ON NUMERICAL EVALUATION OF FINITE TIME RUIN PROBABILITIES

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Summary

In this paper we review three algorithms to calculate the probability of ruin in finite time for the classical risk model. We discuss the computational aspects of these algorithms and consider the question of which algorithm should be preferred.

1. Introduction

In the classical risk model the insurer's surplus at time t is

$$U_c(t) = u + ct - S_c(t)$$

where u is the initial surplus (ie. the surplus at time zero), c is the rate of premium income per unit time and $S_c(t)$ denotes aggregate claims up to time t. The aggregate claims process $\{S_c(t)\}_{t,0}$ is a compound Poisson process, and, without loss of generality, we can let the Poisson parameter equal 1. Without loss of generality we can also let the distribution of individual claim amounts have mean 1. Thus, we can write $c = 1 + \mu$, where μ is the premium loading factor. The probability of survival to time t is defined as

$$\pm_c(u,t) = Pr(U_c(\neg) > 0 \text{ for all } \neg, 0 < \neg \cdot t)$$

In this paper we shall comment on three methods of approximating $\pm_c(u,t)$ which have appeared in the literature in recent times. Each method is based on the same discrete time risk model. In the discrete model, aggregate claims per unit time have a compound Poisson distribution with Poisson parameter $1=(1+\mu)^{-}$, where $^{-}$ is a positive integer, and individual claim amounts are distributed on the non-negative integers with mean $^{-}$. We denote by $G(k)$ and g_k the distribution function and probability function respectively of aggregate claims per unit time. The probability function can be calculated easily using Panjer's (1981) recursion formula. The premium income per unit time in this model is 1. If we let the initial surplus, u, be an integer, then the surplus process moves on the integers and the probability of survival to (integer) time t is

$$\pm(u,t) = Pr(U(\neg) > 0 \text{ for all } \neg, \neg = 1,2,\ldots,t)$$

where $U(\neg)$ denotes the surplus at time $\neg$.

Dickson and Waters (1991) explain why a reasonable approximation to $\pm_c(u,t)$ is

$$\pm_c(u,t) \approx \pm(^{-}u, (1+\mu)^{-}t)$$

In the next section we set out three methods of calculating $\pm(u,t)$. In Section 3 we discuss some computational aspects and in the final section we consider the question of which method should be preferred.

2. Three Methods of Calculating $\psi(u,t)$

2.1. The Method of de Vylder and Goovaerts

De Vylder and Goovaerts (1988) use a slightly different definition of survival to the one given in the previous section. However, this does not affect the application of their method to the discrete time model described in the previous section. Their approach, which considers the function $\tilde{\psi}(u,t) = 1 - \psi(u,t)$, is as follows. For $u = 0,1,2,\ldots$

$$\tilde{\psi}(u,1) = 1 - G(u)$$

i.e. ruin occurs at the end of the first time period if the aggregate claim amount in that time period is strictly greater than u. For $u = 0,1,2,\ldots$ and $t = 2,3,4,\ldots$

$$\tilde{\psi}(u,t) = \tilde{\psi}(u,1) + \sum_{j=0}^{u} g_j \tilde{\psi}(u+1-j, t-1)$$

i.e. if ruin occurs by time t, either the aggregate claim amount in the first time period causes ruin, or else the aggregate claim amount is $j \le u$, and ruin occurs in the next $t-1$ time periods from the resulting surplus $u+1-j$.

The number of computations involved in this algorithm can be reduced by means of a neat truncation procedure. Let $\varepsilon > 0$ be small and let k_0 be the least integer such that $G(k_0) \ge 1 - \varepsilon$. Now define

$$g_j^{\varepsilon} = \begin{cases} g_j & \text{for } j = 0,1,2,\ldots,k_0 \\ 0 & \text{for } j = k_0+1, k_0+2,\ldots \end{cases}$$

and let

$$\tilde{\psi}_{\varepsilon}(u,1) = \begin{cases} \tilde{\psi}(u,1) & \text{if } u \le k_0 \\ 0 & \text{if } u > k_0 \end{cases}$$

For $t = 2,3,4,\ldots$ define

$$\tilde{\psi}_{\varepsilon}(u,t) = \tilde{\psi}_{\varepsilon}(u,1) + \sum_{j=0}^{u} g_j^{\varepsilon} \tilde{\psi}_{\varepsilon}(u+1-j, t-1)$$

provided that the calculated value is at least ε. Otherwise define $\tilde{\psi}_{\varepsilon}(u,t)$ to be 0. We can easily apply the ideas of de Vylder and Goovaerts (1988) to show that

$$\tilde{\psi}(u,t) - 3\varepsilon t \le \tilde{\psi}_{\varepsilon}(u,t) \le \tilde{\psi}(u,t)$$

In our illustrations in the next section, we will calculate $\tilde{\psi}_{\varepsilon}(u,t)$ since we can control the error introduced by the truncation. For a given value of t, we will always set $\varepsilon = 10^{-4}/3t$ so that the error introduced by truncation is no more than 1 in the fourth decimal place. (The rescaling which we will introduce in Section 3 produces

approximations to $\tilde{A}_c(u,t)$ which are mostly correct to 4 decimal places, so this choice of ε should not have much effect on the accuracy of the approximation.)

We comment that the above prescription is not quite that given by de Vylder and Goovaerts (1988). It is a modification of their approach which produces better approximations to $\tilde{A}_c(u,t)$ or $\pm_c(u,t)$. However, all the ideas presented above come from their paper.

2.2. The Method of Dickson and Waters

Dickson and Waters (1991) also consider the aggregate claim amount in the first time period and apply the formula

$$\pm(u-1,t+1) = \sum_{j=0}^{u+1} g_j \pm(u-j,t)$$

for $u > 1$ and $t > 0$ to get

$$\pm(u,t) = g_0^{-1}\left[\pm(u-1,t+1) - \sum_{j=1}^{u+1} g_j \pm(u-j,t)\right]$$

with

$$\pm(1,t) = g_0^{-1}\pm(0,t+1)$$

Recursive calculation of $\pm(u,t)$ is then possible if values of $\pm(0,\tau)$ can be calculated for $\tau = t+1, t+2, \ldots, t+u$. Dickson and Waters (1991) show that

$$\pm(0,\tau) = \sum_{j=0}^{\tau+1}\left[1 - \frac{j}{\tau}\right] g(j,\tau) \qquad (2.1)$$

where $g(j,\tau)$ is the probability that aggregate claims over a time interval of length τ equal j, $j = 0,1,2,\ldots$.

This method requires values of $\pm(0,\tau)$ for $\tau = t+1, t+2, \ldots, t+u$. In turn, for each of these values of τ we require values of $g(j,\tau)$ which is the probability function of a compound Poisson random variable. At first sight it is tempting to compute such values via Panjer's recursion formula. However, this is computationally inefficient. As noted by Dickson and Egídio dos Reis (1996) who consider a related problem, it is better to use convolutions. Suppose we have calculated $\pm(0,\tau)$. This means we have values of $g(j,\tau)$ for $j = 0,1,2,\ldots,\tau-1$. To calculate $\pm(0,\tau+1)$ we require values of $g(j,\tau+1)$ for $j = 0,1,2,\ldots,\tau$. For $i = 0,1,2,\ldots,\tau-1$ we can calculate $g(i,\tau+1)$ as

$$g(i,\tau+1) = \sum_{k=0}^{i} g(k,1)\,g(i-k,\tau)$$

and then we can use Panjer's recursion formula to calculate $g(\tau,\tau+1)$. This approach must involve calculating values of $g(j,1)$, but these can be calculated efficiently by

Panjer's recursion formula and the whole procedure of calculating $g(j,\zeta)$ for successive values of ζ is much more efficient in terms of the number of operations required to calculate the probability function of aggregate claims over successive time periods. Similarly, to start proceedings, we need values of $f(j,t+1)$ to calculate $\phi(0,t+1)$ and these are calculated most efficiently by Panjer's recursion formula.

A disadvantage of this algorithm is that it is numerically unstable, ie. after a large number of computations, it produces computed values of survival probabilities outside the interval $[0,1]$. Dickson and Waters (1991) propose a pragmatic solution to this problem. They note that

$$0 \leq \max\{\phi(u,t+1), \phi(u-1,t)\} \leq \phi(u,t) \leq 1 \qquad (2.2)$$

and constrain the calculated survival probabilities to behave in this way. In our numerical illustrations in the next section we shall apply this constraint to allow a comparison of computer run times. We will comment on the effect of this constraint on approximations to $\phi_\alpha(u,t)$ in Section 4.

2.3. The Prabhu/Seal Formula

We can apply the arguments used to derive the Prabhu/Seal formula for the classical risk model (see Prabhu (1961) and, for example, Seal (1978)) to write down the corresponding formula for our discrete model. If the aggregate claim amount at time t is less than $u+t$ (so that the surplus at time t is greater than 0) then either

(i) the surplus has been above 0 at times $\zeta = 1,2,\ldots,t$, or else

(ii) at some time $j \leq t-1$ the surplus was 0 (the probability of which is $g(u+j,j)$) and in the remaining time period of length $t-j$ the surplus remained above 0 (the probability of which is $\phi(0,t-j)$).

Combining the probabilities of the above events we have

$$G(u+t-1,t) = \phi(u,t) + \sum_{j=1}^{u+t-1} g(u+j,j)\phi(0,t-j)$$

where $G(j,t)$ denotes the distribution function of aggregate claims over a time interval of length t. Hence

$$\phi(u,t) = G(u+t-1,t) - \sum_{j=1}^{u+t-1} g(u+j,j)\phi(0,t-j) \qquad (2.3)$$

with $\phi(0,\zeta)$ given by (2.1).

Kling and Goovaerts (1991) approximate $\phi_\alpha(u,t)$ by discretising the continuous time version of (2.3). However, allowing for rescaling of time units and claim amounts, the formulae from which they calculate their approximations are identical to (2.1) and (2.3).

3. Comparisons

In the previous section we have summarised three algorithms for calculating $\Psi(u,t)$ or $\phi(u,t)$. Each of the formulae are exact, and, in principle, each method should give the same solution as the others. (The algorithm for $\Psi_1(u,t)$ may, of course, give a different solution.) In the tables below we show approximations to $\phi_\alpha(u,t)$ calculated by the Prabhu/Seal formula. We have chosen this method simply because the calculations are unaffected by truncation or by numerical stability.

Table 1, Exponential Claims

		t = 10	t = 30	t = 50	t = 100	t = 500
u = 0	dVG	0	1	3	11	163
	DW	0	0	0	0	3
	P/S	0	0	0	0	3
	$\phi_\alpha(u,t)$	0.2146	0.1480	0.1284	0.1100	0.0925
u = 10	dVG	0	2	4	13	170
	DW	0	2	4	17	383
	P/S	0	3	11	70	7230
	$\phi_\alpha(u,t)$	0.9681	0.8758	0.8163	0.7394	0.6435
u = 20	dVG	0	2	5	14	172
	DW	2	5	11	36	778
	P/S	1	5	16	89	8075
	$\phi_\alpha(u,t)$	0.9996	0.9908	0.9754	0.9396	0.8629
u = 30	dVG	1	3	6	16	175
	DW	4	10	20	59	1163
	P/S	2	9	23	110	8127
	$\phi_\alpha(u,t)$	1	0.9996	0.9978	0.9890	0.9488
u = 40	dVG	1	3	6	17	180
	DW	9	18	32	87	1741
	P/S	3	13	32	133	8400
	$\phi_\alpha(u,t)$	1	1	0.9999	0.9984	0.9815
u = 50	dVG	1	3	6	17	183
	DW	17	29	48	121	2334
	P/S	4	18	42	160	8972
	$\phi_\alpha(u,t)$	1	1	1	0.9998	0.9936

Tables 1 and 2 show computing times in seconds for each of the three algorithms - denoted in the tables by dVG, DW and P/S respectively - for a range of values of u and t. All computations were carried out on a Digital AlphaServer 4000 2/300MHz CPU. For Table 1, the individual claim amount distribution in the classical risk model is exponential, whereas in Table 2 it is Pareto(2,1). In each set of calculations, the value of μ is 0.1. To apply the algorithms of the previous section, we rescaled these

distributions to have mean 20, then discretised the rescaled distributions using the method described by de Vylder and Goovaerts (1988). We chose a mean of 20 as this level of rescaling is sufficient to produce very accurate approximations to $\psi_c(u,t)$ - see Dickson and Waters (1991, Table 5).

Table 2, Pareto Claims

		t = 10	t = 30	t = 50	t = 100	t = 500
u = 0	dVG	0	1	6	46	6524
	DW	0	0	0	0	4
	P/S	0	0	0	0	4
	$\psi_c(u,t)$	0.3061	0.2186	0.1886	0.1568	0.1126
u = 10	dVG	0	3	9	60	6336
	DW	0	2	4	16	49
	P/S	0	3	9	60	6228
	$\psi_c(u,t)$	0.9068	0.7826	0.7117	0.6180	0.4595
u = 20	dVG	1	5	14	76	6764
	DW	2	5	11	36	756
	P/S	1	5	14	77	6410
	$\psi_c(u,t)$	0.9722	0.9143	0.8672	0.7878	0.6136
u = 30	dVG	1	7	20	94	7274
	DW	4	10	20	59	1312
	P/S	1	7	20	94	6728
	$\psi_c(u,t)$	0.9877	0.9591	0.9312	0.8745	0.7127
u = 40	dVG	2	11	27	115	7787
	DW	9	18	32	87	1580
	P/S	3	11	27	115	7108
	$\psi_c(u,t)$	0.9932	0.9773	0.9605	0.9217	0.7814
u = 50	dVG	3	15	36	137	7409
	DW	16	29	47	121	2010
	P/S	3	15	36	138	7806
	$\psi_c(u,t)$	0.9957	0.9858	0.9751	0.9484	0.8308

We make the following points about the computer runtimes in Tables 1 and 2:

(i) When u = 0 it is clearly better to calculate values of $\psi(u,t)$ from formula (2.1). Even with t as large as 500, the runtime is trivial.

(ii) There is a marked difference for the runtimes under the method of de Vylder and Goovaerts in the two tables. The reason for this is simple. The truncation procedure has no impact when the individual claim amount distribution is Pareto. In each calculation, k_0 exceeded the largest value of j for which we had to calculate g_j in order to apply the method. Recall that we have set $\varepsilon = 10^{-4}/3t$.

If we increase " by a factor of 10, the truncation procedure still has no impact, resulting in no change in runtime or accuracy of the approximation.

(iii) In Table 1, it is clear that the method of de Vylder and Goovaerts is the most efficient in terms of runtimes for u > 0. However, this is not the case in Table 2. Their method requires runtimes that are broadly comparable with those for the Prabhu/Seal formula, but are much greater than those required for the method of Dickson and Waters.

4. Conclusions

Based on the numbers presented in the previous section, we can draw the following conclusions:

(i) When u = 0, calculation of $\psi(u,t)$ by formula (2.1) is clearly best.

(ii) When the individual claim amount distribution has a light tail, the method of de Vylder and Goovaerts will be the most efficient computationally, as the truncation procedure will have a reflect.

(iii) There seems to be little reason to use the Prabhu/Seal formula, although it is marginally superior to the method of de Vylder and Goovaerts in Table 2 when t = 500.

(iv) The method of Dickson and Waters appears to be computationally most efficient in Table 2, at least for large values of t. However, Dickson and Waters (1991) observe that the algorithm appears to be unstable for values of u greater than about 30. In Tables 1 and 2, the values shown for $u \leq 30$ arise from the application of the constraint given by (2.2). When u = 30, the calculated values are virtually identical to four decimal places to those shown in the tables. However, as u increases, the error can increase, by as much as 2.5% of the true value of $\psi(u,t)$. We would therefore recommend the use of this algorithm only when $u < 30$. (Recall that we have been working in a framework where the mean individual claim amount is 1).

In choosing which algorithm to apply, we should also consider whether we want to know the value of the survival probability for a given value of u and t, or whether we are interested in a range of values. In calculating $\psi(u,t)$ the method of de Vylder and Goovaerts also gives values of $\psi(!,¿)$ for $1 \cdot ¿ \cdot t ¡ 1$ and for $1 \cdot ! \cdot u + t ¡ ¿$, whereas the method of Dickson and Waters gives values of $\psi(!,¿)$ for $0 \cdot ! \cdot u ¡ 1$ and for $t \cdot ¿ \cdot t + u ¡ !$, excluding $\psi(0;t)$. By contrast, all the Prabhu/Seal formula offers is values of $\psi(0,¿)$ for $¿ = 1,2,:::,t ¡ 1$.

In summary, we would suggest that there is no clear answer to the question of which algorithm should be preferred. The figures in Tables 1 and 2 show that, in terms of

computer run times, there is no clear choice. Perhaps the only real conclusion we can draw is a negative one - there appears to be little reason for using the Prabhu/Seal formula when u > 0.

Finally, we note that the conclusions of this study differ substantially from those of Steenackers and Goovaerts (1991, Section 2.4) who conduct a similar study and conclude that "The best performance is obtained by the method of Kling and Goovaerts", ie. what we have called the Prabhu/Seal method. There are three reasons why they reach a different conclusion to ours. First, they take the definition of survival to be

$$Pr(U(\tau) \geq 0 \text{ for all } \tau, \tau = 1, 2, \ldots, t)$$

in applying the method of Dickson and Waters. This method leads to poorer approximations to $\pm_d(u, t)$ - see Dickson and Waters (1991). Second, their analysis applies the algorithm given by de Vylder and Goovaerts (1988), whereas we have used a modified (improved!) version of it. Third, our analysis has been based on computing times since the algorithms presented in this paper are just different ways of calculating the same quantity. In this paper, numerical accuracy has been an issue only when an algorithm is unstable.

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