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On the Distribution of the Deficit at Ruin When Claims are Phase-Type

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Abstract

We consider the distribution of the deficit at ruin in the Sparre Andersen renewal risk model given that ruin occurs. We show that if the individual claim amounts have a phase-type distribution, then there is a simple phase-type representation for the distribution of the deficit. We illustrate the application of this result with several examples.

Keywords: Phase-type distribution, Sparre Andersen model, Renewal risk model, Deficit at ruin, Ladder height, Maximal aggregate loss, Erlang distribution, Coxian distribution.

1 Introduction

The purpose of this paper is to establish a fairly robust result regarding the distribution of the deficit at ruin in the general renewal risk model, often referred to as the Sparre Andersen model. In this model, the insurer's surplus at time t, which we denote by $U(t)$, is given by

$$U(t) = u + ct - \mathcal{S}(t)$$

where u is the initial surplus, c is the rate of premium income per unit time, and $\mathcal{S}(t)$ denotes the aggregate claim amount up to time t. We assume that the aggregate claims process $\{\mathcal{S}(t)\}_{t\geq 0}$ is comprised of a renewal claim number process $\{\mathcal{N}(t)\}_{t\geq 0}$ whose interclaim times $\{W_1, W_2, \ldots\}$ are generally distributed with common mean $E(W) = 1/\lambda$. The individual claim amounts $X_1, X_2, \ldots$, independent of $\{\mathcal{N}(t)\}_{t\geq 0}$, are positive, independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) $P(t) = 1 - \bar{P}(t) = \Pr(X \leq t)$ and mean $E(X) = \mu$. We assume that $c = (1+\theta)\lambda\mu$, where $\theta > 0$ is the premium loading factor. We define $R = cW$ as the interclaim revenue random variable having cdf $A(y) = \Pr(R \leq y)$ and moment generating function (mgf) $\hat{A}(s) = \int_0^\infty e^{sy} dA(y)$.

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For this surplus process, we define the probability of ultimate ruin as

$$\psi(u) = \Pr(U(t) < 0 \text{ for some } t > 0)$$

or, equivalently, $\psi(u) = \Pr(T < \infty)$ where T denotes the time of ruin and is defined by

$$T = \begin{cases} \inf\{t | U(t) < 0\} \\ \infty \text{ if } U(t) \geq 0 \text{ for all } t > 0. \end{cases}$$

We write $\delta(u) = 1 - \psi(u)$, and for convenience we introduce $\rho = \psi(0)$. (It is well known that $\rho = 1/(1+\theta)$ in the case of a Poisson claim number process; see, for example, Bowers *et al* (1997).) Note that $\delta(x) = \Pr(L \leq x)$ is the cdf of the maximal aggegrate loss L.

As will be seen in the next section, phase-type distributions are characterized by an initial probability vector and a matrix of transition rates. The main result of this paper is given in Section 3 where we show that if claim amounts are phase-distributed, then so too is the deficit, with precisely the same matrix of transition rates. What is particularly remarkable about this result is that it applies for the general Sparre Andersen model. In fact, if the claim number process is not Poisson, the only additional step needed in most cases is a recursive calculation to determine the necessary initial probability vector.

Section 4 contains several numerical examples intended to illustrate the main result, and general comments on the power of the result are contained in Section 5.

2 Mathematical Preliminaries

2.1 Notation

In the sequel, we shall make use of the following two functions:

$$G(u, y) = \Pr(T < \infty,\ U(T) \geq -y) \quad \text{and} \quad \psi(u, y) = \Pr(T < \infty,\ U(T) < -y).$$

Clearly, $\psi(u, y) = \psi(u) - G(u, y)$. Gerber *et al* (1987) introduced the function $G(u, y)$ for the classical risk model, defined as the probability that ruin occurs from initial surplus u with a deficit at ruin no greater than y. $\psi(u, y)$ is the probability that ruin occurs and the deficit at ruin exceeds y. Let

$$g(u, y) = \frac{d}{dy} G(u, y)$$

denote the (defective) probability density function (pdf) of the deficit at ruin if it exists. These functions have been studied in the classical risk model by a number of authors including Dufresne and Gerber (1988), Dickson (1989), and Willmot (2000). Corresponding to these functions, we introduce the proper (non-defective) cdf G_u defined by

$$G_u(y) = 1 - \bar{G}_u(y) = \frac{G(u,y)}{\psi(u)}$$

with pdf

$$g_u(y) = \frac{g(u,y)}{\psi(u)}.$$

We let the random variable Y_u denote the deficit at ruin given that ruin occurs, so that Y_u has cdf G_u and pdf g_u. In the special case of a Poisson claim number process, it is well-known (see, for example, Bowers *et al* (1997)) that the distribution of Y_u when $u = 0$ is the equilibrium distribution of P, defined by $P_e(x) = 1 - \bar{P}_e(x) = \int_0^x \bar{P}(y)dy/\mu$, so that $G_0(y) = P_e(y)$.

2.2 Phase-type distributions

In this section, we summarize key results for phase-type distributions in ruin theory. Proofs of all results quoted in this section can be found in Rolski *et al* (1999, Chapter 8) and Asmussen (2000, Chapter 8). An important early reference on the use of phase-type distributions in a ruin-theoretic context is Asmussen and Rolski (1991). A more detailed description of phase-type distributions and their applications can be found in Neuts (1981) and Latouche and Ramaswami (1999). A brief overview of phase-type distributions and their properties can also be found in Asmussen (1992) and Stanford and Stroiński (1994). We provide a summary below.

Consider a continuous time Markov chain (CTMC) with a single absorbing state 0 and m transient states. Let J_t denote the state of the CTMC at time t. The row vector α contains the probabilities α_j that the process starts in the various transient states $j = 1, 2, \ldots, m$. The components of α need not sum to 1, as the process may start in the absorbing state with probability α_0. However, in all cases where one is using a phase-type distribution to model a purely continuous quantity with no discrete weight at 0, we require $\alpha e^T = 1$. (Here, and in what follows, e^T is a column vector of ones of length m.) The infinitesimal generator Q for the CTMC is given by

$$Q = \begin{bmatrix} 0 & 0 \\ s_0^T & S \end{bmatrix}.$$

In other words, $S = [s_{ij}]$ is the matrix of transition rates among the transient states, and $s_0^T = [s_{i0}]$ is the column vector of absorption rates into state 0 from the transient states. Necessarily, $s_0^T = -Se^T$, and S is an $m \times m$ matrix whose diagonal entries are negative and whose other entries are non-negative. Under these assumptions, the time V until absorption has occurred has mean

$$E(V) = -\alpha S^{-1} e^T,$$

cdf

$$F(x) = 1 - \bar{F}(x) = 1 - \alpha \exp(xS)e^T \quad \text{for} \quad x \geq 0,$$

and pdf

$$f(x) = \alpha \exp(xS)s_0^T \quad \text{for} \quad x > 0,$$

where the matrix exponential is defined by

$$\exp(xS) = \sum_{n=0}^{\infty} \frac{x^n}{n!} S^n.$$

Furthermore, we can interpret the jth component of $\alpha \exp(xS)$ as the probability that absorption has not occurred by time x, and that the process is in transient state j at time x; that is

$$\Pr(V > x, J_x = j; j = 1, 2, \ldots, m) = \alpha \exp(xS).$$

As this distribution is completely determined by α and S, we say either that V is "phase-distributed with representation (α, S)", or write $V \sim PH(\alpha, S)$. Occasionally, we will say that "F has PH representation (α, S)".

We turn now to closure properties of phase-type distributions that will be used in the sequel. First of all, the equilibrium distribution of F, denoted by F_e, is also phase-distributed with representation (π, S) where

$$\pi = -\alpha S^{-1}/E(V)$$

so that $\pi e^T = 1$. In other words, even if F has probability mass at 0, its equilibrium distribution F_e does not. The fact that the equilibrium distribution employs the same matrix S as the original distribution underlies a key fact: any form of conditional distribution based on the time to absorption exceeding a given value necessarily employs the same matrix S. This is because, if absorption has not occurred by time t, say, then at time t the process must be in one of the transient states, and the future evolution of the process is uniquely governed by the transition rate matrix S. This "matrix memorylessness", as we call it, means that such ruin-theoretic quantities as the ladder height distribution employ the same matrix as the claim amount distribution, if the latter is phase-distributed. Therefore, such ruin-theoretic quantities remain within the same subclass of phase-type distributions for well-known families such as mixtures of Erlangs or Coxian distributions.

Next, we consider the distribution of sums of phase-distributed random variables. If $Y \sim PH(\alpha, S)$ is independent of $Z \sim PH(\beta, T)$, then the sum $Y + Z$ is likewise phase-distributed with representation (Δ, C) where the row vector $\Delta = (\alpha, \alpha_0 \beta)$ and

$$C = \begin{bmatrix} S & s_0^T \beta \\ 0 & T \end{bmatrix}.$$

In other words, once the process has left the transient states associated with Y, it moves on to the states associated with Z, from which it is ultimately absorbed. Furthermore, the sum of n iid random variables, $Y(n) = Y_1 + Y_2 + \cdots + Y_n$ where each $Y_i \sim PH(\alpha, S)$, is phase-distributed. In fact, $Y(n) \sim PH(\alpha(n), T(n))$ where $\alpha(n) = (\alpha, \alpha_0\alpha, \alpha_0^2\alpha, \ldots, \alpha_0^{n-1}\alpha)$ and

$$T(n) = \begin{bmatrix} S & s_0^T\alpha & 0 & \cdots & 0 & 0 \\ 0 & S & s_0^T\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S & s_0^T\alpha \\ 0 & 0 & 0 & \cdots & 0 & S \end{bmatrix}.$$

The corresponding column vector of absorption rates is given by

$$s_0^T(n) = (\alpha_0 s_0, \alpha_0 s_0, \alpha_0 s_0, \ldots, \alpha_0 s_0, s_0)^T.$$

Note that if each Y_i is strictly positive, then $\alpha_0 = 0$ and the above quantities simplify considerably.

The last, and most important, closure property we state has implications for the maximal aggregate loss distribution. Let N be a geometrically distributed random variable with $\Pr(N = n) = (1 - \phi)\phi^n, n = 0, 1, \ldots$. A compound geometric sum $L = L_1 + \cdots + L_N$ of N iid phase-distributed random variables $L_i \sim PH(\alpha, S)$ is again phase-distributed. Moreover, its representation is of the same order m and is easily calculated: $L \sim PH(\gamma, B)$ where $\gamma = \phi\alpha$ and

$$B = S + \phi s_0^T \alpha.$$

Figures 1 and 2 help to visualize this key result. Figure 1 illustrates one phase-type representation for L. With probability $1 - \phi$, $N = 0$ so that $L = 0$. This is represented by the arc going to the ultimate absorbing state $\mathcal{A}$. In all other cases, L consists of at least L_1, so with probability ϕ, at least one $PH(\alpha, S)$ random variable is involved. Upon absorption into the absorbing state 0, however, one returns with probability ϕ for another, independent $PH(\alpha, S)$ length of time. It is easily verified that the number of times N through this process is geometrically distributed with $\Pr(N = n) = (1-\phi)\phi^n$. Thus, we can write $L = L_1 + \cdots + L_N$. An alternate representation, however, is displayed in Figure 2 by ignoring the instances of absorption into state 0, and focussing instead on the ultimate absorption into state $\mathcal{A}$. One can move directly from transient state i to transient state j at rate s_{ij}, or indirectly at rate $\phi s_{i0}\alpha_j$ by being absorbed from state i, feeding back, and restarting in state j; that is, at overall rate $b_{ij} = s_{ij} + \phi s_{i0}\alpha_j$. The matrix $B = [b_{ij}]$ contains the sum of these rates.

If we assume that the iid claim amount random variables $X_i \sim PH(\alpha, S)$, then the ruin-theoretic consequences of these above results can be summarized as follows:

- Let F now be the cdf associated with the (non-defective) ladder height distribution. From Propositions 4.1 and 4.3 of Asmussen (2000), pp.229-230, F has PH representation (α_+^*, S) where $\alpha_+^* e^T = 1$. Furthermore, $\alpha_+^* = \alpha_+/\rho$ where the (defective) row vector α_+ is the unique solution of a fixed-point problem; namely, α_+ satisfies $\alpha_+ = \varphi(\alpha_+)$ where

$$\varphi(\alpha_+) = \alpha \hat{A}(S + s_0^T \alpha_+) = \alpha \int_0^\infty \exp(y(S + s_0^T \alpha_+))dA(y). \qquad (1)$$

- Let L now represent the maximal aggregate loss. From Theorem 4.4 in Asmussen (2000), pp.230-231, the probability of ultimate ruin in the general Sparre Andersen model with phase-distributed claim amounts is given by

$$\psi(u) = \Pr(L > u) = \alpha_+ \exp(uB)e^T$$

where $B = S + D$ and $D = s_0^T \alpha_+ = \rho s_0^T \alpha_+^*$. In other words, $L \sim PH(\alpha_+, B)$.

3 The Main Result

Henceforth, define $C = \begin{bmatrix} S & D \\ 0 & S+D \end{bmatrix} = \begin{bmatrix} S & s_0^T \alpha_+ \\ 0 & B \end{bmatrix}$. Furthermore, let $e_*^T = \begin{bmatrix} e^T \\ e^T \end{bmatrix}$. We are now ready to state the main result.

Theorem: *The deficit Y_u is phase-distributed with representation (π_G, S) where*

$$\pi_G = \frac{\alpha_+ \exp(uB)}{\alpha_+ \exp(uB)e^T} = \frac{\alpha_+ \exp(uB)}{\psi(u)}.$$

To establish this result, we present both an analytic and a probabilistic proof. The analytic proof employs established, current results from recent advances in ruin theory, and thus better fits into the body of existing work. On the other hand, the probabilistic proof provides a better intuitive understanding. As each reveals important aspects that the other does not, we have included both for greater exposition. We begin with the analytic proof.

Analytic Proof:

The starting point for the proof is eq.(2.1) of Willmot (2001):

$$\begin{aligned}
\bar{G}_u(y) &= \frac{\rho}{(1-\rho)\psi(u)} \int_0^u \bar{F}(y + u - x)d\delta(x) \\
&= \frac{\rho}{(1-\rho)\psi(u)} \int_0^u \alpha_+^* \exp((y + u - x)S)e^T d\delta(x) \\
&= \left(\frac{\rho}{(1-\rho)\psi(u)} \alpha_+^* \int_0^u \exp((u - x)S)d\delta(x) \right) \exp(yS)e^T \\
&= \gamma_u \exp(yS)e^T, \qquad (2)
\end{aligned}$$

where

$$\gamma_u = \frac{\rho}{(1-\rho)\psi(u)}\alpha_+^* \int_0^u \exp((u-x)S)d\delta(x). \tag{3}$$

Clearly, (2) implies that G_u is differentiable and so g_u is well-defined. As indicated in Latouche and Ramaswami (1999), p.42, the entries of the matrix exponential are necessarily either positive whenever $(u-x)$ is, or identically zero for all $u \geq x$. As a consequence, the components of γ_u are non-negative. Furthermore, it follows using eq.(2.7) of Willmot (2001) that

$$\begin{aligned}
\gamma_u e^T &= \frac{\rho}{(1-\rho)\psi(u)} \int_0^u \alpha_+^* \exp((u-x)S)e^T d\delta(x) \\
&= \frac{\rho}{(1-\rho)\psi(u)} \int_0^u \bar{F}(u-x)d\delta(x) \\
&= \frac{\rho}{(1-\rho)\psi(u)} \cdot \frac{(1-\rho)\psi(u)}{\rho} \\
&= 1,
\end{aligned}$$

so that γ_u is a valid (non-defective) initial probability vector. Therefore, $Y_u \sim PH(\gamma_u, S)$. We now find an alternate expression for γ_u. To do this, we make use of a result which holds for the general Sparre Andersen model. In particular, from eq.(2.13) of Willmot (2001), one has

$$\frac{\psi(u+y)}{\psi(u)} = \bar{G}_u(y) + \int_0^y \psi(y-x)g_u(x)dx,$$

which is clearly equivalent to

$$\delta(u+y) = \delta(u) + \int_0^y g(u,x)\delta(y-x)dx$$

given by Dickson (1989), eq.(1). After straightforward manipulation, we can rewrite either of these as

$$1 - \frac{\psi(u+y)}{\psi(u)} = \int_0^y g_u(x)\delta(y-x)dx. \tag{4}$$

Working with the left hand side (LHS) of (4), note that

$$\begin{aligned}
1 - \frac{\psi(u+y)}{\psi(u)} &= 1 - \frac{\alpha_+ \exp((u+y)B)e^T}{\psi(u)} \\
&= 1 - \frac{\alpha_+ \exp(uB)}{\psi(u)} \exp(yB)e^T \\
&= 1 - \pi_G \exp(yB)e^T. \qquad (5)
\end{aligned}$$

Turning now to the right hand side (RHS) of (4), we observe that the expression

$$\int_0^y g_u(x)\delta(y-x)dx$$

represents the cdf for the sum of a $PH(\gamma_u, S)$ random variable and a $PH(\alpha_+, B)$ random variable. By Theorem 8.2.6 of Rolski *et al* (1999), or Theorem 2.2.2 of Neuts (1981), the convolution of these distributions is again phase-distributed with representation $((\gamma_u, 0), C)$. Therefore,

$$\int_0^y g_u(x)\delta(y-x)dx = 1 - (\gamma_u, 0)\exp(yC)e_*^T. \tag{6}$$

We now directly calculate $\exp(yC) = \sum_{n=0}^{\infty} \frac{y^n}{n!} C^n$. Note that

$$\begin{aligned} C^2 &= \begin{bmatrix} S & D \\ 0 & S+D \end{bmatrix}^2 \\ &= \begin{bmatrix} S^2 & (S+D)^2 - S^2 \\ 0 & (S+D)^2 \end{bmatrix}. \end{aligned}$$

In general, it is not difficult to show by induction that

$$C^n = \begin{bmatrix} S & D \\ 0 & S+D \end{bmatrix}^n = \begin{bmatrix} S^n & (S+D)^n - S^n \\ 0 & (S+D)^n \end{bmatrix} \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Therefore, it immediately follows that

$$\begin{aligned} \exp(yC) &= \sum_{n=0}^{\infty} \frac{y^n}{n!} \begin{bmatrix} S^n & (S+D)^n - S^n \\ 0 & (S+D)^n \end{bmatrix} \\ &= \begin{bmatrix} \exp(yS) & \exp(yB) - \exp(yS) \\ 0 & \exp(yB) \end{bmatrix}. \end{aligned}$$

Substituting this result into (6) yields

$$\begin{aligned} \int_0^y g_u(x)\delta(y-x)dx &= 1 - (\gamma_u, 0)\exp(yC)e_*^T \\ &= 1 - (\gamma_u \exp(yS), \gamma_u \exp(yB) - \gamma_u \exp(yS))e_*^T \\ &= 1 - \gamma_u \exp(yB)e^T. \end{aligned} \tag{7}$$

Comparing the expressions for the LHS and RHS of (4), equations (5) and (7) imply that $\gamma_u \exp(yB)e^T = \pi_G \exp(yB)e^T$ for all $y \geq 0$. This equality can be interpreted as follows: letting $\mathcal{X} \sim PH(\pi_G, S)$ and $\mathcal{Y} \sim PH(\gamma_u, S)$, and assuming that both $\mathcal{X}$ and $\mathcal{Y}$ are independent of the maximal aggregate loss L, we have

$$\Pr(\mathcal{Y} + L > y) = \gamma_u \exp(yB)e^T = \pi_G \exp(yB)e^T = \Pr(\mathcal{X} + L > y) \tag{8}$$

for all non-negative values of u and y. Therefore, (8) implies that $\mathcal{Y} + L$ is equivalent in distribution to $\mathcal{X} + L$, and hence $\mathcal{Y}$ is equivalent in distribution to $\mathcal{X}$, due to the independence of both $\mathcal{X}$ and $\mathcal{Y}$ from L. Therefore, for the purposes of determining the initial probability vector, we can resort to γ_u or π_G. As the latter is notably simpler in form, we determine π_G, and this concludes the analytical proof. □

In the probabilistic proof of the Theorem below, we establish that, not only are the representations $PH(\gamma_u, S)$ and $PH(\pi_G, S)$ equivalent, but $\gamma_u = \pi_G$.

Probabilistic Proof:

This proof starts from the fact that the maximal aggregate loss L follows a compound geometric distribution, with the terms in the random sum being the individual ladder heights L_i (see, for example, Rolski *et al* (1999), Section 6.5). We consider the joint probability $\psi(u, y)$ that ruin occurs and that the deficit upon ruin exceeds y, given an initial surplus of u. Now, ruin occurs if and only if the maximal aggregate loss L exceeds u. Expressing L in its compound geometric form, we obtain

$$\begin{aligned}\psi(u,y) &= \Pr(L > u, U(T) < -y) \\ &= \sum_{n=1}^{\infty}(1-\rho)\rho^n \Pr(L_1 + \cdots + L_n > u, U(T) < -y). \qquad (9)\end{aligned}$$

By the Law of Total Probability where we condition upon the particular ladder height L_k that causes ruin, we can write

$$\begin{aligned}\psi(u,y) &= \sum_{n=1}^{\infty}(1-\rho)\rho^n \sum_{k=1}^{n} \Pr(L_1 + \cdots + L_{k-1} \le u, L_1 + \cdots + L_k > u + y) \\ &= \sum_{k=1}^{\infty} \rho^k \Pr(L_1 + \cdots + L_{k-1} \le u, L_1 + \cdots + L_k > u + y), \qquad (10)\end{aligned}$$

where the latter expression is immediately obtained after interchanging the orders of summation. We note that this expression can be interpreted probabilistically as follows: $\Pr(L_1 + \cdots + L_{k-1} \le u, L_1 + \cdots + L_k > u + y)$ is the probability that the sum of $(k-1)$ iid ladder heights does not cause ruin, but the kth ladder height does, and the sum of the k ladder heights $L_1 + \cdots + L_k$ exceeds the initial surplus by an amount in excess of y. Furthermore, ρ^k represents the probability that the maximal aggregate loss L consists of at least these k ladder heights. Carrying out the required integrations produces the following results:

$$\begin{aligned}\psi(u,y) &= \sum_{k=1}^{\infty} \rho^k \Pr(L_1 + \cdots + L_{k-1} \le u, L_1 + \cdots + L_k > u + y) \\ &= \sum_{k=1}^{\infty} \rho^k \int_0^u \Pr(L_k > u + y - t) d\Pr(L_1 + \cdots + L_{k-1} \le t)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \rho^k \int_0^u \alpha_+^*(k-1)\exp(t\mathcal{T}(k-1))s_0^T(k-1)\alpha_+^*\exp((u-t+y)S)e^T dt \\
&= \left(\sum_{k=1}^{\infty} \rho^k \int_0^u \alpha_+^*(k-1)\exp(t\mathcal{T}(k-1))s_0^T(k-1)\alpha_+^*\exp((u-t)S)dt\right)\exp(yS)e^T \\
&= \gamma_u'\exp(yS)e^T, \qquad (11)
\end{aligned}
$$

where γ_u' is the entire expression within brackets. After a derivation similar to the above, it is not hard to show that $\gamma_u' = \psi(u)\gamma_u$, where γ_u is defined by (3). Therefore, (11) establishes that the joint probability $\psi(u, y)$ possesses the desired phase-type form involving the matrix S; it remains only to show that γ_u' simplifies considerably.

In order to do so, we return to the traditional definition of a phase-type distribution as representing the time to absorption in a transient CTMC. So long as absorption has not occurred by a particular time instant, the process must be in one of the transient states at that time instant. Recalling J_t as the state of the process at time t, assuming that absorption has not occurred, and N as the number of ladder heights comprised in the maximal aggregate loss, we can then interpret the vector γ_u' as follows:

$$
\begin{aligned}
\gamma_u' &= \sum_{k=1}^{\infty} \rho^k \int_0^u \alpha_+^*(k-1)\exp(t\mathcal{T}(k-1))s_0^T(k-1)\alpha_+^*\exp((u-t)S)dt \\
&= \sum_{k=1}^{\infty} \rho^k \Pr(L_1 + \cdots + L_{k-1} \le u, L_1 + \cdots + L_k > u, J_u = j; j = 1, 2, \ldots, m) \\
&= \sum_{k=1}^{\infty} \Pr(L_1 + \cdots + L_{k-1} \le u, L_1 + \cdots + L_k > u, N \ge k, J_u = j; j = 1, 2, \ldots, m) \\
&= \Pr(L > u, J_u = j; j = 1, 2, \ldots, m). \qquad (12)
\end{aligned}
$$

However, this last probability vector is more easily determined according to the form set out in Section 2.2:

$$
\begin{aligned}
\gamma_u' &= \Pr(L > u, J_u = j; j = 1, 2, \ldots, m) \\
&= \alpha_+\exp(uB). \qquad (13)
\end{aligned}
$$

After substituting for γ_u' in (11), and dividing $\psi(u, y)$ by the probability of ruin $\psi(u)$, the same expression is obtained for the initial probability vector γ_u as before; namely, π_G. This completes the probabilistic proof. □

An immediate consequence of equations (11) and (13) is the following result:

Corollary: *The joint probability that ruin occurs and the deficit at ruin exceeds y is given by*

$$\psi(u, y) = \alpha_+\exp(uB)\exp(yS)e^T.$$

Remark: We observe that every component of the vector $\pi_G = \alpha_+ \exp(uB)/\psi(u)$ is positive for $u > 0$ even if the original probability vector α for the claim amount distribution has some null components (as in the case of an Erlang distribution, for example). This can be seen from Latouche and Ramaswami (1999), p.42, or from the following reasoning, by contradiction. In order for some component (say the jth) of π_G to be null, it would have to be the case that there was no way to reach transient state j at time u from *any* transient state at time 0. Using results for CTMCs, if one cannot reach state j at a particular time u, then one cannot reach it at *any* point in time, including time 0. Thus, state j is completely redundant to the phase-type fitting, and can be eliminated, thereby reducing the order of the phase-type fit.

Simplifications in the Compound Poisson Model:

Recall that α_+ is the ***unique*** solution of the fixed-point problem defined by (1). In the classical compound Poisson model, interclaim revenues are exponentially distributed with $A(y) = 1 - e^{-\lambda y/c}$ for $y \geq 0$. Letting $\lambda_* = \lambda/c$, note that

$$\begin{aligned}\int_0^\infty \exp(y(S + s_0^T \alpha_+)) dA(y) &= \int_0^\infty \exp(y(S + s_0^T \alpha_+)) \lambda_* e^{-\lambda_* y} dy \\ &= \lambda_* \int_0^\infty \exp(y(-\lambda_* I_m + S + s_0^T \alpha_+) dy \\ &= \lambda_* (\lambda_* I_m - S - s_0^T \alpha_+)^{-1},\end{aligned}$$

where I_m represents the $m \times m$ identity matrix. Therefore, (1) implies that α_+ satisfies

$$\alpha_+ = \lambda_* \alpha (\lambda_* I_m - S - s_0^T \alpha_+)^{-1},$$

or equivalently,

$$\lambda_* \alpha_+ - \alpha_+ S - \alpha_+ s_0^T \alpha_+ - \lambda_* \alpha = 0. \tag{14}$$

Based on Corollary 3.1 of Asmussen (2000), p.227, we try as the *candidate* solution $\alpha_+ = -\lambda_* \alpha S^{-1}$. Note that the LHS of (14) then becomes

$$\begin{aligned}& \lambda_* \alpha_+ - \alpha_+ S - \alpha_+ s_0^T \alpha_+ - \lambda_* \alpha \\ =\ & -\lambda_*^2 \alpha S^{-1} + \lambda_* \alpha S^{-1} S - \lambda_*^2 \alpha S^{-1} s_0^T \alpha S^{-1} - \lambda_* \alpha \\ =\ & -\lambda_*^2 \alpha S^{-1} + \lambda_* \alpha + \lambda_*^2 \alpha S^{-1} (S e^T) \alpha S^{-1} - \lambda_* \alpha \\ =\ & -\lambda_*^2 \alpha S^{-1} + \lambda_*^2 \alpha e^T \alpha S^{-1} \\ =\ & -\lambda_*^2 \alpha S^{-1} + \lambda_*^2 \alpha S^{-1} \\ =\ & 0.\end{aligned}$$

This leads to the following result:

Corollary: *In the case of a Poisson claim number process, $\alpha_+ = \rho\eta$ where $\eta = -\alpha S^{-1}/\mu$ represents the initial probability vector of the equilibrium claim amount distribution P_e.*

Proof: Direct substitution yields

$$\alpha_+ = -\lambda_* \alpha S^{-1} = -\frac{\lambda}{c}\alpha S^{-1} = \frac{1}{1+\theta}(-\alpha S^{-1}/\mu) = \rho\eta$$

which completes the proof. □

While we are able to obtain an explicit formula for α_+ in this particular model, it is more difficult to do so in general. In this more general situation, we may compute α_+ numerically, as in the iterative procedure described in Asmussen (2000), pp.230-231.

4 Examples

In this section, we illustrate the application of the results of the previous section with four examples. We comment that the computation of matrix exponentials is a simple task with the aid of software. The results in this section can be readily obtained using packages such as *Mathematica* and *Maple.*

Example 1: Our first example is intended to illustrate the computation of the various vectors and the matrix exponential for a well-known simple problem. Example 1 of Gerber *et al* (1987) considers an individual claim amount distribution that is an equal mixture of two exponentials at rates 3 and 7 respectively, with Poisson claims at rate $\lambda = 1$ and a relative security loading $\theta = 0.4$. Revenue is earned at rate $c = 1/3$. It is straightforward to show that $\rho = 5/7, \alpha_+^* = (7/10, 3/10)$, and $\alpha_+ = (1/2, 3/14)$. In a similar fashion,

$$B = \begin{pmatrix} -3/2 & 9/14 \\ 7/2 & -11/2 \end{pmatrix}$$

so that the matrix exponential can be calculated as

$$\exp(uB) = \begin{pmatrix} \frac{9}{10}e^{-u} + \frac{1}{10}e^{-6u} & \frac{9}{70}e^{-u} - \frac{9}{70}e^{-6u} \\ \frac{7}{10}e^{-u} - \frac{7}{10}e^{-6u} & \frac{1}{10}e^{-u} + \frac{9}{10}e^{-6u} \end{pmatrix}.$$

This yields the well-known ruin probability $\psi(u) = (24e^{-u} + e^{-6u})/35$ and the following initial probability vector π_G for the deficit Y_u:

$$\pi_G = \left(\frac{42 - 7e^{-5u}}{48 + 2e^{-5u}}, \frac{6 + 9e^{-5u}}{48 + 2e^{-5u}}\right).$$

The distribution of Y_u is also a mixture of Exp(3) and Exp(7) densities, with the weights above. This is in agreement with Gerber *et al* (1987), p.157, and Willmot (2000), p.69. We observe that u only has an impact on π_G for very small values of u (relative to the mean individual claim amount).

Example 2: Suppose that the individual claim amount distribution is Erlang(3) with mean $\mu = 1$, so that $\alpha = (1, 0, 0)$ and

$$S = \begin{pmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{pmatrix}.$$

Let $\lambda = 1$ and $c = 1.1$ so that $\rho = 10/11$. Then, $\alpha_+ = (10/33, 10/33, 10/33)$ and

$$B = \begin{pmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ \frac{10}{11} & \frac{10}{11} & \frac{-23}{11} \end{pmatrix}.$$

Unlike the preceding example, B has eigenvalues which are complex. However, evaluation by software is nonetheless trivial for a given value of u. Table 1 shows the elements of π_G for some values of u. While Y_u does not belong to the class of Erlang distributions, it does belong to the class of mixtures of Erlangs, as all elements of π_G are positive. The key point here is that the matrix S that describes the *size* of the deficit is unchanged, guaranteeing that the deficit distribution is a mixture of Erlang distributions up to order three. This is in agreement with the conclusions of Willmot (2000), where an approach which differs from our present phase-type method yields an alternative analytic representation for the mixing weights.

Table 1: Elements of π_G in Example 2 for various values of u

u	π_{G1}	π_{G2}	π_{G3}
0	0.33333	0.33333	0.33333
0.5	0.18068	0.35388	0.46544
1	0.16210	0.33580	0.50210
1.5	0.16114	0.33128	0.50758
2	0.16132	0.33064	0.50804
2.5	0.16138	0.33060	0.50802
3	0.16139	0.33060	0.50801

As in the preceding example, it is only when the initial surplus is relatively small that the elements of π_G vary.

We also remark that the phase-type representation of the distribution of Y_u allows us to easily find moments of Y_u. From standard results for phase-type distributions (see, for example, Neuts (1981) or Rolski *et al* (1999)), we find that

$$E(Y_u^k) = (-1)^k k! (\pi_G S^{-k} e^T).$$

Although this formula does not in general yield a simple analytical expression for $E(Y_u^k)$, it does provide a fast and effective method of calculation given a set of parameter values. Table 2 shows moments of Y_u for the same values of u as in Table 1. We observe that these are not monotone functions, but they do converge quickly.

Table 2: Moments of Y_u in Example 2 for various values of u

u	$E(Y_u)$	$E(Y_u^2)$	$E(Y_u^3)$	$E(Y_u^4)$	$E(Y_u^5)$
0	0.66667	0.74074	1.11111	2.07407	4.60905
0.5	0.57175	0.58026	0.81950	1.46520	3.15210
1	0.55333	0.55158	0.77029	1.36671	2.92398
1.5	0.55119	0.54850	0.76536	1.35736	2.90331
2	0.55109	0.54842	0.76529	1.35735	2.90353
2.5	0.55112	0.54847	0.76538	1.35756	2.90402
3	0.55113	0.54848	0.76540	1.35760	2.90413

We note that in computational terms, this approach to calculating moments of Y_u employing the phase-type representation is an efficient alternative to the recursive approach derived in Lin and Willmot (2000).

Example 3: Suppose the individual claim amount distribution is a general Erlang mixture given by the pdf

$$p(y) = \sum_{i=1}^{2}\sum_{j=1}^{3} q_{ij}\frac{\beta_i^j y^{j-1}e^{-\beta_i y}}{(j-1)!} \quad \text{for} \quad y > 0,$$

where $\beta_1 = 1, \beta_2 = 0.5, q_{11} = 0.05, q_{12} = 0.1, q_{13} = 0.5, q_{21} = 0.05, q_{22} = 0.2,$ and $q_{23} = 0.1$. Written in phase-type form, we have

$$\alpha = (0.5, 0.1, 0.05, 0.1, 0.2, 0.05)$$

and

$$S = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & -0.5 \end{bmatrix}.$$

For this distribution, $\mu = 3.25$. Let $\lambda = 1$ and $\theta = 1$ so that $c = 6.5$. Then, one obtains (to 5 decimal places of accuracy)

$$\alpha_+ = (0.07692,\ 0.09231,\ 0.10000,\ 0.03077,\ 0.09231,\ 0.10769)$$

and

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$$B = \begin{bmatrix} -1.00000 & 1.00000 & 0 & 0 & 0 & 0 \\ 0 & -1.00000 & 1.00000 & 0 & 0 & 0 \\ 0.07692 & 0.09231 & -0.90000 & 0.03077 & 0.09231 & 0.10769 \\ 0 & 0 & 0 & -0.50000 & 0.50000 & 0 \\ 0 & 0 & 0 & 0 & -0.50000 & 0.50000 \\ 0.03846 & 0.04615 & 0.05000 & 0.01539 & 0.04615 & -0.44615 \end{bmatrix}.$$

As B has eigenvalues which are complex, we can find expressions for the mixing weights π_G in terms of trigonometric functions. This is an enhancement to the approach employed by Willmot (2000), where it is established that the deficit pdf g_u is of the same form as p. However, unlike the previous example, Willmot (2000) does not in general obtain a simple analytic form for the specific mixing weights.

From $\psi(u) = \alpha_+ \exp(uB)e^T$, we get that

$$\begin{aligned}\psi(u) = {} & 0.01518e^{-0.6154u} + 0.51614e^{-0.2005u} \\ & + 0.03296e^{-0.5495u}\text{Cos}[0.14324u] - 0.06429e^{-1.2157u}\text{Cos}[0.30563u] \\ & - 0.06059e^{-0.5495u}\text{Sin}[0.14324u] - 0.05064e^{-1.2157u}\text{Sin}[0.30563u],\end{aligned}$$

and the elements of π_G work out to be:

$$\begin{aligned}\pi_{G1} = {} & \tfrac{1}{\psi(u)}(0.00374e^{-0.6154u} + 0.01991e^{-0.2005u} \\ & + 0.01113e^{-0.5495u}\text{Cos}[0.14324u] + 0.04215e^{-1.2157u}\text{Cos}[0.30563u] \\ & - 0.00622e^{-0.5495u}\text{Sin}[0.14324u] - 0.00180e^{-1.2157u}\text{Sin}[0.30563u]),\end{aligned}$$

$$\begin{aligned}\pi_{G2} = {} & \tfrac{1}{\psi(u)}(0.01420e^{-0.6154u} + 0.04879e^{-0.2005u} \\ & + 0.03977e^{-0.5495u}\text{Cos}[0.14324u] - 0.01045e^{-1.2157u}\text{Cos}[0.30563u] \\ & - 0.01287e^{-0.5495u}\text{Sin}[0.14324u] + 0.09269e^{-1.2157u}\text{Sin}[0.30563u]),\end{aligned}$$

$$\begin{aligned}\pi_{G3} = {} & \tfrac{1}{\psi(u)}(0.04177e^{-0.6154u} + 0.08690e^{-0.2005u} \\ & + 0.10289e^{-0.5495u}\text{Cos}[0.14324u] - 0.13156e^{-1.2157u}\text{Cos}[0.30563u] \\ & - 0.00854e^{-0.5495u}\text{Sin}[0.14324u] - 0.16803e^{-1.2157u}\text{Sin}[0.30563u]),\end{aligned}$$

$$\begin{aligned}\pi_{G4} = {} & \tfrac{1}{\psi(u)}(- 0.00498e^{-0.6154u} + 0.02126e^{-0.2005u} \\ & + 0.00741e^{-0.5495u}\text{Cos}[0.14324u] + 0.00708e^{-1.2157u}\text{Cos}[0.30563u] \\ & + 0.01407e^{-0.5495u}\text{Sin}[0.14324u] + 0.00396e^{-1.2157u}\text{Sin}[0.30563u]),\end{aligned}$$

$$\begin{aligned}\pi_{G5} = {} & \tfrac{1}{\psi(u)}(0.00664e^{-0.6154u} + 0.09924e^{-0.2005u} \\ & - 0.02963e^{-0.5495u}\text{Cos}[0.14324u] + 0.01606e^{-1.2157u}\text{Cos}[0.30563u] \\ & + 0.05018e^{-0.5495u}\text{Sin}[0.14324u] + 0.01133e^{-1.2157u}\text{Sin}[0.30563u]),\end{aligned}$$

$$\pi_{G6} = \frac{1}{\psi(u)}\big(- 0.04619e^{-0.6154u} + 0.24005e^{-0.2005u}$$
$$- 0.09861e^{-0.5495u}\text{Cos}[0.14324u] + 0.01243e^{-1.2157u}\text{Cos}[0.30563u]$$
$$-0.09721e^{-0.5495u}\text{Sin}[0.14324u] + 0.01122e^{-1.2157u}\text{Sin}[0.30563u]\big).$$

Table 3 shows the elements of π_G (to 5 decimal places of accuracy) for several choices of u.

Table 3: Elements of π_G in Example 3 for various values of u

u	π_{G1}	π_{G2}	π_{G3}	π_{G4}	π_{G5}	π_{G6}
0	0.15385	0.18462	0.20000	0.06154	0.18462	0.21538
0.5	0.11147	0.18672	0.22148	0.05660	0.18334	0.24040
1	0.08516	0.17683	0.23611	0.05324	0.18391	0.26475
2	0.05849	0.15024	0.24308	0.04942	0.18802	0.31076
5	0.04102	0.10660	0.20119	0.04513	0.19802	0.40805
10	0.03852	0.09480	0.17047	0.04203	0.19572	0.45847
20	0.03855	0.09446	0.16822	0.04118	0.19237	0.46522

We observe that the initial surplus u continues to influence the initial probability vector for much larger values of u than in the previous two examples.

Example 4: Our final example has been chosen to show that the computations are still relatively straightforward when the claim number process is not Poisson: the only additional complexity is the determination of the row vector α_+. Suppose that $c = 1$ and the interclaim time distribution is a mixture of 3 Erlangs with $\lambda = 0.5$, initial probability vector $\nu = (0.4,\ 0.2,\ 0.4)$, and transition matrix

$$H = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Clearly, in the above, we have that $h_0^T = (0,0,1)^T$. Let the individual claim amounts be distributed according to a feedforward Coxian distribution (see Horváth and Telek (2000), p.195, Figure 1, which can be shown to be equivalent to Kleinrock (1975), p.14, but with time reversed) with PH representation $\alpha = (0.2,\ 0.3,\ 0.4,\ 0.1)$ and

$$S = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

The claim amount mean is $\mu = 16/15$, and the pdf can be expressed as

$$p(y) = \frac{4}{5}e^{-y} + \frac{14}{5}e^{-2y} - \frac{24}{5}e^{-3y} + \frac{8}{5}e^{-4y} \quad \text{for} \quad y > 0. \tag{15}$$

The pdf (15) is sometimes referred to as a linear combination of exponentials.

To determine α_+, we employ the numerical procedure described in Asmussen (2000), p.231. However, since the interclaim time distribution is itself phase-type, we may also make use of Proposition 1.7 in Asmussen (2000), p.221, to obtain that α_+ satisfies

$$\alpha_+ = \alpha(\nu \otimes I_4)[-(H \oplus B)]^{-1}(h_0^T \otimes I_4), \tag{16}$$

where "$\otimes$" and "$\oplus$" represent the Kronecker product and Kronecker sum respectively (see Asmussen (2000), pp.346-347, for further details). As in eq.(4.3) of Asmussen (2000), p.231, the result of computing α_+ by iteration of (16) is (to 5 decimal places of accuracy)

$$\alpha_+ = (0.09007,\ 0.07254,\ 0.20063,\ 0.11384),$$

from which it follows that

$$B = \begin{bmatrix} -1.00000 & 1.00000 & 0 & 0 \\ 0 & -3.00000 & 3.00000 & 0 \\ 0 & 0 & -2.00000 & 2.00000 \\ 0.36026 & 0.29016 & 0.80252 & -3.54463 \end{bmatrix}.$$

Furthermore, we can once again obtain expressions for π_G in terms of the initial surplus u. In particular, from $\psi(u) = \alpha_+ \exp(uB)e^T$, we get that

$$\psi(u) = 0.23381e^{-3.6926u} - 0.26698e^{-3.5987u} + 0.04517e^{-1.6967u} + 0.46507e^{-0.5566u},$$

and the elements of π_G work out to be:

$$\pi_{G1} = \frac{1}{\psi(u)}(-0.05523e^{-3.6926u} + 0.06368e^{-3.5987u} - 0.01894e^{-1.6967u} + 0.10056e^{-0.5566u}),$$

$$\pi_{G2} = \frac{1}{\psi(u)}(-0.09319e^{-3.6926u} + 0.11626e^{-3.5987u} - 0.00638e^{-1.6967u} + 0.05585e^{-0.5566u}),$$

$$\pi_{G3} = \frac{1}{\psi(u)}(-0.03054e^{-3.6926u} + 0.01242e^{-3.5987u} + 0.03385e^{-1.6967u} + 0.18490e^{-0.5566u}),$$

$$\pi_{G4} = \frac{1}{\psi(u)}(0.41277e^{-3.6926u} - 0.45932e^{-3.5987u} + 0.03664e^{-1.6967u} + 0.12376e^{-0.5566u}).$$

For several choices of u, Table 4 displays the elements of π_G (to 5 decimal places of accuracy).

Table 4: Elements of π_G in Example 4 for various values of u

u	π_{G1}	π_{G2}	π_{G3}	π_{G4}
0	0.18878	0.15205	0.42054	0.23862
0.5	0.19177	0.12103	0.41659	0.27061
1	0.19949	0.11598	0.40884	0.27570
3	0.21422	0.11928	0.39869	0.26781
5	0.21602	0.12001	0.39769	0.26629
10	0.21622	0.12010	0.39757	0.26611

5 Concluding Remarks

In this paper, we have demonstrated that in the Sparre Andersen risk model the conditional distribution of the deficit at ruin (given that ruin occurs) is of phase-type if the distribution of the individual claim sizes is of phase-type. A key component of the derivation is the identification by Asmussen (1992) of the distribution of the ladder height random variable in this setting, or equivalently the distribution of the amount of a drop in surplus given that a drop does occur. Separate analytic and probabilistic proofs are provided, due to the fact that each approach provides separate insight into the problem.

A remarkable feature of the result is the fact that the matrix of transition rates S is exactly the same for both the claim size distribution and the distribution of the deficit at ruin, and the two distributions differ only insofar as their initial probability vectors. While this result is in agreement with our probabilistic intuition, as discussed in the body of the paper, its importance cannot be overemphasized. It implies not only that the phase-type representation is invariant under the deficit at ruin mapping, but that this invariance property also extends to subclasses of the phase-type family of distributions. Examination of the phase-type representation (and the matrix of transition rates in particular) for such classes of distributions as the exponential, combinations and mixtures of exponentials, and mixtures of Erlangs reveals immediately that the distribution of the deficit remains within the same class. This result provides additional insight into the conclusions in Willmot (2000) which were obtained through a mixture representation. It is worth noting that for the purposes of the present analysis the ordinary Erlang distribution should be viewed as being within the class of Erlang mixtures with a particular initial probability vector, as is illustrated by Example 2.

Finally, we note that in addition to the result being quite general in nature, it is normally quite tractable computationally using standard software packages currently available, as we have demonstrated through our examples.

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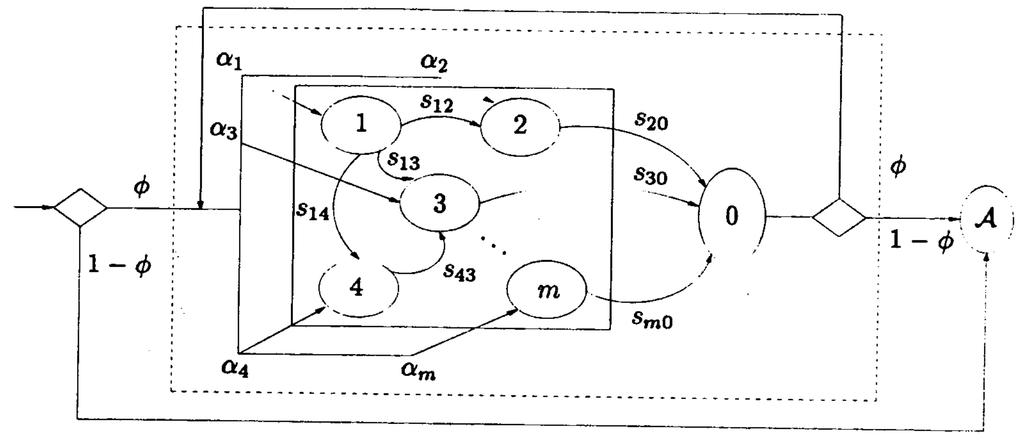

Figure 1: Phase-type formulation for a compound geometric distribution

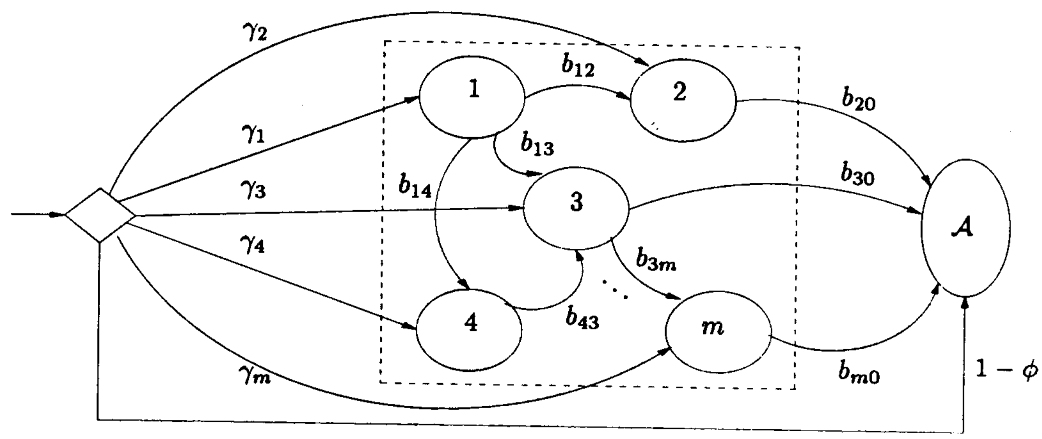

Figure 2: Phase-type formulation when feedback is eliminated

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No.	Date	Subject	Author
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No.	Date	Subject	Author
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No.	Date	Subject	Author
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