The multivariate De Pril transform

Bjørn Sundt
University of Bergen & University of Melbourne

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Abstract

In the present paper we extend the definition of the De Pril transform to multivariate functions with a positive mass at $\mathbf{0}$. We show that, like in the univariate case, this De Pril transform is additive for convolutions. Furthermore we discuss De Pril transforms of compound functions and higher order functions. Finally we introduce the Dhaene-De Pril transform of a multivariate function with a positive mass at $\mathbf{0}$ and discuss advantages and disadvantages of this transform compared to the De Pril transform.

1 Introduction

1A. During the last two decades there has grown up an extensive literature on recursive exact and approximate evaluation of aggregate claims distributions, starting with Panjer (1980, 1981). Most of the literature has been confined to univariate distributions. However, in insurance, multivariate aggregate claims distributions can also be of interest, e.g. when considering the joint distribution of aggregate claims of bodily injury and material damage in motor insurance. Further applications are discussed in Sundt (1998*b*).

Hesselager (1996) and Sundt (1998*b*) discuss in different directions multivariate extensions of Panjer's (1981) recursion for compound distributions.

Hesselager (1996) assumes that the counting distribution is multivariate, but that the severities are univariate. This can be interpreted as if the numbers of different sorts of claims are dependent. However he assumes that the severities of a fixed type of claims are mutually independent and identically distributed and independent of the severities of the other types and of the number of claims of the various types.

Sundt (1998*b*) keeps Panjer's univariate counting distribution, but assumes that each claim event generates a vector of severities. For instance, a car accident can generate payments for both bodily injury and material damage. He points out an intersection between his model and one of the models studied by Hesselager (1996). He also indicates how his approach can be applied to other classes of counting distributions. As an example he presents a multivariate extension of a recursion deduced by Sundt (1992) in the univariate case.

1B. Inspired by in particular De Pril (1989) and Dhaene & De Pril (1994), Sundt (1995) defined and discussed the De Pril transform of a univariate distribution as a tool for recursive evaluation of aggregate claims distributions. In Dhaene & Sundt (1998) the definition was extended to more general univariate functions in connection with approximate evaluation of distributions. Properties of the De Pril transform are also discussed in Sundt (1998a), Sundt, Dhaene, & De Pril (1998), Dhaene, Willmot, & Sundt (1996), and Sundt & Ekuma (1998).

1C. In the present paper we shall extend the definition of the De Pril transform to multivariate functions. In Section 2 we recapitulate some definitions and results from the univariate case in a form that will facilitate the multivariate extension performed in Section 3. As a starting point for the multivariate extension we apply the extension of Sundt (1998a) to the recursion of Sundt (1992). We show that, like in the univariate case, the multivariate De Pril transform is additive for convolutions, and discuss De Pril transforms of compound functions and higher order functions. Finally, in Section 4 we introduce the Dhaene-De Pril transform as a multivariate version of a function discussed by Dhaene & De Pril (1994) in the univariate case, and discuss its advantages and disadvantages compared to the De Pril transform.

1D. In the present paper we shall work on distributions in the representation of their probability functions. Therefore, we shall for simplicity refer to probability functions as distributions.

2 The univariate De Pril transform

2A. Let $\mathbb{N}_1$ denote the set of non-negative integers and $\mathcal{P}_{10}$ the class of distributions on $\mathbb{N}_1$ with a positive mass at zero. For a positive integer k and functions a and b we define $R_k[a,b]$ to be the distribution $p \in \mathcal{P}_{10}$ given by the recursion

$$p(n) = \sum_{i=1}^{k} \left(a(i) + \frac{b(i)}{n} \right) p(n-i) \qquad (n = 1, 2, \ldots) \tag{2.1}$$

with $p(n) = 0$ for all $n < 0$. Following Sundt (1992), we let $\mathcal{R}_k$ denote the class of all such distributions for faxed k; we allow $k = \infty$. We have that $\mathcal{R}_\infty = \mathcal{P}_{10}$

2B. Let $\mathbb{N}_{1+}$ denote the set of positive integers and $\mathcal{P}_{1+}$ the class of distributions on $\mathbb{N}_{1+}$. For $p \in \mathcal{P}_{10}$ and $h \in \mathcal{P}_{1+}$ the compound distribution $p \vee h$ is given by

$$p \vee h = \sum_{n=0}^{\infty} p(n) h^{n*}. \tag{2.2}$$

As

$$h^{n*}(x) = 0, \qquad (x \in \mathbb{N}_1;\ n = x+1, x+2, \ldots) \tag{2.3}$$

we obtain

$$(p \vee h)(x) = \sum_{n=0}^{x} p(n) h^{n*}(x), \qquad (x \in \mathbb{N}_1)$$

where the infinite summation in (2.2) has been reduced to a finite summation. In particular we have that $(p \vee h)(0) = p(0) > 0$. Thus $p \vee h \in \mathcal{P}_{10}$.

Sundt (1992) showed that if p is $R_k[a, b]$, then

$$(p \vee h)(x) = \sum_{y=1}^{x} (p \vee h)(x-y) \sum_{i=1}^{k} \left(a(i) + \frac{b(i)}{i} \frac{y}{x} \right) h^{i*}(y). \quad (x \in \mathbb{N}_{1+}) \tag{2.4}$$

This result means that $p \vee h$ is $R_k[c, d]$ with

$$c(y) = \sum_{i=1}^{k} a(i) h^{i*}(y); \quad d(y) = y \sum_{i=1}^{k} \frac{b(i)}{i} h^{i*}(y). \quad (y \in \mathbb{N}_{1+}) \tag{2.5}$$

2C. From (2.1) we see that the distribution $R_\infty[0, b]$ satisfies the recursion

$$p(n) = \frac{1}{n} \sum_{i=1}^{n} b(i) p(n-i). \qquad (n \in \mathbb{N}_{1+}) \tag{2.6}$$

By solving (2.6) for $b(n)$ we obtain

$$b(n) = \frac{1}{p(0)} \left(np(n) - \sum_{i=1}^{n-1} b(i) p(n-i) \right). \qquad (n \in \mathbb{N}_{1+}) \tag{2.7}$$

Thus, for any $p \in \mathcal{R}_\infty$ there exists a unique function b such that p can be represented as $R_\infty[0, b]$. We call this b the *De Pril transform* of p and denote it by φ_p. For convenience we extend the definition of the De Pril transform with $\varphi_p(0) = 0$. Thus we can rewrite (2.6) and (2.7) as

$$p(n) = \frac{1}{n} \sum_{i=1}^{n} \varphi_p(i)\, p(n-i) \qquad (n \in \mathbb{N}_{1+}) \tag{2.8}$$

$$\varphi_p(n) = \frac{1}{p(0)} \left(np(n) - \sum_{i=1}^{n-1} \varphi_p(i)\, p(n-i) \right) \qquad (n \in \mathbb{N}_1) \tag{2.9}$$

(we make the convention that $\sum_{i=r}^{s} = 0$ when $s < r$). The term *De Pril transform* was introduced by Sundt (1995).

>From (2.5) we see that if $p \in \mathcal{P}_{10}$ and $h \in \mathcal{P}_{1+}$, then

$$\varphi_{p \vee h}(y) = y \sum_{i=1}^{x} \frac{\varphi_p(i)}{i} h^{i*}(y), \qquad (y \in \mathbb{N}_{1+}) \tag{2.10}$$

which was given by Sundt (1995). He also showed that for $f_1, f_2, \ldots, f_m \in \mathcal{P}_{10}$

$$\varphi_{*_{j=1}^{r} f_j} = \sum_{j=1}^{r} \varphi_{f_j}. \tag{2.11}$$

Let p be $R_k[a, b]$. Sundt (1995) showed that then

$$\varphi_p(n) = na(n) + b(n) + \sum_{i=1}^{k} a(i)\, \varphi_p(n-i) \qquad (n \in \mathbb{N}_{1+}) \tag{2.12}$$

with $a(n) = b(n) = 0$ for all $n > k$ and $\varphi_p(n) = 0$ for all negative n. Sundt & Ekuma (1998) used this and (2.5) to show that

$$\varphi_{p \vee h}(x) = x \sum_{y=1}^{k} \left(a(y) + \frac{b(y)}{y} \right) h^{y*}(x) + \sum_{y=1}^{x-1} \varphi_{p \vee h}(x-y) \sum_{z=1}^{k} a(z)\, h^{z*}(y). \qquad (n \in \mathbb{N}_{1+}) \tag{2.13}$$

Sundt (1995) showed that a distribution in $\mathcal{P}_{10}$ has a non-negative De Pril transform if and only if it is infinitely divisible, or, equivalently, that it can be represented as a compound Poisson distribution.

2D. Formulae (2.9), (2.11), and (2.8) can be applied for numerical evaluation of convolutions of distributions in $\mathcal{P}_{10}$. For each of the distributions we can evaluate its De Pril transform by the recursion (2.9). Then we use (2.11) to find the De Pril transform of the convolution, and finally we apply this De Pril transform in (2.8) for recursive evaluation of the convolution. In practice the individual distributions in the convolutions will often be represented as compound distributions. In that case their De Pril transforms can be evaluated by (2.10).

Unfortunately, for large x, numerical evaluation of (2.10) can be rather time-consuming. Therefore methods have been developed under which the De Pril transform of the counting distribution is replaced with a function that is equal to zero for all values of the argument larger than some integer r. Such approximations were first discussed within the terminology of De Pril transforms by Dhaene & Sundt (1998). Of earlier references, we mention De Pril (1989) and Dhaene & De Pril (1994). With such approximations to De Pril transforms the resulting approximations to distributions are not necessarily distributions themselves. Dhaene & Sundt (1998) therefore extended the definition (2.9) of the De Pril transform to functions in $\mathcal{F}_{10}$, the class of functions on $\mathbb{N}_1$ with a positive mass at zero, and they discussed properties of the De Pril transform within this framework. In particular they showed that the additivity property (2.11) still holds for functions in $\mathcal{F}_{10}$, and that (2.10) holds when $p \in \mathcal{F}_{10}$ and $h \in \mathcal{F}_{1+}$, being the class of functions on $\mathbb{N}_{1+}$; also in this case we define $p \vee h$ by (2.2).

>From (2.8) we see that the De Pril transform of a function in $\mathcal{F}_{10}$ determines the function only up to a multiplicative constant. However, a distribution in $\mathcal{P}_{10}$ is uniquely determined by its De Pril transform as it should sum to one.

As the recursion (2.1) determines a function $p \in \mathcal{F}_{10}$ only up to a multiplicative constant, Dhaene & Sundt (1998) defined p to be in the form $R_k\left[a, b\right]$ if it satisfies that recursion, and showed that (2.12) still holds for such functions. Then the extension of (2.13) to the case when $p \in \mathcal{F}_{10}$ is in the form $R_k\left[a, b\right]$ and $h \in \mathcal{F}_{1+}$, is trivial.

2E. Recursions like (2.4) and (2.8) were originally developed for probability functions. Dhaene, Willmot, & Sundt (1996) discuss how one from recursions for probability functions can deduce recursions for cumulations like cumulative distribution functions and functions of even higher order. In this connection they defined for functions $f \in \mathcal{F}_{10}$ the cumulation operator Γ by

$$\Gamma f\left(x\right) = \sum_{y=0}^{x} f\left(y\right). \qquad \left(x \in \mathbb{N}_1\right)$$

We see that we have $\Gamma f = u * f$ where $u \in \mathcal{F}_{10}$ is defined by

$$u\left(x\right) = 1. \qquad \left(x \in \mathbb{N}_1\right)$$

$>$From (2.9) we obtain that

$$\varphi_u(x) = 1. \qquad (x \in \mathbb{N}_1)$$

Application of (2.11) gives

$$\varphi_{\Gamma^t f}(x) = \varphi_f(x) + t, \qquad (x \in \mathbb{N}_{1+},\, t \in \mathbb{N}_1) \tag{2.14}$$

which was proved by Dhaene, Willmot, & Sundt (1996).

3 The multivariate De Pril transform

3A. In this section we shall extend the definition of the De Pril transform to multivariate functions. As indicated in Section 2, the univariate De Pril transform could be developed from the framework of distributions in the $\mathcal{R}_k$ classes. We shall apply this framework as the basis also for the development of the multivariate De Pril transform. We shall start within the framework of multivariate distributions. However, as also in the multivariate case it seems desirable to be able to work with approximations to distributions where the approximations are not necessarily distributions themselves, we shall extend the framework to more general functions. Such approximations are further discussed in Sundt (1998*c*).

3B. For the following we shall need to introduce some notation.

Let m be a positive integer. We denote a column vector by a bold-face letter and its elements by the corresponding italic with the number of the element indicated by a subscript; furthermore, we let the italic with subscript $\cdot$ denote the sum of the elements, e.g. $\mathbf{x} = (x_1, \ldots, x_m)'$ and $x_{\cdot} = \sum_{j=1}^m x_j$. For two $m \times 1$ vectors $\mathbf{x}$ and $\mathbf{y}$, by $\mathbf{y} \leq \mathbf{x}$ we shall mean that $y_j \leq x_j$ for $j = 1, \ldots, m$, and by $\mathbf{y} < \mathbf{x}$ that $y_j \leq x_j$ for $j = 1, \ldots, m$ with strict inequality for at least one j. For $j = 1, \ldots, m$ we define $\mathbf{e}_j$ to be the $m \times 1$ vector whose jth element is 1 and all the other elements are 0. By $\mathbf{0}$ we shall mean the vector of which all elements are equal to 0. Strictly speaking we should also have indicated the dimension m in the notation for $\mathbf{e}_j$ and $\mathbf{0}$, but as the dimension will normally be clear, we shall drop that.

We shall now define m-variate versions of some classes defined in Section 2 in the univariate case. We introduce

$$\mathbb{N}_m = \left\{\mathbf{x} = (x_1, \ldots, x_m)' : x_j \in \mathbb{N}_1;\ j = 1, \ldots, m\right\}$$

$$\mathbb{N}_{m+} = \{\mathbf{x} \in \mathbb{N}_m : \mathbf{x} > \mathbf{0}\}.$$

Furthermore we let $\mathcal{P}_{m0}$ and $\mathcal{F}_{m0}$ denote the classes of respectively distributions and functions on $\mathbb{N}_m$ with a positive mass at $\mathbf{0}$ and $\mathcal{P}_{m+}$ and $\mathcal{F}_{m+}$ the classes of respectively distributions and functions on $\mathbb{N}_{m+}$.

3C. For functions $p \in \mathcal{F}_{10}$ and $h \in \mathcal{F}_{m+}$, the compound function $p \vee h$ is still given by (2.2). In connection with (2.2), we pointed out that although that formula involved an infinite summation for the function $p \vee h$, this summation became finite for values of the function. Although somewhat more obscure, this is also the case in the multivariate situation. Corresponding to (2.3) we have

$$h^{n*}(\mathbf{x}) = 0, \qquad (\mathbf{x} \in \mathbb{N}_m;\ n = x. + 1, x. + 2, \ldots) \tag{3.1}$$

and insertion in (2.2) gives

$$(p \vee h)(\mathbf{x}) = \sum_{n=0}^{x.} p(n)\, h^{n*}(\mathbf{x}). \qquad (\mathbf{x} \in \mathbb{N}_m)$$

As $(p \vee h)(\mathbf{0}) = p(0) > 0$, we see that $p \vee h \in \mathcal{F}_{m0}$.

3D. Sundt (1998*b*) pointed out that if p is $R_k[a, b]$ and $h \in \mathcal{P}_{m+}$, then $f = p \vee h$ satisfies the recursions

$$f(\mathbf{x}) = \sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} f(\mathbf{x} - \mathbf{y}) \sum_{i=1}^{k} \left(a(i) + \frac{b(i)}{i} \frac{y_j}{x_j} \right) h^{i*}(\mathbf{y}) \quad \left(\mathbf{x} \geq \mathbf{e}_j;\ j = 1, \ldots, m\right) \tag{3.2}$$

$$f(\mathbf{x}) = \sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} f(\mathbf{x} - \mathbf{y}) \sum_{i=1}^{k} \left(a(i) + \frac{b(i)}{i} \frac{y.}{x.} \right) h^{i*}(\mathbf{y}). \quad (\mathbf{x} \in \mathbb{N}_{m+}) \tag{3.3}$$

Both these recursions reduce to (2.4) when $m = 1$.

To develop a recursion for $p \vee h$ based on the De Pril transform of p, we can let $k = \infty$, $a = 0$, and $b = \varphi_p$. Because of (3.1), letting $k = \infty$ does not create any problem as we still have a finite number of non-zero terms in the inner sums. We obtain that for all $p \in \mathcal{P}_{10}$ and $h \in \mathcal{P}_{m+}$ we have

$$f(\mathbf{x}) = \frac{1}{x_j} \sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} f(\mathbf{x} - \mathbf{y})\, y_j \sum_{i=1}^{y.} \frac{\varphi_p(i)}{i} h^{i*}(\mathbf{y}) \quad \left(\mathbf{x} \geq \mathbf{e}_j;\ j = 1, \ldots, m\right) \tag{3.4}$$

$$f(\mathbf{x}) = \frac{1}{x.} \sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} f(\mathbf{x} - \mathbf{y})\, y. \sum_{i=1}^{y.} \frac{\varphi_p(i)}{i} h^{i*}(\mathbf{y}). \quad (\mathbf{x} \in \mathbb{N}_{m+}) \tag{3.5}$$

3E. We see that both (3.4) and (3.5) are very similar to (2.8). In particular, if we in (3.5) let

$$\varphi_f(\mathbf{y}) = y_\cdot \sum_{i=1}^{y_\cdot} \frac{\varphi_p(i)}{i} h^{i*}(\mathbf{y}), \qquad (\mathbf{y} \in \mathbb{N}_{m+}) \tag{3.6}$$

we obtain

$$f(\mathbf{x}) = \frac{1}{x_\cdot} \sum_{\mathbf{0}<\mathbf{y}\leq\mathbf{x}} \varphi_f(\mathbf{y}) f(\mathbf{x}-\mathbf{y}), \qquad (\mathbf{x} \in \mathbb{N}_m) \tag{3.7}$$

which seems to be an appropriate m-variate extension of (2.8). Analogously we could have developed an m-variate De Pril transform from (3.4). However, then both the m-variate De Pril transform and the recursions would have depended on j. In particular, this would have implied that if $f(\mathbf{x})$ is symmetric in the elements of $\mathbf{x}$, that would have not been the case with $\varphi_f(\mathbf{x})$, and that does not seem logical. This will be further discussed in Section 4.

By letting $\varphi_f(\mathbf{0}) = 0$ and solving (3.7) for $\varphi_f(\mathbf{x})$ we obtain

$$\varphi_f(\mathbf{x}) = \frac{1}{f(\mathbf{0})}\left(x_\cdot f(\mathbf{x}) - \sum_{\mathbf{0}<\mathbf{y}<\mathbf{x}} \varphi_f(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\right), \qquad (\mathbf{x} \in \mathbb{N}_m) \tag{3.8}$$

which represents an m-variate extension of (2.9). Like in the univariate case we see that φ_f is uniquely determined by (3.7).

Any $f \in \mathcal{P}_{m0}$ can be represented as $p \vee h$ with $p \in \mathcal{P}_{10}$ and $h \in \mathcal{P}_{m1}$ given by

$$p(1) = 1 - p(0) = \pi = 1 - f(\mathbf{0}) \tag{3.9}$$

$$h(\mathbf{x}) = \frac{f(\mathbf{x})}{\pi}. \qquad (\mathbf{x} \in \mathbb{N}_{m+}) \tag{3.10}$$

Thus we have defined φ_f for all $f \in \mathcal{P}_{m0}$.

Let us more generally consider (3.8) as definition of the De Pril transform of a function $f \in \mathcal{F}_{m0}$. Then (3.7) is satisfied. Like in the univariate case we see that the De Pril transform determines f up to a multiplicative constant, and if f is a distribution, then it is uniquely determined by its De Pril transform.

When considering whether (3.8) makes sense as a definition of the m-variate De Pril transform, we have to check to what extent the properties of the De Pril transform in the univariate case extend to the m-variate case. In particular we want the additivity property (2.11) to hold. Furthermore, we wonder whether (3.6) still holds in the more general case when $p \in \mathcal{F}_{10}$ and

$h \in \mathcal{F}_{m+}$. In the following two subsections we shall prove that the answer to each of these questions is affirmative.

3F. To prove the additivity property we extend the deductions in subsection 5A of Dhaene & Sundt (1998).

Let $\mathcal{F}_m$ denote the set of functions on $\mathbb{N}_m$. The convolution $h_1 * h_2$ of two functions h_1 and h_2 in $\mathcal{F}_m$ is defined by

$$(h_1 * h_2)(\mathbf{x}) = \sum_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}} h_1(\mathbf{y}) h_2(\mathbf{x} - \mathbf{y}). \qquad (\mathbf{x} \in \mathbb{N}_m)$$

For a function $h \in \mathcal{F}_m$ we introduce the transform $\overline{h} \in \mathcal{F}_m$ given by

$$\overline{h}(\mathbf{x}) = x.h(\mathbf{x}). \qquad (\mathbf{x} \in \mathbb{N}_m)$$

For $f \in \mathcal{F}_{m0}$ we can now reformulate (3.7) as

$$\overline{f} = \varphi_f * f.$$

This relation determines φ_f uniquely.

Lemma 1. *For $h_1, h_2 \in \mathcal{F}_m$ we have*

$$\overline{h_1 * h_2} = \overline{h_1} * h_2 + h_1 * \overline{h_2.}$$

Proof. For any $\mathbf{x} \in \mathbb{N}_m$ we have

$$\overline{h_1 * h_2}(\mathbf{x}) = x.(h_1 * h_2)(\mathbf{x}) = x. \sum_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}} h_1(\mathbf{y}) h_2(\mathbf{x} - \mathbf{y}) =$$
$$\sum_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}} y.h_1(\mathbf{y}) h_2(\mathbf{x} - \mathbf{y}) + \sum_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}} h_1(\mathbf{y})(x. - y.) h_2(\mathbf{x} - \mathbf{y}) =$$
$$\sum_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}} \overline{h_1}(\mathbf{y}) h_2(\mathbf{x} - \mathbf{y}) + \sum_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}} h_1(\mathbf{y}) \overline{h_2}(\mathbf{x} - \mathbf{y}) =$$
$$\left(\overline{h_1} * h_2\right)(\mathbf{x}) + \left(h_1 * \overline{h_2}\right)(\mathbf{x}) = \left(\overline{h_1} * h_2 + h_1 * \overline{h_2.}\right)(\mathbf{x}),$$

which proves the lemma. Q.E.D.

The additivity property (2.11) for functions in $\mathcal{F}_{m0}$ is now easily proved in the same way as in the proof of Theorem 5.1 in Dhaene & Sundt (1998) under application of Lemma 1.

3G. Lemma 6.1 in Dhaene & Sundt (1998) says that if $h \in \mathcal{F}_1$, then

$$\overline{h^{n*}} = \frac{n}{k} \overline{h^{k*}} * h^{(n-k)*}. \qquad (n \in \mathbb{N}_{1+};\ k = 1, \ldots, n) \tag{3.11}$$

With application of Lemma 1 the proof trivially extends to $h \in \mathcal{F}_m$.

The proof of the following theorem is a modification of the proof of Theorem 3.1 in Sundt, Dhaene, & De Pril (1998) for the univariate case.

Theorem 1. *If $f = p \vee h$ with $h \in \mathcal{F}_{m+}$ and $p \in \mathcal{F}_{10}$ in the form $R_k[a, b]$, then* (3.3) *and* (3.2) *hold.*

Proof. For $\mathbf{x} \in \mathbb{N}_{m+}$ and $j = 1, \ldots, m$ we have

$$f(\mathbf{x}) = \sum_{n=1}^{x.} p(n) h^{n*}(\mathbf{x}) = \sum_{n=1}^{x.} \sum_{i=1}^{k} \left(a(i) + \frac{b(i)}{n} \right) p(n-i) h^{n*}(\mathbf{x}) =$$
$$\sum_{i=1}^{x.} \sum_{n=i}^{x.} \left(a(i) + \frac{b(i)}{n} \right) p(n-i) h^{n*}(\mathbf{x}) =$$
$$\sum_{i=1}^{x.} \sum_{n=0}^{x.} \left(a(i) + \frac{b(i)}{n+i} \right) p(n) h^{(n+i)*}(\mathbf{x}).$$

$>$From (3.11) we obtain

$$h^{(n+i)*}(\mathbf{x}) = \frac{1}{x.} \frac{n+i}{i} \sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} y. h^{i*}(\mathbf{y}) h^{n*}(\mathbf{x} - \mathbf{y}).$$

Thus

$$f(\mathbf{x}) = \sum_{i=1}^{x.} \sum_{n=0}^{x.} \sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} \left(a(i) + \frac{b(i)}{i} \frac{y.}{x.} \right) p(n) h^{i*}(\mathbf{y}) h^{n*}(\mathbf{x} - \mathbf{y}) =$$
$$\sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} \sum_{i=1}^{x.} \left(a(i) + \frac{b(i)}{i} \frac{y.}{x.} \right) h^{i*}(\mathbf{y}) \sum_{n=0}^{x.} p(n) h^{n*}(\mathbf{x} - \mathbf{y}) =$$
$$\sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} \sum_{i=1}^{x.} \left(a(i) + \frac{b(i)}{i} \frac{y.}{x.} \right) h^{i*}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) =$$
$$\sum_{\mathbf{0} < \mathbf{y} \leq \mathbf{x}} f(\mathbf{x} - \mathbf{y}) \sum_{i=1}^{k} \left(a(i) + \frac{b(i)}{i} \frac{y.}{x.} \right) h^{i*}(\mathbf{y}),$$

that is, (3.3) holds.

The proof of (3.2) goes analogously. However, there we will also have to prove Lemma 1 and (3.11) with $x.h(x)$ replaced with $x_j h(x)$ in the definition of $\overline{h}$. These modifications are trivial.

This completes the proof of Theorem 1. Q.E.D.

$>$From Theorem 1 we obtain immediately that (3.4) and (3.5) hold for $f = p \vee h$ with $p \in \mathcal{F}_{10}$ and $h \in \mathcal{F}_{m+}$, and thus φ_f is given by (3.6).

The following theorem gives a multivariate version of (2.13).

Theorem 2. *If $f = p \vee h$ with $h \in \mathcal{F}_{m+}$ and $p \in \mathcal{F}_{10}$ in the form $R_k[a,b]$, then*

$$\varphi_f(\mathbf{x}) = x.\sum_{i=1}^{k}\left(a(i)+\frac{b(i)}{i}\right)h^{i*}(\mathbf{x}) + \sum_{\mathbf{0}<\mathbf{y}<\mathbf{x}}\varphi_f(\mathbf{x}-\mathbf{y})\sum_{i=1}^{k}a(i)h^{i*}(\mathbf{y}). \quad (\mathbf{x}\in\mathbb{N}_{m+}) \tag{3.12}$$

with $a(n) = b(n) = 0$ for all $n > k$ and $\varphi_p(n) = 0$ for all negative n.

Proof. We shall prove (3.12) by induction on $\mathbf{x}$.
For $j = 1, \ldots, m$ (3.6) gives

$$\varphi_f(\mathbf{e}_j) = \varphi_p(1)\,h(\mathbf{e}_j),$$

and by insertion of (2.12) we obtain

$$\varphi_f(\mathbf{e}_j) = (a(1)+b(1))\,h(\mathbf{e}_j).$$

Thus (3.12) holds for $\mathbf{x} = \mathbf{e}_j$.

Now let $\mathbf{v} \in \mathbb{N}_{m+}$ and assume that (3.12) holds for all $\mathbf{x}$ such that $\mathbf{0} < \mathbf{x} < \mathbf{v}$. From (3.8) we obtain

$$\varphi_f(\mathbf{v}) = \frac{1}{f(\mathbf{0})}\left(v.f(\mathbf{v}) - \sum_{\mathbf{0}<\mathbf{y}<\mathbf{v}} f(\mathbf{v}-\mathbf{y})\varphi_f(\mathbf{y})\right).$$

Insertion of (3.3) and (3.12) gives

$$\varphi_f(\mathbf{v}) = \frac{1}{f(\mathbf{0})}\Bigg\{\sum_{\mathbf{0}<\mathbf{y}\leq\mathbf{v}} f(\mathbf{v}-\mathbf{y})\sum_{i=1}^{k}\left(v.a(i)+\frac{b(i)}{i}y.\right)h^{i*}(\mathbf{y}) - \sum_{\mathbf{0}<\mathbf{y}<\mathbf{v}} f(\mathbf{v}-\mathbf{y})\Bigg[y.\sum_{i=1}^{k}\left(a(i)+\frac{b(i)}{i}\right)h^{i*}(\mathbf{y}) + \sum_{\mathbf{0}<\mathbf{z}<\mathbf{y}}\varphi_f(\mathbf{y}-\mathbf{z})\sum_{i=1}^{k}a(i)h^{i*}(\mathbf{z})\Bigg]\Bigg\},$$

from which we obtain

$$\varphi_f(\mathbf{v}) = v.\sum_{i=1}^{k}\left(a(i)+\frac{b(i)}{i}\right)h^{i*}(\mathbf{v}) + \frac{S}{f(\mathbf{0})} \tag{3.13}$$

with

$$S =$$

$$\sum_{\mathbf{0}<\mathbf{y}<\mathbf{v}} f(\mathbf{v}-\mathbf{y}) \left\{ \sum_{i=1}^{k} (v. - y.)\, a(i)\, h^{i*}(\mathbf{y}) - \sum_{\mathbf{0}<\mathbf{z}<\mathbf{y}} \varphi_f(\mathbf{y}-\mathbf{z}) \sum_{i=1}^{k} a(i)\, h^{i*}(\mathbf{z}) \right\} =$$

$$\sum_{i=1}^{k} a(i) \left\{ \sum_{\mathbf{0}<\mathbf{y}<\mathbf{v}} h^{i*}(\mathbf{y})\,(v. - y.)\, f(\mathbf{v}-\mathbf{y}) - \right.$$

$$\left. \sum_{\mathbf{0}<\mathbf{z}<\mathbf{v}} h^{i*}(\mathbf{z}) \sum_{\mathbf{z}<\mathbf{y}<\mathbf{v}} \varphi_f(\mathbf{y}-\mathbf{z})\, f(\mathbf{v}-\mathbf{y}) \right\} =$$

$$\sum_{i=1}^{k} a(i) \sum_{\mathbf{0}<\mathbf{y}<\mathbf{v}} h^{i*}(\mathbf{y}) \left\{ (v. - y.)\, f(\mathbf{v}-\mathbf{y}) - \sum_{\mathbf{0}<\mathbf{t}<\mathbf{v}-\mathbf{y}} \varphi_f(\mathbf{t})\, f(\mathbf{v}-\mathbf{y}-\mathbf{t}) \right\} =$$

$$\sum_{i=1}^{k} a(i) \sum_{\mathbf{0}<\mathbf{y}<\mathbf{v}} h^{i*}(\mathbf{y})\, \varphi_f(\mathbf{v}-\mathbf{y})\, f(\mathbf{0}).$$

By insertion in (3.13) we see that (3.12) holds for $\mathbf{x} = \mathbf{v}$, and by induction it holds for all $\mathbf{x} \in \mathbb{N}_{m+}$. Q.E.D.

For $k = 1$ (3.12) reduces to

$$\varphi_f(\mathbf{x}) = x.\,(a+b)\, h(\mathbf{x}) + a \sum_{\mathbf{0}<\mathbf{y}<\mathbf{x}} \varphi_f(\mathbf{x}-\mathbf{y})\, h(\mathbf{y}). \qquad (\mathbf{x} \in \mathbb{N}_{m+}) \quad (3.14)$$

In the univariate case this recursion is discussed by Sundt & Ekuma (1998).

3H. In this subsection we shall extend the characterisation of distributions with non-negative De Pril transforms in terms of compound Poisson distributions from the univariate case to the multivariate case. Our deduction is parallel to the deduction of Sundt (1995) for the univariate case.

We shall need the following lemma.

Lemma 2. *A distribution $f \in \mathcal{P}_{m0}$ with a non-negative De Pril transform satisfies the condition*

$$\sum_{\mathbf{x}\in\mathbb{N}_{m+}} \frac{\varphi_f(\mathbf{x})}{x.} < \infty.$$

Proof. As φ_f is non-negative, we obtain from (3.8) that for all $\mathbf{x} \in \mathbb{N}_{m+}$

$$\frac{\varphi_f(\mathbf{x})}{x.} = \frac{1}{f(\mathbf{0})} \left(f(\mathbf{x}) - \frac{1}{x.} \sum_{\mathbf{0}<\mathbf{y}<\mathbf{x}} \varphi_f(\mathbf{y})\, f(\mathbf{x}-\mathbf{y}) \right) \leq \frac{f(\mathbf{x})}{f(\mathbf{0})},$$

and summation over $\mathbf{x}$ gives

$$\sum_{\mathbf{x}\in\mathbb{N}_{m+}} \frac{\varphi_f(\mathbf{x})}{x.} \leq \sum_{\mathbf{x}\in\mathbb{N}_{m+}} \frac{f(\mathbf{x})}{f(\mathbf{0})} = \frac{1-f(\mathbf{0})}{f(\mathbf{0})} < \infty. \qquad \text{Q.E.D.}$$

Theorem 3. *A distribution in $\mathcal{P}_{m0}$ has a non-negative De Pril transform if and only if it can be represented as a compound distribution $p \vee h$ where $h \in \mathcal{P}_{m+}$ and p is a Poisson distribution.*

Proof. We first assume that $f = p \vee h$ with $h \in \mathcal{P}_{m+}$ and p is the Poisson distribution given by

$$p(n) = \frac{\lambda^n}{n!}e^{-\lambda}. \qquad (n \in \mathbb{N}_1;\ \lambda > 0)$$

It is easily shown that p is $R_1[0,\lambda]$, and thus (3.14) gives

$$\varphi_f(\mathbf{x}) = x.\lambda h(x) \geq 0, \qquad (\mathbf{x} \in \mathbb{N}_{m+}) \tag{3.15}$$

that is, φ_f is non-negative.

Now, let us assume that φ_f is non-negative. From Lemma 2 we obtain that

$$\lambda = \sum_{\mathbf{x}\in\mathbb{N}_{m+}} \frac{\varphi_f(\mathbf{x})}{x.}$$

is finite. We define $h \in \mathcal{P}_{m+}$ by

$$h(\mathbf{x}) = \frac{\varphi_f(\mathbf{x})}{\lambda x.}. \qquad (\mathbf{x} \in \mathbb{N}_{m+})$$

As (3.15) holds and a distribution in $\mathcal{P}_{m0}$ is uniquely determined by its De Pril transform, we must have that $f = p \vee h$, that is, every distribution in P_{m0} can be represented in the form $f = p \vee h$ where $h \in \mathcal{P}_{m+}$ and p is a Poisson distribution.

This completes the proof of Theorem 3. Q.E.D.

>From Theorem 2.1 in Horn & Steutel (1978) follows that a distribution in $\mathcal{P}_{m0}$ is infinitely divisible if anf only if it can be represented in the form $p \vee h$ where $h \in \mathcal{F}_{m+}$ and p is a Poisson distribution. Together with Theorem 3 this implies that a distribution in $\mathcal{P}_{m0}$ is infinitely divisible if and only if its De Pril transform is non-negative. This result is closely related to Corollary 2.2 in Horn & Steutel (1978).

3I. In this subsection we shall generalise (2.14) to the multivariate case. For $f \in \mathcal{F}_m$ we define the cumulation operator Γ by

$$\Gamma f(\mathbf{x}) = \sum_{\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}} f(\mathbf{y}) . \qquad (\mathbf{x} \in \mathbb{N}_m)$$

We see that $\Gamma f = u * f$ with $u \in \mathcal{F}_{m0}$ defined by

$$u(\mathbf{x}) = 1. \qquad (\mathbf{x} \in \mathbb{N}_m)$$

>From (3.8) we obtain that for $\mathbf{x} \in \mathbb{N}_{m+}$

$$\varphi_u(\mathbf{x}) = x. - \#\{\mathbf{y} : \mathbf{0} < \mathbf{y} < \mathbf{x}\} = x. + 2 - \prod_{j=1}^{m} (x_j + 1) .$$

Application of (2.11) gives

$$\varphi_{\Gamma^t f}(\mathbf{x}) = \varphi_f(\mathbf{x}) + t \left(x. + 2 - \prod_{j=1}^{m} (x_j + 1) \right), \qquad (\mathbf{x} \in \mathbb{N}_{m+},\ t \in \mathbb{N}_1)$$

which looks much more complicated than in the univariate case. In the bivariate case we obtain

$$\varphi_{\Gamma^t f}(\mathbf{x}) = \varphi_f(\mathbf{x}) + t(1 - x_1 x_2) . \qquad (\mathbf{x} \in \mathbb{N}_{2+},\ t \in \mathbb{N}_1)$$

4 The multivariate Dhaene-De Pril transform

4A. For a function $f \in \mathcal{F}_{10}$, Dhaene & De Pril (1994) considered instead of the De Pril transform the function ψ_f given by

$$\psi_f(x) = \begin{cases} \ln f(0) & (x = 0) \\ \dfrac{\varphi_f(x)}{x}. & (x \in \mathbb{N}_{1+}) \end{cases} \tag{4.1}$$

Let us call ψ_f the *Dhaene-De Pril transform* of f.

Advantages and disadvantages of applying the De Pril transform instead of the Dhaene-De Pril transform have been discussed in Dhaene & Sundt (1998) and Sundt (1998*a*). When Sundt (1995) chose to study the De Pril transform instead of the Dhaene-De Pril transform, he considered the transform mainly as a computational tool, and therefore he wanted to keep the number of algebraic operations at a minimum. He considered the recursions (2.8) and (2.9) as the most important tools when applying the transform for numerical calculations, and by using the De Pril transform instead of the Dhaene-De Pril transform one avoids some multiplications. Furthermore, the De Pril transform seems to more in line with the representation (2.1) of $R_k[a,b]$ as introduced in Sundt (1992).

On the other hand, the Dhaene-De Pril transform also has advantages. In particular the value of $\psi_f(0)$ ensures that the transform determines functions in $\mathcal{F}_{10}$ uniquely. Also some formulae and deductions look more tidy with the Dhaene-De Pril transform. In the next subsection we shall extend the definition of the Dhaene-De Pril transform to the multivariate case, and we shall see that in that case it has an additional advantage that does not appear in the univariate case.

4B. Any function $f \in \mathcal{F}_{m0}$ with $f(\mathbf{0}) < 1$ can be represented as $p \vee h$ with $p \in \mathcal{P}_{10}$ and $h \in \mathcal{F}_{m+}$ given by (3.9) and (3.10). Application of (4.1) in (3.4) and (3.5) gives

$$f(\mathbf{x}) = \frac{1}{x_j} \sum_{\mathbf{0}<\mathbf{y}\leq\mathbf{x}} f(\mathbf{x}-\mathbf{y})\, y_j \sum_{i=1}^{y.} \psi_p(i)\, h^{i*}(\mathbf{y}) \quad (\mathbf{x} \geq \mathbf{e}_j;\ j=1,\dots,m) \tag{4.2}$$

$$f(\mathbf{x}) = \frac{1}{x.} \sum_{\mathbf{0}<\mathbf{y}\leq\mathbf{x}} f(\mathbf{x}-\mathbf{y})\, y. \sum_{i=1}^{y.} \psi_p(i)\, h^{i*}(\mathbf{y}). \quad (\mathbf{x} \in \mathbb{N}_{m+}) \tag{4.3}$$

We let

$$\psi_f(\mathbf{y}) = \begin{cases} \ln f(\mathbf{0}) & (\mathbf{y} = \mathbf{0}) \\ \sum_{i=1}^{y.} \psi_p(i)\, h^{i*}(\mathbf{y}). & (\mathbf{y} \in \mathbb{N}_{m+}) \end{cases}$$

Insertion in (4.2) and (4.3) gives

$$f(\mathbf{x}) = \frac{1}{x_j} \sum_{\mathbf{0}<\mathbf{y}\leq\mathbf{x}} y_j \psi_f(\mathbf{y})\, f(\mathbf{x}-\mathbf{y}) \quad (\mathbf{x} \geq \mathbf{e}_j;\ j=1,\dots,m) \tag{4.4}$$

$$f(\mathbf{x}) = \frac{1}{x.} \sum_{\mathbf{0}<\mathbf{y}\leq\mathbf{x}} y. \psi_f(\mathbf{y})\, f(\mathbf{x}-\mathbf{y}), \quad (\mathbf{x} \in \mathbb{N}_{m+}) \tag{4.5}$$

which can be solved for $\psi_f(\mathbf{x})$ by respectively

$$\psi_f(\mathbf{x}) = \frac{1}{f(\mathbf{0})} \left(f(\mathbf{x}) - \frac{1}{x_j} \sum_{\mathbf{0}<\mathbf{y}<\mathbf{x}} y_j \psi_f(\mathbf{y})\, f(\mathbf{x}-\mathbf{y}) \right) \quad (\mathbf{x} \geq \mathbf{e}_j;\ j=1,\dots,m) \tag{4.6}$$

$$\psi_f(\mathbf{x}) = \frac{1}{f(\mathbf{0})} \left(f(\mathbf{x}) - \frac{1}{x.} \sum_{\mathbf{0}<\mathbf{y}<\mathbf{x}} y. \psi_f(\mathbf{y})\, f(\mathbf{x}-\mathbf{y}) \right). \quad (\mathbf{x} \in \mathbb{N}_{m+}) \tag{4.7}$$

By rescaling f we obtain that (4.4)–(4.7) also hold when $f(\mathbf{0}) \geq 1$. Thus we see that in the multivariate case the Dhaene-De Pril transform gives more flexibility with regard to how to recursively evaluate the function from the transform and the transform from the function.

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Bjørn Sundt
Department of Mathematics
University of Bergen
Johannes Bruns gate 12
N–5008 Bergen
NORWAY