

# Confidence Intervals for Ratios: Econometric Examples in Stata and R

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## Abstract

Ratios of parameter estimates are often used in econometric applications. However, the test of these ratios when estimated can cause difficulties since the ratio of asymptotically normally distributed random variables have a Cauchy distribution for which there are no finite moments.

This paper presents a method for the estimation of confidence intervals based on the Fieller approach that has been shown to be preferable to the usual Delta method. Using example applications in both Stata and R, we demonstrate that a few extra steps in the examination of the estimate of the ratio may provide a confidence interval with superior coverage.

Key words: turning points, quadratic specification, interaction terms, long-run elasticities, willingness-to-pay, NAIRU, 2SLS & ILS, Stata, R

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## 1. Introduction

Many econometric applications draw inferences from a parameter of interest that is defined as the ratio of regression coefficients. However, the properties of the ratio of estimates can be problematic. This paper examines eight examples of econometric models where inferences for ratios are used. These examples include; the interpretation of a dummy variable in terms of changes in a continuous regressor, the location of a turning point in a quadratic specification where the marginal impact of a regressor changes sign, the interpretation of marginal effect of one regressor when interacted with another, the estimation of the long-run effect in a dynamic model, the determination of the “Taylor Rule” for dynamic macro models, the determination of the non-accelerating inflation rate of unemployment (NAIRU), the willingness to pay, and the structural parameter in an exactly identified system of equations as estimated by the two-stage least squares method. We present analysis of applications of each of these cases along with the corresponding Stata code to obtain these results.

In this paper, we emphasise the use of confidence intervals (CIs) instead of the use of p-values. Because confidence intervals are expressed in the units of the parameter of interest instead of as a probabilistic abstraction, we concentrate on the nature of these intervals. The traditional approach for constructing CIs is based on the Delta method where a first order Taylor-series expansion is used to approximate a linear relationship between the estimated parameters and the ratio. This method is the standard approach to the estimation of CIs and tests of hypothesis for non-linear functions of regression parameter estimates and is available in most econometric/statistics software such as Stata, SAS, EViews, and SPSS.

An alternative method for the construction of CIs for ratios is Fieller’s 1954 proposal. This approach has been shown to be superior to the application of the Delta method in several applications (e.g. Hirschberg and Lye 2010c). This superiority has been found in the coverage of the resulting tests where the estimated  $100(1 - \alpha)\%$  interval from the Fieller method more closely coincides with the theoretical interval than the alternate Delta interval.<sup>1</sup> In addition, Fieller intervals are not forced to be symmetric as are the Delta intervals. However, formally the ratio of two asymptotically normally distributed random variables (as are regression coefficient estimates) is distributed as a non-central Cauchy distribution for which there are no finite moments. Thus, the traditional approach to constructing a CI with the estimate of the expected value and variance - by adding and subtracting a multiple of the square-root of the variance - may not be possible. However, we may still be able to make finite probability statements concerning the ratio over a limited range of probabilities. In some cases, the Fieller approach may not result in a finite CI for

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<sup>1</sup> Where  $\alpha$  is the significance level of the test statistic.

some values of  $\alpha$ . In such cases the resulting CI may also be the complement of a finite interval or the whole real line. The advantage to the graphical approach presented here is that it can provide an indication of when this may be the case and can provide partial information that can be used to construct open ended intervals. This is shown in some of the examples below.

Another attraction to the Fieller method is the ease of computation. We demonstrate that this method can be computed using a direct computational method similar to the Delta and does not require the use of simulations and sampling strategies as would be needed when employing a Bootstrap or Bayesian method. The Stata code and the R script for all the examples used here are available in the appendices of this paper.

In the next section, we outline the aspects of the theory involved for the Fieller as it can be applied to the ratios of regression coefficients. Due to the facility of using Stata for these analyses we emphasise the syntax of Stata commands to construct the CIs. In Section 3 we provide eight examples of econometric applications where the result of interest is a ratio of parameters or the ratio of linear combinations of parameters as estimated from standard regressions, probit models, systems of equations, and non-linear models.

## 2. Confidence Intervals for a ratio of regression coefficients

### 2.1 Ratios of Regression Parameters

Consider the general linear model,

$$\mathbf{Y} = \mathbf{X}(\mathbf{B}) + \mathbf{e} \quad (2.1.1)$$

where  $\mathbf{Y}$  is an  $(N \times 1)$  vector of observations on the dependent variable,  $\mathbf{X}(\mathbf{B})$  is a  $(N \times 1)$  function of an  $(N \times K)$  matrix  $\mathbf{X}$  of regressors and a  $(K \times 1)$  vector  $\mathbf{B}$  of unknown parameters, and  $\mathbf{e}$  is an  $(N \times 1)$  vector of disturbances with mean 0 and covariance matrix  $\sigma^2 \mathbf{I}$ .<sup>2</sup> Suppose interest lies in the ratio of regression coefficients defined by,

$$\theta = \frac{\rho}{\phi} = \frac{K'\mathbf{B} + k}{L'\mathbf{B} + l}, \quad (2.1.2)$$

where  $K$  and  $L$  are  $(K \times 1)$  vectors of known constants thus  $K'\mathbf{B}$  and  $L'\mathbf{B}$  are linear combinations of the parameters and  $k$  and  $l$  are constants. The usual estimate of the ratio  $\theta$  in (2.1.2) is,

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<sup>2</sup> For most of the examples used here we assume a linear model such that  $\mathbf{X}(\mathbf{B}) = \mathbf{X}\mathbf{B}$ . Note that the assumption of heteroskedastic and serially independent errors can be relaxed by using robust standard errors such as Huber-White standard errors, clustered standard errors or Newey-West standard errors.

$$\tilde{\theta} = \frac{\hat{\rho}}{\hat{\phi}} = \frac{K\hat{\mathbf{B}} + k}{L\hat{\mathbf{B}} + l}, \quad (2.1.3)$$

where  $\hat{\mathbf{B}}$  is the  $(K \times 1)$  vector of the estimates of  $\mathbf{B}$  in (2.1.1).

The models we consider here assume the estimates of  $\mathbf{B}$  are normally distributed with a covariance matrix that we can approximate. In general, regression, maximum likelihood estimation and non-linear least squares all result in estimates of the parameters under these assumptions.

## 2.2 The Delta Method for CIs

The estimated variance of  $\tilde{\theta}$  based on the Delta method is defined for the ratio of the parameter estimates. We assume that the parameter estimates are bivariate normally distributed.<sup>3</sup>

$$\begin{bmatrix} \hat{\rho} \\ \hat{\phi} \end{bmatrix} \sim N \left\{ \begin{bmatrix} \rho \\ \phi \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \right\} \quad (2.2.1)$$

and  $\hat{\theta} = \hat{\rho} / \hat{\phi}$ .

The estimate of the variance of the ratio of normally distributed random variables based on the Delta method is defined as: (see example 5.5.27 in Casella and Berger 2002)

$$\begin{aligned} \text{var}(\hat{\theta}) &\approx (\hat{\rho} / \hat{\phi})^2 \left[ (\sigma_1^2 / \hat{\rho}^2) + (\sigma_2^2 / \hat{\phi}^2) - 2(\sigma_{12} / \hat{\phi}\hat{\rho}) \right] \\ &\approx \hat{\phi}^{-2} \left[ \sigma_1^2 + \hat{\theta}^2 \sigma_2^2 - 2\hat{\theta}\sigma_{12} \right] \end{aligned} \quad (2.2.2)$$

When using estimates of the variance covariance matrix, the Delta method  $100(1-\alpha)\%$  CI for  $\hat{\theta}$  is:

$$\text{CI}(\hat{\theta})_{\text{Delta}} = \hat{\theta} \pm t_{\alpha/2} \left( \hat{\phi}^{-1} \sqrt{\hat{\sigma}_1^2 + \hat{\theta}^2 \hat{\sigma}_2^2 - 2\hat{\theta}\hat{\sigma}_{12}} \right), \quad (2.2.3)$$

where  $t_{\alpha/2}$  is the value from the  $t$  distribution with  $T-k$  degrees of freedom.

## 2.3 The Fieller Method for CIs

The Fieller method (Fieller 1932, 1954, Rao 1973, pp 241-242) provides a general procedure for constructing confidence limits for statistics defined as ratios. A  $100(1-\alpha)\%$  Fieller CI for  $\theta$  can be found by inverting the  $t$ -test associated with the null hypothesis based on a linear combination of the parameters defined as:  $H_0 : \rho - \phi\tilde{\theta} = 0$  for different possible values of  $\theta$  as designated by  $\tilde{\theta}$ . Based on our assumption that the parameter estimates are normally distributed

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<sup>3</sup> This assumption may be relaxed to allow for the parameters to have an asymptotically normal distribution however the methods discussed here will then be subject to an additional degree of approximation.

we find that the linear combination of the parameter estimates for any  $\tilde{\theta}$  is also normally distributed:

$$(\hat{\rho} - \hat{\phi}\tilde{\theta}) \sim N\left((\rho - \phi\tilde{\theta}), (\rho^2\sigma_1^2 - 2\rho\phi\sigma_{12} + \phi^2\sigma_2^2)\right) \quad (2.3.1)$$

Or we can rewrite this expression in terms of a standard normal distribution as:

$$\frac{(\hat{\rho} - \hat{\phi}\tilde{\theta})}{\sqrt{(\hat{\rho}^2\sigma_1^2 - 2\hat{\rho}\hat{\phi}\sigma_{12} + \hat{\phi}^2\sigma_2^2)}} \sim N(0,1) \quad (2.3.2)$$

By squaring both sides and using the estimated elements of the covariance matrix we can specify (2.3.2) in terms of the square of a  $t$ -distribution for a  $100(1 - \alpha)\%$  CI as:

$$\frac{(\hat{\rho} - \hat{\phi}\tilde{\theta})^2}{(\hat{\rho}^2\hat{\sigma}_1^2 - 2\hat{\rho}\hat{\phi}\hat{\sigma}_{12} + \hat{\phi}^2\hat{\sigma}_2^2)} = t_{\alpha/2}^2, \quad (2.3.3)$$

The limiting values of  $\theta$  that would satisfy (2.3.3) define the confidence interval - in this way we have “inverted” the test statistic to define the confidence interval.

By moving the denominator in (2.3.3) to the right hand side and combining terms this expression can be written as a quadratic equation in  $\tilde{\theta}$  as:  $a\tilde{\theta}^2 + b\tilde{\theta} + c = 0$ , where  $a = \hat{\phi}^2 - t_{\alpha/2}^2\hat{\sigma}_2^2$ ,  $b = 2\left[t_{\alpha/2}^2\hat{\sigma}_{12} - \hat{\phi}\hat{\rho}\right]$  and  $c = \hat{\rho}^2 - t_{\alpha/2}^2\hat{\sigma}_1^2$ . The two roots of the quadratic equation,

$$\text{CI}(\hat{\theta})_{\text{Fieller}} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}, \quad (2.3.4)$$

define the Fieller  $100(1 - \alpha)\%$  CI for  $\hat{\theta}$ . In general,  $\frac{-b}{2a} \neq \hat{\theta}$  thus unlike the Delta CI, the Fieller CI would tend not be symmetric about the estimate of the ratio. Two finite bounds can be derived if the hypothesis  $H_0 : \phi \neq 0$  can be rejected when  $\alpha$  is the level of significance (Buonaccorsi 1979). In addition to the finite interval case, the Fieller CI may be the complement of a finite interval when  $(b^2 - 4ac > 0, a < 0)$  or of the whole real line when  $b^2 - 4ac < 0, a < 0$ . These conditions are discussed in Scheffé (1970) and Zerbe (1982).

Hirschberg and Lye (2010a) present a line plot representation of the  $100(1 - \alpha)\%$  Fieller CI for  $\theta$ . The linear combination  $(\hat{\rho} - \hat{\phi}\theta)$  is plotted along with its corresponding  $100(1 - \alpha)\%$  confidence bounds defined as:

$$(\hat{\rho} - \hat{\phi}\theta) \pm t_{\alpha/2}^2 \sqrt{(\hat{\rho}^2\hat{\sigma}_1^2 - 2\hat{\rho}\hat{\phi}\hat{\sigma}_{12} + \hat{\phi}^2\hat{\sigma}_2^2)}, \quad (2.3.5)$$

The estimate  $\hat{\theta}$  is found by determining where  $(\hat{\rho} - \hat{\phi}\tilde{\theta}) = 0$ . Similarly, the bounds of the  $100(1 - \alpha)\%$   $CI_{\text{Fieller}}(\hat{\theta})$  can be found by determining for what values of  $\tilde{\theta}$  the two bounding functions defined in (2.3.5) cut the zero line.

## 2.4 Stata Commands<sup>4</sup>

As an example, suppose the regression model is,

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon \quad (2.4.1)$$

and the ratio of interest is defined as,

$$\theta = \frac{\rho}{\phi} \quad (2.4.2)$$

where  $\rho = \beta_1 + 2\beta_2 + k$ ,  $k$  is a constant in this case = 0.5 and  $\phi = 1 - \beta_3$ .

We first estimate the model using an appropriate estimation technique and standard errors. For example, suppose we estimate the model using OLS and store the results for later use with the name “reg”.

```
regress y x2 x3
estimates store reg
```

To find the limits of the delta 95% CI we use the Stata command `nlcom`. Within this command we need to define the numerator  $\hat{\rho}$ , the denominator  $\hat{\phi}$  and the ratio  $\hat{\theta} = \hat{\rho}/\hat{\phi}$ , where in this case we define the formula for each in Stata by:  $\hat{\rho} = (_b[_cons] + 2*_b[x2] + .5)$  and  $\hat{\phi} = (1 - _b[x3])$ . Note that `_b[_cons]` indicates the estimate for the parameter  $\beta_1$ , `_b[x2]` indicates the estimate for the parameter  $\beta_2$  and `_b[x3]` indicates the estimate for the parameter  $\beta_3$ ). The Stata command `nlcom` can be used to define  $\hat{\rho}$ ,  $\hat{\phi}$  and  $\hat{\theta}$  with the command syntax:

```
nlcom (rho: (_b[_cons] + 2*_b[x2] + .5) ) (phi: (1 - _b[x3])) ///
(theta: _b[_cons] + 2*_b[x2] + .5) / (1 - _b[x3])), post
```

The sequence of these definitions is such that we can retrieve the variance and covariance matrix of the numerator and denominator as the first two rows and columns of the covariance matrix stored when using `nlcom` with the `post` option (here defined as the matrix V). To find the limits of the Fieller 95% CI by solving the two roots of the quadratic equation given in (2.3.3) we first define  $\hat{\rho}$ ,  $\hat{\phi}$ ,  $\rho$  and  $\phi$  and their variances `v11`, `v22` and their covariance `v12` by the following:

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<sup>4</sup> These commands are similar to the ones used for the examples presented in Section 3. The complete program used for each example application is included in the appendix.



```

matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar rho = _b[rho]
scalar phi = _b[phi]

```

Where the variance of  $\hat{\rho} = v11$ , the variance of  $\hat{\phi} = v22$  and the covariance between  $\hat{\rho}$  and  $\hat{\phi} = v12$  also need to be defined as scalars.

```

scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)
scalar lm2 = (-bb - rad) / (2 * aa)

```

The roots of the quadratic are then found by first defining the parameters of the quadratic as scalars and then using the solution to the quadratic equations to solve for the roots as `lm1` and `lm2`. The result would be a finite CI if the quadratic has real roots (in this case if `rad` is not missing).

For a graphical representation of the  $100(1 - \alpha)\%$  Fieller CI for  $\theta$ , we use first bring back the regression results with the `restore` option on the `estimates` command then we use the `margins` command to calculate the line  $(\hat{\rho} - \hat{\phi}\theta)$  and its corresponding 95% CIs over a range of possible values of  $\theta$ . In the command to obtain a range of possible values of  $\theta$  we use one of the variables in the model say  $x_3$ ,<sup>5</sup>

```

estimates restore reg
quiet margins, expression (_b[_cons] + 2 * _b[x2] + .5) ///
- (1 - _b[x3]) * x3 at (x3=(-15(1)15)) level(95)

```

In this case we have considered possible values for theta between -15 and 15 with increments of 1. To plot the calculated functions from margins we use the Stata command `marginsplot`,

```

marginsplot, yline(0) recast(line) recastci(rline) title(Fieller CI )

```

---

<sup>5</sup> Note that we are not changing the values of  $x_3$  we are just using this as a temporary name – this name is required to be one of the regressors in the equation.

## 2.4 R Commands<sup>6</sup>

With the same example expression as in (2.3.1) we assume the regression model is,

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon \quad (2.4.1)$$

and again the ratio of interest is defined as,

$$\theta = \frac{\rho}{\phi} \quad (2.4.2)$$

where  $\rho = \beta_1 + 2\beta_2 + h$ ,  $h$  is a constant in this case = 0.5 and  $\phi = 1 - \beta_3$ .

We first estimate the model using an appropriate estimation technique and standard errors.

Again we estimate the model using OLS and store the results for later use with the name “*regresults*” and where “*dataname*” describes which data are used to fit model,

```
reg = (lm(y~x2+x3,data=dataname))
regresults = summary(reg)
```

After these commands we can store the coefficient estimates in a vector *b* and the variance-covariance matrix in a matrix *V*

```
bcoeffs = regresults$coefficients
b = bcoeffs[ ,1]
V = vcov(reg)
```

To find the limits of the delta 95% CI we use the command “*deltamethod*” which is available in the “*msm*” package. This package will need to be installed before this command can be used.

Within this command we need to define the numerator  $\hat{\rho}$  and the denominator  $\hat{\phi}$ . The *deltamethod* command requires the construction of a column vector *R* which contains the numerator as the first element and the denominator as the second element. *R* is constructed using matrix algebra where  $R = Tb + A = \begin{bmatrix} \hat{\rho} \\ \hat{\phi} \end{bmatrix}$ . In this case  $T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$  and  $A = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ . We also need the variance-covariance

matrix of *R* which is equal to  $VR = TRT'$ . The delta standard error of *R* is called *sdm* and we can then construct the upper and lower bounds of the 95% CI based on the delta method called *dfub* and *dflb* in the syntax. This requires the appropriate critical value *t2*.

```
library("msm")
T=matrix(c(1,0,2,0,0,-1),nrow=2,ncol=3)
A=matrix(c(0.5,1),nrow=2,ncol=1)
R = T%*%b + A
VR = (T%*%V)%*%t(T)
sdm = deltamethod(~ x1/x2,R,VR)
df = regresults$df
nalpha = 0.05
```

<sup>6</sup> The complete program used for each example application is included in the appendix.

```
t2 = qt(1-nalpha/2, df[2])
dfub = tp + t2*sdm
df1b = tp - t2*sdm
```

To find the limits of the Fieller 95% CI by solving the two roots of the quadratic defined in (2.3.4) we first define the parameters of the quadratic and then use the solution to the quadratic equations to solve for the roots as  $lm1$  and  $lm2$ . The result would be a finite CI if the quadratic has real roots (in this case if  $rad$  is not missing).

```
aa = (R[2,1]^2) - (t2^2*VR[2,2])
bb = (2*t2^2*VR[1,2]) - (2*R[1,1]*R[2,1])
cc = (R[1,1]^2) - (t2^2*VR[1,1])
rad = sqrt(bb^2 - 4*aa*cc)
lm1 = (-bb + rad) / (2 * aa)
lm2 = (-bb - rad) / (2 * aa)
```

For a graphical representation of the  $100(1 - \alpha)\%$  Fieller CI for  $\theta$ , we first obtain a range of values for  $\theta_j$ . In this case we have considered possible values for  $\theta_j = -15$  to  $\theta_j = 15$  and called this sequence  $x$ .

```
n1 = -15
n2 = 15
x1 = seq(n1, n2, by = sign(n2-n1)*1)
x = matrix(x1, nrow=NROW(x1), ncol = 1)
```

We then construct the linear combination  $(\hat{\rho} - \hat{\phi}\theta_j)$  which we call  $p$  and its corresponding variance  $vp$ . The 95% confidence bounds are given by  $pu$  and  $pl$ ,

```
p = R[1,1] - R[2,1]*x
vp = VR[1,1] - 2*VR[1,2]*x + VR[2,2]*x^2
pu = p + t2*sqrt(vp)
pl = p - t2*sqrt(vp)
```

To plot the linear combination and corresponding confidence bounds we construct  $y$  which contains  $p$ ,  $pu$  and  $pl$  using the `cbind` function and then use the `matplot` command,

```
y = cbind(p,pu,pl)
matplot(x,y,type='l',col="blue")
```

To plot lines representing the delta upper and lower bounds as black lines, the Fieller upper and lower bounds as red lines, and the location of the estimated ratio as a violet line can be added to the plot with the syntax,

```
abline(v = df1b, col = "black")
abline(v = dfub, col = "black")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = p, col = "violet")
```

In order to use the R script included here it is necessary to load the appropriate libraries to access the programs used. The following script should be included for all of these examples.

```
library(haven) # A package to read new Stata data sets
library(foreign) # A package to read Stata data sets
library("msm") # for calculation delta standard errors
library(ggplot2) # advanced graphics
library(lmtest) # testing linear regression models
library(systemfit) # estimation of sur models
library(AER) # Applied Econometrics with R
library(car) # companion to Applied Regression
library("plm") # linear models for panel data
library(sandwich) # different robust covariance matrix estimators
```

### 3. Econometric Applications

In this section, we demonstrate the use of the Fieller method in eight econometric applications where a function of interest is a ratio of parameters or ratio of linear combinations of parameters. All the code used for the estimations reported here use Stata and are listed in the appendix.<sup>7</sup>

The first example is the interpretation of a dummy variable in terms of changes in a continuous regressor. This technique is related to the slope ratio assay method originally used in pharmacology. The application presented here employs the border phenomenon observed when smoking laws in one state influenced demand for gambling in an adjacent state.

In the second example, we demonstrate how one can determine the turning point in a quadratic specification where the marginal impact of a regressor changes sign. It is frequently the case that non-linearity in a regression is modelled with a quadratic specification. One aspect of this form of specification is that the marginal impact of the regressor is not constant and may change sign within the range of the data. The application considered here is the case of the environmental Kuznets curve that postulates that countries with higher national incomes pollute at decreasing rates when they approach a turning point in the quadratic relationship between pollution and national income.

Our third example demonstrates how the limits of the marginal effect of a variable that is interacted with another can be estimated. In this example, we show how the marginal effects of regressors that are interacted can be determined. And over what range of the interacted regressor will lead to a significant marginal effect.

The fourth example, demonstrates how the long-run marginal effects in a dynamic model can be viewed as a ratio. In this example, we use aggregate energy demand that is modelled with a

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<sup>7</sup> The do files and data for these examples can be found in the Appendix and online at [https://www.online.fbe.unimelb.edu.au/t\\_drive/FTS](https://www.online.fbe.unimelb.edu.au/t_drive/FTS).

dynamic model to estimate both the short-run and long-run price elasticities. This example also shows how the Stata code can be modified to account for a non-linear estimation model.

The fifth example is another dynamic model where we investigate the “Taylor Rule” for dynamic macro models. In this case the specification is more complex than in the fourth example and demonstrates how the CI for the long-run effect in a model with multiple lags can be estimated.

Our sixth example is the determination of the non-accelerating inflation rate of unemployment (or NAIRU). In this example a Phillips curve relationship is estimated where the NAIRU is found as a ratio of parameter estimates.

In the seventh application, we consider the measure of the willingness-to-pay model that is also related to the 50% dose model used in pharmacology. Here a binomial dependent variable is modelled as a function of a continuous regressor. The ratio of interest is the value of the continuous regressor that results in a predicted probability of .5. In the willingness-to-pay literature this value is the minimum someone is willing to pay when the dependent variable is the purchase and price is the continuous variable.

In the eighth example, we demonstrate how the structural parameter in an exactly identified system of equations can be estimated via the indirect least squares method which is equivalent to the two-stage least squares approach. This method employs the ratio of parameters for the structural parameter estimation.

### 3.1 Parallel-line assay: A compensation interpretation of a dummy variable.<sup>8</sup>

Finney (1978) considers the slope-ratio assay in which the potency of a test drug in comparison to a standard drug is required. In the assay, amounts of a standard drug are used on some patients while other amounts of a test drug are used on other patients. The assumed model is,

$$y_i = \beta_0 + \phi x_i + \rho d_i + \varepsilon_i, \quad (3.1.1)$$

where  $y$  is the response,  $x$  is the dosage of the drug,  $d$  is an indicator of 0 for standard drug and 1 for test drug and  $\rho = (\phi\theta)$ . The parameter of interest is  $\theta$  which is the ratio of parameters  $\theta = \frac{\rho}{\phi}$ . This can be interpreted as the amount of the standard drug required to give an equivalent response to a unit dose of the test drug.

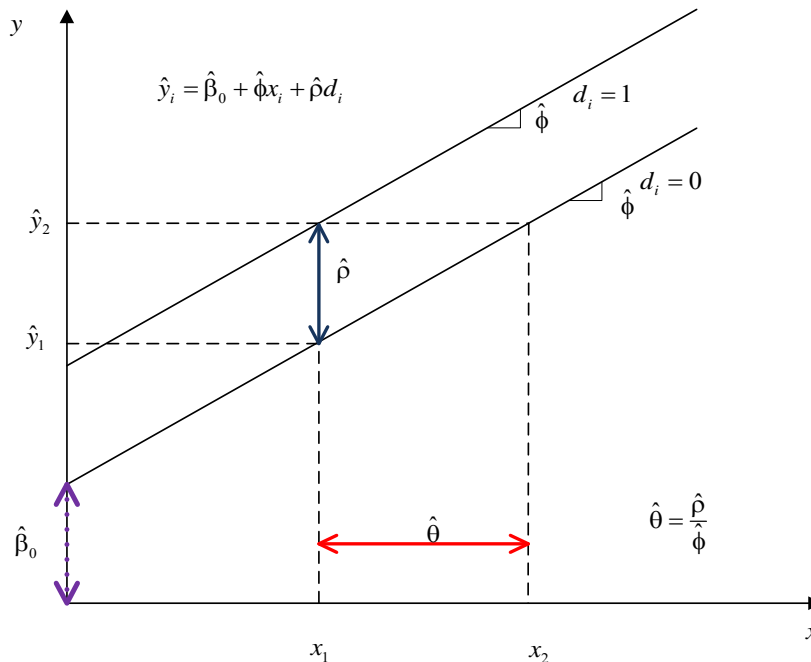
This method can be generalized to any case where we fit a regression with continuous regressors as well as dummy variables. This technique can provide an interpretation of the value of the coefficient on the dummy variable in terms of the change in the value of a continuous variable.

---

<sup>8</sup> See `parallelassay_pokies.do` in Appendix A.1 for the code to generate the example results shown here.

Thus, the usual interpretation of the dummy variable in the units of the dependent variable can be changed to be in the units of one of the independent variables.

**Figure 3.1.1** The relationship between the estimated dummy variable parameter  $\hat{\rho}$  and the translated version  $\hat{\theta}$  in the simple case.



For example, if we fit a model with the specification as in (3.1.1) we would have a predicted relationship as shown in Figure 3.1.1. Where the dummy variable  $d$  shifts the linear relationship between  $y$  and  $x$  by an amount  $\hat{\rho}$  for any value of  $x$  and the slope of the parallel lines is  $\hat{\phi}$ .<sup>9</sup> This means that we can interpret  $\hat{\rho}$  as the increased amount of  $x$  needed to compensate for when  $d = 0$ . Thus, from Figure 3.1.1 if the level of  $x$  is  $x_1$  and  $d = 1$  then the predicted value of  $y$  is given as  $\hat{y}_2$ . However, if the dummy is equal to zero then the predicted value of  $y$  is  $\hat{y}_1$  where  $\hat{y}_1 = \hat{y}_2 - \hat{\rho}$  and the usual interpretation of  $\hat{\rho}$  is in the units of  $y$ . Alternatively, we can convert the value of  $\hat{\rho}$  in terms of the equivalent change in  $x$  to have the same effect as the case when the dummy variable is equal to one. This amount is determined by  $\hat{\theta} = \frac{\hat{\rho}}{\hat{\phi}}$  which interprets the impact of the dummy in the corresponding value of the regressor. However, this translation of units requires the ratio of the estimated values of  $\phi$  and  $\rho$  which are stochastic terms and thus complicates the interpretation of  $\hat{\theta}$  and the construction of its corresponding CI.

In this example, we apply this approach to determine the impact of the proximity to the state border on a smoking ban in gaming areas of licensed premises that was instituted on 1<sup>st</sup> September 2002 in Victoria, Australia. The objective of this analysis is to determine the increase in the

<sup>9</sup> Where all parameters are assumed to be estimated as positive values.

number of EGMs that venues in communities on the border with New South Wales would need to compensate them for the loss in revenue they incurred due to the institution of smoking bans in Victoria and not in New South Wales. Hirschberg and Lye (2010d) investigate the nature of the pattern of falls in local electronic gaming machine (EGM) revenue that occurred due to this ban. The ban had a differential impact on different regions based on their proximity to the border with New South Wales, where there were no smoking bans in effect at the time. Victoria is subdivided into local government areas (LGAs). Using LGA-level annual data, Hirschberg and Lye (2010d) regress the %change in EGM Expenditure 2002-2003 in \$2002 ( $\% \Delta EGMExpend$ ) on the percentage change in EGM numbers ( $\% \Delta EGM$ ), a measure of social-economic status of the LGA ( $IRSED$ ) and a dummy variable ( $nborder$ ) that takes the value 1 if the LGA does not border the state of New South Wales. The specification is given in (3.1.2).

$$\% \Delta EGMExpend_i = \beta_0 + \beta_1 IRSED + \phi \% \Delta EGM_i + \rho nborder_i + \varepsilon_i \quad (3.1.2)$$

**Table 3.1.1.** Impact of Local Smoking Bans regression results (with robust standard errors), where:

$fall = \% \Delta EGMExpend$ ,  $irsed = IRSED$ ,  $cegm = \% \Delta EGM$ , and  $nborder = nborder$ .

Linear regression		Number of obs	=	60	
		F(3, 56)	=	5.25	
		Prob > F	=	0.0029	
		R-squared	=	0.2325	
		Root MSE	=	3.7914	
fall	Coef.	Robust Std. Err.	t	P> t	[95% Conf. Interval]
nborder	4.445393	1.836437	2.42	0.019	.766568 8.124218
cegm	.5034429	.1880271	2.68	0.010	.1267793 .8801065
irsed	-.035189	.0112757	-3.12	0.003	-.0577769 -.012601
_cons	18.99533	11.30684	1.68	0.099	-3.654991 41.64565

The regression results are reported in Table 3.1.1. The parameter of interest is  $\theta$  which is the ratio of parameters on the dummy for no border ( $nborder$ ) and the percent change in the number of EGMs ( $\% \Delta EGM$ ) is reported in Table 3.1.2 as 8.830. This implies that the number of EGMs within the LGA would need to increase by this percentage to compensate for the location influence of the smoking bans, when also considering the social-economic status of the LGA. The mean of  $\% \Delta EGM$  is -0.208 and it ranges from a minimum of -6.75 and a maximum of 4.69.

**Table 3.1.2** The estimates of  $\rho$ ,  $\phi$  and  $\theta$ . The Delta CI for  $\hat{\theta}$

rho: <code>_b[nborder]</code>						
phi: <code>_b[cegm]</code>						
theta: <code>_b[nborder] / _b[cegm]</code>						
fall	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
rho	4.445393	1.836437	2.42	0.015	.846043	8.044743
phi	.5034429	.1880271	2.68	0.007	.1349165	.8719693
theta	8.829985	4.463193	1.98	0.048	.0822874	17.57768

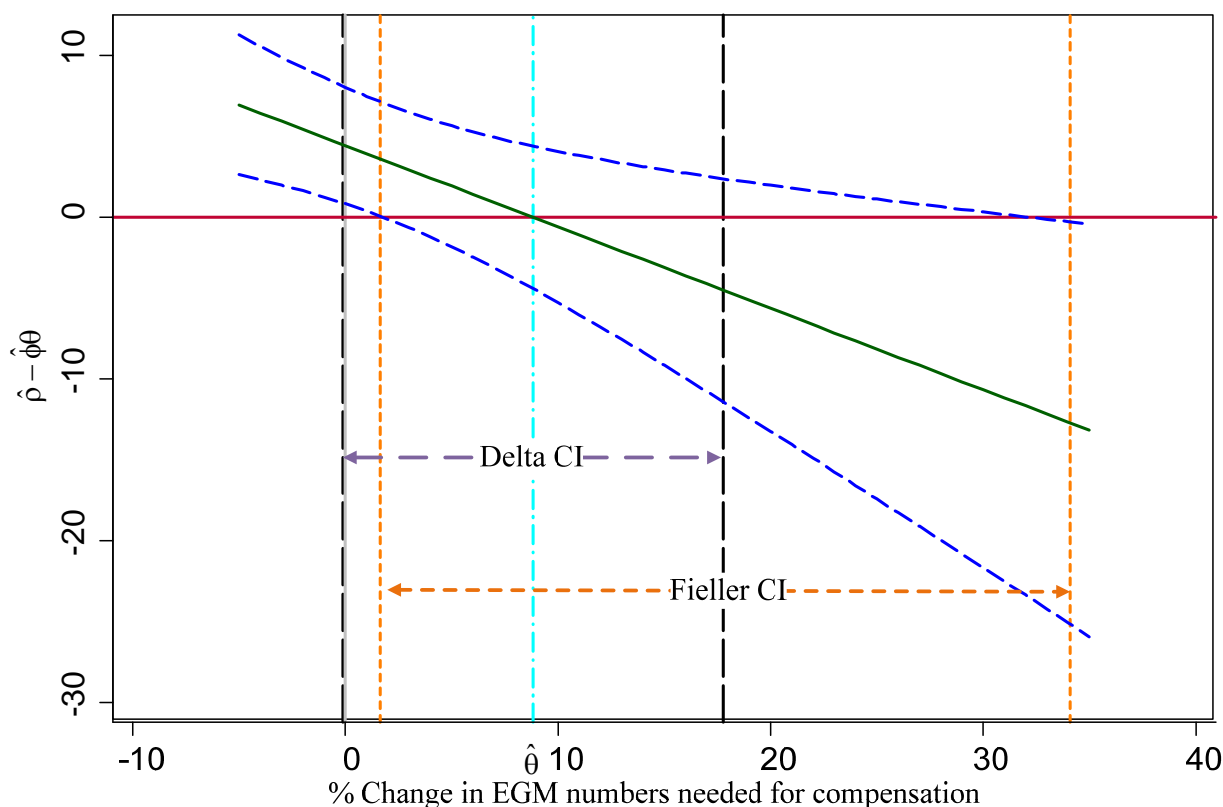
The Delta CI can be computed using Stata routine *nlcom* the 95% interval from this routine is defined as [.082 to 17.578] as reported in Table 3.1.2. Furthermore, the estimated value of  $\hat{\theta} = 8.83$  is higher than any of the % changes of EGMs for any observation in the data.

In addition, we obtained the 95% Fieller CI as [1.638 to 34.077] from the solution to the quadratic equation as specified in (2.3.4). In this case, we find that although the Delta interval indicates that the estimate of the equivalence of a change in EGMs is only significant at the 5% level the Fieller interval indicates that the lower bound is further from zero.

The line-plot for the linear combination along with the CI as defined in (2.3.5) is given in Figure 3.1.2. From the line plot we note that the upper bound is almost double the level of the Delta upper bound. In addition, the upper Fieller bound of more than 34% would require a change in the number of EGMs to be way beyond a viable number for any year. This implies that to compensate the venues on the border for the advent of a smoking ban to insure they are not disadvantaged more than other Victorian venues, would require an increase of on average 8.83 % more EGMs and to insure compensation with 95% confidence would require over a 34% increase. An increase that is way beyond the maximum that was observed over the time and local communities observed up to then.



**Figure 3.1.2** The linear combination  $\hat{\rho} - \hat{\phi}\theta$  with the 95% CI<sup>10</sup>



### 3.2 The Turning Point: An Environmental Kuznets Curve.<sup>11</sup>

The estimation of non-linear relationships with a polynomial specification is often used in empirical economic research. Adding a polynomial in a regressor to a model specification will allow the marginal effect of the regressor to vary with the value of the regressor. The most commonly employed polynomial is the quadratic form where the relationship estimated is either a U-shaped or inverted U-shaped curve. In many cases the curvature of the estimated relationship will not imply a sign change in the marginal effect within the range of the data. However, when such a sign change does occur within the range of the regressor values, there is an indication of a turning point – the limiting value of the regressor to have a marginal effect of a certain sign. A common example of such a relationship is the Kuznets (1955) curve that proposes that the relationship between income inequality and income, can be represented by an inverted U-shaped curve. Whereby income inequality increases with income then reaches a maximum and then falls with increased income.<sup>12</sup>

<sup>10</sup> All figures in the main body of this paper were prepared with Stata and MS Visio.

<sup>11</sup> See harbaughs.do in Appendix A.2 for the Stata do file to generate the results in this example.

<sup>12</sup> This method for inverse tests is more fully developed for the turning points in higher order polynomials in Lye and Hirschberg (2012).

The Environmental Kuznets Curve (EKC) hypothesizes an inverted U-shaped relationship between per capita income and pollution. In this case, it is hypothesized that pollution increases with economic growth up to a point where it will start to diminish when an economy reaches a specific income. We use data from Harbaugh et al. (2002) that contains measures of ambient sulfur dioxide (SO<sub>2</sub>) concentrations as an indicator of pollution in 102 cities located in 45 countries between 1971 and 1992 to estimate an EKC. The specification is,

$$\log(SO2)_{it} = \beta_1 G_{it} + \beta_2 G_{it}^2 + \mathbf{X}'_{it} \boldsymbol{\lambda} + \mu_i + v_{it} \quad (3.2.1)$$

where  $SO2_{it}$  is the mean annual sulfur dioxide measured by each monitor,  $G_{it}$  is per capita gross domestic product (in 1,000 \$) at time  $t$  for the country in which monitoring site  $i$  is located,  $\mathbf{X}_{it}$  are country and site-specific covariates and year dummy variables and  $\mu_i$  is a random site-specific effect estimated via random effects and clustered standard errors by site. The turning point is given by:

$$\theta = \frac{\rho}{\phi} = \frac{-\beta_1}{2\beta_2} \quad (3.2.2)$$

The model was estimated with random effects for each site and fixed effects by year. The number of observations=2,314 with 267 clusters. The  $\mathbf{X}_{it}$  variables include a measure of national trade intensity (TRADE), an index of democratic government (DEMO), and population density (DENSAT). Indicator variable for each year are also included (i . IYEAR). The  $z$ -statistics  $p$  values are adjusted for the number of clusters defined by site. The Stata estimation results are in Table 3.2.1:

**Table 3.2.1** Environmental Kuznets Curve regression estimate where:

$\ln\text{mean} = \log(\text{SO}_2)$ ,  $\text{GDP} = G, c.\text{GDP}\#c.\text{GDP} = G^2$ ,  $[\text{DENS\text{NAT}, \text{TRADE}, \text{DEMO}] = \mathbf{X}$ , and  $\text{IYEAR} = t$ .

```

.      xtreg lnMEAN c.GDP c.GDP#c.GDP i.IYEAR DENS\text{NAT} TRADE DEMO, re vce(cluster ISITE)

Random-effects GLS regression              Number of obs   =       2,314
Group variable: ISITE                     Number of groups =        267

R-sq:                                     Obs per group:
  within = 0.2038                          min =           1
  between = 0.2207                         avg =           8.7
  overall = 0.2557                          max =          21

corr(u_i, X) = 0 (assumed)                 Wald chi2(26)   =       498.67
                                           Prob > chi2     =       0.0000

                                           (Std. Err. adjusted for 267 clusters in ISITE)

```

lnMEAN	Coef.	Robust Std. Err.	z	P> z	[95% Conf. Interval]	
GDP	.0976685	.0355064	2.75	0.006	.0280772	.1672598
c.GDP#c.GDP	-.0038141	.0016201	-2.35	0.019	-.0069895	-.0006387
IYEAR						
1972	.1363376	.0848848	1.61	0.108	-.0300336	.3027087
1973	-.2216577	.0749604	-2.96	0.003	-.3685774	-.0747381
(Lines removed)						
1992	-.9241317	.1287244	-7.18	0.000	-1.176427	-.6718365
DENS\text{NAT}	1.849771	.213952	8.65	0.000	1.430433	2.269109
TRADE	-.0118193	.0018923	-6.25	0.000	-.0155282	-.0081105
DEMO	-.059991	.0098674	-6.08	0.000	-.0793307	-.0406514
_cons	3.950391	.1790951	22.06	0.000	3.599371	4.301411
sigma_u	.76503045					
sigma_e	.37484179					
rho	.80640585	(fraction of variance due to u_i)				

The ratio gives the estimates of the turning point and the corresponding Delta method CI for the Kuznets curve as obtained by the call to `n1com` as shown in Table 3.2.2

**Table 3.2.2** Environmental Kuznets Curve estimates of  $\rho$ ,  $\phi$ , and  $\theta$

```

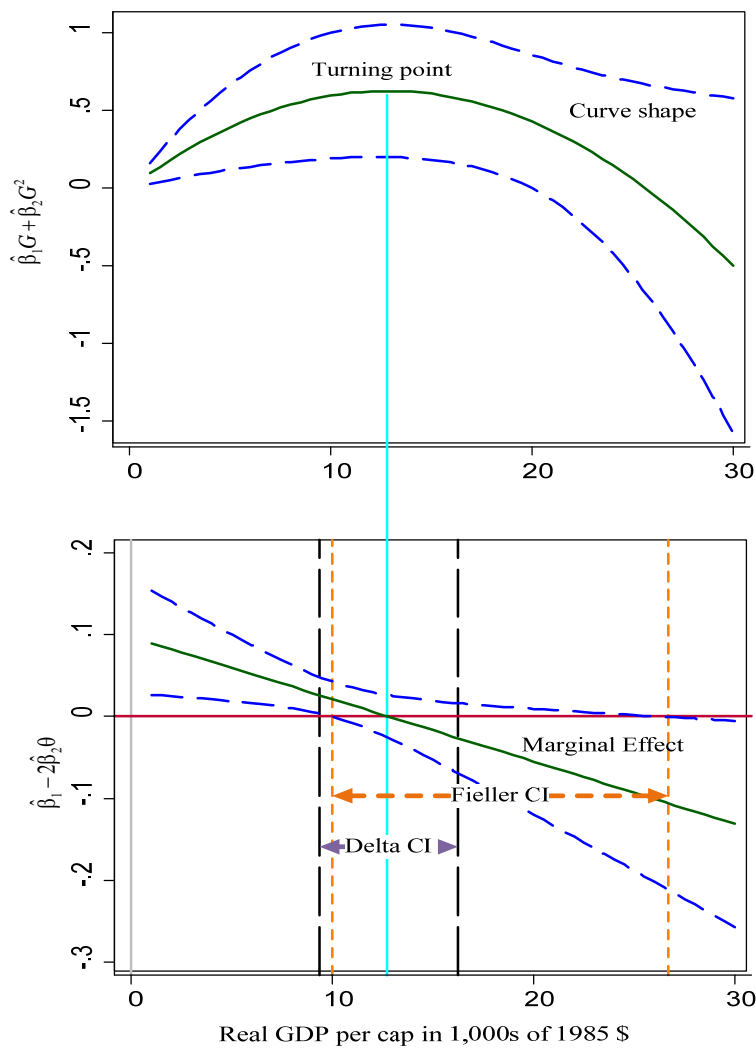
rho:  _b[GDP]
phi:  -2*_b[c.GDP#c.GDP]
theta: _b[GDP] / (-2*_b[c.GDP#c.GDP] )

```

lnMEAN	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
rho	.0976685	.0355064	2.75	0.006	.0280772	.1672598
phi	.0076283	.0032402	2.35	0.019	.0012775	.013979
theta	12.80351	1.748275	7.32	0.000	9.376958	16.23007

From Table 3.2.2 we note that the estimate of the Delta CI for the turning point is 9.376 to 16.230. To determine the shape of the quadratic relationship we can plot the curve defined by the impact of variations in  $G$  as  $\hat{\beta}_1 G + \hat{\beta}_2 G^2$ . Figure 3.2.1 is the plot of the implied curve and the first derivative function along with their corresponding 95% CI.

**Figure 3.2.1** The shape of the quadratic relationship and the Fieller interval as determined where the bounds of the first derivative function equals zero.



The Fieller interval as calculated from the solution to the quadratic equation is given as 9.965 to 26.691. From Figure 3.2.1 the wide upper bound of the Fieller as compared to the Delta upper bound corresponds to the flattening out of the upper bound of the 95% CI in the top graph of the shape. Note that in this data the GDP per capita has a maximum of 18.095 which would mean that the relationship never would reach the point where the marginal effect of income becomes negative if we use the Fieller interval instead of the Delta interval. This is quite apparent from the curve shape plot in Figure 3.2.1. From this plot we can see that the Delta upper bound at 16.230 is at a point where the curve is fairly flat while the Fieller interval would indicate that pollution would not be predicted to fall with higher GDP until it reached 26.691.

### 3.3 Interactions: Journal subscriptions as influenced by age and price<sup>13</sup>

Interaction terms are widely used in econometric models when the influence of two regressors are assumed to have a combination effect. For example, in the model specified by (3.3.1):

$$y = \beta_1 + \beta_2 x + \beta_3 w + \beta_4 xw + \beta_5 v_1 + \dots + \beta_k v_{k-4} + \varepsilon \quad (3.3.1)$$

when both  $x$  and  $w$  are both continuous variables we define the partial influence function as the marginal effect of a change in  $x$ :<sup>14</sup>

$$\frac{\partial E(y/x,w)}{\partial x} = \beta_2 + \beta_4 w \quad (3.3.2)$$

If  $x$  is a dummy variable and  $w$  is a continuous variable, the difference in the regression equation when  $x = 1$  and when  $x = 0$  is defined as:

$$\frac{\Delta E(y/x,w)}{\Delta x} = \beta_2 + \beta_4 w \quad (3.3.3)$$

However, both the expressions in (3.3.2) and (3.3.3) introduce the difficulty of an appropriate choice of the other regressor at which to evaluate these margins.

One approach is to set the value of this other regressor to a value such as the mean and then make the computation. Alternatively, one can determine at what value of the other regressor do these definitions go to zero or change sign. Thus, we are searching for the limiting value of the interacted regressor that forces the influence of the regressor of interest to go to some value. In specification (3.3.1) this would be the value of  $w$  that results in  $\frac{\partial E(y/x,w)}{\partial x} = c$  or  $\frac{\Delta E(y/x,w)}{\Delta x} = c$  where typically  $c = 0$  or some other value. From (3.3.2) and (3.3.3) we would find this value as  $w^* = \frac{c - \beta_2}{\beta_4}$  and is estimated by  $\hat{w}^* = \frac{c - \hat{\beta}_2}{\hat{\beta}_4}$  where  $\hat{\beta}_i$  are the OLS estimates of the  $\beta_i$ ,  $i=2,4$ . The question of interest is whether the value of  $w$  would ever be close to  $w^*$  or not. We propose that this may be investigated via the use of the partial influence function that maps how  $\frac{\partial E(y/x,w)}{\partial x}$  or  $\frac{\Delta E(y/x,w)}{\Delta x}$  vary with values of  $w$ .

For an example of the application of the partial influence function we use an analysis from Stock and Watson (2003, pg. 227) in which they estimate the relationship between the number of subscriptions to a journal at US libraries ( $Y$ ) and its library subscriptions price. The data are for the year 2000 and cover 180 economics journals. *Price* is measured in prices per citation which is used as a proxy variable for dollars per idea. Other explanatory variables include the age of journal (*Age*) and the number of characters printed per year in the journal (*Char*) as a measure of the

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<sup>13</sup> This Section follows the techniques presented in Hirschberg and Lye (2010b). See the Stata do file journal.do in Appendix A.3 for the code used to generate the results presented here.

<sup>14</sup> Note that a similar result can be found for  $w$ .

content of the journal. Here we interact the price variable with age to allow for the price elasticity to vary with the age of the journal. The estimated regression equation is specified as:

$$\begin{aligned} \text{Log}(Y) = & \beta_1 + \beta_2 \text{Log}(Age) + \beta_3 \text{Log}(Price) \\ & + \beta_4 (\text{Log}(Price) \times \text{Log}(Age)) + \beta_5 \text{Log}(Char) + \varepsilon \end{aligned} \quad (3.3.4)$$

The regression results are reported in Table 3.3.1:

**Table 3.3.1:** Estimated regression results for Library Journal Subscriptions. Where:  $\text{loclc} = \text{Log}(Y)$ ,  $\text{lage} = \text{Log}(Age)$ ,  $\text{lprice} = \text{Log}(Price)$ ,  $\text{lpla} = (\text{Log}(Price) \times \text{Log}(Age))$  and  $\text{lchar} = \text{Log}(Char)$ .

. reg loclc lprice lpla lage lchar						
Source	SS	df	MS	Number of obs	=	180
Model	143.252483	4	35.8131208	F(4, 175)	=	75.75
Residual	82.7375673	175	.472786099	Prob > F	=	0.0000
Total	225.990051	179	1.26251425	R-squared	=	0.6339
				Adj R-squared	=	0.6255
				Root MSE	=	.68759
loclc	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
lprice	-.8989097	.1615082	-5.57	0.000	-1.217664	-.5801551
lpla	.1409591	.0448348	3.14	0.002	.0524725	.2294457
lage	.3735148	.0889742	4.20	0.000	.1979143	.5491153
lchar	.229466	.1051262	2.18	0.030	.0219876	.4369445
_cons	3.433521	.314857	10.91	0.000	2.812115	4.054926

Based on these estimates, we find that the estimated price elasticity is defined as:

$$-\hat{\beta}_2 + \hat{\beta}_4 \times \text{Log}(Age) = -0.899 + 0.141 \times \text{Log}(Age) \quad (3.3.5)$$

(3.3.5) implies that the price elasticity of a journal will be smaller in magnitude (less negative) with age – in other words: libraries are less sensitive to price changes for journals that have been around longer. The ratio  $(\frac{0.899}{0.141})$  defines the value of  $\text{Log}(Age)$  at which  $-0.899 + 0.141 \times \text{Log}(Age) = 0$ .

Figure 3.3.1 plots the estimated price elasticity and its corresponding confidence bound. Since the actual values of  $Age$  in the data has a minimum value of 4 ( $\text{Log}(Age) = 0.60$ ) and a maximum value of 156 ( $\text{Log}(Age) = 5.05$ ) we conclude from Figure 3.3.1 that the price elasticity is significantly different from 0 for all the journal ages in this data set.

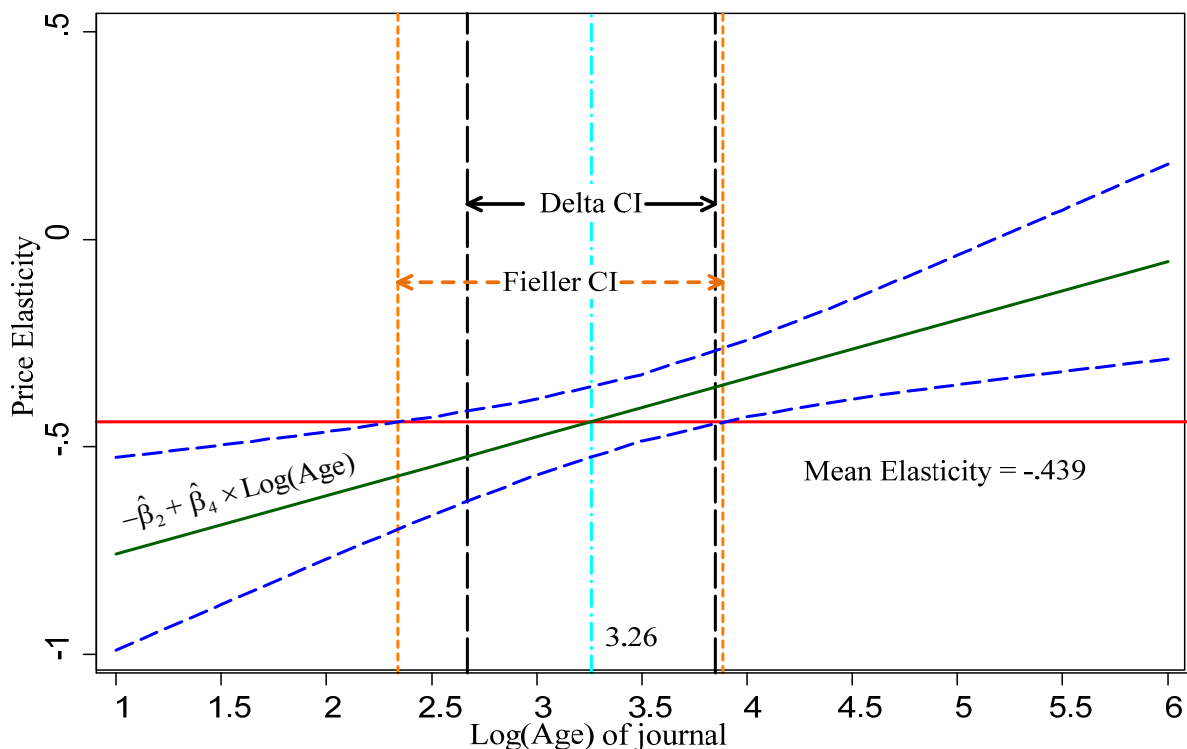
It is common practice to report a summary price elasticity such as the value of the mean of all the price elasticities which in this case equals  $-0.439$ . To find the 95% Delta and Fieller intervals for this case define  $\tilde{\theta} = \frac{\hat{\rho}}{\hat{\phi}} = \frac{-0.439 - \beta_2}{\beta_4}$ . Figure 3.3.1 plots a reference line at  $-0.439$  from which we can determine the values of  $Age$  for which the elasticity is significantly different from the mean of all the price elasticities (defined by  $m\_elast$ ). We see that this is either for young or old journals, that is when  $Age$  is less than 10.4 ( $\text{Log}(Age) = 2.34$ ) or  $Age$  is greater than 48.7 ( $\text{Log}(Age)$

= 3.88). We can compute this value of  $\tilde{\theta}$  using the Delta method as shown in Table 3.3.2 below. From Table 3.3.2 we find the limits are when Age is less than 14.5 ( $\text{Log}(\text{Age}) = 2.67$ ) or Age is greater than 46.8 ( $\text{Log}(\text{Age}) = 3.85$ ).

**Table 3.3.2** The Delta estimate for the ratio of rho and phi. (m\_elast is the mean of the elasticities (-0.439))

rho: ( m_elast-_b[lprice])						
phi: _b[lpla]						
theta: ( ( m_elast-_b[lprice]) /_b[lpla])						
loc lc	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
rho	.459526	.1615082	2.85	0.004	.1429758	.7760762
phi	.1409591	.0448348	3.14	0.002	.0530844	.2288337
theta	3.259996	.2990973	10.90	0.000	2.673776	3.846216

**Figure 3.3.1:** Price Elasticity Function for the Demand for Economics Journals



The 95% Fieller CI can be computed based on the solution to (2.3.4) to be 2.342 to 3.886 which results in lower values for both the lower bound and upper bound. Figure 3.3.1 demonstrates this result where the price elasticity is plotted as a function of the log of the age of the journal to find the limits of the log of the age at which the elasticity is the same as the mean elasticity based on where the 95% confidence interval cuts the line at the mean elasticity of -.439.

### 3.4 The Long-run: A non-linear specification for energy demand<sup>15</sup>

Dynamic model specifications are widely employed for the analysis of time-series data. A dynamic specification includes a lag or multi-period lags of the dependent variable as regressors. One of the consequences of this type of model is the need for the interpretation of the parameter estimates of other regressors in relation to the estimated coefficients on the lagged dependent variables.

A simple dynamic specification with one lag can be defined by (3.4.1).

$$y_t = \lambda y_{t-1} + \beta x_t + \varepsilon_t \quad (3.4.1)$$

In this model  $y_t$  is the dependent variable at time  $t$ ,  $x_t$  is an explanatory variable at time  $t$  and  $\varepsilon_t$  is the error at time  $t$ .<sup>16</sup> In the long-run it is assumed that the values of  $y$  will tend to a specific value (designated as  $\tilde{y}$ ). Thus,  $y_T = y_{T-1} = \tilde{y}$  when  $t$  reaches the long-run at time  $T$ . Under the assumption of a stable process (that  $|\lambda| < 1$ ), the long-run model can be written as  $\tilde{y} - \lambda\tilde{y} = \beta x_t + \varepsilon_t$  or  $\tilde{y} = (1 - \lambda)^{-1} (\beta x_t + \varepsilon_t)$  and the long-run marginal effect of  $x$  on  $\tilde{y}$  or  $\frac{\partial \tilde{y}}{\partial x}$  is defined as a ratio  $\theta = \frac{\rho}{\phi} = \frac{\beta}{1-\lambda}$ . Since the estimate of  $\lambda$  in most economic applications is positive we find that the marginal impact in the long-run of a change in  $x$  is usually greater in magnitude than the short-run marginal effect estimated by  $\beta$ .

Bernard et al (2007) employ a non-linear dynamic specification in their estimate of an energy demand model for the commercial sector of the Québec economy using annual data from 1962 to 2000. Sector specific total energy demand measured in terajoules ( $TE$ ) is modelled as a function of its own lag, real energy price ( $PE$ ) real income ( $IN$ ), heating degree days as a measure of heating demand caused by low temperatures ( $HDD$ ) and an error term  $\varepsilon$ . To ensure that temperature effects for year  $t$  are restricted to year  $t$ , the coefficient of  $HDD_{t-1}$  is set to  $-\lambda\pi_3$ . The model is specified in (3.4.2).

$$\begin{aligned} \text{Log}(TE_t) = & \pi_0 + \lambda \text{Log}(TE_{t-1}) + \pi_1 \text{Log}(PE_t) + \pi_2 \text{Log}(IN_t) \\ & + \pi_3 [\text{Log}(HDD_t) - \lambda \text{Log}(HDD_{t-1})] + \varepsilon_t \end{aligned} \quad (3.4.2)$$

where the long run elasticity is:  $\theta = \frac{\rho}{\phi} = \frac{\pi_1}{1-\lambda}$ .

We estimate the model and report the results in Table 3.4.1. From these results, we find that the model appears to fit quite well with the parameter on the lagged dependent variable and the

<sup>15</sup> See commercial.do in Appendix A.4 for the code used to generate the results presented in this example.

<sup>16</sup> Note that an assumption in the estimation of dynamic models of this form with OLS is that the errors are not subject to autocorrelation.



lagged heating degree days as an auto regressive parameter of .401. And the short-run price elasticity estimate of -.319. Table 3.4.2 provides the estimates of the long-run elasticity with the corresponding CI based on the Delta method. From this table, we can read the long-run elasticity is -.532. This result would be in keeping with the usual assumption that the long-run estimates are greater in magnitude than the short-run case.

**Table 3.4.1:** Energy demand model for the commercial sector of the Québec economy as estimated via non-linear least squares. Where:

$$\log teg = \text{Log}(TE_t), \log lteg = \text{Log}(TE_{t-1}), \log pep = \text{Log}(PE_t), \log gdp = \text{Log}(IN_t),$$

$$\log hdd = \text{Log}(HDD_t), \log lhdd = \text{Log}(HDD_{t-1}), b_0 = \hat{\pi}_0, b_1 = \hat{\lambda}, b_2 = \hat{\pi}_1, b_3 = \hat{\pi}_2, \text{ and } b_4 = \hat{\pi}_3.$$

```

.      nl (logteg = {b0} + {b1}*loglteg + {b2}*logpep + {b3}*loggdp + ///
>      {b4}*(loghdd - {b1}*loglhdd)), variables(loglteg logpep loggdp loghdd loglhdd)
(obs = 38)

```

Source	SS	df	MS			
Model	1.9148677	4	.47871692	Number of obs =	38	
Residual	.10000236	33	.003030374	R-squared =	0.9504	
Total	2.01487	37	.054455947	Adj R-squared =	0.9444	
				Root MSE =	.0550488	
				Res. dev. =	-117.8863	

logteg	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
/b0	1.83286	.411209	4.46	0.000	.9962492	2.669471
/b1	.4006196	.0966556	4.14	0.000	.2039723	.5972669
/b2	-.3187478	.076197	-4.18	0.000	-.4737719	-.1637238
/b3	.5463033	.1223594	4.46	0.000	.2973612	.7952454
/b4	.4603591	.1698961	2.71	0.011	.114703	.8060152

Parameter b0 taken as constant term in model & ANOVA table

**Table 3.4.2** The Delta CI for the long-run price elasticity

```

rho: /b2
phi: (1 - /b1)
theta: /b2 /(1 - /b1)

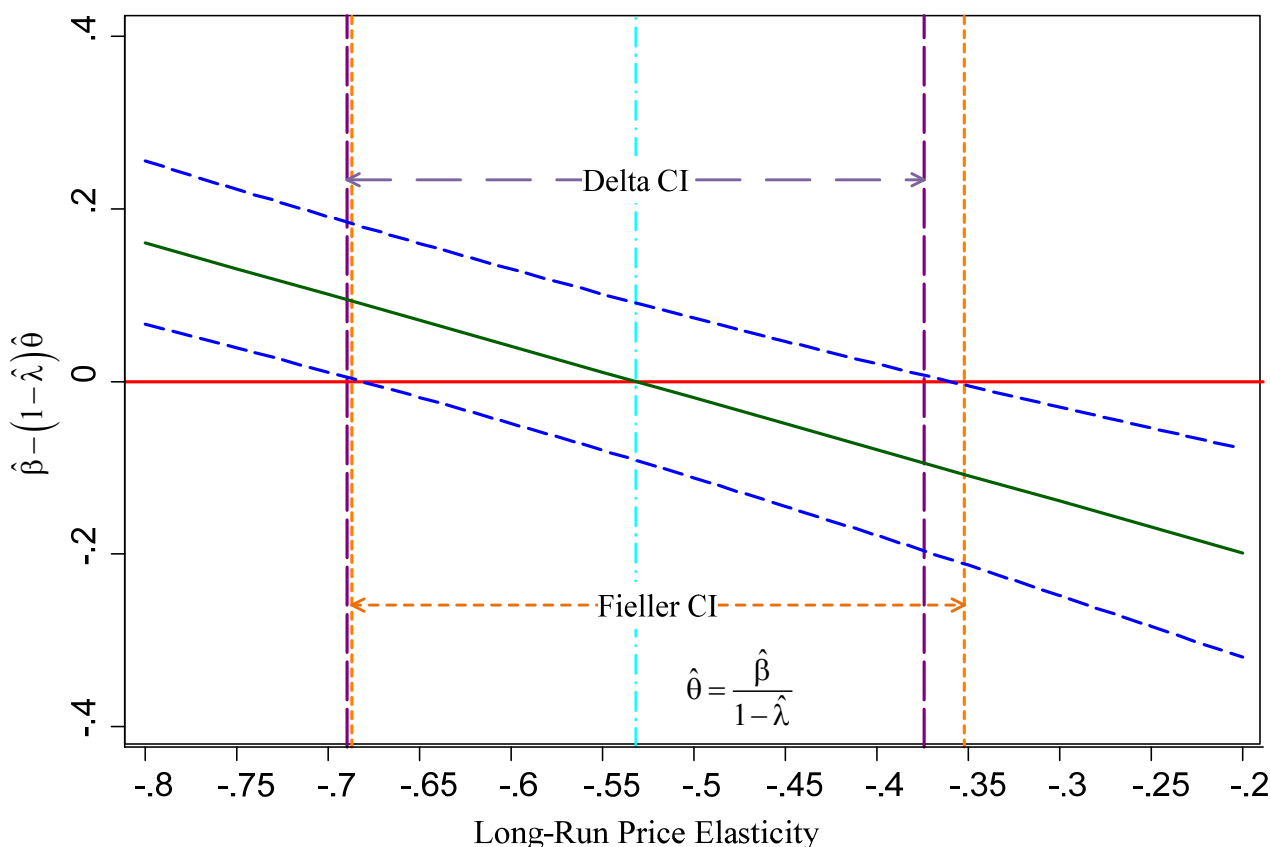
```

logteg	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
rho	-.3187478	.076197	-4.18	0.000	-.4680913	-.1694044
phi	.5993804	.0966556	6.20	0.000	.4099389	.7888219
theta	-.5317955	.0775872	-6.85	0.000	-.6838636	-.3797275

The Fieller interval can be determined from Figure 3.4.1 or from the solution to the quadratic equation as -.687 to -.352 – both would indicate that the lower bound is lower but the upper bound is higher than the Delta result. However, since the value of  $\hat{\phi}$  has a very large t-statistic of -6.20

and the correlation between the numerator and denominator is  $-.802$  (large and the same sign as the long-run elasticity) it is not surprising that the two methods agree on the CI so closely.<sup>17</sup>

**Figure 3.4.1** The Graphic representation of the Fieller CI



### 3.5 A Multiple period lag model: Taylor Rule Coefficients<sup>18</sup>

Another application of a long run effect is the estimation of Taylor Rules (see Taylor (1993)) with interest rate smoothing and forward looking data. Coibion and Gorodnichenko (2011) estimate a dynamic specification with one and two periods lagged dependent variables. The logic for determining the long-run impact of the other regressors is the same as in the case of a one-period lag dynamic specification except in this case we have need to consider the sum of the parameters on two lagged dependent variables. The specification is of the form,

$$r_t = \alpha + \lambda_1 r_{t-1} + \lambda_2 r_{t-2} + \beta_1 E_t \pi_{t+j} + \beta_2 E_t g y_{t+j} + \beta_3 E_t x_{t+j} + \varepsilon_t \quad (3.5.1)$$

where  $r_t$  is the nominal interest rate in quarter  $t$ ,  $g y_{t+j}$  is output growth predicted  $j$  quarters in the future from quarter  $t$ ,  $\pi_{t+j}$  is inflation predicted  $j$  quarters in the future from quarter  $t$ , and  $x_{t+j}$  is the output gap predicted  $j$  quarters in the future from quarter  $t$ . The long-run parameter of interest is

<sup>17</sup> See Hirschberg and Lye (2010c) for more details on the nature of the relationship between the CIs for both methods. In general, when the denominator is significantly different from zero and the correlation between the denominator and the numerator is the same sign as the ratio the Delta and the Fieller will tend to coincide.

<sup>18</sup> The Stata routine to perform this estimation is given in greenbook.do in Appendix A.5.

defined as:  $\theta = \frac{\rho}{\phi} = \frac{\beta_1}{1 - \lambda_1 - \lambda_2}$ . The interpretation of this parameter is the long run response of

interest rates to a permanent 1-percentage point increase in inflation. The specification allows for interest smoothing of order 2. The model is estimated using real-time data using the Greenbook forecasts from the Philadelphia Fed of future macroeconomic variables.<sup>19</sup> In Table 3.5.1 we report the results of their baseline model for the 1983-2002 period where the standard errors are estimated using a Newey-West error specification with 5 lags.

**Table 3.5.1** Estimates of the Taylor Rule dynamic regression, where:

$ffr = r_t$ ,  $ffr1 = r_{t-1}$ ,  $ffr2 = r_{t-2}$ ,  $pi\_tp = E_t \pi_{t+j}$ ,  $gry\_tp = E_t g_{t+j}$ , and  $gap\_t03 = E_t x_{t+j}$ .

Regression with Newey-West standard errors		Number of obs	=		158	
maximum lag: 5		F( 5, 152)	=		2015.17	
		Prob > F	=		0.0000	
ffr	Newey-West		t	P> t	[95% Conf. Interval]	
	Coef.	Std. Err.				
lffr1	1.279632	.0834952	15.33	0.000	1.114671	1.444593
lffr2	-.3393612	.0814875	-4.16	0.000	-.5003556	-.1783669
pi_tp	.1513119	.0588523	2.57	0.011	.0350378	.2675861
gry_tp	.1302064	.0339801	3.83	0.000	.0630721	.1973408
gap_t03	.0354846	.0129752	2.73	0.007	.0098496	.0611196
_cons	-.3927445	.1483813	-2.65	0.009	-.6859006	-.0995885

The estimates of  $\theta$ ,  $\rho$  and  $\phi$  are listed in Table 3.5.2.

**Table 3.5.2** The estimates of of  $\theta$ ,  $\rho$  and  $\phi$ .

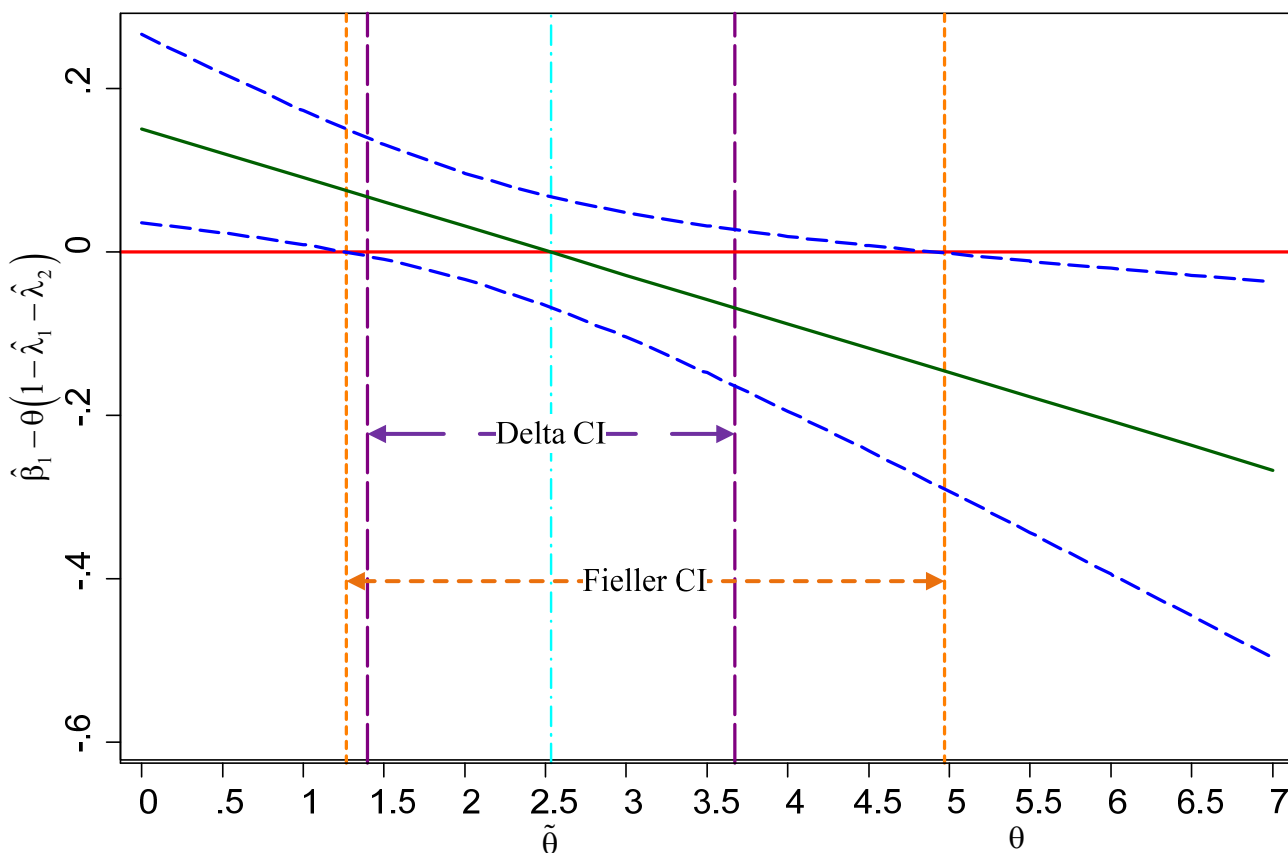
rho: $\_b[pi\_tp]$						
phi: $(1 - \_b[lffr1] - \_b[lffr2])$						
theta: $\_b[pi\_tp] / (1 - \_b[lffr1] - \_b[lffr2])$						
ffr	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
rho	.1513119	.0588523	2.57	0.010	.0359635	.2666603
phi	.0597292	.0230663	2.59	0.010	.0145201	.1049384
theta	2.533297	.5752197	4.40	0.000	1.405887	3.660707

From Table 3.5.2 we find that the 95% Delta CI is 1.406 to 3.661, and the estimate of  $\theta$  is 2.533. To plot the linear combination of the parameters that can be used to define the Fieller interval we plot  $\hat{\beta}_1 - \theta(1 - \hat{\lambda}_1 - \hat{\lambda}_2)$  with different values of  $\theta$  to find where this linear combination is equal to zero along with the 95% CI as shown below in Figure 3.5.1. Note that the Fieller CI has

<sup>19</sup> See <https://www.philadelphiafed.org/research-and-data/real-time-center/greenbook-data/>.

both lower and higher limits than the corresponding Delta CI. The Fieller CI is defined as: 1.265 to 4.971.

**Figure 3.5.1** The Fieller and Delta CIs for the long-run



### 3.6 The Non-Accelerating Inflation Rate of Unemployment (NAIRU) <sup>20</sup>

The Phillips (1958) Curve is the proposed relationship between inflation and unemployment. The non-accelerating inflation rate of unemployment (NAIRU see Modigliani and Papademos (1975)) is a version of the natural rate of unemployment designed to account for expectations. Typically, these models specify a measure of inflation as the dependent variable with unemployment as a regressor along with other regressors to account for unexpected changes in prices. Thus, allowing one to solve for the implied level of unemployment when inflation is equal to zero and the unexpected changes are zero as well by the ratio of the intercept to minus the parameter on unemployment.

For example, Gruen et al (1999) use Australian data to estimate a model of the rate of wage inflation, measured by the rate of change of unit labour costs, as a function of the level of unemployment, the change in the rate of unemployment and the expected rate of inflation. They

<sup>20</sup> This section is based on materials from Hirschberg and Lye (2010a). The Stata routine for this analysis can be found in `nairu.do` in Appendix A.6.

also choose to model annual movements and include a lagged dependent variable to determine the long run process. We write this specification as:

$$\begin{aligned} \Delta_4 \ln ULC_t - \Delta_4 \ln P_{t-1} = & \alpha_0 + \alpha_1 (\Delta_4 \ln P_t^* - \Delta_4 \ln P_{t-1}) + \alpha_2 U_t \\ & + \alpha_3 \Delta U_{t-1} + \alpha_4 (\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln P_{t-2}) \\ & + \alpha_5 (\Delta_4 \ln ULC_{t-1} - \Delta_4 \ln ULC_{t-4}) + \varepsilon_t \end{aligned} \quad (3.6.1)$$

where  $ULC$  = unit labour costs per person, and is equal to wages per person divided by non-farm productivity per person;  $P$  = CPI,  $P^*$  = expected price level;  $U$  = rate of unemployment;  $\Delta = 1$  quarter change; and  $\Delta_4 = 4$  quarter (yearly) change. An estimate of the NAIRU, (the value of  $U$  when  $\Delta_4 \ln ULC_t - \Delta_4 \ln P_{t-1} = 0$ ), is given as the ratio of the intercept and minus the parameter on the unemployment level:  $\hat{\theta} = \frac{\hat{\rho}}{\hat{\phi}} = \frac{\hat{\alpha}_0}{-\hat{\alpha}_2}$ , where  $\hat{\alpha}_0$ ,  $\hat{\alpha}_2$  are the corresponding estimates of the parameters in (3.6.1). We report the results in Table 3.6.1.

**Table 3.6.1:** Phillips curve estimates for Australia 1985:1-2003:4 using Newey-West standard errors with 4 lags where:  $y = \Delta_4 \ln ULC_t - \Delta_4 \ln P_{t-1}$ ,  $x2 = \Delta_4 \ln P_t^* - \Delta_4 \ln P_{t-1}$ ,  $x3 = U_t$ ,  $x4 = \Delta U_{t-1}$ ,  $x5 = \Delta_4 \ln ULC_{t-1} - \Delta_4 \ln P_{t-2}$ , and  $x6 = \Delta_4 \ln ULC_{t-1} - \Delta_4 \ln ULC_{t-4}$ .

newey y x2 x3 x4 x5 x6 , lag(4)							
Regression with Newey-West standard errors					Number of obs	=	76
maximum lag: 4					F( 5, 70)	=	45.53
					Prob > F	=	0.0000
y	Newey-West		t	P> t	[95% Conf. Interval]		
	Coef.	Std. Err.					
x2	.1671618	.093689	1.78	0.079	-.0196951	.3540187	
x3	-.2458883	.1144283	-2.15	0.035	-.4741084	-.0176682	
x4	-.2800783	.4402248	-0.64	0.527	-1.158079	.5979224	
x5	.5843068	.1108769	5.27	0.000	.3631699	.8054438	
x6	.5562282	.0751941	7.40	0.000	.4062583	.706198	
_cons	1.327986	.7940696	1.67	0.099	-.2557362	2.911708	

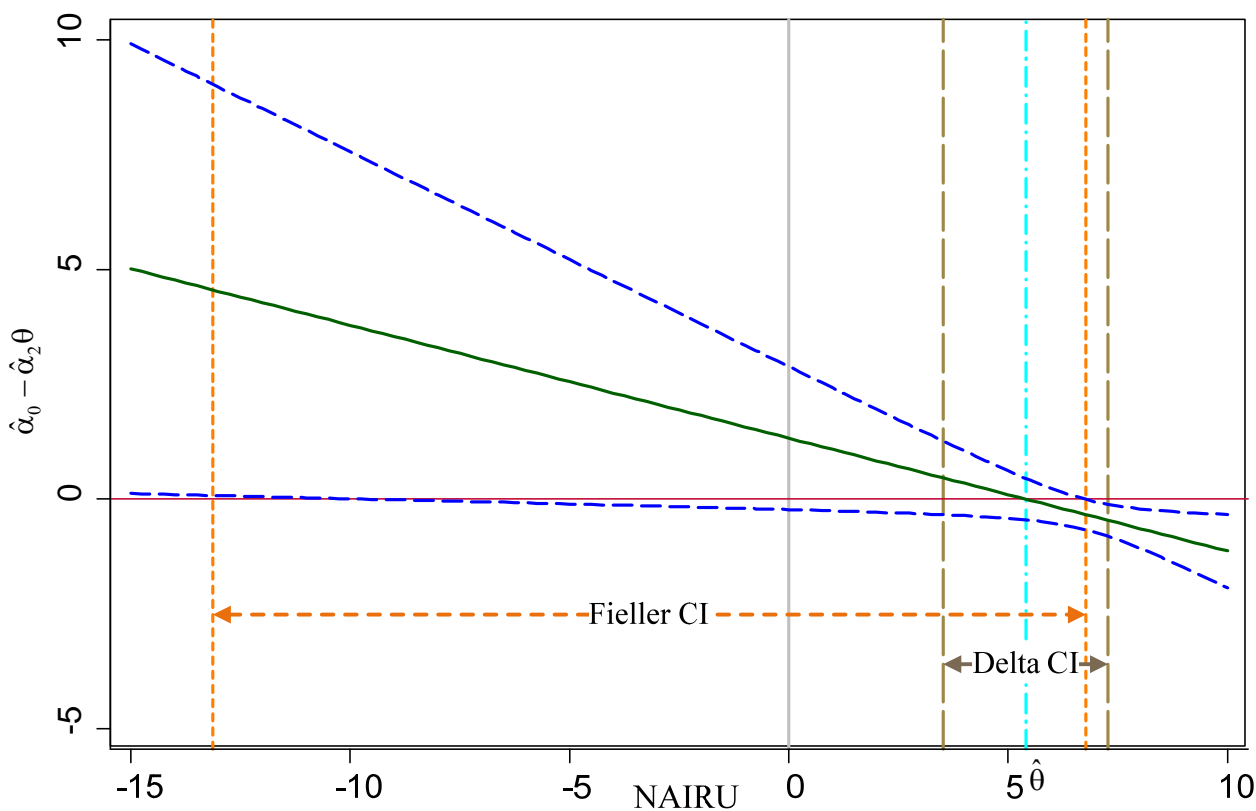
The results of the Delta CI are listed in Table 3.6.2 and from this table we find that the estimated value of the NAIRU ( $\hat{\theta}$ ) is determined as 5.401 with a 95% CI of 3.159 to 7.643. This implies that we could reject the null hypothesis that the NAIRU is equal to zero.

**Table 3.6.2:** The estimate of the NAIRU with the corresponding Delta CI.

	rho:	-_b[_cons]					
	phi:	_b[x3]					
	theta:	-_b[_cons] / _b[x3]					
	y	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
	rho	-1.327986	.8337527	-1.59	0.111	-2.962111	.3061395
	phi	-.2458883	.1111786	-2.21	0.027	-.4637943	-.0279822
	theta	5.400769	1.143771	4.72	0.000	3.159019	7.642519

However, this is a case where the Fieller CI implies a very different result from the Delta estimate. In this case the 95% Fieller interval is estimated as -12.794 to 6.932. Thus, we have estimated that the higher limit is less than the corresponding Delta one while the lower limit is much lower and would imply that a test of the hypothesis that the NAIRU is equal to zero could not be rejected at the 95% level based on the Fieller interval. Note that this would also be true for the 90% interval when  $\alpha = .10$  (in this case the Fieller interval would be -.957 to 6.725). In contrast, from the p-value reported for the Delta in Table 3.6.2 we note that we can reject the null with an  $\alpha < .001$ . The contrast between the Delta and Fieller CIs is most evident from the line plot shown in Figure 3.6.1.

**Figure 3.6.1** The 95% Fieller and Delta CIs for the NAIRU.



### 3.7 The 50% dose and Willingness-to-Pay.

The 50% dose problem was the original application that Fieller (1944) considered when he first investigated the CI of the ratio of parameter estimates. In the pharmacology literature, the 50% dose refers to the limits of the dose of a drug that would result in a greater than 50% chance that it would act on the subject. In the economics literature, the willingness-to-pay is defined as the maximum price someone will pay to purchase the product or service (Hole 2007 presents an extensive review). In this type of analysis, the data consists of a dichotomous response to a continuous stimulus – i. e. the drug dose or a price. Typically, this problem is modelled by a probability of occurrence/purchase using a binary dependent variable model such as a logit or probit estimator,

$$P(Y_t = 1) = \alpha + \gamma X_t + \sum_{i=1}^k \beta_i W_{it} + \varepsilon_t \quad (3.7.1)$$

where  $X$  is the variable dose/price and the  $W$ 's are other covariates. The question posed by the 50% dose is: What value of  $X$  (designated as  $\theta$ ) makes this relationship true?

$$.5 = \alpha + \gamma\theta + \sum_{i=1}^k \beta_i W_{it} + \varepsilon_t \quad (3.7.2)$$

Solving for the limiting value of  $\theta$  we have the ratio given by:

$$\theta = \frac{\rho}{\phi} = \frac{.5 - \left( \alpha + \sum_{i=1}^k \beta_i W_{it} \right)}{\gamma} \quad (3.7.3)$$

The ratio can be evaluated at specific values of  $W$  for cases of interest or at the means of these other regressors.

As an example, we use data on the survivors of the Titanic disaster where the dose/price is the fare paid for the trip in pounds (see Frey et al 2010) and the binomial result survival of the disaster.<sup>21</sup>

$$P(\text{survive}) = \alpha + \gamma \text{fare}_i + \beta_1 \text{female} + \beta_2 \text{age} + \varepsilon_i \quad (3.7.4)$$

We use a probit model to estimate the specification defined in (3.7.4). The results of this estimation are given in Table 3.7.1.

---

<sup>21</sup> Since the passengers did not know what the possibility of the disaster was prior to purchasing the ticket this is not a traditional willingness-to-pay problem but more like a 50% dose model. The Stata do file for this example is `will_to_pay.do` as listed in Appendix A.7.

**Table 3.7.1** Probit results for willingness-to-pay with data from the Titanic.

probit survive fare age female		Number of obs		=	1,033
Probit regression		LR chi2(3)		=	345.42
Log likelihood = -525.5508		Prob > chi2		=	0.0000
		Pseudo R2		=	0.2473

survived	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
fare	.0059587	.0010804	5.52	0.000	.0038411 .0080763
age	-.006552	.0031211	-2.10	0.036	-.0126692 -.0004348
female	1.440331	.0910842	15.81	0.000	1.261809 1.618852
_cons	-.8152731	.1074258	-7.59	0.000	-1.025824 -.6047223

The Delta method CI for the 50% fare for the average age person ( $m\_age$ ) and assuming 50% gender split, results in the estimated CI on the 50% dose/willingness-to-pay defined by the post-estimation results given in Table 3.7.2. The numerator is defined by  $\hat{\rho} = -(\hat{\alpha} + .5\hat{\beta}_1 + \hat{\beta}_2 m\_age)$  with a value of .290 and the denominator  $\hat{\phi} = \hat{\gamma}$  with a value of .006, and the expected willingness-to-pay as 48.7 pounds with an approximate Delta 95% interval of (32.82 to 64.54).

**Table 3.7.2** The estimated willingness-to-pay and the corresponding Delta CI.

rho: $-(\_b[\_cons] + \_b[age]*m\_age + .5*\_b[female])$						
phi: $(\_b[fare])$						
theta: $(-(\_b[\_cons] + \_b[age]*m\_age + \_b[female] * .5) / \_b[fare] )$						

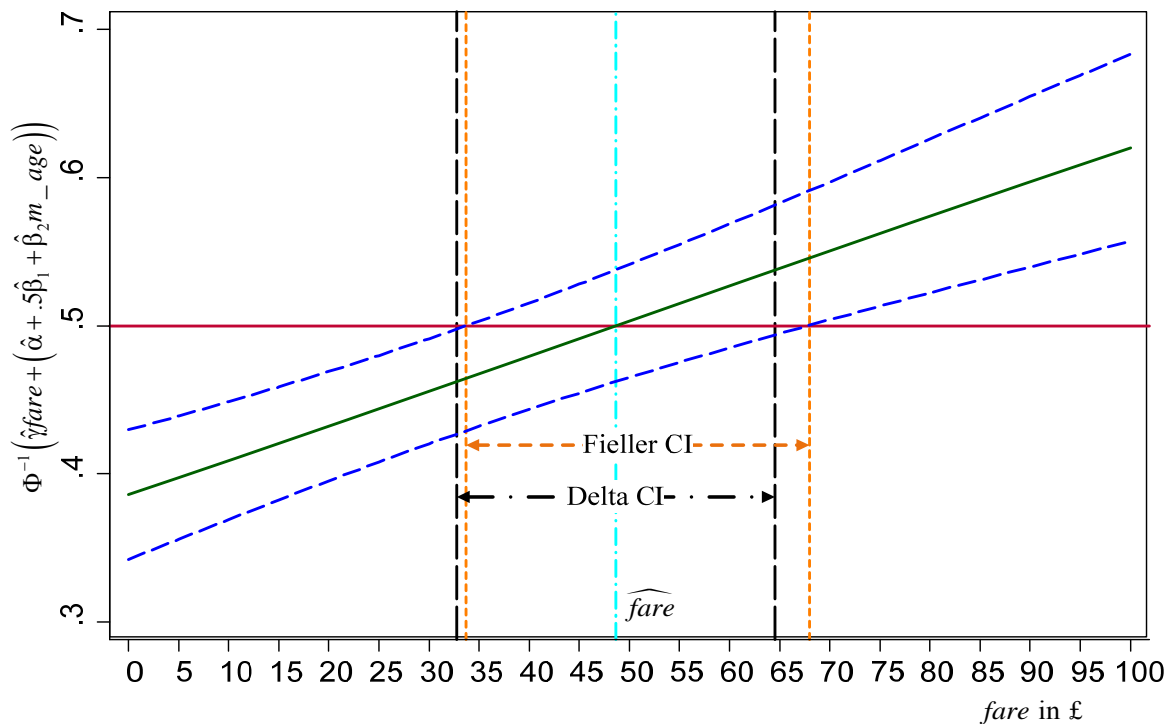
  

survived	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]
rho	.2900579	.0583259	4.97	0.000	.1757412 .4043745
phi	.0059587	.0010804	5.52	0.000	.0038411 .0080763
theta	48.67789	8.092075	6.02	0.000	32.81771 64.53806

To estimate the Fieller interval we use the `margins` command to predict the probability of survival followed by the `marginplot` command where the linear combination of the parameters given by  $\Phi^{-1}(\hat{\phi}\theta - \hat{\rho}) = \Phi^{-1}(\hat{\gamma}fare + (\hat{\alpha} + .5\hat{\beta}_1 + \hat{\beta}_2 m\_age))$  with the corresponding CI, where  $\Phi^{-1}$  indicates the inverse of the cumulative normal distribution defined by the `normal()` function in Stata.



**Figure 3.7.1** The Fieller and Delta CIs for the 50% level of the willingness to pay for survival on the Titanic



In this case the Delta and Fieller intervals coincide quite closely since the  $t$ -statistic on fare is 5.52 although the Fieller interval is limited at higher fares than the Delta. Note that in Figure 3.7.1 the reference line for the predicted probability of survival is put at .5. However, alternative levels could have been chosen or ranges of values i. e. between .4 and .6 instead.

### 3.8 Indirect Least Squares as Two Stage Least Squares

When a model is exactly identified with one endogenous regressor, the Two-Stage Least Squares (2SLS) and the instrumental variable (IV) estimates can be shown to be equivalent to the indirect least-squares (ILS) estimate of the structural parameter. Furthermore, when the ILS estimator is the ratio of two parameter estimates from equations that are estimated using seemingly unrelated regressions (SUR) the Delta CI is equivalent to the CI from the usual 2SLS estimation procedure. Hirschberg and Lye (2017) examine the Delta and Fieller CIs for the 2SLS/IV estimator of the coefficient of the endogenous regressor under these conditions.

Suppose the equation of interest is,

$$Y_i = \beta_0 + \beta_1 X_i + \gamma_1 W_{1i} + \gamma_2 W_{2i} + \dots + \gamma_K W_{Ki} + u_i \quad (3.8.1)$$

where  $Y$  is the dependent variable of interest,  $X$  is an endogenous regressor vector,  $W_k$  are the predetermined or exogenous regressors and  $u$  is the error term. 2SLS/IV estimation requires  $r$  instruments  $Z_1 \dots Z_r$  ( $r \geq 1$ ) that satisfy two conditions: (i) they are uncorrelated with  $u$  and (ii) are

correlated with  $X$ . When  $r = 1$  we say that (3.8.1) is exactly identified. In this case the reduced form equation of  $X$  is,

$$X_i = \pi_{10} + \pi_{11}Z_{1i} + \psi_{11}W_{1i} + \psi_{12}W_{2i} + \dots + \psi_{1K}W_{Ki} + v_{1i} \quad (3.8.2)$$

and similarly, the reduced-form equation of  $Y$  is defined by:

$$Y_i = \pi_{20} + \pi_{21}Z_{1i} + \psi_{21}W_{1i} + \psi_{22}W_{2i} + \dots + \psi_{2K}W_{Ki} + v_{2i} \quad (3.8.3)$$

The ILS estimator of  $\beta_1$  in (3.8.1) is the ratio:

$$\tilde{\theta} = \frac{\hat{\rho}}{\hat{\phi}} = \frac{\hat{\pi}_{21}}{\hat{\pi}_{11}} \quad (3.8.4)$$

where  $\hat{\pi}_{21}$  is the OLS estimator of  $\pi_{21}$  in (3.8.3),  $\hat{\pi}_{11}$  is the OLS estimator of  $\pi_{11}$  in (3.8.2) and the covariance of these two parameter estimates is estimated via the application of the SUR technique.

For example, Acemoglu, etal (2001) examine the effect of institutions on economic performance. In this case the structural equation is,

$$\log(y_i) = \mu + \beta R_i + \mathbf{X}'_i \theta + u_i \quad (3.8.5)$$

where  $y_i$  is income per capita in country  $i$ ,  $R_i$  is the protection against expropriation measures and  $\mathbf{X}_i$  is a vector of other covariates that includes dummy variables for the different continents on which the countries are located and the absolute value of their latitude.<sup>22</sup>

Alternately we can consider the two reduced form equations. The protection against expropriation variable  $R_i$  is treated as endogenous, with reduced form defined by:

$$R_i = \pi_{10} + \pi_{11} \log(M_i) + \mathbf{X}'_i \delta_1 + v_{1i} \quad (3.8.6)$$

where  $M_i$  is the settler mortality rate in 1,000 mean strength. The reduced form equation for log income per capita is

$$\log(y_i) = \pi_{20} + \pi_{21} \log(M_i) + \mathbf{X}'_i \delta_2 + v_{2i} \quad (3.8.7)$$

This model is just-identified with  $\log(M_i)$  being excluded from (3.8.5). In this case the indirect least squares estimate of the structural parameter ( $\tilde{\beta}_{ILS}$ ) is defined as the ratio of the parameters:

$$\tilde{\beta}_{ILS} = \frac{\hat{\pi}_{21}}{\hat{\pi}_{11}}. \quad (3.8.8)$$

To demonstrate this equivalence of the 2SLS/IV and the ILS approaches, we provide the 2SLS/IV estimates for the model specified in (3.8.5) where  $\log(M_i)$  is used as the instrument in

---

<sup>22</sup> The Stata do file for this example is ils.do as listed in Appendix A.8.

Table 3.8.1. From this result, we note that the estimate for the structural parameter ( $\beta$ ) is given as 1.107 with a 95% CI of .242 to 1.972 and would result in the rejection of the null hypothesis that  $\beta = 0$  at the 5% level of significance.

**Table 3.8.1** The Stata estimation results for the 2SLS application to the model where:  $\logpgdp95 = \log(y_i)$ ,  $\logrm4 = \log(M_i)$ ,  $avexpr = R_i$ , and  $africa, asia, lat\_abst, other\_cont = X_i$

Instrumental variables (2SLS) regression						
					Number of obs	= 64
					Wald chi2(5)	= 37.78
					Prob > chi2	= 0.0000
					R-squared	= 0.0108
					Root MSE	= 1.0296
logpgp95	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
avexpr	1.107077	.4413078	2.51	0.012	.2421296	1.972024
africa	-.4372669	.403835	-1.08	0.279	-1.228769	.3542352
asia	-1.047085	.499367	-2.10	0.036	-2.025826	-.0683438
lat_abst	-1.178178	1.671129	-0.71	0.481	-4.453531	2.097174
other_cont	-.9904014	.9500526	-1.04	0.297	-2.85247	.8716674
_cons	1.440453	2.703206	0.53	0.594	-3.857732	6.738639
Instrumented: avexpr						
Instruments: africa asia lat_abst other_cont logem4						

**Table 3.8.2** The Stata estimated parameters for the two reduced form equations via SUR.

Seemingly unrelated regression						
Equation	Obs	Parms	RMSE	"R-sq"	F-Stat	P
logpgp95	64	5	.7014402	0.5839	17.96	0.0000
avexpr	64	5	1.255011	0.3277	6.24	0.0000
	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
<b>logpgp95</b>						
logem4	-.3767581	.0974069	-3.87	0.000	-.5696847	-.1838314
lat_abst	1.046296	.7400363	1.41	0.160	-.419439	2.512031
africa	-.7228463	.2179443	-3.32	0.001	-1.154512	-.2911802
asia	-.5245579	.267608	-1.96	0.052	-1.054589	.0054734
other_cont	.1847984	.4489049	0.41	0.681	-.7043144	1.073911
_cons	9.997378	.5091475	19.64	0.000	8.988947	11.00581
<b>avexpr</b>						
logem4	-.3403178	.1742797	-1.95	0.053	-.6855008	.0048651
lat_abst	2.009322	1.324067	1.52	0.132	-.61316	4.631805
africa	-.2579581	.3899443	-0.66	0.510	-1.030292	.5143757
asia	.4719882	.4788021	0.99	0.326	-.4763397	1.420316
other_cont	1.061534	.8031773	1.32	0.189	-.5292599	2.652328
_cons	7.729295	.9109629	8.48	0.000	5.925018	9.533572

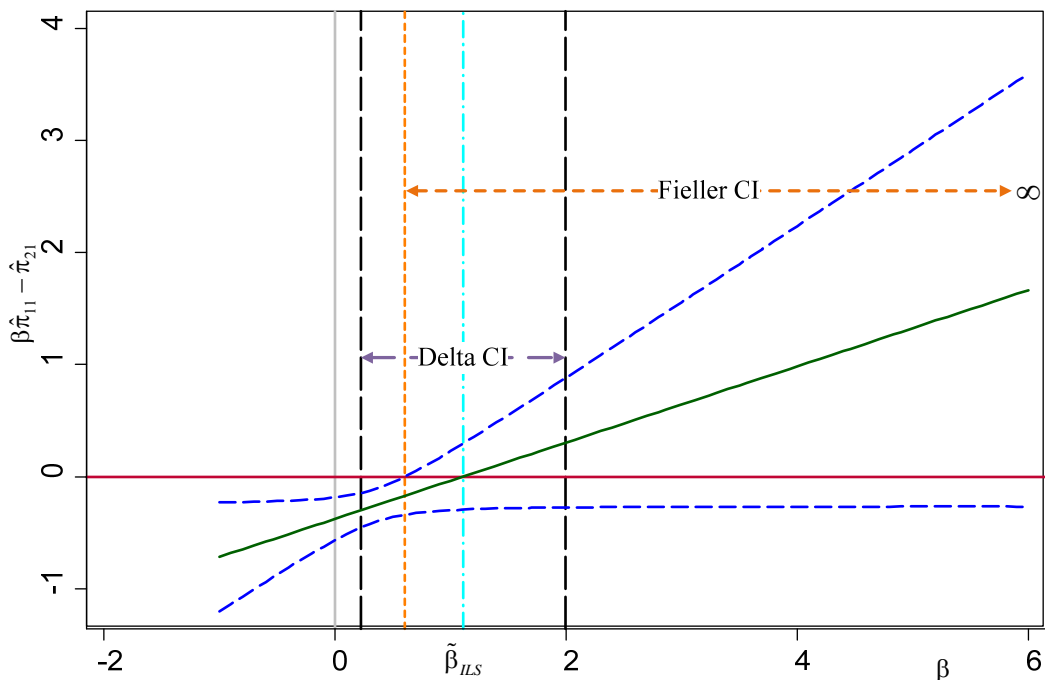
In Table 3.8.2 we present the results of the SUR estimation for the two reduced form regressions. From (3.8.8) the ILS approach results in the estimate of the structural parameter as the ratio of the two parameters estimated for the logem4 regressor in the two equations. The Delta 95% CI for the ratio of these two parameter estimates is given in Table 3.8.3. From this table, we note

that the ILS estimate of the structural parameter and its CI is identical to the 2SLS/IV estimate reported in Table 3.8.1.<sup>23</sup>

**Table 3.8.3** The Stata results of the estimation of the ratio defined by  $\tilde{\beta}_{ILS} = \frac{\hat{\pi}_{21}}{\hat{\pi}_{11}}$

	Coef.	Std. Err.	z	P> z	[95% Conf. Interval]	
rho:	_b[logppp95:logem4]					
phi:	_b[avexpr:logem4]					
theta:	_b[logppp95:logem4] / _b[avexpr:logem4]					
rho	-.3767581	.0974069	-3.87	0.000	-.5676721	-.185844
phi	-.3403178	.1742797	-1.95	0.051	-.6818998	.0012641
theta	1.107077	.4413078	2.51	0.012	.2421296	1.972024

**Figure 3.8.1** Fieller 95% Confidence Interval for the structural parameter.



Following the derivation of the Fieller interval as discussed in Section 2.3, we find that there are two roots for the quadratic equation (2.3.4) as  $-29.313$  to  $.602$ . We note that these two roots bear little similarity to the Delta bounds. This is an example of the Fieller confidence interval resulting in the complement of a finite interval since in (2.3.4)  $(b^2 - 4ac) = .031 > 0$  and  $a = -.006 < 0$ . If we construct the line plot representation in the range of the Delta CI as shown in Figure 3.8.1, we find the appropriate interpretation of the Fieller CI is  $(.602 \text{ to } \infty)$  or that the upper

<sup>23</sup> Note that since the regressors in these equations are the same the estimated parameters from the SUR are identical to the OLS results if they were estimated separately. However, the SUR approach provides an estimate of the covariance between the parameter estimates of the two equations that is required for the correct estimation of the Delta and Fieller CIs.

bound of the 95% Fieller CI is not defined. This would imply that we can reject the hypothesis that the parameter estimate is less than or equal to zero but there is no limit to the parameter estimate in the positive direction. A more thorough discussion of this result is available in Hirschberg and Lye (2017).

## 4.0 Summary and Conclusions

In this paper, we have presented a comparison between the Fieller and Delta methods for the approximation of the CI for cases where the quantity of interest is the ratio of a linear combination of parameter estimates. We present this method with emphasis on the application of the Stata statistical software routine. The examples of the consideration of ratios has been shown in various econometric applications.

One of the conclusions that can be drawn from these examples is that although the Fieller and the Delta CIs coincide in many cases, there are some situations where the use of the Fieller and Delta result in far different conclusions. Recent studies of the coverage of the CIs of the two methods has found that the Fieller CI conforms much more closely to the true interval when Delta and the Fieller result in very different outcomes. Using the simple graphical method demonstrated in this paper, the Fieller can be used in any application which employs a ratio of parameter estimates.

A general caveat that applies for both the Delta and the Fieller is the assumption that the parameter estimates are normally distributed or asymptotically normally distributed. This assumption is dependent on many of the other assumptions implied by the correct specification of the functional forms and the error generating process.

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# Appendix



## Appendix A. Example Stata do-files for the paper

The Stata do files for each example application as listed in Table A.1. The code for each file are listed here as well. The do files and data are in the *do\_files* folder and the data for these are in the *data* folder. The R-script for all examples are provided in one file. The R-scripts read the Stata files used. The data are available at: [https://www.online.fbe.unimelb.edu.au/t\\_drive/FTS/data](https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data).

**Table A.1** The Stata do files and data for each example.

<i>Appendix</i>	<i>do file</i>	<i>dta file</i>
A.1 Parallel Line Assay	parallelassay_pokies.do	new_fall.dta
A.2 Turning Point	harbaughs.do	HLW.dta
A.3 Interactions	journal.do	journal.dta
A.4 Long-run – Nonlinear	commercial.do	commercial.dta
A.5 Long-run – Multi-lags	greenbook.do	greenbook1.dta
A.6 NAIRU	nairu.do	phillips.dta
A.7 50% Dose & Willingness-to-pay	will_to_pay.do	titanic1.dta
A.8 2SLS/IV & ILS	ils.do	maketable4.dta

### A.1 Parallel Line Assay

```

/*
  parallelassay_pokies.do

  Program to perform Assay analysis to estimate the equivalent value for a
  dummy variable. Use the pokies data for smoking bans near the border to
  determine what change in the number of EGMs is equivalent to being on the
  border based on the data from Hirschberg & Lye (2010d).
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/new_fall.dta", clear
/*
  Run a regression on the change in the change in EGM expenditures (fall)
  as a function of the dummy for the border (nborder), the change in the
  number of EGMs (cegm) and the measure of socioeconomic status of the population
  (irsed).
*/
reg fall nborder cegm irsed , robust
estimates store reg
scalar df = e(df_r)
/*
  Define rho as the parameter estimate for the numerator of the ratio and phi
  as the denominator and theta as the ratio. Make sure they are defined in the
  order that the numerator is first then the denominator with the ratio as
  last.
*/
nlcom (rho:_b[nborder]) (phi:_b[cegm] ) (theta: _b[nborder] /_b[cegm]), ///
level(95) post
/*
  Use the variances and covariances to define the quadratic for the exact Fieller.

```

Note that the order matters here so that v11 is the variance of the numerator and v22 is the variance of the denominator. With v12 as the covariance. Note that the Fieller and Delta will tend to agree when the sign of the correlation is the same as the sign of the ratio and visa versa.

```

*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalpha/2))
display _newline(2) _skip(5) "corr = " corr ", v12 = " v12 ", v22 = " ///
v22 ", v11 = " v11 ", t-stat = " t2 ", alpha = " nalpha
/*
Find solution to quadratic equation for the exact Fieller results.
Define the rho as the numerator and phi as the denominator.
The t2 is the critical t-stat for the level of alpha.
*/
scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)
scalar lm2 = (-bb - rad) / (2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Plot out the values of the linear combination of the parameter estimates.
Vary the value of cegm to evaluate the linear combination of the parameters
for nborder and cegm to range from -5 to 35 using the margins statement.
This is the translated value of the parameter on the nborder dummy.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]
estimates restore reg
quiet margins, expression (_b[nborder] - _b[cegm]*cegm) at ///
(cegm = (-5(1)35)) level(95)
/*
Plot out the linear combination with the Delta 95% confidence interval
*/
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2

marginsplot, yline(0) recast(line) recastci(rline) ///
xline(`low' `upb', lcolor(violet) lpattern(longdash) ) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium) ) ///
xline(`ratio' , lcolor(cyan) lpattern(dash_dot) lwidth(wide) ) ///
xline(0 , lcolor(gs12) ) name(Assay2, replace) ciopts( lcolor(blue) ///
lpattern(dash dash) ) title( 95% Fieller & Delta CI) xtitle(Change in EGMs) ///
ytitle(Translated dummy)

```

## A.2 Turning Point

```
/*
harbaughs.do

Example of the location of the turning point in a quadratic equation.
Estimate the Environmental Kuznets Curve based on the data from
Harbaugh et al. (2002)
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/HLW.dta", clear
/*
Limit the data to only sulfur dioxide data
*/
keep if POLL == "SO2"
/*
Define the data as panel
*/
iis ISITE
tis IYEAR
/*
Estimate model with random effects for site, fixed effects for year
and clustered standard errors by site. Retrieve the degrees of freedom
(the number of clusters) and store the results to use latter in the routine.
*/
xtreg lnMEAN c.GDP c.GDP#c.GDP i.IYEAR DENSAT TRADE DEMO, re vce(cluster ISITE)
scalar df =e(N_g)
estimates store reg
/*
Define rho as the parameter estimate for the numerator of the ratio and phi
as the denominator and theta as the ratio. Make sure they are defined in the
order that the numerator is first then the denominator with the ratio as
last.
*/
nlcom (rho:_b[GDP]) (phi:-2*_b[c.GDP#c.GDP] ) ///
(theta: _b[GDP] /(-2*_b[c.GDP#c.GDP] ) ), level(95) post
/*
Use the variances and covariances to define the quadratic for the exact Fieller.
Note that the order matters here so that v11 is the variance of the numerator
and v22 is the variance of the denominator. With v12 as the covariance.
Note that the Fieller and Delta will tend to agree when the sign of the
correlation is the same as the sign of the ratio and visa versa.
*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalpha/2))
display _newline(2) _skip(5) "corr = " corr " , v12 = " v12 " , v22 = " ///
v22 " , v11 = " v11 " , t-stat = " t2 " , alpha = " nalpha
/*
Find solution to quadratic equation for the exact Fieller results.
Define the rho as the numerator and phi as the denominator.
The t2 is the critical t-stat for the level of alpha.
*/
```

```

scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)
scalar lm2 = (-bb - rad) / (2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Plot out the values of the linear combination of the parameter estimates.
at different values of GDP to range from 1 to 30 using the margins statement.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]
/*
Restore the regression results and plot out the shape of the predicted
quadratic function of SO2 by GDP
*/
estimates restore reg
quiet margins, expression (_b[GDP]*GDP + _b[c.GDP#c.GDP]*GDP*GDP) at ///
(GDP = (1(.5)30)) level(95)
local ratio = theta
marginsplot, recast(line) recastci(rline) ciopts( lcolor(blue) ///
xline(`ratio' , lcolor(cyan) lpattern(solid) lwidth(medium) ) ///
lpattern(dash dash ) name(E_Kuznets_shape, replace) title(Curve shape ) ///
ytitle(SO2) xtitle(GDP)
/*
Plot the line plot of the inverse test of the Fieller interval.
In this case it is the same as the first derivative function.
Thus, we use the margins routine to evaluate the first derivative function.
Then, plot the values with a reference line on the y-axis at zero.
Plot out the linear combination with the Delta 95% confidence interval
*/
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2

quiet margins, dydx(GDP) at (GDP = (1(.5)30)) level(95)
marginsplot, yline(0) recast(line) recastci(rline) ///
xline(`low' `upb', lcolor(purple) lpattern(longdash ) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium) ) ///
xline(`ratio' , lcolor(cyan) lpattern(solid) lwidth(medium) ) ///
xline(0 , lcolor(gs12) ) ///
name(E_Kuznets2, replace) ciopts( lcolor(blue) lpattern(dash dash ) ) ///
title( 95% Fieller & Delta CI) xtitle(GDP) ytitle(Marginal Effect)
/*
Combine the two marginsplot so that one can see how the shape of the function
determines where the turning point is and the first derivative function can
find the turning point along with the corresponding Fieller CI
*/
graph combine E_Kuznets_shape E_Kuznets2, col(1) iscale(.5) xcommon ///
fxsize(50) name(E_Kuznets_Total, replace)

```

### A.3 Interactions

```
/*
    journal.do

    Example of the marginal effect of a regressor when interacted with another.
    Program to model demand for Journals based on age and price
    from Stock and Watson (2003, pg. 227) where price and age are interacted.
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/journal.dta", clear
/*
    Define variables needed for this analysis
*/
gen loclc = log(oclc)
gen lprice = log(price)
gen lage = log(age)
gen lchar = log(char)
gen lpla = lprice*lage
/*
    Estimate the regression, store it and define the degrees of freedom
*/
reg loclc lprice lpla lage lchar
estimates store reg
scalar df = e(df_r)
/*
    Define the mean of the elasticities and call the scalar m_elast
*/
quiet summarize lage
scalar m_lage = r(mean)
scalar m_elast = _b[lprice] + _b[lpla] * m_lage
/*
    Define rho as the parameter estimate for the numerator of the ratio and phi
    as the denominator and theta as the ratio. Make sure they are defined in the
    order that the numerator is first then the denominator with the ratio as
    last.
*/
nlcom ( rho: ( m_elast-_b[lprice]) ) ( phi: _b[lpla] ) ///
      (theta: ( ( m_elast-_b[lprice]) /_b[lpla]) ) , level(95) post
/*
    Use the variances and covariances to define the quadratic for the exact Fieller.
    Note that the order matters here so that v11 is the variance of the numerator
    and v22 is the variance of the denominator. With v12 as the covariance.
    Note that the Fieller and Delta will tend to agree when the sign of the
    correlation is the same as the sign of the ratio and visa versa.
*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalpha/2))
display _newline(2) _skip(5) "corr = " corr " , v12 = " v12 " , v22 = " ///
      v22 " , v11 = " v11 " , t-stat = " t2 " , alpha = " nalpha
/*
    Find solution to quadratic equation for the exact Fieller results.
```

```

Define the rho as the numerator and phi as the denominator.
The t2 is the critical t-stat for the level of alpha.
*/
scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)
scalar lm2 = (-bb - rad) / (2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Compute the Delta interval for alpha.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]
/*
Plot out the values of the price elasticities as they vary with age
Define the price elasticity estimates as they differ from the mean price
elasticity estimates to range from 1 to 6 using the margins statement.
*/
estimates restore reg
quiet margins, expression (_b[lprice] + _b[lpla]*lage) at ///
(lage = (1(.5)6)) level(95)
/*
Plot out the elasticities with the Delta 95% confidence interval
*/
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2
local _m_elast = m_elast
local _m_lage = m_lage

marginsplot, yline(`_m_elast', lcolor(red) lpattern(solid)) ///
recast(line) recastci(rline) ///
xline(`low' `upb', lcolor(purple) lpattern(longdash)) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium)) ///
name(Elast, replace) ciopts( lcolor(blue) lpattern(dash dash)) ///
xline(`_m_lage' , lcolor(cyan) lpattern(dash_dot) lwidth(medium)) ///
title(Price Elasticity for Journals) xtitle( Log(Age) ) ytitle(Elasticity)

```

## A.4 Long-run - Nonlinear

```
/*
commercial.do

Example of a dynamic model estimated using the non-linear estimation.
Program to model demand for Energy in Quebec from Bernard et al (2007)
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/commercial.dta"
/*
generate the log of the data series
*/
gen logteg = log(teg)
gen loglteg = log(lteg)
gen logep = log(ep)
gen loggdp = log(gdp)
gen loghdd = log(hdd)
gen loglhdd = log(lhdd)
/*
Estimate the dynamic demand model. Note that with non-linear estimation it
is necessary to list the variables using in the model with the variables()
option to the nl statement to be able to use the margins statement as a
post-estimation procedure.
*/
nl (logteg = {b0} + {b1}*loglteg + {b2}*logep + {b3}*loggdp + ///
{b4}*(loghdd - {b1}*loglhdd)), variables(loglteg logep loggdp loghdd loglhdd)
estimates store reg
/*
Fieller 95% confidence interval graphical approach
*/
scalar df =e(df_r)
/*
Define rho as the parameter estimate for the numerator of the ratio and phi
as the denominator and theta as the ratio. Make sure they are defined in the
order that the numerator is first then the denominator with the ratio as
last.
*/
nlcom (rho:/b2) (phi:(1 - /b1) ) (theta: /b2 /(1 - /b1) ), level(95) post
/*
Use the variances and covariances to define the quadratic for the exact Fieller.
Note that the order matters here so that v11 is the variance of the numerator
and v22 is the variance of the denominator. With v12 as the covariance.
Note that the Fieller and Delta will tend to agree when the sign of the
correlation is the same as the sign of the ratio and visa versa.
And when the t-statistic on the denominator (phi) is significantly different
from zero.
*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalpha/2))
display _newline(2) _skip(5) "corr = " corr " , v12 = " v12 " , v22 = " ///
v22 " , v11 = " v11 " , t-stat = " t2 " , alpha = " nalpha
```

```

/*
Find solution to quadratic equation for the exact Fieller results.
Define the rho as the numerator and phi as the denominator.
The t2 is the critical t-stat for the level of alpha.
*/
scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)
scalar lm2 = (-bb - rad) / (2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Compute the Delta interval for alpha.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]
/*
Plot out the values of the linear combination of the parameter estimates.
Use the variable logep to evaluate the linear combination of the parameters
estimates to range from -.8 to -.2 using the margins statement.
This is the value of the LR elasticity.
*/
estimates restore reg

quiet margins, expression (_b[/b2] - (1-_b[/b1])*logep) at ///
(logep = (-.8(.05)-.2)) level(95) saving(elast, replace)
/*
Plot out the linear combination with the Delta 95% confidence interval
*/
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2

marginsplot, yline(0, lcolor(red) lpattern(solid)) ///
recast(line) recastci(rline) ///
xline(`low' `upb', lcolor(purple) lpattern(longdash)) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium)) ///
xline(`ratio' , lcolor(cyan) lpattern(dash_dot) lwidth(medthin)) ///
name(Elect_Elast, replace) ciopts( lcolor(blue) lpattern(dash dash)) ///
title(Long-Run Elasticities) xtitle(Price Elasticities) ytitle(Linear Function)

```



## A.5 Long-run – Multi-lags

```
/*
greenbook.do

Example of a dynamic specification with two lagged values.
Program to model Taylor Rule Coefficients using the specification in
Coibion and Gorodnichenko (2011)
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/greenbook1.dta"
/*
Set the timeseries indicator for variable a
*/
tsset a
/*
Generate new variables as the lags
*/
gen lffr1 = ffr[_n-1]
gen lffr2 = ffr[_n-2]
/*
Estimate the model using Newey-West standard errors to account for unspecified
hetroskedasticity and autocorrelation
*/
newey ffr lffr1 lffr2 pi_tp gry_tp gap_t03 if dum82>0, lag(5)
scalar df =e(df_r)
/*
Define rho as the parameter estimate for the numerator of the ratio and phi
as the denominator and theta as the ratio. Make sure they are defined in the
order that the numerator is first then the denominator with the ratio as
last.
*/
nlcom (rho:_b[pi_tp]) (phi: (1 - _b[lffr1] - _b[lffr2]) ) ///
(theta: _b[pi_tp] / (1 - _b[lffr1] - _b[lffr2]) ), level(95) post
/*
Use the variances and covariances to define the quadratic for the exact Fieller.
Note that the order matters here so that v11 is the variance of the numerator
and v22 is the variance of the denominator. With v12 as the covariance.
Note that the Fieller and Delta will tend to agree when the sign of the
correlation is the same as the sign of the ratio and visa versa.
And when the t-statistic on the denominator (phi) is significantly different
from zero.
*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalpha/2))
display _newline(2) _skip(5) "corr = " corr " , v12 = " v12 " , v22 = " ///
v22 " , v11 = " v11 " , t-stat = " t2 " , alpha = " nalpha
/*
Find solution to quadratic equation for the exact Fieller results.
Define the rho as the numerator and phi as the denominator.
The t2 is the critical t-stat for the level of alpha.
*/
```

```

scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)
scalar lm2 = (-bb - rad) / (2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Compute the Delta interval for alpha.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]
/*
Plot out the values of the linear combination of the parameter estimates.
Use the variable pi_tp to evaluate the linear combination of the parameters
estimates to range from 0 to 7 using the margins statement.
Note it is necessary to re-run the regression using the newey program since
we cannot run margins from the stored results.
*/
quiet newey ffr lffr1 lffr2 pi_tp gry_tp gap_t03 if dum82>0, lag(5)
quiet margins, expression (_b[pi_tp] - (1 - _b[lffr1] - _b[lffr2])*pi_tp) ///
at (pi_tp = (0(.5)7)) level(95)
/*
Plot out the linear combination with the Delta 95% confidence interval
*/
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2

marginsplot, yline(0, lcolor(red) lpattern(solid)) ///
recast(line) recastci(rline) ///
xline(`low' `upb', lcolor(purple) lpattern(longdash)) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium)) ///
xline(`ratio' , lcolor(cyan) lpattern(dash_dot) lwidth(medthin)) ///
name(Taylor, replace) ciopts( lcolor(blue) lpattern(dash dash)) ///
title(Taylor Rule) xtitle(Long-Run Responses) ytitle(Linear Function)

```

## A.6 NAIRU

```

/*
    nairu.do

    Program to estimate the Non-accelerating inflation rate of unemployment NAIRU
    The based on Gruen et al (1999) as specified by Hirschberg and Lye (2010a)
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/phillips.dta"
/*
    Estimate model using Newey-West standard errors where.
    See the text for the variable definitions.
*/
gen time = _n
tsset time

newey y x2 x3 x4 x5 x6 , lag(4)
scalar df =e(df_r)
estimates store reg
/*
    Define rho as the parameter estimate for the numerator of the ratio and phi
    as the denominator and theta as the ratio. Make sure they are defined in the
    order that the numerator is first then the denominator with the ratio as
    last.
*/
nlcom (rho:-_b[_cons]) (phi: _b[x3]) ///
(theta: -_b[_cons] /_b[x3] ), level(95) post
/*
    Use the variances and covariances to define the quadratic for the Fieller by
    retrieving the covariance matrix of the numerator and the denominator.
    Note that the order matters here so that v11 is the variance of the numerator,
    v22 is the variance of the denominator, and v12 as the covariance.
    Note that the Fieller and Delta will tend to agree when the sign of the
    correlation is the same as the sign of the ratio and visa versa.
*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalpha/2))
display _newline(2) _skip(5) "corr = " corr ", v12 = " v12 ", v22 = " ///
v22 ", v11 = " v11 ", t-stat = " t2 ", alpha = " nalpha
/*
    Find solution to quadratic equation for the exact Fieller results.
    Define the rho as the numerator and phi as the denominator.
    The t2 is the critical t-stat for the level of alpha.
*/
scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)

```

```

scalar lm2 = (-bb - rad ) / ( 2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Compute the Delta interval for alpha.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]

/*
Compute the linear combination with the Delta 95% confidence interval
set limits to the margins evaluation use the x3 variable to scale the marginal
effect. This may need some fine tuning. Note that we need to rerun the
estimation
instead of just restore it since the newey program does not store the results
in a form that can be used by the margins program.
*/
quiet newey y x2 x3 x4 x5 x6 , lag(4)
local plowb = min(-15,lw,lm1,lm2)
local pupb = max(lm1,lm2,10)
quiet margins, expression (_b[_cons] + _b[x3]*x3) at ///
(x3 = (`plowb'(.25)`pupb')) level(95)

/*
Plot the linear combination to demonstrate how the Fieller interval
relates to the Delta.
In this example we plot the CIs based on both the Fieller and the Delta.
*/
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2
marginsplot, yline(0, lwidth(vthin) ) recast(line) recastci(rline) ///
xline(0, lcolor(gs12) ) ///
xline(`low' `upb', lcolor(brown) lpattern(longdash) lwidth(medium) ) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium) ) ///
xline(`ratio' , lcolor(cyan) lpattern(dash_dot) lwidth(wide) ) ///
name(NAIRU, replace) ciopts( lcolor(blue) lpattern(dash dash) ) ///
title(Fieller & Delta CI) xtitle(NAIRU) ytitle(Linear Combination)

```

## A.7 50% Dose & Willingness-to-pay

```

/*
    will_to_pay.do

    Program to provide Fieller and Delta intervals for application of a
    willingness-to-pay model to the survival of the Titanic and the fares paid.
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/titanic1.dta"
/*
    Estimate the mean value of age
*/
quiet summarize age, meanonly
scalar m_age = r(mean)
/*
    Estimate model store the result and save the degrees of freedom
*/
probit survive fare age female
estimates store reg
scalar df = e(N) - e(k)
/*
    Define rho as the parameter estimate for the numerator of the ratio and phi
    as the denominator and theta as the ratio. Make sure they are defined in the
    order that the numerator is first then the denominator with the ratio as
    last.
*/
nlcom ( rho: -(_b[_cons]+ _b[age]*m_age + .5*_b[female]) ) ///
(phi: (_b[fare]) ) ///
(theta: (-(_b[_cons]+ _b[age]*m_age + _b[female] * .5) / _b[fare] ) ), level(95)
post
/*
    Use the variances and covariances to define the quadratic for the exact Fieller.
    Note that the order matters here so that v11 is the variance of the numerator
    and v22 is the variance of the denominator. With v12 as the covariance.
    Note that the Fieller and Delta will tend to agree when the sign of the
    correlation is the same as the sign of the ratio and visa versa.
*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalphi/2))
display _newline(2) _skip(5) "corr = " corr " , v12 = " v12 " , v22 = " ///
v22 " , v11 = " v11 " , t-stat = " t2 " , alpha = " nalphi
/*
    Find solution to quadratic equation for the exact Fieller results.
    Define the rho as the numerator and phi as the denominator.
    The t2 is the critical t-stat for the level of alpha.
*/
scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)

```

```

scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad ) / ( 2 * aa)
scalar lm2 = (-bb - rad ) / ( 2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Compute the Delta interval for alpha.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]
/*
Plot out the values of the probability of survival where female=.5 and the
mean of age by the fare paid. Use the normal() function to evaluate the
linear combination of the probit parameters as a probability for fares from
0 to 100 pounds.
*/
estimates restore reg

quiet margins, expression ( normal( _b[_cons]+_b[fare]*fare + _b[age]*age + ///
_b[female] * female ) ) at(age (mean) female = .5 fare=(0(5)100)) level(95)
/*
Plot out the linear combination with the Delta & Fieller 95% confidence interval
*/
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2

marginsplot, yline(.5) recast(line) recastci(rline) ///
xline(`low' `upb', lcolor(purple) lpattern(longdash) ) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium) ) ///
xline(`ratio' , lcolor(cyan) lpattern(dash_dot) lwidth(medium) ) ///
name(will_to_pay, replace) ciopts( lcolor(blue) lpattern(dash dash) ) ///
title(Fieller & Delta CI) xtitle(Fare) ytitle(P(survive))

```

## A.8 2SLS/IV & ILS

```

/*
  ils.do

  Program to provide Fieller and Delta intervals for the IV estimate of the
  structural parameter by the use of indirect least squares when the model
  is exactly identified with one instrument. This model uses the model
  estimated in Hirschberg and Lye(2017) using data from Acemoglu, Johnson
  and Robinson (2001)
*/
clear all
use "https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/maketable4.dta"
drop if baseco ~= 1
/*
  Use the standard two-stage Least squares approach to estimate the model
  where avexpr is instrumented by logem4
*/
ivregress 2sls logpgp95 (avexpr = logem4) africa asia lat_abst other_cont
/*
  Estimate two reduced-form equations using Seemingly unrelated regressions
  and define the degrees of freedom as the N-k/2 where k is the total number
  of parameters estimated in both equations.
*/
sureg (logpgp95 avexpr = logem4 lat_abst africa asia other_cont) , small
scalar df = e(N) - e(k)/2
estimates store sur
/*
  Define rho as the parameter estimate for the numerator of the ratio and phi
  as the denominator and theta as the ratio. Make sure they are defined in the
  order that the numerator is first then the denominator with the ratio as
  last. Note that the estimated theta is the same as the IV estimated value
*/
nlcom (rho: _b[logpgp95:logem4] ) (phi: _b[avexpr:logem4] ) ///
(theta: _b[logpgp95:logem4] / _b[avexpr:logem4] ), level(95) post
/*
  Use the variances and covariances to define the quadratic for the Fieller CI.
  Note that the order matters here so that v11 is the variance of the numerator
  and v22 is the variance of the denominator. With v12 as the covariance.
  The Fieller and Delta will tend to agree when the sign of the
  correlation is the same as the sign of the ratio and visa versa and both
  numerator and denominator are significantly different from zero.
*/
matrix V = e(V)
scalar v12 = V[1,2]
scalar v11 = V[1,1]
scalar v22 = V[2,2]
scalar corr = v12/sqrt(v11*v22)
scalar nalpha = .05
scalar t2 = invttail(df,scalar(nalpha/2))
display _newline(2) _skip(5) "corr = " corr " , v12 = " v12 " , v22 = " ///
v22 " , v11 = " v11 " , t-stat = " t2 " , alpha = " nalpha
/*
  Find solution to quadratic equation for the exact Fieller results.
  Define the rho as the numerator and phi as the denominator.
  The t2 is the critical t-stat for the level of alpha.
*/

```

```

scalar rho = _b[rho]
scalar phi = _b[phi]
scalar aa = (phi*phi) - (t2*t2*v22)
scalar bb = (2*t2*t2*v12) - (2*rho*phi)
scalar cc = (rho*rho) - (t2*t2*v11)
scalar rad = sqrt(bb*bb - 4*aa*cc)
scalar lm1 = (-bb + rad) / (2 * aa)
scalar lm2 = (-bb - rad) / (2 * aa)
display _newline(2) _skip(5) "Fieller Limits " lm1 " to " lm2
/*
Compute the Delta interval for alpha.
*/
scalar up = _b[theta] + t2*sqrt(V[3,3])
scalar lw = _b[theta] - t2*sqrt(V[3,3])
scalar theta = _b[theta]
/*
Restore the results of the seemingly unrelated regression.
Plot out the values of the structural parameter by redefining the variable
logem4 to evaluate the linear combination of the parameter estimates.
First restore the sur results.
Plot the widest set of values to account for the extreme Fieller interval
*/
estimates restore sur
/*
Draw the Delta and only the single Fieller bound that coincides the parameter
estimate by using a more appropriate range of values.
*/
quiet margins, expression (_b[logpgp95:logem4]-_b[avexpr:logem4]*logem4) ///
at (logem4 = (-1(.1)6)) level(95)
local ratio = theta
local low = lw
local upb = up
local flm1 = lm1
local flm2 = lm2

marginsplot, yline(0) recast(line) recastci(rline) ///
xline(`low' `upb', lcolor(violet) lpattern(longdash) ) ///
xline(`flm1' `flm2', lcolor(orange) lpattern(shortdash) lwidth(medium) ) ///
xline(`ratio' , lcolor(cyan) lpattern(dash_dot) lwidth(wide) ) ///
xline(0 , lcolor(gs12) ) ///
name(IV2, replace) ciopts( lcolor(blue) lpattern(dash dash) ) ///
title( 95% Delta CI & Fieller lower bound) xtitle(Structural Parameter) ///
ytitle(Linear Combination)

```



## Appendix B. R Code to replicate the results

### B.1 Parallel Line Assay

```
#
# Install these packages first
#

library(haven) # A package to read Stata data sets
library(foreign) # A package to read Stata data sets
library("msm") # for calculation delta standard errors
library(ggplot2) # advanced graphics
library(lmtest) # testing linear regression models
library(systemfit) # estimation of sur models
library(AER) # Applied Econometrics with R
library(car) # companion to Applied Regression
library("plm") # linear models for panel data
library(sandwich) # different robust covariance matrix estimators

#=====
# 3.1 Parallel-line assay: A compensation interpretation of a dummy variable.
#=====
# read txt file into data set fall
#

fall= read.dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/new_fall.dta")

#
# Run a regression on the change in the change in EGM expenditures (fall)
# as a function of the dummy for the border (nborder),
# the change in the number of EGMS (cegm) and the measure of socioeconomic status
# of the population (irsed).
# save the results in sures
#
reg2 = (lm(fall~irsed+nborder+cegm,data=fall))
summary(reg2)
sures = summary(reg2)
#
# Use robust or White standard errors type 1
#
rreg2 = coeftest(reg2,vcov=hccm(reg2,type="hc1"))
#
# save df = degrees of freedom; B = estimated coefficients and the robust
# variance-covariance matrix V
#
df = sures$df
b = sures$coefficients
B = b[,1]

V = vcovHC(reg2, type = "HC1")
#
# Choose the significance level and find the appropriate critical value
#
nalpha = 0.05
t2 = qt(1-nalpha/2, df[2])
```

```

#
# Define the transformation matrix for this case to pick the last two
# coefficients
#

T = matrix(0, nrow = 2, ncol = 4)
T[1,3] = 1
T[2,4] = 1

#
# Set up for Delta standard errors
# use transformation matrix to define the vector of the numerator and denominator
# where x1 = estimated coefficient of nborder and
# x2 = estimated coefficient of cegm
# let B2 define the coefficients x1 and x2 and VV be their corresponding covariance
#

B2 = T%*%B
VV = (T%*%V)%*%t(T)
#
# call deltamethod routine to compute the approximate standard error of the ratio =
sdm
#

sdm = deltamethod(~ x1/x2,B2,VV)
tp = B2[1,1]/B2[2,1]
dfub = tp + t2*sdm
dflb = tp - t2*sdm
message("Delta Ratio = ", tp ,", Upper = ", dfub, ", Lower = ", dflb)
#
# Quadratic Solution for the Fieller
#
aa = (B2[2,1]^2) - (t2^2*VV[2,2])
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])
cc = (B2[1,1]^2) - (t2^2*VV[1,1])
rad = sqrt(bb^2 - 4*aa*cc)
lm1 = (-bb + rad) / (2 * aa)
lm2 = (-bb - rad) / (2 * aa)
message("Fieller Ratio = ", tp ,", Upper = ", lm1, ", Lower = ", lm2)

#
# Plot out p = the linear combination and its corresponding confidence
# bounds pu and pl for the graphical Fieller where n1 and n2 are the bounds
# of the possible values
#
n1 = -10
n2 = 40
x1 = seq(n1, n2, by = sign(n2-n1)*5)
x = matrix(x1,nrow=NROW(x1), ncol = 1)
p = B2[1,1] - B2[2,1]*x
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2
pu = p + t2*sqrt(vp)
pl = p - t2*sqrt(vp)

y = cbind(p,pu,pl)
matplot(x,y,type='l',col="blue",main="Border Influence on Pokies")
abline(h = 0, col = "gray60")

```

```
abline(v = dflb, col = "black")
abline(v = dfub, col = "black")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,10, "Ratio", col = "violet", adj = c(-.1, -.1))
```

## B.2 Turning Point

```
#####  
# 3.2 The Turning Point: An Environmental Kuznets Curve.  
#####  
  
panel= read.dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/HLW.dta")  
  
panel = panel[panel$POLL == "SO2",]  
  
harbp = pdata.frame(panel,index=c("ISITE","IYEAR"))  
  
reg2=(plm(lnMEAN~GDP+I(GDP^2)+DENSAT+TRADE+DEMO+factor(IYEAR),data=harbp,  
          model="random",random.method="swar"))  
summary(reg2)  
  
sures = summary(reg2)  
  
coeftest(reg2,vcovHC)  
b = sures$coefficients  
B = b[,1]  
V = vcovHC(reg2)  
  
nalpha = 0.05  
t2 = qt(1-nalpha/2, df[2])  
  
# Delta  
  
T=matrix(0,nrow=2,ncol=27)  
T[1,2] = -1  
T[2,3] = 2  
  
B2 = T%*%B  
VV = (T%*%V)%*%t(T)  
sdm = deltamethod(~ x1/x2,B2,VV)  
tp = B2[1,1]/B2[2,1]  
dfub = tp + t2*sdm  
dflb = tp - t2*sdm  
message("Delta Ratio = ", tp ,", Upper = ", dfub, ", Lower = ", dflb)  
  
# Quadratic Fieller  
  
aa = (B2[2,1]^2) - (t2^2*VV[2,2])  
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])  
cc = (B2[1,1]^2) - (t2^2*VV[1,1])  
rad = sqrt(bb^2 - 4*aa*cc)  
lm1 = (-bb + rad) / (2 * aa)  
lm2 = (-bb - rad) / (2 * aa)  
message("Fieller Ratio = ", tp ,", Upper = ", lm1, ", Lower = ", lm2)  
  
# Graphical Fieller  
  
n1 = 0.1  
n2 = 30.1  
x1 = seq(n1, n2, by = sign(n2-n1)*1)
```

```
x = matrix(x1, nrow=NROW(x1) , ncol = 1)

p = B2[1,1] - B2[2,1]*x
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2
pu = p + t2*sqrt(vp)
pl = p - t2*sqrt(vp)

y = cbind(p,pu,pl)
matplot(x,y,type='l',col="blue",main="Turning Point")
abline(h = 0, col = "gray80")
abline(v = dflb, col = "black")
abline(v = dfub, col = "black")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,.2, "Ratio", col = "violet", adj = c(-.1, -.1))
```

## B.3 Interactions

```
#####  
# 3.3 Interactions: Journal subscriptions as influenced by age and price  
#####  
  
#read txt file  
journals=  
read.dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/journal.dta")  
  
reg2 = (lm(log(oclc)~log(char)+log(price)*log(age),data=journals))  
summary(reg2)  
sures = summary(reg2)  
  
df = sures$df  
b = sures$coefficients  
vb = vcov(reg2)  
nalpha = 0.05  
t2 = qt(1-nalpha/2, df[2])  
  
# Define the mean elasticity at the mean of the age  
  
m_lage = mean(log(journals$age))  
m_elast = b[3] + b[5] * m_lage  
  
B = b[,1]  
V = vb  
  
# Delta standad errors  
# 1st define the covariance of the numerator and denominator  
  
T=matrix(0,nrow=2,ncol=5)  
T[1,3] = -1  
T[2,5] = 1  
  
A = matrix(c(m_elast,0),nrow=2,ncol=1)  
B2 = A + T%*%B  
VV = (T%*%V)%*%t(T)  
sdm = deltamethod(~ x1/x2,B2,VV)  
tp = B2[1,1]/B2[2,1]  
dfub = tp + t2*sdm  
dflb = tp - t2*sdm  
message("Delta Ratio = ", tp ,", Upper = ", dfub, ", Lower = ", dflb)  
  
# Quadratic  
  
aa = (B2[2,1]^2) - (t2^2*VV[2,2])  
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])  
cc = (B2[1,1]^2) - (t2^2*VV[1,1])  
rad = sqrt(bb^2 - 4*aa*cc)  
lm1 = (-bb + rad) / (2 * aa)  
lm2 = (-bb - rad) / (2 * aa)  
message("Fieller Ratio = ", tp ,", Upper = ", lm1, ", Lower = ", lm2)  
  
# Graphical
```

```

n1 = 1
n2 = 6
x1 = seq(n1, n2, by = 0.5)
x = matrix(x1,nrow=NROW(x1), ncol = 1)

p = -(B2[1,1]-A[1,1]) +B2[2,1]*x
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2
pu = p + t2*sqrt(vp)
pl = p - t2*sqrt(vp)

y = cbind(p,pu,pl)
matplot(x,y,type='l',col="blue",main="Journal Subscriptions")
abline(h = 0, col = "gray80")
abline(h = m_elast, col = "gray60")
abline(v = dflb, col = "black")
abline(v = dfub, col = "black")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,.15, "Ratio", col = "violet", adj = c(-.1, -.1))

```

## B.4 Long-run – Nonlinear

```
#####  
#3.4 The Long-run: A non-linear specification for energy demand  
#####  
  
commercial=  
read.dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/commercial.dta")  
  
reg2 = nls(log(teg)-b0+b1*log(lteg)+b2*log(ep)+b3*log(gdp)+b4*(log(hdd)-b1*log(lhdd)),  
          data=commercial)  
summary(reg2)  
sures=summary(reg2)  
  
df = sures$df  
b = sures$coefficients  
vb = vcov(reg2)  
  
B = b[,1]  
V = vb  
t2 = 1.96  
  
# Delta  
  
T=matrix(c(0,0,0,-1,1,0,0,0,0,0),nrow=2,ncol=5)  
A=matrix(c(0,1),nrow=2,ncol=1)  
B2 = A + T%%B  
VV = (T%%V)%%t(T)  
sdm = deltamethod(~ x1/x2,B2,VV)  
tp = B2[1,1]/B2[2,1]  
dfub = tp + t2*sdm  
dflb = tp - t2*sdm  
message("Delta Ratio = ", tp, ", Upper = ", dfub, ", Lower = ", dflb)  
  
# Quadratic Fieller  
  
aa = (B2[2,1]^2) - (t2^2*VV[2,2])  
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])  
cc = (B2[1,1]^2) - (t2^2*VV[1,1])  
rad = sqrt(bb^2 - 4*aa*cc)  
lm1 = (-bb + rad) / (2 * aa)  
lm2 = (-bb - rad) / (2 * aa)  
message("Fieller Ratio = ", tp, ", Upper = ", lm1, ", Lower = ", lm2)  
  
# Graphical Fieller  
  
n1 = -1  
n2 = 0  
x1 = seq(n1, n2, by = sign(n2-n1)*0.05)  
x = matrix(x1, nrow=NROW(x1), ncol = 1)  
  
p = B2[1,1] - B2[2,1]*x  
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2  
pu = p + t2*sqrt(vp)  
pl = p - t2*sqrt(vp)
```



```
y = cbind(p,pu,pl)
matplot(x,y,type='l',col="blue",main="Non-linear Model of Energy Demand")
abline(h = 0, col = "gray60")
abline(v = dflb, col = "black")
abline(v = dfub, col = "black")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,.3, "Ratio", col = "violet", adj = c(-.1, -.1))
```

## B.5 Long-run – Multi-lags

```
#####  
#3.5 A Multiple period lag model: Taylor Rule Coefficients  
#####  
  
greenbook=  
read.dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/greenbook2.dta")  
  
reg2 = (lm(ffr~lffr1+lffr2+pi_tp+gry_tp+gap_t03,data=greenbook))  
summary(reg2)  
sures = summary(reg2)  
  
df = sures$df  
b = sures$coefficients  
vb =NeweyWest(reg2, lag = 5, prewhite = FALSE)  
nalp = 0.05  
t2 = qt(1-nalp/2, df[2])  
  
B = b[,1]  
V = vb  
  
# Delta  
  
T=matrix(c(0,0,0,-1,0,-1,1,0,0,0,0,0),nrow=2,ncol=6)  
A=matrix(c(0,1),nrow=2,ncol=1)  
B2 = A + T%*%B  
VV = (T%*%V)%*%t(T)  
sdm = deltamethod(~ x1/x2,B2,VV)  
tp = B2[1,1]/B2[2,1]  
dfub = tp + t2*sdm  
dflb = tp - t2*sdm  
message("Delta Ratio = ", tp, ", Upper = ", dfub, ", Lower = ", dflb)  
  
# Quadratic Fieller  
  
aa = (B2[2,1]^2) - (t2^2*VV[2,2])  
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])  
cc = (B2[1,1]^2) - (t2^2*VV[1,1])  
rad = sqrt(bb^2 - 4*aa*cc)  
lm1 = (-bb + rad) / (2 * aa)  
lm2 = (-bb - rad) / (2 * aa)  
message("Fieller Ratio = ", tp, ", Upper = ", lm1, ", Lower = ", lm2)  
  
# Graphical Fieller  
  
n1 = 0.1  
n2 = 7.1  
x1 = seq(n1, n2, by = sign(n2-n1)*0.5)  
x = matrix(x1, nrow=NROW(x1), ncol = 1)  
  
p = B2[1,1] - B2[2,1]*x  
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2  
pu = p + t2*sqrt(vp)  
pl = p - t2*sqrt(vp)
```

```
y = cbind(p,pu,pl)
matplot(x,y,type='l',col="blue",main="Taylor-Rule Coefficients")
abline(h = 0, col = "gray60")
abline(v = dflb, col = "gray60")
abline(v = dfub, col = "gray60")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,.1, "Ratio", col = "violet", adj = c(-.1, -.1))
```

## B.6 NAIRU

```
#####  
# 3.6 The Non-Accelerating Inflation Rate of Unemployment (NAIRU)  
#####  
  
phillips=  
read.dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/phillips.dta")  
  
reg2 = (lm(y~x2+x3+x4+x5+x6,data=phillips))  
summary(reg2)  
sures = summary(reg2)  
  
df = sures$df  
b = sures$coefficients  
vb =NeweyWest(reg2, lag = 4, prewhite = FALSE)  
nalpha = 0.05  
t2 = qt(1-nalpha/2, df[2])  
  
B = b[,1]  
V = vb  
  
# Delta  
  
T=matrix(c(-1,0,0,0,0,1,0,0,0,0,0,0),nrow=2,ncol=6)  
B2 = T%*%B  
VV = (T%*%V)%*%t(T)  
sdm = deltamethod(~ x1/x2,B2,VV)  
tp = B2[1,1]/B2[2,1]  
dfub = tp + t2*sdm  
dflb = tp - t2*sdm  
message("Delta Ratio = ", tp ,", Upper = ", dfub, ", Lower = ", dflb)  
  
# Quadratic Fieller  
  
aa = (B2[2,1]^2) - (t2^2*VV[2,2])  
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])  
cc = (B2[1,1]^2) - (t2^2*VV[1,1])  
rad = sqrt(bb^2 - 4*aa*cc)  
lm1 = (-bb + rad) / (2 * aa)  
lm2 = (-bb - rad) / (2 * aa)  
message("Fieller Ratio = ", tp ,", Upper = ", lm1, ", Lower = ", lm2)  
  
# Graphical Fieller  
  
n1 = -15  
n2 = 10  
x1 = seq(n1, n2, by = sign(n2-n1)*1)  
x = matrix(x1,nrow=NROW(x1), ncol = 1)  
  
p = -B2[1,1] + B2[2,1]*x  
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2  
pu = p + t2*sqrt(vp)  
pl = p - t2*sqrt(vp)  
  
y = cbind(p,pu,pl)
```

```
matplot(x,y,type='l',col="blue",main="NAIRU")
abline(h = 0, col = "gray60")
abline(v = dflb, col = "black")
abline(v = dfub, col = "black")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,1, "Ratio", col = "violet", adj = c(-.1, -.1))
```

## B.7 50% Dose & Willingness-to-pay

```
#####  
# 3.7 The 50% dose and Willingness-to-Pay.  
#####  
  
will =  
read_dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/titanic1.dta")  
  
reg2 = (glm(survived-fare+age+female, family = binomial(link=probit), data=will))  
summary(reg2)  
sures = summary(reg2)  
  
b = sures$coefficients  
B=b[ ,1]  
V = vcov(reg2)  
  
t2 = 1.96  
m_age = mean(will$age)  
  
# Delta  
  
T = matrix(c(-1,0,0,1,-m_age,0,-0.5,0),nrow=2,ncol=4)  
B2 = T%*%B  
VV = (T%*%V)%*%t(T)  
sdm = deltamethod(~ x1/x2,B2,VV)  
tp = B2[1,1]/B2[2,1]  
dfub = tp + t2*sdm  
dflb = tp - t2*sdm  
message("Delta Ratio = ", tp ,", Upper = ", dfub, ", Lower = ", dflb)  
  
# Quadratic Fieller  
  
aa = (B2[2,1]^2) - (t2^2*VV[2,2])  
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])  
cc = (B2[1,1]^2) - (t2^2*VV[1,1])  
rad = sqrt(bb^2 - 4*aa*cc)  
lm1 = (-bb + rad) / (2 * aa)  
lm2 = (-bb - rad) / (2 * aa)  
message("Fieller Ratio = ", tp ,", Upper = ", lm1, ", Lower = ", lm2)  
  
# Graphical Fieller  
  
n1 = 0.1  
n2 = 101  
x1 = seq(n1, n2, by = sign(n2-n1)*5)  
x = matrix(x1, nrow=NROW(x1), ncol = 1)  
  
p = B2[1,1] - B2[2,1]*x  
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2  
pu = p + t2*sqrt(vp)  
pl = p - t2*sqrt(vp)  
  
y = cbind(p,pu,pl)  
matplot(x,y,type='l',col="blue",main="Willingness to Pay")  
abline(h = 0, col = "gray80")
```

```
abline(v = dflb, col = "gray60")
abline(v = dfub, col = "gray60")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,.3, "Ratio", col = "violet", adj = c(-.1, -.1))
```

## B.8 2SLS/IV & ILS

```
#####  
# 3.8 Indirect Least Squares as Two Stage Least Squares  
#####  
  
ils =  
read.dta("https://www.online.fbe.unimelb.edu.au/t_drive/FTS/data/maketable4.dta")  
  
ils = ils[ils$baseco == 1,]  
  
#iv estimation using a 2SLS routine  
  
iv = ivreg(logpgp95~avexpr+lat_abst+africa+asia+other_cont|  
          logem4+lat_abst+africa+asia+other_cont,data=ils)  
summary(iv)  
  
#Use an SUR regression to perform the Indirect Least Squares estimate  
  
r1 = avexpr~logem4+lat_abst+africa+asia+other_cont  
  
r2 = logpgp95~logem4+lat_abst+africa+asia+other_cont  
  
fitsur = systemfit(list(reg1 = r1, reg2 = r2), method="SUR", data=ils)  
  
summary(fitsur)  
sures = summary(fitsur,rcovformula= 1)  
  
b = sures$coefficients  
B = b[,1]  
V = sures$coefCov  
V = V*((64-6)/64)  
t2 = 1.96  
  
# Delta Construct the covariance matrix for the application of the Delta Method  
  
T=matrix(0,nrow=2,ncol=12)  
T[1,8] = 1  
T[2,2] = 1  
  
B2 = T%*%B  
VV = (T%*%V)%*%t(T)  
sdm = deltamethod(~ x1/x2,B2,VV)  
tp = B2[1,1]/B2[2,1]  
dfub = tp + t2*sdm  
dflb = tp - t2*sdm  
message("Delta Ratio = ", tp, ", Upper = ", dfub, ", Lower = ", dflb)  
  
# Quadratic Fieller  
  
aa = (B2[2,1]^2) - (t2^2*VV[2,2])  
bb = (2*t2^2*VV[1,2]) - (2*B2[1,1]*B2[2,1])  
cc = (B2[1,1]^2) - (t2^2*VV[1,1])  
rad = sqrt(bb^2 - 4*aa*cc)  
lm1 = (-bb + rad) / (2 * aa)  
lm2 = (-bb - rad) / (2 * aa)
```



```

message("Fieller Ratio = ", tp ,", Upper = ", lm1, ", Lower = ", lm2)

# Graphical Fieller
n1 = -2
n2 = 6
x1 = seq(n1, n2, by = sign(n2-n1)*.2)
x = matrix(x1,nrow=NROW(x1), ncol = 1)

p = B2[1,1] - B2[2,1]*x
vp = VV[1,1] - 2*VV[1,2]*x + VV[2,2]*x^2
pu = p + t2*sqrt(vp)
pl = p - t2*sqrt(vp)

y = cbind(p,pu,pl)
matplot(x,y,type='l',col="blue",main="2SLS and Indirect Least Squares")
abline(h = 0, col = "gray80")
abline(v = dflb, col = "black")
abline(v = dfub, col = "black")
abline(v = lm1, col = "red")
abline(v = lm2, col = "red")
abline(v = tp, col = "violet")
text(tp,3.5, "Ratio", col = "violet", adj = c(-.1, -.1))

```

## Appendix C. Results of the R-code

Note that these results are similar to those reported in the paper however there are some slight differences due to the variation in computational algorithms employed by the routines used in R and Stata.

### C.1 Parallel-line assay: A compensation interpretation of a dummy variable.

```
Call:
lm(formula = fall ~ irsed + nborder + cegm, data = fall)

Residuals:
    Min       1Q   Median       3Q      Max
-9.4510 -2.7817 -0.3157  2.4242 10.2174

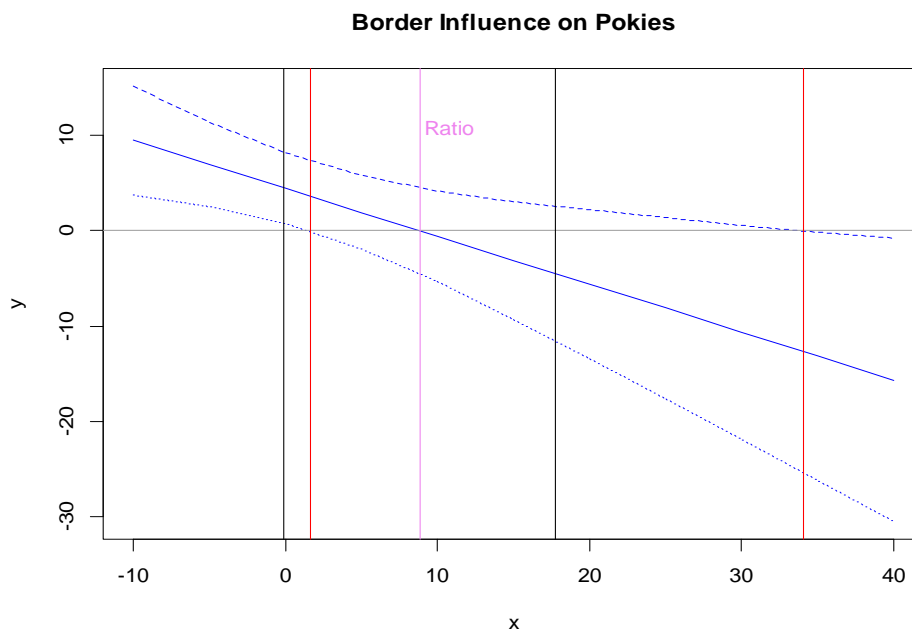
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  18.99533   10.98411    1.729  0.08926 .
    irsed     -0.03519    0.01101   -3.196  0.00229 **
    nborder    4.44539    1.41211    3.148  0.00263 **
    cegm       0.50344    0.30801    1.635  0.10776

---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.791 on 56 degrees of freedom
Multiple R-squared:  0.2325, Adjusted R-squared:  0.1913
F-statistic: 5.653 on 3 and 56 DF, p-value: 0.001865
```

Delta Ratio = 8.82998490823155, Upper = 17.7708348770884, Lower = -0.110865060625279

Fieller Ratio = 8.82998490823155, Upper = 34.0774781991443, Lower = 1.6378768354804



## C.2 The Turning Point: An Environmental Kuznets Curve.

```

Call:
lm(formula = lnMEAN ~ GDP + I(GDP^2) + DENSAT + TRADE + DEMO
    factor(1YEAR), data = harbp, model = "random", random.method = "GLS")

Unbalanced Panel: n = 267, T = 1-21, N = 2314

Effects:
              var std.dev share
idiosyncratic 0.1405 0.3748 0.268
individual    0.3842 0.6199 0.732
theta:
  Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
 0.4825 0.8024 0.8346 0.8124 0.8505 0.8692

Residuals:
  Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
-2.0883 -0.1781 0.0473 0.0120 0.2432 1.7698

Coefficients:
              Estimate Std. Error t-value Pr(>|t|)
(Intercept)  3.9574835  0.3108721 12.7303 < 2.2e-16 ***
GDP           0.1043870  0.0228283  4.5727 5.072e-06 ***
I(GDP^2)     -0.0040445  0.0010613 -3.8110 0.000142 ***
DENSAT       1.7162096  0.1716775  9.9967 < 2.2e-16 ***
TRADE        -0.0114835  0.0012246 -9.3772 < 2.2e-16 ***
DEMO         -0.0636345  0.0078049 -8.1532 5.776e-16 ***
factor(1YEAR)1972 0.1492774  0.2997525  0.4980 0.618530
factor(1YEAR)1973 -0.2183800  0.2979858 -0.7329 0.463723
factor(1YEAR)1974 -0.2383359  0.2978428 -0.8002 0.423674
factor(1YEAR)1975 -0.2901636  0.2966532 -0.9781 0.328117
factor(1YEAR)1976 -0.2637995  0.2956207 -0.8924 0.372295
factor(1YEAR)1977 -0.3186591  0.2954778 -1.0785 0.280945
factor(1YEAR)1978 -0.3749915  0.2950875 -1.2708 0.203936
factor(1YEAR)1979 -0.3836101  0.2952551 -1.2992 0.193989
factor(1YEAR)1980 -0.4806125  0.2953162 -1.6275 0.103779
factor(1YEAR)1981 -0.4959254  0.2953532 -1.6791 0.093271 .
factor(1YEAR)1982 -0.5036038  0.2949314 -1.7075 0.087860 .
factor(1YEAR)1983 -0.6452449  0.2953724 -2.1845 0.029026 *
factor(1YEAR)1984 -0.5922295  0.2958170 -2.0020 0.045401 *
factor(1YEAR)1985 -0.6223916  0.2963136 -2.1004 0.035799 *
factor(1YEAR)1986 -0.5915269  0.2965173 -1.9949 0.046171 *
factor(1YEAR)1987 -0.6740175  0.2969677 -2.2697 0.023321 *
factor(1YEAR)1988 -0.6814608  0.2977566 -2.2887 0.022190 *
factor(1YEAR)1989 -0.6411864  0.2984960 -2.1481 0.031814 *
factor(1YEAR)1990 -0.6375782  0.2989144 -2.1330 0.033033 *
factor(1YEAR)1991 -0.7023400  0.2983610 -2.3540 0.018657 *
factor(1YEAR)1992 -0.9176766  0.3000440 -3.0585 0.002250 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Total Sum of Squares:    504.81
Residual Sum of Squares: 356.75
R-Squared:               0.2974
Adj. R-Squared:         0.28941
F-statistic: 36.5076 on 26 and 2287 DF, p-value: < 2.22e-16

```

t test of coefficients:

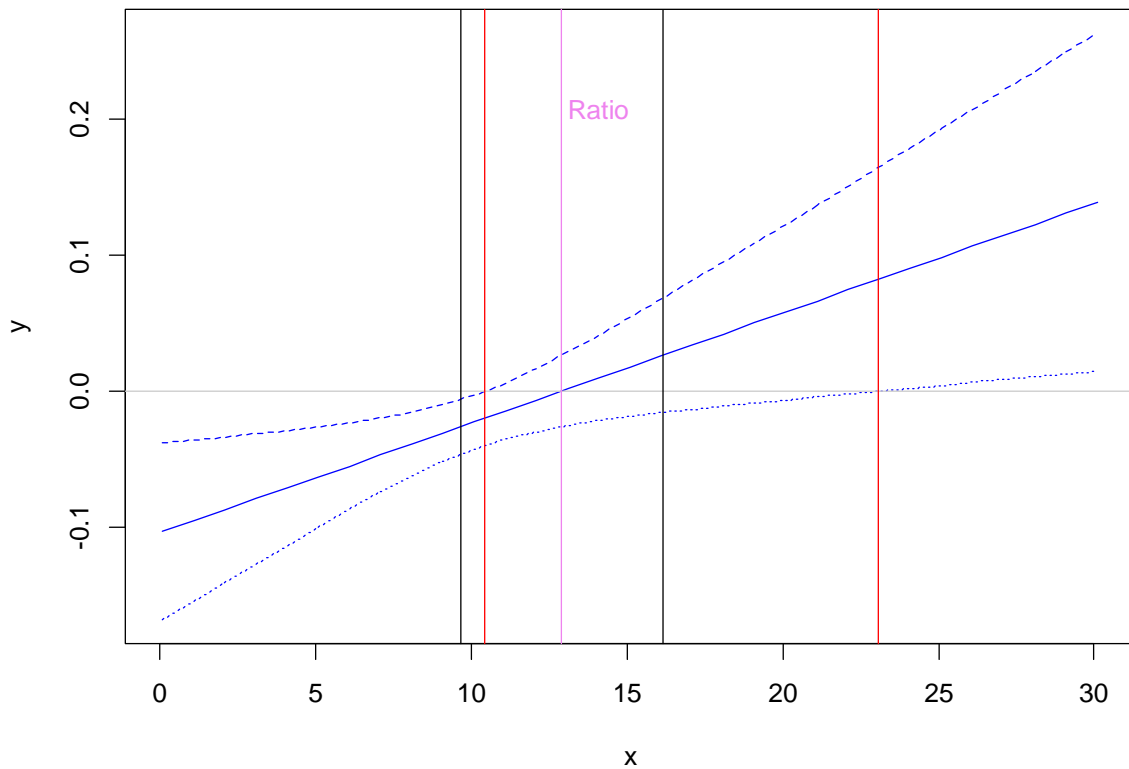
	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	3.9574835	0.1672150	23.6670	< 2.2e-16	***
GDP	0.1043870	0.0327833	3.1841	0.001471	**
I (GDP^2)	-0.0040445	0.0015400	-2.6263	0.008690	**
DENS NAT	1.7162096	0.1958956	8.7608	< 2.2e-16	***
TRADE	-0.0114835	0.0017637	-6.5109	9.156e-11	***
DEMO	-0.0636345	0.0096796	-6.5741	6.044e-11	***
factor(I YEAR) 1972	0.1492774	0.0884269	1.6881	0.091520	.
factor(I YEAR) 1973	-0.2183800	0.0800416	-2.7283	0.006414	**
factor(I YEAR) 1974	-0.2383359	0.0868050	-2.7456	0.006086	**
factor(I YEAR) 1975	-0.2901636	0.0871674	-3.3288	0.000886	***
factor(I YEAR) 1976	-0.2637995	0.0815608	-3.2344	0.001236	**
factor(I YEAR) 1977	-0.3186591	0.0815041	-3.9097	9.509e-05	***
factor(I YEAR) 1978	-0.3749915	0.0808196	-4.6399	3.683e-06	***
factor(I YEAR) 1979	-0.3836101	0.0824185	-4.6544	3.435e-06	***
factor(I YEAR) 1980	-0.4806125	0.0868985	-5.5307	3.552e-08	***
factor(I YEAR) 1981	-0.4959254	0.0877616	-5.6508	1.795e-08	***
factor(I YEAR) 1982	-0.5036038	0.0863480	-5.8323	6.243e-09	***
factor(I YEAR) 1983	-0.6452449	0.0886748	-7.2765	4.685e-13	***
factor(I YEAR) 1984	-0.5922295	0.0939825	-6.3015	3.526e-10	***
factor(I YEAR) 1985	-0.6223916	0.0995285	-6.2534	4.778e-10	***
factor(I YEAR) 1986	-0.5915269	0.0989979	-5.9751	2.659e-09	***
factor(I YEAR) 1987	-0.6740175	0.1021767	-6.5966	5.210e-11	***
factor(I YEAR) 1988	-0.6814608	0.1130740	-6.0267	1.945e-09	***
factor(I YEAR) 1989	-0.6411864	0.1194571	-5.3675	8.789e-08	***
factor(I YEAR) 1990	-0.6375782	0.1224223	-5.2080	2.079e-07	***
factor(I YEAR) 1991	-0.7023400	0.1264195	-5.5556	3.087e-08	***
factor(I YEAR) 1992	-0.9176766	0.1308926	-7.0109	3.104e-12	***

---  
 Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Delta Ratio = 12.9048459139564, Upper = 16.149712559329, Lower = 9.65997926858378

Fieller Ratio = 12.9048459139564, Upper = 23.0723086627591, Lower = 10.4285053715352

### Turning Point



### C.3 Interactions: Journal subscriptions as influenced by age and price

```
Call:
lm(formula = log(oclc) ~ log(char) + log(price) * log(age), data = journals)

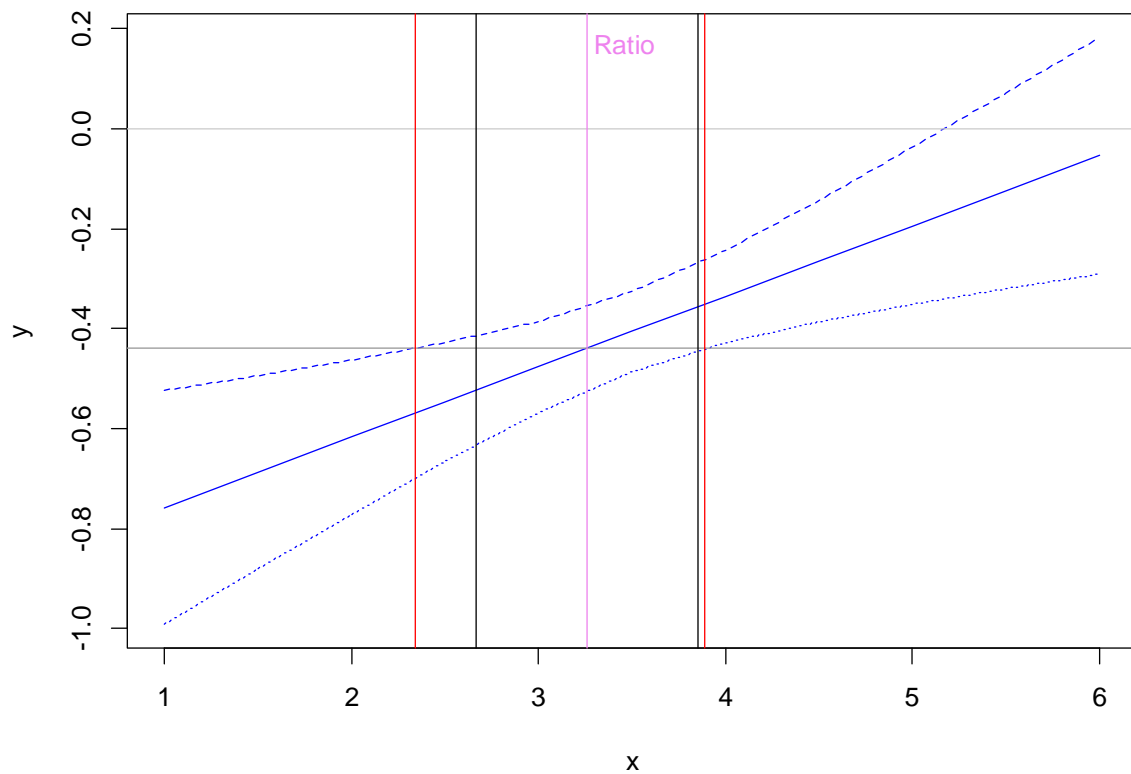
Residuals:
    Min       1Q   Median       3Q      Max
-2.19007 -0.43541  0.05901  0.39276  1.75026

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    3.43352    0.31486   10.905 < 2e-16 ***
log(char)       0.22947    0.10513    2.183  0.03038 *
log(price)     -0.89891    0.16151   -5.566 9.66e-08 ***
log(age)        0.37351    0.08897    4.198 4.28e-05 ***
log(price):log(age) 0.14096    0.04483    3.144  0.00196 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.6876 on 175 degrees of freedom
Multiple R-squared:  0.6339, Adjusted R-squared:  0.6255
F-statistic: 75.75 on 4 and 175 DF, p-value: < 2.2e-16
```

Delta Ratio = 3.25999621904176, Upper = 3.85029839682713, Lower = 2.66969404125639  
 Fieller Ratio = 3.25999621904176, Upper = 3.88618136536722, Lower = 2.34161981074442

**Journal Subscriptions**



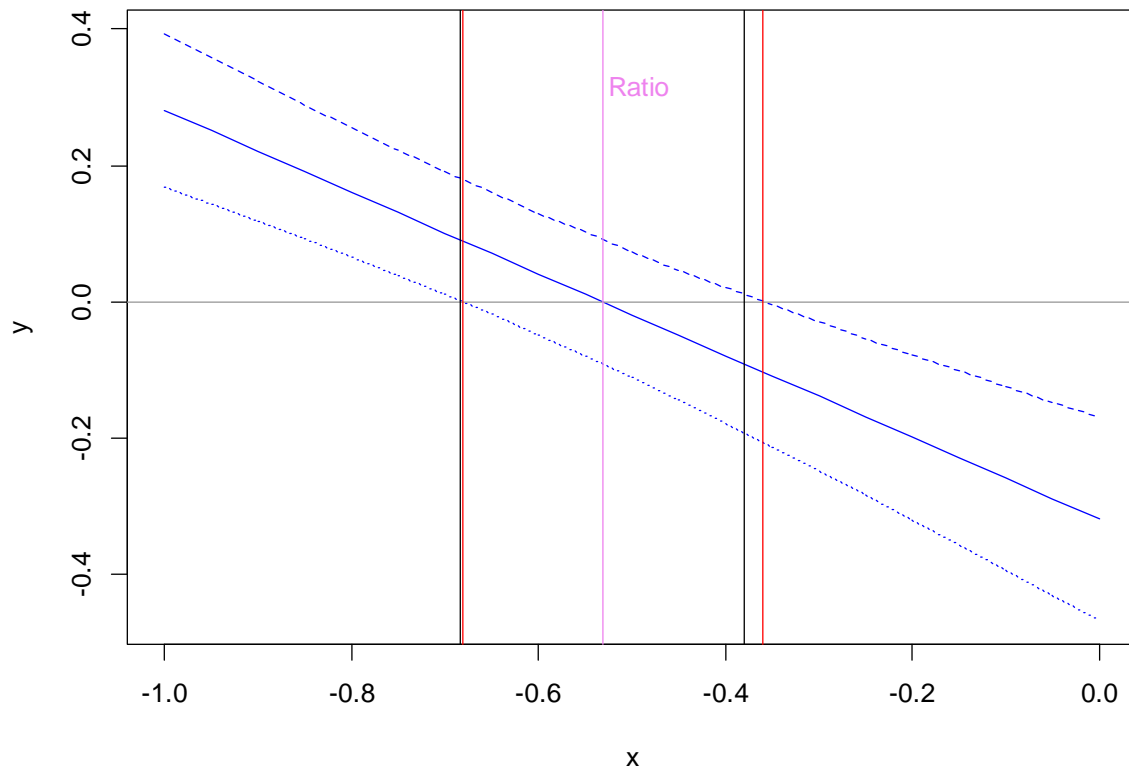
## C.4 The Long-run: A non-linear specification for energy demand

Formula: $\log(\text{teg}) \sim b_0 + b_1 * \log(\text{l teg}) + b_2 * \log(\text{ep}) + b_3 * \log(\text{gdp}) + b_4 * (\log(\text{hdd}) - b_1 * \log(\text{l hdd}))$					
Parameters:					
	Estimate	Std. Error	t value	Pr(> t )	
b0	1.83286	0.41121	4.457	9.04e-05	***
b1	0.40062	0.09666	4.145	0.000223	***
b2	-0.31875	0.07620	-4.183	0.000199	***
b3	0.54630	0.12236	4.465	8.85e-05	***
b4	0.46036	0.16990	2.710	0.010598	*
---					
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1					
Residual standard error: 0.05505 on 33 degrees of freedom					
Number of iterations to convergence: 3					
Achieved convergence tolerance: 7.465e-06					

Delta Ratio = -0.531795574610913, Upper = -0.379724120962985, Lower = -0.68386702825884

Fieller Ratio = -0.531795574610913, Upper = -0.359933282023237, Lower = -0.681289642962219

**Non-linear Model of Energy Demand**



## C.5 A Multiple period lag model: Taylor Rule Coefficients

```

Call:
lm(formula = ffr ~ lffr1 + lffr2 + pi_tp + gry_tp + gap_t03,
    data = greenbook)

Residuals:
    Min       1Q   Median       3Q      Max
-1.25825 -0.15402 -0.00395  0.14497  1.08349

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.39274    0.11398   -3.446  0.000736 ***
lffr1        1.27963    0.07357  17.393 < 2e-16 ***
lffr2       -0.33936    0.07165   -4.736  4.96e-06 ***
pi_tp        0.15131    0.04459    3.393  0.000881 ***
gry_tp       0.13021    0.02901    4.488  1.41e-05 ***
gap_t03      0.03548    0.01131    3.136  0.002054 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

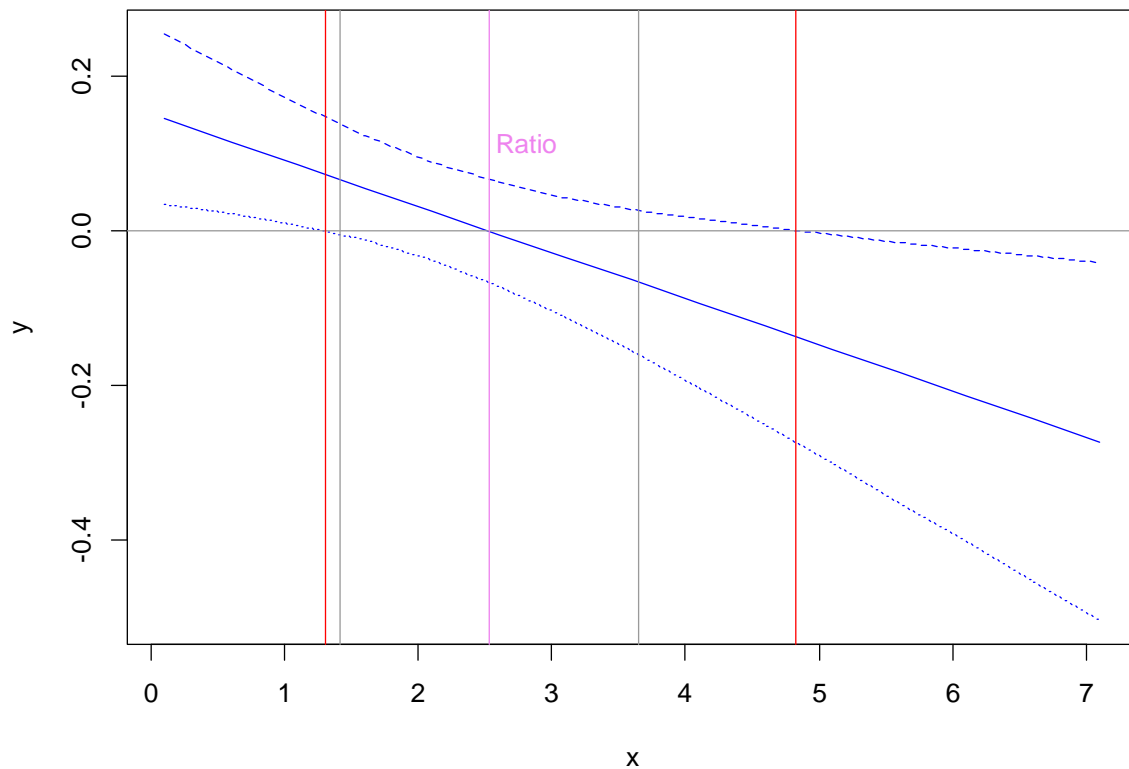
Residual standard error: 0.3082 on 152 degrees of freedom
Multiple R-squared:  0.9817, Adjusted R-squared:  0.9811
F-statistic: 1635 on 5 and 152 DF,  p-value: < 2.2e-16

```

Delta Ratio = 2.53329686184319, Upper = 3.64796783895622, Lower = 1.41862588473017

Fieller Ratio = 2.53329686184319, Upper = 4.83084367757339, Lower = 1.30415497218433

**Taylor-Rule Coefficients**



## C.6 The Non-Accelerating Inflation Rate of Unemployment (NAIRU)

```

Call:
lm(formula = y ~ x2 + x3 + x4 + x5 + x6, data = philips)

Residuals:
    Min       1Q   Median       3Q      Max
-3.3404 -0.6421  0.0033  0.8064  2.2232

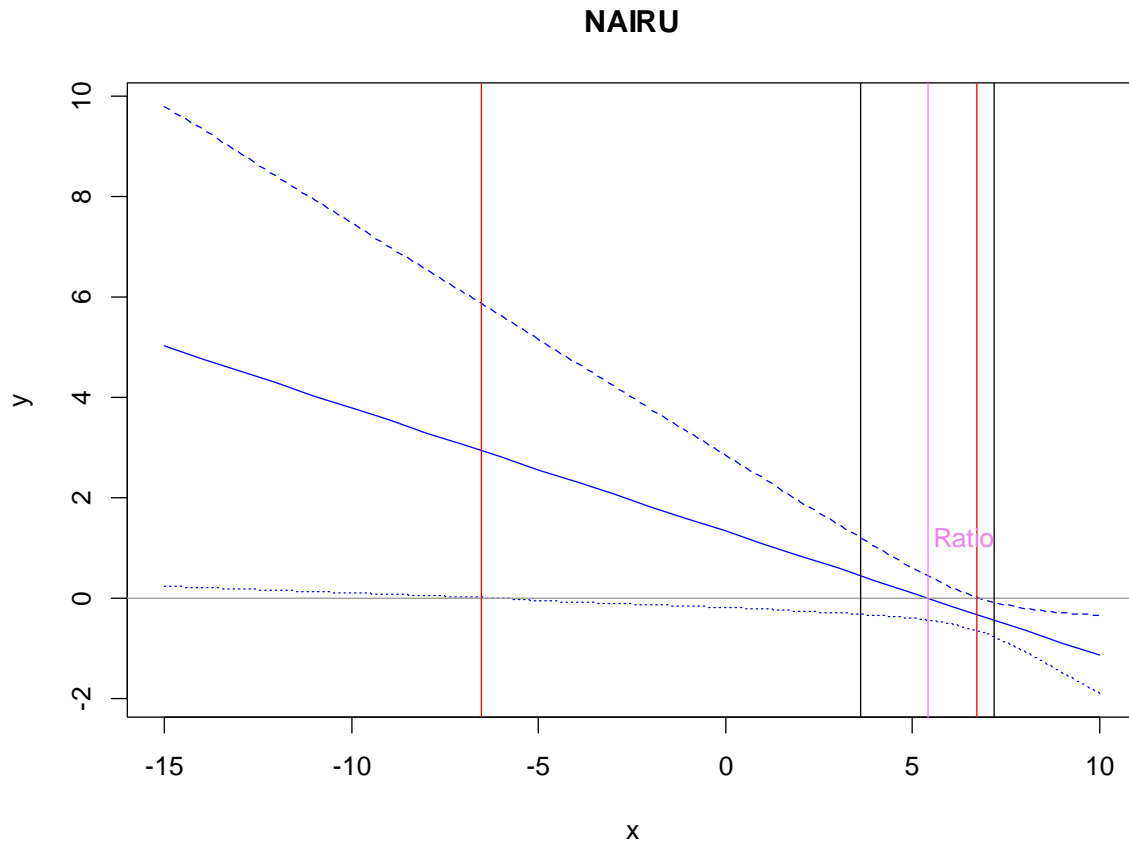
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.32799    0.83375   1.593   0.1157
x2           0.16716    0.07790   2.146   0.0354 *
x3          -0.24589    0.11118  -2.212   0.0303 *
x4          -0.28008    0.47844  -0.585   0.5602
x5           0.58431    0.07351   7.948  2.27e-11 ***
x6           0.55623    0.10064   5.527  5.24e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.151 on 70 degrees of freedom
Multiple R-squared:  0.6937, Adjusted R-squared:  0.6718
F-statistic: 31.71 on 5 and 70 DF, p-value: < 2.2e-16

```

Delta Ratio = 5.40076886862696, Upper = 7.19837251143356, Lower = 3.60316522582036

Fieller Ratio = 5.40076886862696, Upper = 6.71311464385227, Lower = -6.51997031531996





## C.7 The 50% dose and Willingness-to-Pay.

```

Call:
glm(formula = survived ~ fare + age + female, family = binomial(link = probit),
     data = will)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.3546 -0.6480 -0.5934  0.7935  2.0477

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.815273   0.108903  -7.486 7.09e-14 ***
fare         0.005959   0.001097   5.431 5.60e-08 ***
age        -0.006552   0.003156  -2.076  0.0379 *
female      1.440329   0.091132  15.805 < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

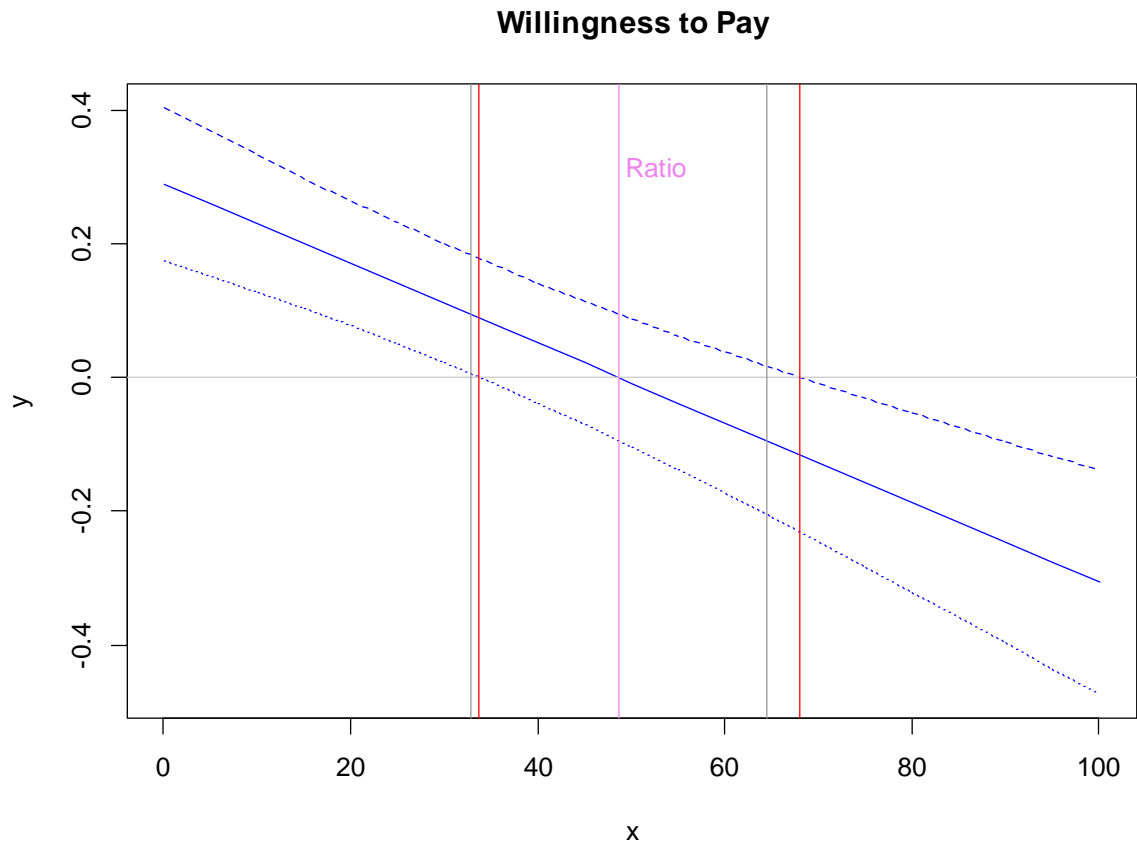
(Dispersion parameter for binomial family taken to be 1)

Null deviance: 1396.5 on 1032 degrees of freedom
Residual deviance: 1051.1 on 1029 degrees of freedom
AIC: 1059.1

Number of Fisher Scoring iterations: 4
    
```

Delta Ratio = 48.6780190053305, Upper = 64.5370334401383, Lower = 32.8190045705227

Fieller Ratio = 48.6780190053305, Upper = 68.050784734777, Lower = 33.7514498861174



## C.8 Indirect Least Squares as Two Stage Least Squares

```

Call:
ivreg(formula = logppp95 ~ avexpr + lat_abst + africa + asia +
      other_cont | logem4 + lat_abst + africa + asia + other_cont,
      data = iis)

Residuals:
    Min       1Q   Median       3Q      Max
-2.7155 -0.6381 -0.1535  0.8188  2.0714

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   1.4405     2.8396   0.507  0.6139
avexpr        1.1071     0.4636   2.388  0.0202 *
lat_abst      -1.1782     1.7554  -0.671  0.5048
africa        -0.4373     0.4242  -1.031  0.3069
asia          -1.0471     0.5246  -1.996  0.0506 .
other_cont    -0.9904     0.9980  -0.992  0.3251
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.082 on 58 degrees of freedom
Multiple R-Squared:  0.01082, Adjusted R-squared:  -0.07445
Wald test: 6.847 on 5 and 58 DF, p-value: 4.418e-05

```

```

systemfit results
method: SUR

              N DF      SSR detRCov  OLS-R2  McElroy-R2
system 128 116 119.89 0.454815 0.413648 0.442259

              N DF      SSR      MSE      RMSE      R2  Adj R2
reg1  64  58  91.3531 1.575054 1.25501 0.327724 0.269769
reg2  64  58  28.5371 0.492018 0.70144 0.583897 0.548026

The covariance matrix of the residuals used for estimation
      reg1      reg2
reg1 1.575054 0.565809
reg2 0.565809 0.492018

The covariance matrix of the residuals
      reg1      reg2
reg1 1.575054 0.565809
reg2 0.565809 0.492018

The correlations of the residuals
      reg1      reg2
reg1 1.000000 0.642735
reg2 0.642735 1.000000

SUR estimates for 'reg1' (equation 1)
Model Formula: avexpr ~ logem4 + lat_abst + africa + asia + other_cont

              Estimate Std. Error t value Pr(>|t|)
(Intercept)  7.729295   0.956922  8.07724 4.5465e-11 ***
logem4       -0.340318   0.183072 -1.85892 0.068114 .
lat_abst     2.009322   1.390869  1.44465 0.153936
africa       -0.257958   0.409618 -0.62975 0.531329
asia         0.471988   0.502958  0.93842 0.351918
other_cont   1.061534   0.843699  1.25819 0.213364
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.255011 on 58 degrees of freedom
Number of observations: 64 Degrees of Freedom: 58
SSR: 91.353114 MSE: 1.575054 Root MSE: 1.255011
Multiple R-Squared: 0.327724 Adjusted R-Squared: 0.269769

SUR estimates for 'reg2' (equation 2)
Model Formula: logppp95 ~ logem4 + lat_abst + africa + asia + other_cont

              Estimate Std. Error t value Pr(>|t|)
(Intercept)  9.997378   0.534835 18.69246 < 2.22e-16 ***
logem4       -0.376758   0.102321 -3.68211 0.00050938 ***
lat_abst     1.046296   0.777372  1.34594 0.18355952
africa       -0.722846   0.228940 -3.15736 0.00252645 **
asia         -0.524558   0.281109 -1.86603 0.06709644 .
other_cont   0.184798   0.471553  0.39189 0.69657337
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.70144 on 58 degrees of freedom
Number of observations: 64 Degrees of Freedom: 58
SSR: 28.537062 MSE: 0.492018 Root MSE: 0.70144
Multiple R-Squared: 0.583897 Adjusted R-Squared: 0.548026

```

Delta Ratio = 1.10707698974693, Upper = 1.97204025006263, Lower = 0.24211372943123

Fieller Ratio = 1.10707698974693, Upper = -199.84868142035, Lower = 0.60934959494866

### 2SLS and Indirect Least Squares

