_k} \right) \qquad (44) \\ &= \left(\frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \right)^2 \chi_{t-1}^2 \left(\frac{1}{n^t \varphi_t} + \frac{1}{\chi_{t-1}} \right) \\ &= \left(\frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \right)^2 \frac{\chi_{t-1} \chi_t}{n^t \varphi_t}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\frac{\tau_\varepsilon}{\tau_{xt}}}{\frac{\tau_\varepsilon}{\tau_{xt-1}}} &= \frac{\left(\frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \right)^2 \frac{\chi_{t-1} \chi_t}{n^t \varphi_t}}{\left(\frac{\Gamma_{T-(t-1)}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_{t-1}} \right)^2 \frac{\chi_{t-2} \chi_{t-1}}{n^{t-1} \varphi_{t-1}}} \\ &= \left(\frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \right)^2 \frac{1}{n} \frac{\chi_t}{\chi_{t-2}} \frac{\varphi_{t-1}}{\varphi_t}. \end{aligned}$$

By (30),

$$\frac{\varphi_{t-1}}{\varphi_t} \left(\frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} \frac{\frac{\tau_v}{\tau_\varepsilon} + \chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon} + \chi_t} \right)^2 = \left\{ \frac{\chi_{t-2} + n^{t-1} \left(\varphi_{t-1} - \varphi_t \right)}{\chi_{t-2}} \right\}^{-1}.$$

Hence,

$$\begin{aligned}
\frac{1}{\tau_{xt}} &= \frac{1}{\tau_{xt-1}}\frac{1}{n}\frac{\chi_t}{\chi_{t-2}}\frac{\chi_{t-2}}{\chi_{t-2}+n^{t-1}\left(\varphi_{t-1}-\varphi_t\right)} \\
&= \frac{1}{\tau_{xt-1}}\frac{1}{n}\frac{\chi_{t-1}+n^t\varphi_t}{\chi_{t-1}-n^{t-1}\varphi_t} \\
&= \frac{1}{\tau_{xt-1}}\frac{1+\frac{n^t\varphi_t}{\chi_{t-1}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}. \qquad \blacksquare
\end{aligned}$$

Proof of Proposition 3.
From **Proposition 2**, $\frac{1}{\tau_{xt}} > \frac{1}{\tau_{xt-1}}$ if and only if

$$1+\frac{n^t\varphi_t}{\chi_{t-1}} > n-\frac{n^t\varphi_t}{\chi_{t-1}} \Leftrightarrow \frac{n^t\varphi_t}{\chi_{t-1}} > \frac{n-1}{2}.$$

From the proof of **Lemma 4** below, $E\left[|q_{it-1}|\right] < E\left[|q_{it}|\right]$ if and only if

$$1 < \frac{1+\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}} \Leftrightarrow n-1 < \frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}+\frac{n^t\varphi_t}{\chi_{t-1}}. \qquad \blacksquare$$

Proof of Lemma 4.
From **Proposition 0**,

$$\begin{aligned}
q_{it}\left(p\right) &= \frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\sum_{k=0}^{t-1} n^k\varphi_k s_{ik} - x_{it-1} - \frac{1}{1-\gamma\mathbf{1}\left\{t<T\right\}}\frac{\chi_{t-1}}{\chi_t}p, \\
q_{it}\left(p_{it}\right) &= \frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\sum_{k=0}^{t-1} n^k\varphi_k\left(s_{ik}-\overline{s}_{ik}\right) - \left(x_{it-1}-\overline{x}_{it-1}\right) \\
&= \frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\sum_{k=0}^{t-1} n^k\varphi_k\left(\varepsilon_{ik}-\overline{\varepsilon}_{ik}\right) - \left(x_{it-1}-\overline{x}_{it-1}\right).
\end{aligned}$$

We write $q_{it} = q_{it}\left(p_{it}\right)$. Since this is normally distributed,

$$E\left[|q_{it}|\right] = \sqrt{\frac{2}{\pi}Var\left[q_{it}\right]}.$$

First,

$$\begin{aligned}
Var\left[x_{it-1}-\overline{x}_{it-1}\right] &= \frac{n+1}{n}\frac{1}{\tau_{xt-1}}, \\
Var\left[\varepsilon_{it-1}-\overline{\varepsilon}_{it-1}\right] &= \frac{n+1}{n}\frac{1}{\tau_\varepsilon n^k\varphi_k}
\end{aligned}$$

by Property II. Therefore,

$$
\begin{aligned}
Var\left[q_{it}\right] &= \frac{n+1}{n}\left\{\left(\frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\right)^2\sum_{k=0}^{t-1}\frac{\left(n^k\varphi_k\right)^2}{\tau_\varepsilon n^k\varphi_k}+\frac{1}{\tau_{xt-1}}\right\} \\
&= \frac{n+1}{n\tau_\varepsilon}\left\{\left(\frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\right)^2\chi_{t-1}+\frac{\tau_\varepsilon}{\tau_{xt-1}}\right\}.
\end{aligned}
$$

By (44),

$$
\frac{\tau_\varepsilon}{\tau_{xt}}=\left(\frac{\Gamma_{T-t}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\right)^2\frac{\chi_{t-1}\chi_t}{n^t\varphi_t}.
$$

Using this,

$$
Var\left[q_{it}\right]=\frac{n+1}{n\tau_\varepsilon}\left\{\frac{\tau_\varepsilon}{\tau_{xt}}\frac{n^t\varphi_t}{\chi_t}+\frac{\tau_\varepsilon}{\tau_{xt-1}}\right\}=\frac{n+1}{n}\left\{\frac{1}{\tau_{xt}}\frac{n^t\varphi_t}{\chi_t}+\frac{1}{\tau_{xt-1}}\right\}.
$$

Finally, use $\frac{1}{\tau_{xt}}=\frac{1+\frac{n^t\varphi_t}{\chi_{t-1}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}\frac{1}{\tau_{xt-1}}$ from **Proposition 1** to write

$$
\begin{aligned}
Var\left[q_{it}\right] &= \frac{n+1}{n}\left\{\frac{1+\frac{n^t\varphi_t}{\chi_{t-1}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}\frac{n^t\varphi_t}{\chi_t}+1\right\}\frac{1}{\tau_{xt-1}} \\
&= \frac{n+1}{n}\left\{\frac{n\chi_{t-1}}{n\chi_{t-1}-n^t\varphi_t}\right\}\frac{1}{\tau_{xt-1}} \\
&= \frac{n+1}{\tau_{xt-1}}\frac{1}{n-\frac{n^t\varphi_t}{\chi_{t-1}}},
\end{aligned}
\tag{45}
$$

or

$$
\begin{aligned}
Var\left[q_{it}\right] &= \frac{n+1}{n}\left\{\frac{n^t\varphi_t}{\chi_t}+\frac{n-\frac{n^t\varphi_t}{\chi_{t-1}}}{1+\frac{n^t\varphi_t}{\chi_{t-1}}}\right\}\frac{1}{\tau_{xt}} \\
&= \frac{n+1}{n}\left\{\frac{n^t\varphi_t+n\chi_{t-1}-n^t\varphi_t}{\chi_t}\right\}\frac{1}{\tau_{xt}} \\
&= \frac{n+1}{\tau_{xt}}\frac{\chi_{t-1}}{\chi_t}.
\end{aligned}
\tag{46}
$$

If we use (45),

$$\frac{E[|q_{it}|]}{E[|q_{it-1}|]} = \sqrt{\frac{Var[x_{it-1}]}{Var[x_{it-2}]}\frac{n-\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}} = \sqrt{\frac{1+\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}{n-\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}\frac{n-\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}} = \sqrt{\frac{1+\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}}.$$

If we use (46),

$$\frac{E[|q_{it}|]}{E[|q_{it-1}|]} = \sqrt{\frac{Var[x_{it}]}{Var[x_{it-1}]}\frac{\chi_{t-1}^2}{\chi_{t-2}\chi_t}} = \sqrt{\frac{1+\frac{n^t\varphi_t}{\chi_{t-1}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}\frac{\chi_{t-1}^2}{\chi_{t-2}\chi_t}} = \sqrt{\frac{1}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}\frac{\chi_{t-1}}{\chi_{t-2}}} = \sqrt{\frac{1+\frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}}}{n-\frac{n^t\varphi_t}{\chi_{t-1}}}}. \qquad \blacksquare$$

Proof of Lemma 5.
Because $\Gamma_{T-t} \in \left[1, \frac{1}{1-\gamma}\right]$ is bounded, it suffices to consider

$$\frac{\varphi_1}{1-\varphi_1} = \frac{\tau_x}{\tau_\varepsilon}\left(\frac{1}{\frac{\tau_v}{\tau_\varepsilon}+\chi_1}\right)^2, \tag{47}$$

$$\varphi_t = \left(1+\frac{n^{t-1}}{\chi_{t-2}}\left(\varphi_{t-1}-\varphi_t\right)\right)\left(\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_{t-1}}{\frac{\tau_v}{\tau_\varepsilon}+\chi_t}\right)^2\varphi_{t-1} \text{ for } t \geq 2. \tag{48}$$

Note $\chi_1 = 1 + n\varphi_1$ in (47). Suppose $\varphi_1 \sim n^{\delta_1}$, $\delta_1 \in \mathbb{R}$. First, $\delta_1 \leq 0$, because otherwise the left hand side becomes negative for sufficiently large n. Second, suppose that the left hand side converges to some positive constant, i.e. $\delta_1 = 0$. But then the left hand side must converge zero, so this is not possible. Therefore, we have $\delta_1 < 0$. For both sides of (47) to have the same order

$$\delta_1 = -2(1+\delta_1) \Leftrightarrow \delta_1 = -\frac{2}{3}.$$

Similarly, using

$$\varphi_2 = (1+n(\varphi_1-\varphi_2))\left(\frac{\frac{\tau_v}{\tau_\varepsilon}+\chi_1}{\frac{\tau_v}{\tau_\varepsilon}+\chi_2}\right)^2\varphi_1,$$

it can be easily verified that $\delta_2 = -\frac{11}{9}$.

Next, suppose $\varphi_t \sim n^{\delta_t}$, $\delta_t \in \mathbb{R}$. First, $\delta_t < \delta_{t-1}$, because otherwise (47) does not have a solution for a large n. Conjecture that for all t, χ_t has the same order with $n^t \varphi_t$. This means that

$$t + \delta_t > t - 1 + \delta_{t-1} \Leftrightarrow 1 > \delta_{t-1} - \delta_t. \tag{49}$$

The conjecture is true for $t = 1, 2$, with $\delta_0 = 0$. Suppose the conjecture is true up to $t - 1$. From (47),

$$\begin{aligned}
\delta_t + 2(t + \delta_t) &= t - 1 + \delta_{t-1} - (t - 2 + \delta_{t-2}) + 2(t - 1 + \delta_{t-1}) + \delta_{t-1} \\
&\Leftrightarrow 3\delta_t = -1 + 4\delta_{t-1} - \delta_{t-2} \\
&\Leftrightarrow 3(\delta_t - \delta_{t-1}) = -1 + \delta_{t-1} - \delta_{t-2} \\
&\Leftrightarrow \delta_t - \delta_{t-1} + \frac{1}{2} = \frac{1}{3}\left(\delta_{t-1} - \delta_{t-2} + \frac{1}{2}\right).
\end{aligned}$$

Therefore, the conjecture must be true for t. This verifies (49). Next,

$$\begin{aligned}
\delta_t - \delta_{t-1} &= -\frac{1}{2} + \left(\frac{1}{3}\right)^{t-1}\left(\delta_1 - \delta_0 + \frac{1}{2}\right) \\
&= -\frac{1}{2} - \left(\frac{1}{3}\right)^{t-1}\frac{1}{6} \\
&= -\frac{1}{2}\left\{1 + \left(\frac{1}{3}\right)^t\right\}.
\end{aligned}$$

and $\frac{n^t \varphi_t}{\chi_{t-1}}$ has an order

$$\begin{aligned}
1 - (\delta_{t-1} - \delta_t) &= 1 - \frac{1}{2}\left\{1 + \left(\frac{1}{3}\right)^t\right\} \\
&= \frac{1}{2}\left\{1 - \left(\frac{1}{3}\right)^t\right\} > 0.
\end{aligned}$$

This is increasing in t but bounded above by $\frac{1}{2}$. Recall the conditions in **Proposition 3**:

$$Var[x_{it-1}] < Var[x_{it}] \Leftrightarrow \frac{n-1}{2} < \frac{n^t \varphi_t}{\chi_{t-1}},$$

$$E[|q_{it-1}|] < E[|q_{it}|] \Leftrightarrow n - 1 < \frac{n^{t-1}\varphi_{t-1}}{\chi_{t-2}} + \frac{n^t \varphi_t}{\chi_{t-1}}.$$

Neither can be satisfied for sufficiently large n. ■

References

[1] Aıt-Sahalia, Y. and M. Saglam (2014): "High Frequency Traders: Taking Advantage of Speed," *Working paper.*

[2] Amador, M. and P. O. Weill (2012): "Learning from Private and Public Observations of Others' Actions," *Journal of Economic Theory*, 147, 910–940.

[3] Biais, B., T. Foucault, and S. Moinas (2914): "Equilibrium Fast Trading," *Working paper.*

[4] Bernhardt, D. and J. Miao (2004): "Informed Trading When Information Becomes Stale," *Journal of Finance*, 59, 1, 339-390.

[5] Bernhardt, D., P. Seiler, and B. Taub (2010): "Speculative Dynamics," *Economic Theory*, 44, 1–52.

[6] Bessembinder, H., K. Chan and P. J. Seguin (1996): "An Empirical Examination of Information, Differences of Opinion, and Trading Activity," *Journal of Financial Economics*, 40, 105–134.

[7] Chae, J (2005): "Trading Volume, Information Asymmetry, and Timing Information," *Journal of Finance*, 60, 1, 413-442.

[8] Cochrane, J. H. (2013): "Finance: Function Matters, Not Size," *Journal of Economic Perspective*, 27, 2, 29-50.

[9] Comerton-Forde, C., T. Hendershott, C. M. Jones, P. C. Moulton, and M. S. Seasholes (2010): "Time Variation in Liquidity: The Role of Market-Maker Inventories and Revenues," *Journal of Finance*, 65, 1, 295-331.

[10] Duffie, D., S. Malamud and G. Manso (2009): "Information Percolation with Equilibrium Search Dynamics," *Econometrica*, 77, 5, 1513–1574.

[11] ——, ——, and —— (2013): "Information Percolation in Segmented Markets," Working Paper.

[12] Engelberg J.E. and C. A. Parsons (2011): "The Causal Impact of Media in Financial Markets," *Journal of Finance*, 66, 1, 67-97.

[13] Foster, F.D. and S. Viswanathan (1993): "Variations in Trading Volume, Return Volatility, and Trading Costs: Evidence on Recent Price Formation Models," *Journal of Finance*, 48, 1, 187-211.

[14] Graham, J.R., J.L. Koski and U. Loewenstein (2006): "Information Flow and Liquidity around Anticipated and Unanticipated Dividend," *Journal of Business*, 79, 5, 2301-2336.

[15] Golosov, M., G. Lorenzoni and A. Tsyvinski (2013): "Decentralized Trading with Private Information," Working Paper.

[16] Hellwig, M. (1980): "On the Aggregation of Information in Competitive Markets," *Journal of Economic Theory*, 22, 477-498.

[17] Hong, H. and J.C. Stein (2007): "Disagreement and the Stock Market," *Journal of Economic Perspective*, 21, 2, 109-128.

[18] Iyer, K., R. Johari, and C.C. Moallemi (2011): "Information Aggregation and Allocative Efficiency in Smooth Markets," Working Paper.

[19] Kyle, A. S. (1985): "Continuous Auctions and Insider Trading," *Econometrica*, 53, 6, 1315-1335.

[20] —— (1989): "Informed Speculation with Imperfect Competition," *Review of Economic Studies*, 56, 317-356.

[21] Ostrovsky, M. (2012): "Information Aggregation in Dynamic Markets with Strategic Traders," *Econometrica*, 80, 6, 2595-2647.

[22] Peress (2014): "The Media and the Diffusion of Information in Financial Markets: Evidence from Newspaper Strikes" *Journal of Finance*, 69, 5, 2007-2043.

[23] Rostek, M. and M. Weretka (2011): "Dynamic Thin Markets," Working Paper.

[24] —— and —— (2012): "Price Inference in Small Markets," *Econometrica*, 80, 2, 687-711.

[25] Vives, X. (2011): "Strategic Supply Function Competition with Private Information," *Econometrica*, 79, 6, 1919-1966.