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The Statistical Distribution of Incurred Losses and Its Evolution Over Time

I: Non-Parametric Models

by

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1. Introduction and Background

This paper is written at the request of, and is partly funded by, the **Casualty Actuarial Society's** Committee on Theory of Risk. It is the first of a trio of papers whose purpose is to answer the following question, posed by the Committee:

> *Assume you know the aggregate loss distribution at policy inception and you have expected patterns of claims reporting, losses emerging and losses paid and other pertinent information, how do you modify the distribution as the policy matures and more information becomes available? Actuaries have historically dealt with the problem of modifying the expectation conditional on emerged information. This expands the problem to continuously modifying the whole distribution from inception until it decays to a point. One might expect that there are at least two separate states that are important. There is the exposure state. It is during this period that claims can attach to the policy. Once this period is over no new claims can attach. The second state is the discovery or development state. In this state claims that already attached to the policy can become known and their value can begin developing. These two states may have to be treated separately.*

In general terms, this brief requires the extension of conventional point estimation of incurred losses to their companion distributions. Specifically, the evolution of this distribution over time is required as the relevant period of origin matures.

Expressed in this way, the problem takes on a natural Bayesian form. For any particular year of origin (the generic name for an accident year, underwriting year, etc), one begins with a **prior distribution** of incurred losses which applies in advance of data collection. As the period of origin develops, loss data accumulate, and may be used for progressive Bayesian revision of the prior.

When the period of origin is fully mature, the amount of incurred losses is known with certainty. The Bayesian revision of the prior is then a single point distribution. The present paper addresses the question of how the Bayesian revision of the prior evolves over time from the prior itself to the final degenerate distribution.

This evolution can take two distinct forms. On the one hand, one may impose no restrictions on the posterior distributions arising from the Bayesian revisions. These posterior distributions will depend on the empirical distributions of certain observations. Such models are **non-parametric**.

Alternatively, the posterior distributions may be assumed to come from some defined family. For example, it may be assumed that the posterior-to-data distribution of incurred losses, as assessed at a particular point of development of the period of origin, is log normal. Any estimation questions must relate to the parameters which define the distribution within the chosen family.

These are **parametric models**. They are, in certain respects, more flexible than non-parametric, but lead to quite different estimation procedures.

When a period of origin is characterised by a set of parameters in this way, it is possible that those parameters change from one period of origin to the next. Models with these properties are called **dynamic models**. If there is a specific linkage between successive period of origin, they are **evolutionary models**.

The present paper deals with non-parametric models only, two future papers dealing with the others.

2. Motivational Example

For motivation, an unrealistically simple example is chosen, its data represented in Table 2.1.

Table 2.1 Data for Motivational Example

Accident Year	Ultimate Number Of Claims	Paid losses ($m) in development year				
		0	1	2	3	4
1994	1,011	1.080	4.295	1.838	0.430	0.217
1995	1,235	1.276	4.812	2.629	0.612	
1996	1,348	1.534	5.017	2.511		
1997	1,329	1.496	5.263			
1998	1,501	1.374				

For the purpose of the present example it is assumed that:

- The ultimate claim count is known with certainty
- No paid losses occur beyond development year 4
- There is no inflation.

Division of each row of paid losses in Table 2.1 by the associated ultimate number of claims produces the **payments per claim incurred** (PPCI) (see eg, Taylor, 1999, pages 88-96) displayed in Table 2.2.

Table 2.2 Payments per Claim Incurred

Accident Year	PPCI ($) in Development Year				
	0	1	2	3	4
1994	1,068	4,248	1,818	425	215
1995	1,033	3,896	2,129	496	
1996	1,138	3,722	1,863		
1997	1,126	3,960			
1998	915				

Let cell (i,j) represent development year j of accident year i, and let $X(i,j)$ denote the PPCI in respect of that cell.

Assume that, prior to the collection of any data,

$$X(i,j) \sim \text{Gamma} \tag{2.1}$$

with

$$\text{E}\,X(i,j) = \theta(j) \tag{2.2}$$

$$\text{V}\,X(i,j) = \tau^2(j), \tag{2.3}$$

with θ(j) and τ^2(j) independent of i.

Suppose that the $X(i,j)$ form a mutually stochastically independent set and that $\theta(j)$ is a sampling from a hyperdistribution with d.f. $F_j(.)$. Suppose the $\theta(j)$ are also stochastically independent. Let $x(i,j)$ denote the realised value of $X(i,j)$ where this observation has been made.

Consider accident year 1996, for example. At its commencement, its total incurred losses per claim had the unknown value

$$\sum_{j=0}^{4} X(1996, j). \tag{2.4}$$

with d.f. $G_0 * G_1 * G_2 * G_3 * G_4$, where the star denotes convolution and $G_j(.)$ is the unconditional d.f. of $X(i,j)$ derived from the gamma distribution in (2.1) and the prior $F_j(.)$.

By the end of 1998, the situation represented in Table 2.2, the observations $x(1996,j)$, j=0,1,2 have been made. The quantity (2.4) therefore becomes

$$\sum_{j=0}^{2} x(1996, j) + \sum_{j=3}^{4} X(1996, j). \tag{2.4a}$$

Note that the best estimate of the d.f. of the second summand in (2.4a) is no longer $G_3 * G_4$ because accident years 1994 and 1995 have provided some data in respect of development years 3 and 4. It is possible to form the Bayesian revision of this d.f.

This causes $G_3(x)$ to be replaced by

Prob $[X(i,3) \leqslant x \mid \{x(k,3), k = 1994, 1995\}]$ for $i \geqslant 1996$,

and similarly for $G_4(.)$.

In this way the d.f. of the initial variable (2.4) can be revised year by year, as data accumulates, until finally the experience of that accident year is complete and (2.4) is replaced by the known quantity (ie single point distribution).

$$\sum_{j=0}^{4} x(1996, j). \tag{2.4b}$$

The remainder of this paper will be concerned with the application of **credibility theory**, itself a Bayesian theory, to the estimation of the distribution of quantities like

$$\sum_{j=0}^{k} x(i, j) + \sum_{j=k+1}^{4} X(i, j) \tag{2.5}$$

as they evolve from $k = -1$ to $k = 4$, under the convention that

$$\sum_{j=0}^{-1} (\text{anything}) = 0. \tag{2.6}$$

3. Bayesian Framework

The example of Section 2 is generalised as follows.

Let $X(i,j)$ denote some variable that is indexed by year of origin i and development year j, $i \geqslant 0$, $0 \leqslant j \leqslant J$ for fixed $J > 0$.

Let $k = i + j$. If the $X(i,j)$ are set out in a rectangular array with i and j labelling rows and columns respectively, then k labels diagonals. Each diagonal represents an **experience year**, ie the calendar period containing year of origin k, as well as development year 1 of year of origin k-1, etc.

Data accumulate over time by the addition of diagonals. At the end of year k, the available data set will be

$$X(k) = \{X(i,j) : i \geq 0,\ 0 \leq j \leq J,\ 0 \leq i + j \leq k\} \tag{3.1}$$

The case $J = 4$, $k = 4$ defines a triangle such as in Table 2.1.

Let $\Theta(j)$ be an abstract parameter applying to development year j and characterising the distribution of $X(i,j)$. Suppose that $\Theta(j)$ is an unobservable random variable on a probability space $\mathcal{P} = (S, \mathcal{A}, F)$. The realisation of $\Theta(j)$ is denoted by $\theta(j)$. It is supposed that $\theta(0)$, …, $\theta(J)$ are iid samplings from $\mathcal{P}$.

Now suppose $X(i,j)$, $i \geq 0$ to be some stochastic quantity dependent on $\theta(j)$. Suppose that the $X(i,j),| \theta(j)$ are stochastically independent and, for fixed j, they are iid.

Let $G(\bullet | \theta)$ denote the d.f. of $X(i,j) | \theta$. For fixed j, this is $G(\bullet | \theta(j))$, which may be conveniently denoted by $G_j^{(\theta)}(\bullet)$, the upper θ indicating conditioning on that variable.

Write

$$G_j(x) = \int G_j^{(\theta)}(x)\ dF(\theta) \tag{3.2}$$

which represents the average of $G_j^{(\theta)}(x)$ over the conditioning parameter, ie the expectation of $G_j^{(\theta)}(x)$ in the absence of any data.

Once data have accumulated, one may calculate the **Bayesian revision** of $G_j(\bullet)$:

$$G_j^{(\theta)}(\bullet| X(k)) = E[G_j^{(\theta)}(\bullet) | X(k)], \tag{3.3}$$

which is an unbiased posterior-to-data estimate of $G_j^{(\theta)}(\bullet)$.

Subsequent sections will be concerned with credibility theory approximations to (3.3).

4. Credibility Theory

4.1 Basic Credibility Theory

Let $Y(i,j)$ be a variable dependent on $\theta(j)$, defined in the same way as $X(i,j)$. The quantities $X(i_1,j_1)\,|\,\theta(j_1)$ and $Y(i_2,j_2)\,|\,\theta(j_2)$ are stochastically independent if $(i_1,j_1)\neq(i_2,j_2)$.

Suppose one seeks a forecast of $Y(i,k+1-i)$, ie relating to experience period k+1, given data $X(k)$. The most efficient forecast is the Bayesian expectation $E[Y(i,k+1-i)\,|\,X(k)]$.

Credibility theory is a **linearised Bayes** theory in which this last expectation is approximated by a quantity that is linear in the data. Specifically, $Y(i,k+1-i)$ is forecast by:

$$Y^*(i,k+1-i)=a+\sum_{h,j} b_{hj}X(h,j) \qquad (4.1)$$

with a and b_{hj} constants, and h,j varying over the set of values such that the $X(h,j)$ form $X(k)$ defined by (3.1).

The forecast $Y^*(i,k+1-i)$ is chosen according to the **least squares criterion**:

$$E[Y^*(i,k+1-i)-Y(i,k+1-i)]^2=\min!, \qquad (4.2)$$

where here and elsewhere in this paper an expectation operator E without a suffix indicates **unconditional expectation**. For example,

$$E[Y(i,j)]=E_{\theta(j)}E[Y(i,j)\,|\,\theta(j)]. \qquad (4.3)$$

Now the forecast (4.1) may be simplified a good deal before the details of (4.2) are worked out. By the symmetry of the $X(i,j)$ for fixed j, arising from the identity of distribution of the $X(i,j)\,|\,\theta(j)$, (4.1) may be written in this form:

$$Y^*(i,k+1-i)=a+\sum_{j} b_j\overline{X}(j), \qquad (4.1a)$$

where

$$\overline{X}(j)=\sum_{h=0}^{k-j} X(h,j)/(k-j+1), \qquad (4.4)$$

and the b_j are constants.

The conditions governing independence:

(i) between the X's and Y's; and
(ii) between the $\theta(j)$;

cause (4.1a) to simplify further:

$$Y^*(i,k+1-i) = a + b\,\overline{X}(k+1-i), \tag{4.1b}$$

with b constant. In other words, the only data that have any predictive value for $Y(i,k+1-i)$ are the $X(h,k+1-i)$.

The calculation of $Y^*(i,k+1-i)$ becomes a simple exercise when (4.1b) is substituted in (4.2). The solution, with $k+1-i$ conveniently abbreviated to just j, is:

$$b = \text{Cov}\,[Y(i,j), \overline{X}(j)] / V[\overline{X}(j)] \tag{4.5}$$

$$a = \nu(j) - b\mu(j), \tag{4.6}$$

where

$$\mu(j) = E\,X(i,j) \tag{4.7}$$
$$\nu(j) = E\;Y(i,j) \tag{4.8}$$

and the variance and covariance in (4.5) are unconditional.

The numerator and denominator of (4.5) may be simplified further, taking account of the above independence assumptions:

$$b = \frac{V_{\theta(j)}E[X(i,j)\,|\,\theta(j)]}{V_{\theta(j)}E[X(i,j)\,|\,\theta(j)] + n_j^{-1}E_{\theta(j)}V[X(i,j)\,|\,\theta(j)]}, \tag{4.9}$$

where n_j is the number of observations $X(i,j)$ in $\overline{X}(j)$. Equivalently,

$$b = n_j / (n_j + K), \tag{4.10}$$

with

$$K = \frac{E_{\theta(j)}V[X(i,j)\,|\,\theta(j)]}{V_{\theta(j)}E[X(i,j)\,|\,\theta(j)]}. \tag{4.11}$$

This last quantity K is sometimes called the **time constant**. The final credibility formula is obtained by substitution of (4.6) in (4.1b) and replacement of b by the more conventional symbol z:

$$Y^*(i,j) = [\nu(j) - \mu(j)] + (1-z)\mu(j) + z\overline{X}(j), \tag{4.12}$$

with $j = k+1-i$ and z (ie b) given by (4.10) and (4.11). Since $X(i,j)$ and $Y(i,j)$ are identically distributed, $\mu(j) = \nu(j)$, and so the square bracketed term in (4.12) vanishes.

This is a representation of the essentials (expressed a little differently) of the original paper on credibility theory (Bühlmann, 1967). A useful and relatively up-to-date survey of the theory is given by Goovaerts and Hoogstad (1987).

4.2 Credible Distribution

Jewell (1974) considered the case in which

$$Y(i,j) = G_j^{(\theta)}(y) = \text{Prob}\,[X(i,j) \le y \mid \theta(j)], \tag{4.13}$$

for some fixed but arbitrary value of y. The "observations" which served as inputs to this model were not the raw $X(i,j)$ but their empirical distribution equivalents. That is, $X(i,j)$ was replaced by

$$\begin{aligned} I_{X(i,j)}(y) &= 0 \text{ if } y < X(i,j) \\ &= 1 \text{ if } y \ge X(i,j). \end{aligned} \tag{4.14}$$

It will be convenient to abbreviate $I_{X(i,j)}(y)$ to $I_{ij}(y)$.

Application of the credibility theory set out in Section 4.1 then leads to a forecast $Y^*(i,k+1-i)$ which is the linearised form of:

$$E\{\text{Prob}\,[X(i,k+1-i) \le y] \mid X(k)\}, \tag{4.15}$$

the linearisation involving the terms $I_{ij}(y)$.

This is a Bayesian forecast of the **distribution** of $X(i,k+1-i)$ and was referred to by Jewell as the **credible distribution**. In terms of the example given in Section 2, it amounts to forecasting the distribution of any entry on the next diagonal of the paid loss triangle, conditional on the triangle observed to date.

The basic credibility formula (4.12) may now be re-interpreted within this new context. First note that, according to the definition of $Y(i,j)$ in (4.13), and making use of (4.8),

$$\begin{aligned} \nu(j) &= E_{\theta(j)} G_j^{(\theta)}(y) \\ &= G_j(y) \end{aligned} \tag{4.16}$$

Note that $G_j(\bullet)$ is effectively the **prior** d.f. on the $X(i,j)$ for the nominated j.

Also, by (4.7) and recalling the replacement of *X(i,j)* by $I_{ij}(y)$,

$$\begin{aligned}\mu(j) &= E\, I_{ij}(y)\\ &= E_{\theta(j)}E[I_{ij}(y)\,|\,\theta(j)]\\ &= E_{\theta(j)}\,\text{Prob}[X(i,j)\le y\,|\,\theta(j)] \qquad [\text{by } (4.14)]\\ &= E_{\theta(j)}G_j^{(\theta)}(y),\end{aligned} \tag{4.17}$$

by the definition of $G_j^{(\theta)}(\bullet)$ in (4.13).

By (4.16) and (4.17),

$$\mu(j) = \nu(j) = G_j(y), \tag{4.18}$$

as was noted more generally at the end of Section 4.1.

This simplifies the credibility formula (4.12) to the following:

$$Y^*(i,j) = (1-z)G_j(y) + z\,\bar{I}_j(y),\ j = k+1-i \tag{4.19}$$

where *Y*(i,j)* is the forecast discussed in (4.15) and $\bar{I}_j(y)$ is the **empirical distribution of observations** *X(i,j)* for the fixed *j* under consideration:

$$\bar{I}_j(y) = n_j^{-1}\sum_i I_{ij}(y). \tag{4.20}$$

An examination of the definition of $I_{ij}(y)$ in (4.14) indicates that $\bar{I}_{ij}(y)$ is the proportion of observations *X(i,j)*, for the fixed *j*, which are less than or equal to *y*.

The credibility *z* is still given by (4.10) with *z* in place of *b*. It remains to interpret the time constant *K* in the present context. This is done by replacing *X(i,j)* by $\bar{I}_{ij}(y)$ in (4.11).

The denominator of (4.11) can be evaluated by the same reasoning as led to (4.17):

$$V_{\theta(j)}E[I_{ij}(y)\,|\,\theta(j)] = V_{\theta(j)}G_j^{(\theta)}(y). \tag{4.21}$$

The variance of $I_{ij}(y)$ in the numerator of (4.11) is a single observation binomial variance, and so the numerator may be written:

$$\begin{aligned}E_{\theta(j)}V\left[I_{ij}(y)\,|\,\theta(j)\right] &= E_{\theta(j)}G_j^{(\theta)}(y)\left[1-G_j^{(\theta)}(y)\right]\\ &= G_j(y) - E_{\theta(j)}\left[G_j^{(\theta)}(y)\right]^2,\end{aligned} \tag{4.22}$$

by (4.16).

The final member of (4.22) may be simplified further:

$$E_{\theta(j)}\left[G_j^{(\theta)}(y)\right]^2 = E_{\theta(j)}\left\{G_j(y)+\left[G_j^{(\theta)}(y)-G_j(y)\right]\right\}^2$$
$$=\left[G_j(y)\right]^2+V_{\theta(j)}\left[G_j^{(\theta)}(y)\right]. \tag{4.23}$$

The quantity K may now be evaluated by means of (4.11) by applying (4.21) as the denominator, and by substituting (4.23) in (4.22) and applying the result as the numerator:

$$K=\frac{G_j(y)\left[1-G_j(y)\right]}{V_{\theta(j)}G_j^{(\theta)}(y)}-1. \tag{4.24}$$

To summarise, $\text{Prob}\left[X(i,j)\le y\right]$ for $j=k+1-i$ is forecast by (4.19) with quantities therein defined by (4.20), (4.10) (with b replaced by z) and (4.24).

By assumption, the $Y(i,j)\,|\,\theta(j)$ are iid for $Y(i,j)$ defined by (4.13) and fixed j, and so the same reasoning may be applied to the forecast of $\text{Prob}\left[X(i,j)\le y\right]$ for all $j=k+1-i, k+2-i$, etc. The formula (4.19) continues to apply.

5. The Forecast Cell Distribution

Section 4 gives us the credibility forecast of $\text{Prob}[X(i,j) \le y]$ for a particular value of y. The collection of these forecasts for all y is a forecast of the entire distribution $G_j(\bullet) \mid X(k)$, which may be denoted $G^*_{j(k)}(\bullet)$, or just $G^*_j(\bullet)$ when the value of k is clear from the context.

Then by (4.19),

$$G^*_j(y) = [1 - z_j(y)]G_j(y) + z_j(y)\bar{I}_j(y), \quad j = k+1-i, k+2-i, \text{ etc} \tag{5.1}$$

where the dependence of z on j and y has been recognised explicitly:

$$z_j(y) = n_j / [n_j + K_j(y)], \tag{5.2}$$

$$K_j(y) = \frac{G_j(y)[1-G_j(y)]}{V_{\theta(j)}G_j^{(\theta)}(y)} - 1. \tag{5.3}$$

It is of interest to observe that $K_j(y)$, and therefore $z_j(y)$ is independent of y in the special case $V_{\theta(j)}G_j^{(\theta)}(y)$ proportional to $G_j(y)[1-G_j(y)]$ for varying y. This result may be put in a more general form as follows.

Proposition. If, for local variations of y, $V_{\theta(j)}G_j^{(\theta)}(y)$ is an increasing (resp. decreasing) function of $G_j(y)[1-G_j(y)]$, then $z_j(y)$ is also (locally) an increasing (resp. decreasing) function of $G_j(y)[1-G_j(y)]$.

Example. Consider the case in which

$$V_{\theta(j)}G_j^{(\theta)}(y) = c\{G_j(y)[1-G_j(y)]\}^{1+\alpha}, \tag{5.4}$$

where $\alpha \ge 0$ and $c \le 4^\alpha$ are constants.

Then (5.3) yields

$$K_j(y) = c^{-1}\{G_j(y)[1-G_j(y)]\}^{-\alpha} - 1 \tag{5.5}$$

If $\alpha = 0$, (5.5) reduces to $K_j(y) = c^{-1} - 1$, and

$$z_j(y) = n_j / (n_j + c^{-1} - 1),$$

which is independent of y.

In the case $\alpha > 0$, (5.5) decreases as $G_j(y)\left[1 - G_j(y)\right]$ increases. It takes a minimum value of $4^{\alpha}/c - 1$ when $G_j(y) = \frac{1}{2}$, and increases without limit as $G_j(y)$ approaches 0 or 1.

This means that the credibility assigned to $\bar{I}_j(y)$ in (5.1) declines toward zero in the tails of the prior distribution $G_j(y)$.

6. Combining Cell Forecasts

Returning to the motivational example of Section 2, note that outstanding losses in respect of accident year 1995 relate to just the single cell (1995, 4). Their distribution is forecast by (5.1) with j=4.

However, outstanding losses in respect of accident year 1996 relate to the two cells with j=3,4 respectively. The distribution in each of the cells is forecast by (5.1). The distribution of outstanding losses is forecast by the convolution $G_3^* * G_4^*(\bullet)$.

In the more general framework of Sections 3 and 4, let $G_{j+}^*(y)$ denote the forecast of $\text{Prob}\left[\sum_{h=j}^{J} X(i,h) \le y\right]$, where J is the maximum value of j considered. Then

$$G_{j+}^*(y) = G_j^* * G_{j+1}^* * \ldots * G_J^*(y) \tag{6.1}$$

By (5.1), $G_j^*(y)$ will typically be a mixed distribution, since $\bar{I}(y)$ is discrete but (typically) $G_j(y)$ will be continuous. Analytical evaluation of convolutions like (6.1) will therefore be awkward in most cases, and best dealt with numerically.

7. Application to Motivational Example

Consider the example set out in Section 2, and specifically outstanding losses in respect of accident year 1997. This requires the forecast $G_{2+}^*(y)$ defined in Section 6.

This is given by (6.1):

$$G_{2+}^*(y) = G_2^* * G_3^* * G_4^*(y), \tag{7.1}$$

with $G_j^*(\bullet)$, $j = 2,3,4$ given by (5.1) to (5.3).

The input parameters required for this evaluation are $G_j(\bullet), V_{\theta(j)}\, G_j^{(\theta)}(y)$ for j=2,3,4. Suppose that the $G_j(\bullet)$ are gamma d.f.'s:

$$dG_j(y)/dy = \left[\Gamma\left(\alpha_j\right)\right]^{-1} c_j^{\alpha_j} y^{\alpha_j - 1} \exp\left(-c_j y\right), \; y > 0, \tag{7.2}$$

with α_j, c_j as in Table 7.1.

Table 7.1

Parameters for Gamma Distributions

j	α_j	c_j	mean	s.d.
2	16.0	0.0080	2,000	500
3	11.11	0.0222	500	150
4	4.0	0.0200	200	100

Table 7.1 also includes, for each j, the gamma distribution's mean $\left(= \alpha_j / c_j\right)$ and s.d. $\left(= \alpha_j^{\frac{1}{2}} / c_j\right)$.

Suppose further that (compare (5.4))

$$V_{\theta(j)} G_j^{(\theta)}(y) = \tfrac{1}{2} G_j(y)\left[1 - G_j(y)\right] \tag{7.3}$$

Then, by (5.3)

$$K_j(y) = 1. \tag{7.4}$$

By (5.2),

$$z_2(y) = \tfrac{3}{4},\; z_3(y) = \tfrac{2}{3},\; z_4(y) = \tfrac{1}{2}. \tag{7.5}$$

By (5.1),

$$G_2^*(y) = 0.25\, G_2(y) + 0.75\bar{I}_2(y)\,, \tag{7.6}$$

where $\bar{I}_2(y)$ is the d.f. consisting of three jumps of probability 1/3 each at y=1818, 1863, 2129 respectively.

Similar formulas evaluate $G_3^*(y)$ and $G_4^*(y)$ respectively.

Figures 7.1 to 7.3 illustrate the computation of $G_j^*(y)$, j=2,3,4. Each of these plots includes $G_j(y)$, $\bar{I}_j(y)$ and $G_j^*(y)$. Figure 7.4 then plots $G_{2+}^*(y)$, given by (7.1). For comparison, it also plots the corresponding prior $G_2 * G_3 * G_4$.

Fig 7.1
Development year 2

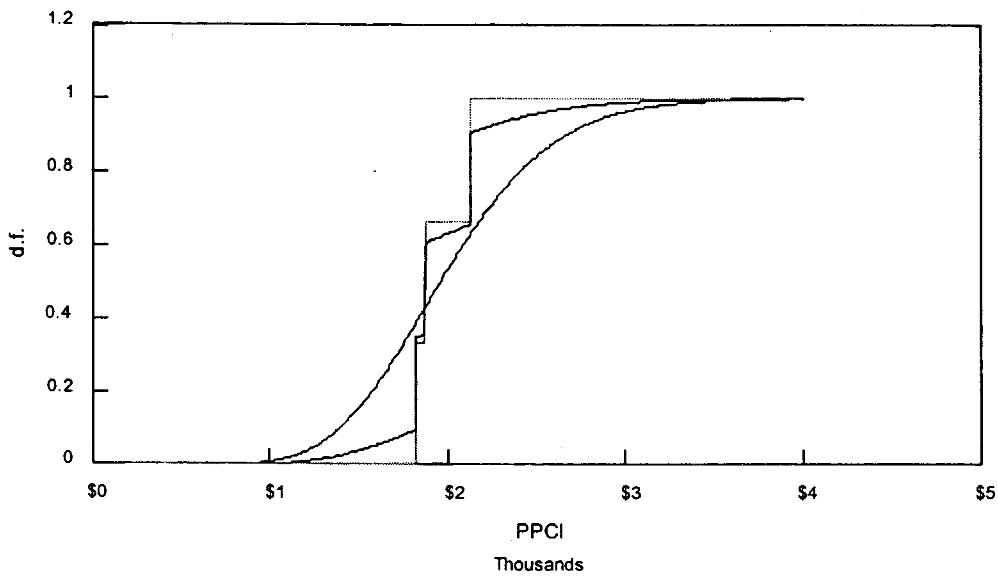


Fig 7.2
Development year 3

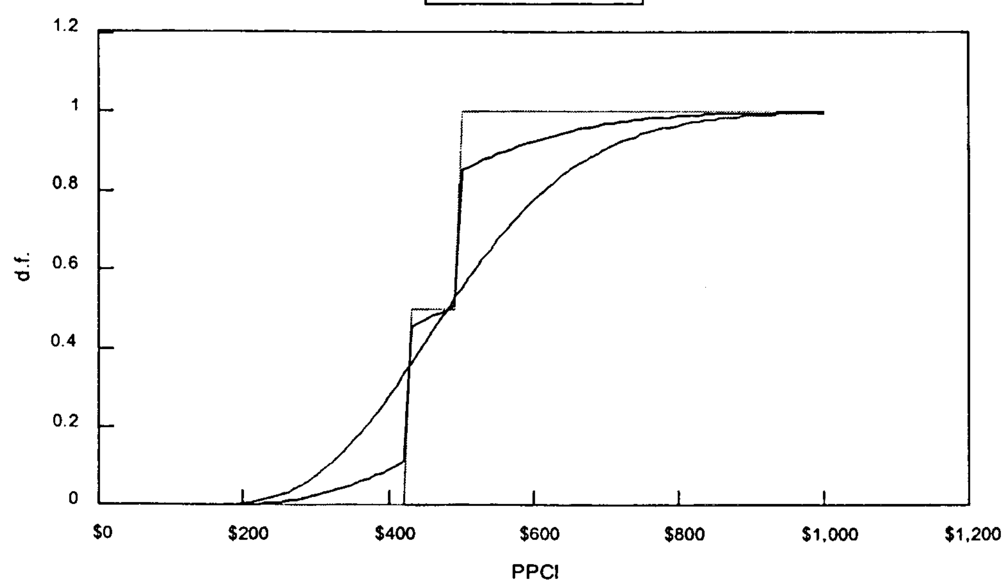


Fig 7.3
Development year 4

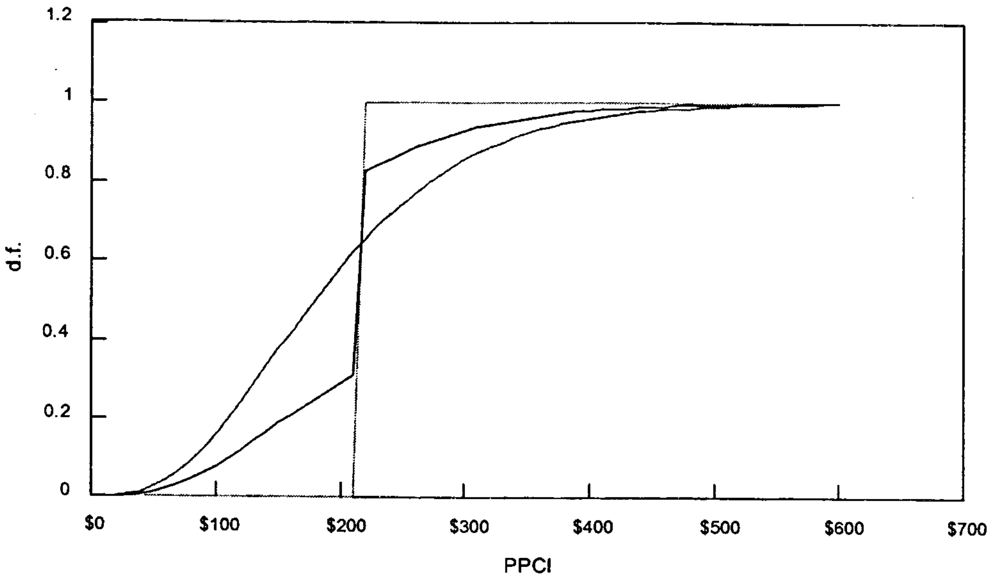


Fig 7.4
Development years 2 to 4

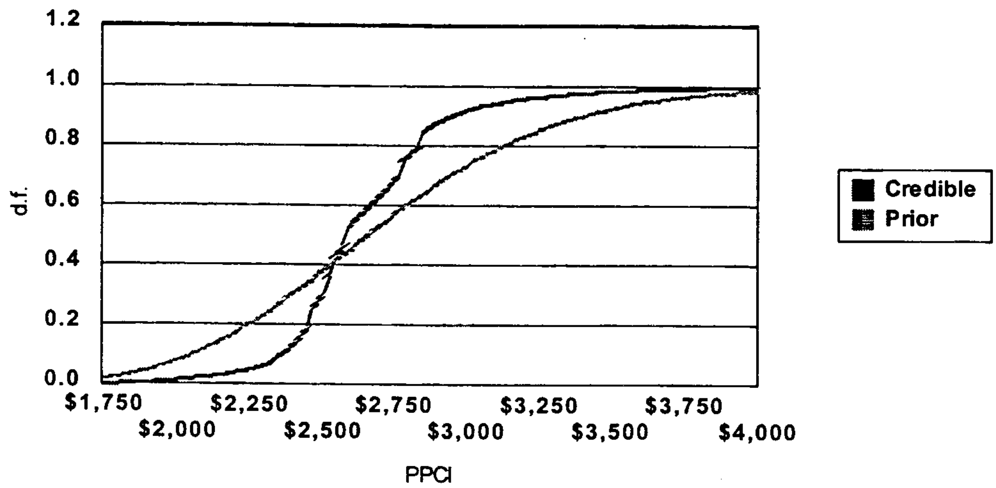


Figure 7.4 shows a reasonable correspondence of G_{2+}^{*} with its prior. This is due to the consistency of $\bar{I}_j(\bullet)$ with $G_j(\bullet)$ for each j, ie the consistency of the observations in Table 2.2 with their prior means (Table 7.1).

8. Other Additive Forms of Outstanding Losses

Sections 6 and 7 were concerned with the outstanding losses of each accident year; equivalently, the outstanding PPCI. Thus, for example, (6.1) provides a forecast of Prob$\left[\sum_{h=j}^{J} X(i,h) \le y\right]$.

The key to this is that the outstanding losses of any accident year are just the summation of a number of the quantities *X(i,j)* whose distributions were forecast in Section 5. The relation between the *X(i,j)* and outstanding losses can be generalised without disturbing the essentials of this structure.

Let $L_k(i)$ denote outstanding losses in respect of accident year *i*, as at the end of experience year *k*. Suppose that

$$L_k(i) = f\left(\sum_{j=k+1-i}^{J} X(i,j) \right), \tag{8.1}$$

for some one-one function *f*. In this framework, the *X(i,j)* may be any quantities satisfying the assumptions made in Section 3.

The forecast distribution of outstanding losses is related to the forecasts of the *X(i,j)* through (8.1).

Since

$$\text{Prob}\left[L_k(i) \le y\right] = \text{Prob}\left[\sum_{j=k+1-i}^{J} X(i,j) \le f^{-1}(y) \right], \tag{8.2}$$

for $f(\bullet)$ increasing (the $\le$ is changed to $\ge$ on the right side of (8.2) if $f(\bullet)$ is decreasing), the left side of (8.2) is forecast by $G_{j+}^{*}\left(f^{-1}(y)\right)$ for $j = k+1-i$, as defined by (6.1).

As an example of (8.1),

$$f(x) = e^x C(i,k) \tag{8.3}$$

$$X(i,j) = \log\left[C(i,j)/C(i,j-1)\right], \tag{8.4}$$

with $C(i,j)$ = cumulative paid losses to end of development year *j* in respect of accident year *i*.

The definitions (8.3) and (8.4) produce a chain ladder analysis (Taylor, 1999, Chapters 2 and 3) with logged age-to-age factors *X(i,j)* . The factor e^x in (8.3) is the age-to-ultimate factor.

In this case, (8.2) becomes

$$\text{Prob}\left[L_k(i) \le y\right] = \text{Prob}\left[\sum_{j=k+1-i}^{J} X(i,j) \le \log\left[y / C(i,k)\right]\right] = G^*_{(k+1-i)+}\left(\log\left[y / C(i,k)\right]\right), \tag{8.5}$$

with $G^*_{(k+1-i)+}$ defined by (6.1).

9. A More Realistic Example

The numerical example of Section 7 was invented for motivational purposes. The present section applies the results of this paper to an example based on real data.

The data, in the form of incremental paid losses are set out in Table 9.1. They are extracted from an Australian Auto Bodily Injury portfolio.

Table 9.2 displays the logged age-to-age factors *X(i,j)*. It also displays the sample mean and standard deviation of these quantities for each *j*. Table 9.2 appears as Table 7.2 in Taylor (1999) as part of a stochastic chain ladder analysis attributed to Hertig (1985).

For this example, it is assumed that each $G_j(\bullet)$ is normal with parameters μ_j given in Table 9.3, and

$$\sigma_j = 0.19 \times 0.8^j \tag{9.1}$$

Table 9.1 Incurred Losses

Period of origin	Incurred losses to end of development year n=																	
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000	$000
1978	9,268	18,263	20,182	22,383	22,782	26,348	26,172	26,184	25,455	25,740	25,711	25,452	25,460	25,422	25,386	25,520	25,646	25,469
1979	9,848	16,123	17,099	18,544	20,534	21,554	23,219	22,381	21,584	21,408	20,857	21,163	20,482	19,971	19,958	19,947	19,991	
1980	13,990	22,484	24,950	33,255	33,295	34,308	34,022	34,023	33,842	33,933	33,570	31,881	32,203	32,345	32,250	32,168		
1981	16,550	28,056	39,995	42,459	42,797	42,755	42,435	42,302	42,095	41,606	40,440	40,432	40,326	40,337	40,096			
1982	11,100	31,620	40,852	38,831	39,516	39,870	40,358	40,355	40,116	39,888	39,898	40,147	39,827	40,200				
1983	15,677	33,074	35,592	35,721	38,652	39,418	39,223	39,696	37,769	37,894	37,369	37,345	37,075					
1984	20,375	33,555	41,756	45,125	47,284	51,710	52,147	51,187	51,950	50,967	51,461	51,382						
1985	9,800	24,663	36,061	37,927	40,042	40,562	40,362	40,884	40,597	41,304	42,378							
1986	11,380	26,843	34,931	37,805	41,277	44,901	45,867	45,404	45,347	44,383								
1987	10,226	20,511	26,882	32,326	35,257	40,557	43,753	44,609	44,196									
1988	8,170	18,567	26,472	33,002	36,321	37,047	39,675	40,398										
1989	10,433	19,484	32,103	38,936	45,851	45,133	45,501											
1990	9,661	23,808	32,966	42,907	46,930	49,300												
1991	14,275	25,551	33,754	38,674	41,132													
1992	13,245	29,206	36,987	44,075														
1993	14,711	27,082	34,230															
1994	12,476	23,126																
1995	9,715																	

Table 9.2 Logged incurred loss age to age factors

Period of origin	Logged age to age factor from development year n to n+1 development year n= 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1978	0.678	0.100	0.104	0.018	0.145	-0.007	0.000	-0.028	0.011	-0.001	-0.010	0.000	-0.001	-0.001	0.005	0.005	-0.007
1979	0.493	0.059	0.081	0.102	0.048	0.074	-0.037	-0.036	-0.008	-0.026	0.015	-0.033	-0.025	-0.001	-0.001	0.002	
1980	0.474	0.104	0.287	0.001	0.030	-0.008	0.000	-0.005	0.003	-0.011	-0.052	0.010	0.004	-0.003	-0.003		
1981	0.528	0.355	0.060	0.008	-0.001	-0.008	-0.003	-0.005	-0.012	-0.028	-0.000	-0.003	0.000	-0.006			
1982	1.047	0.256	-0.051	0.017	0.009	0.012	-0.000	-0.006	-0.006	0.000	0.006	-0.008	0.009				
1983	0.747	0.073	0.004	0.079	0.020	-0.005	0.012	-0.050	0.003	-0.014	-0.001	-0.007					
1984	0.499	0.219	0.078	0.047	0.089	0.008	-0.019	0.015	-0.019	0.010	-0.002						
1985	0.923	0.380	0.050	0.054	0.013	-0.005	0.013	-0.007	0.017	0.026							
1986	0.858	0.263	0.079	0.088	0.084	0.021	-0.010	-0.001	-0.022								
1987	0.696	0.270	0.184	0.087	0.140	0.076	0.019	-0.009									
1988	0.821	0.355	0.220	0.096	0.020	0.069	0.018										
1989	0.625	0.499	0.193	0.163	-0.016	0.008											
1990	0.902	0.325	0.264	0.090	0.049												
1991	0.582	0.278	0.136	0.062													
1992	0.791	0.236	0.175														
1993	0.610	0.234															
1994	0.617																
Average	0.699	0.250	0.124	0.065	0.049	0.020	-0.001	-0.013	-0.004	-0.006	-0.006	-0.007	-0.003	-0.003	0.001	0.004	-0.007
Standard deviation	0.169	0.121	0.095	0.045	0.052	0.033	0.017	0.019	0.013	0.018	0.021	0.014	0.013	0.002	0.004	0.002	

Table 9.3 Parameters of prior distributions

j	μ_j	σ_j
1	0.60	0.152
2	0.20	0.122
3	0.10	0.097
4	0.05	0.078
5	0.03	0.062
6	0.02	0.050
7 and later	0.00	

Table 9.3 also displays values of σ_j, calculated from (9.1), for comparison with the sample values in Table 9.2.

Finally, it is assumed that

$$V_{\theta(j)}G_j^{(\theta)}(y) = 0.1G(y)[1-G(y)], \tag{9.2}$$

so that (5.3) yields

$$K_j(y) = 9. \tag{9.3}$$

While Table 9.2 displays the full data triangle of dimension 17, the example examines estimates of the form (5.1) as the dimension $k+1$ of the triangle grows from 1 to 17. By (5.2) and (9.3),

$$z_j(y) = (k+1-j)/(k+10-j) \tag{9.4}$$

for given k.

Now restore the full notation $G^*_{j(k)}(\bullet)$ for (5.1). This will yield forecast distributions for $j(k) = k+1-i,\ k+2-i,\ \ldots,\ k$ in respect of underwriting year $i = 0, 1, \ldots, k$.

There are no data for $j > k$, and so the forecast distributions $G^*_{j(k)}(\bullet)$ must be taken as the priors $G_{j(k)}(\bullet)$ for $j(k) = k+1,\ k+2,$ etc.

With this understanding, (6.1) is applied to yield

$$G^*_{j(k)+}(y) = G^*_{j(k)} * \ldots * G^*_k * G_{k+1} * \ldots * G_J(y). \tag{9.5}$$

The corresponding prior is

$$\begin{aligned} G_{j+}(y) &= G_j * \ldots * G_J(y) \\ &= \Phi\left(y; \sum_{h=j}^{J} \mu_h, \sum_{h=j}^{J} \sigma_j^2\right) \end{aligned} \tag{9.6}$$

where $\Phi(\bullet; \mu, \sigma^2)$ denotes the normal d.f. with mean μ and variance σ^2.

Result (9.5) gives the d.f. of the logged age-to-ultimate factor that is applied to incurred losses at end of development year *j*.

Let $W(i, k-i)$ denote incurred losses in respect of underwriting year *i*, as measured at end of experience year *k* (ie development year *k* – *i*). Then estimated ultimate incurred losses are given by

$$W^*(i, J+1) = W(i, k-i)\exp(f), \tag{9.7}$$

for logged age-to-ultimate factor *f*.

Then

$$\begin{aligned} \text{Prob}\left[W^*(i, J+1) \le w\right] &= \text{Prob}\left[W(i, k-i)\exp(f) \le w\right] \\ &= \text{Prob}\left[f \le \log\left[w / W(i, k-i)\right]\right] \\ &= G^*_{(k-i)(k)+}\left[\log\left[w / W(i, k-i)\right]\right]. \end{aligned} \tag{9.8}$$

Note that, by (9.4), the credibility factors involved in $G^*_{(k-i)(k)+}$ are

$$\frac{i+1}{i+10}, \frac{i}{i+9}, \frac{i-1}{i+8}, \ldots, \frac{1}{10}, \tag{9.9}$$

which do not depend on the size of the data triangle.

The quantity (9.8) is the forecast distribution of ultimate incurred losses for underwriting year, based on data up to and including experience year *k*. By (9.6), it compares with a prior

$$G_{(k-i)+}\left[\log\left[w / W(i, k-i)\right]\right]. \tag{9.10}$$

Note that this is the prior **conditional on actual losses** incurred to the end of development year *k* – *i*. Specifically, it is **not** the original prior for the underwriting year, ie at end of development year 0, which is

$$G_{0+}\left[\log\left[w / W(i, o)\right]\right].$$

Figures 9.1 to 9.5 display the forecast d.f. in (9.8) and the corresponding prior (9.10) for underwriting year 1980 (i = 2) at the various points of development, corresponding to k = 1, 2, 4, 9, 16 ie 1980, 1981, 1983, 1988, 1995.

Figure 9.1

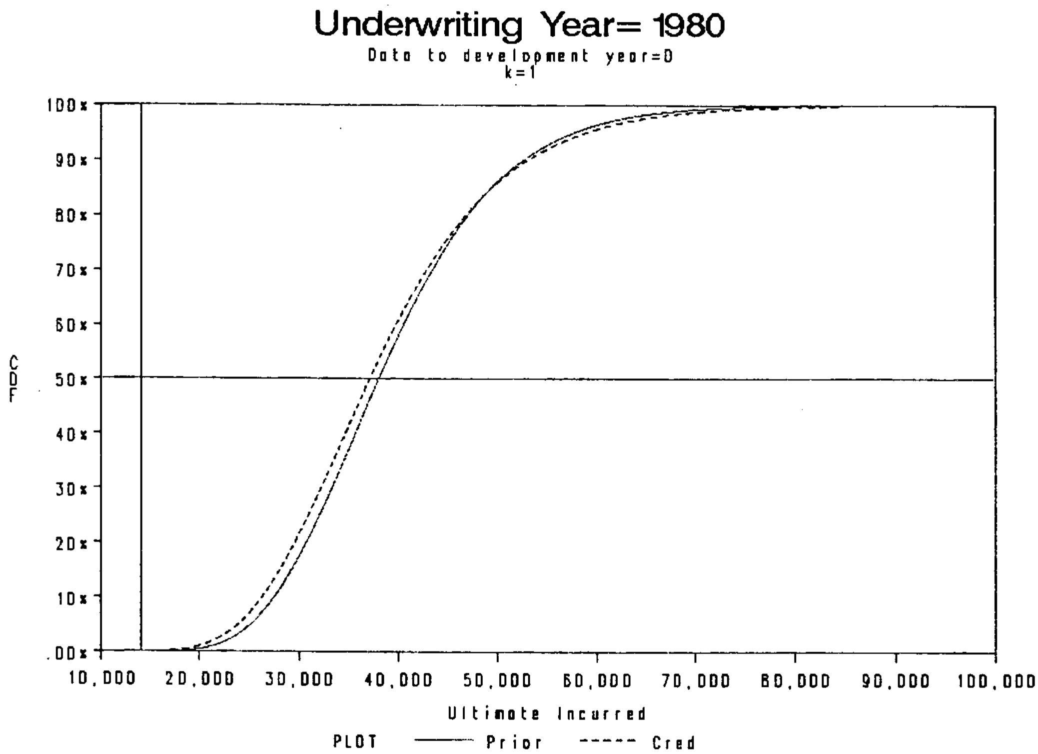


Figure 9.2

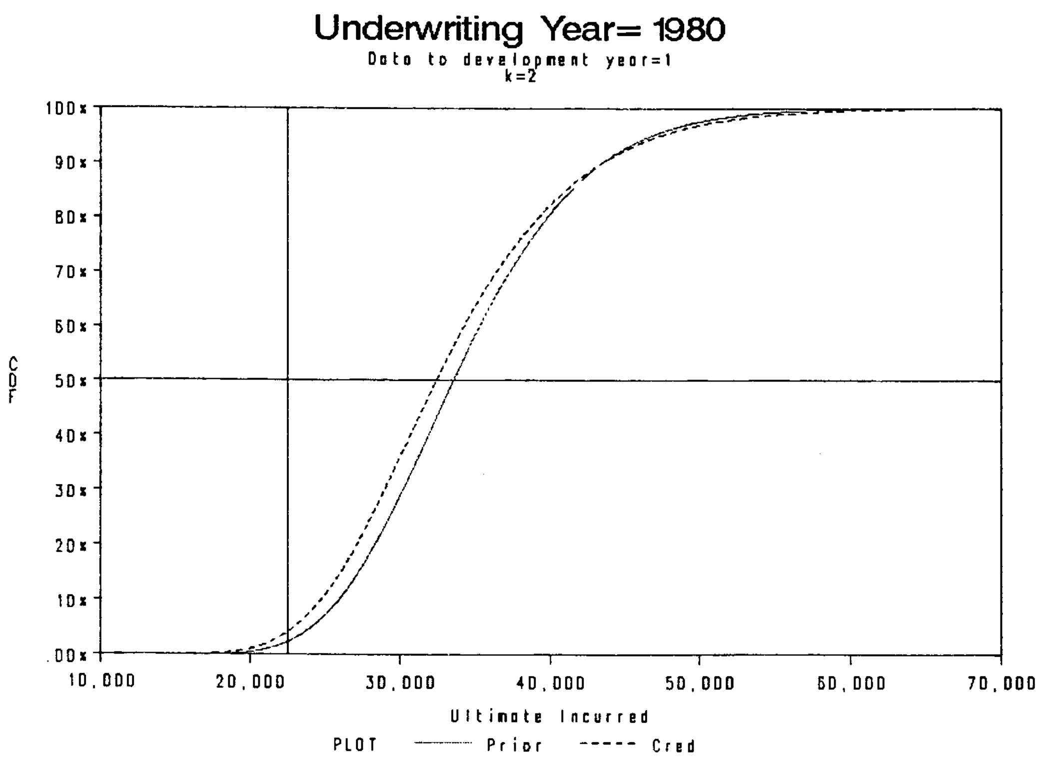


Figure 9.3

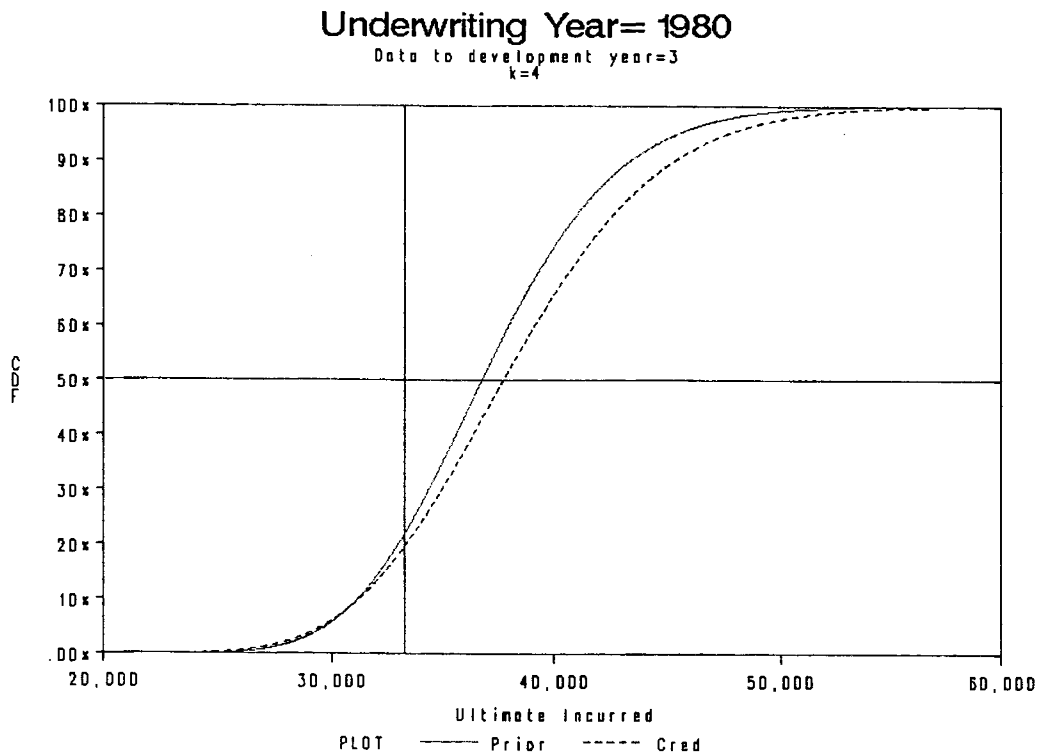


Figure 9.4

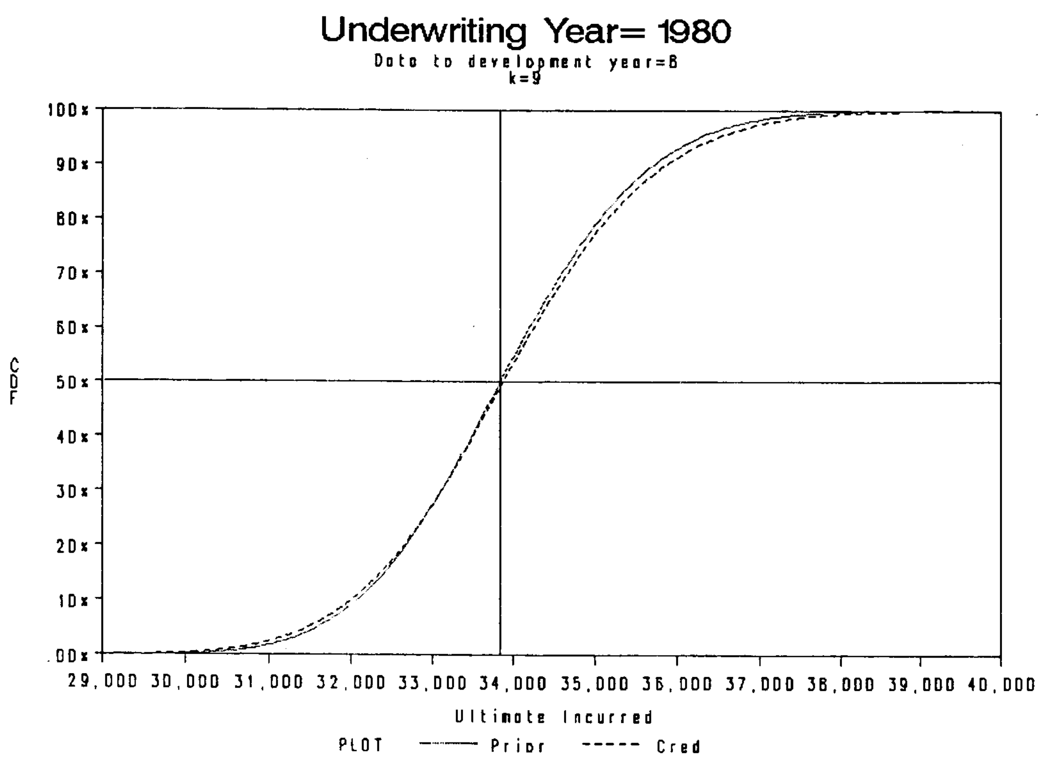
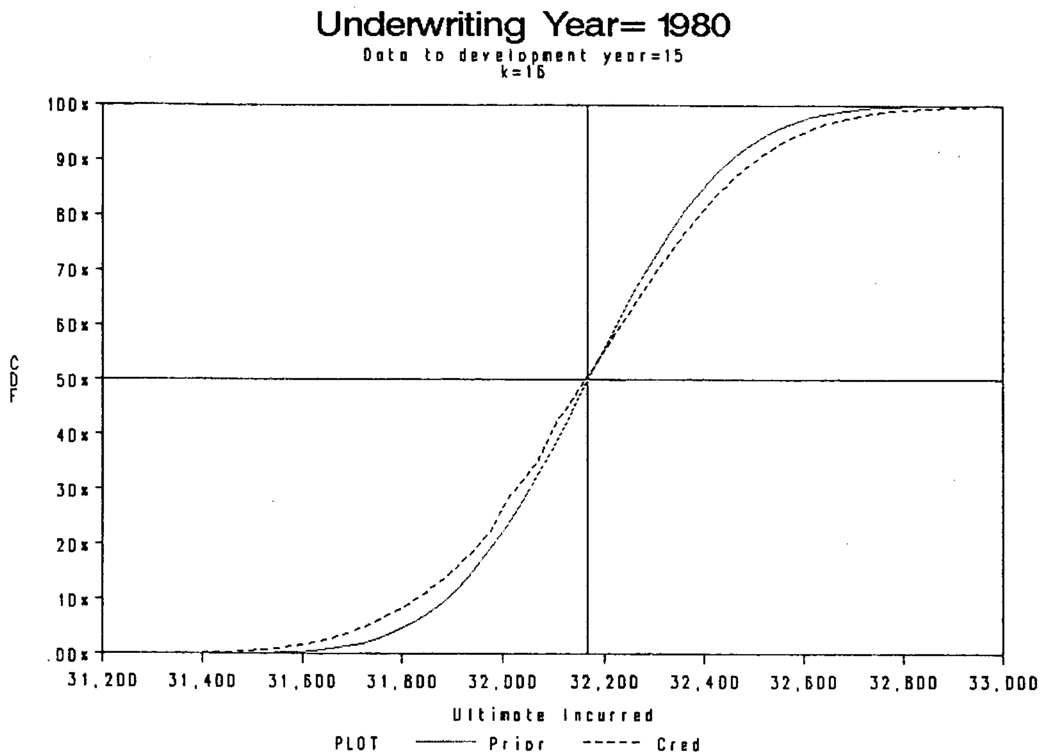


Figure 9.5



The vertical bar in each plot represents incurred losses to date $W(i,k-i)$.

Observations to be made on the plots are:

- The forecast (credible) distribution tends to converge to the prior with increasing development year, due to the reducing number of distributions in convolution (9.5) as j increases.
- The centre (specifically the median) follows the vertical bar for $j \geq 7$ (since then $\mu_j = 0$).
- The forecast distribution loses smoothness at the highest development years, where it is based on only a handful of data points.

10. Acknowledgment

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