The Markovian regime-switching risk model with a threshold dividend strategy

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Abstract

In this paper, we study a regime-switching risk model with a threshold dividend strategy in which the rate for the Poisson claim arrivals and the distribution of the claim amounts are driven by an underlying (external) Markov jump process. The purpose of this paper is to study the unified Gerber-Shiu discounted penalty function and the moments of the total dividend payments until ruin. We adopt an approach which is akin to the one used in Lin and Pavlova (2006) to extend the results for the classical risk model with a threshold dividend strategy to our model. The matrix form of systems of integro-differential equations is presented and the analytical solutions to these systems are derived. Finally, numerical illustrations with exponential claim amounts are also given.

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1. Introduction

In recent years, the risk models with dividend strategies are of particular interest to some researches. The threshold strategy is the one under which the dividends are paid at a rate that is less than the premium rate when the surplus exceeds a constant level (threshold) and no dividends are paid otherwise. Some recent references in this area include Albrecher et al. (2005), Asmussen (2000), Gerber and Shiu (2006), Lin and Pavlova (2006), Zhu and Yang (2007), and references therein. Lin and Pavlova (2006) studied the Gerber-Shiu function and related problems for the classical compound Poisson model with such a dividend strategy. In this paper we consider a Markovian regime-switching risk model with a fixed threshold dividend strategy which is a natural extension of the classical risk model. In the model, we assume that both the frequency of the claim arrivals and the distribution of the claim amounts are influenced by an external environment process. This type of process is also known as the Markov-modulated risk process and was studied in Asmussen (1989) and Reinhard (1984) two decades ago. The primary motivation for this generalization is

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to enhance the flexibility of the model parameter settings for the classical risk process. The examples usually given are weather conditions and epidemic outbreaks, even though seasonality would play a role and can probably not be modeled by a Markovian regime-switching model. Zhu and Yang (2007) referred to states of the environment process as economic circumstances or political regime switchings. It is therefore theoretically appealing to include in the classical risk process assumptions the variation in both claim frequencies and claim severities as a result of external environmental factors. The modeling framework that is advocated in this paper achieves this.

Zhu and Yang (2007) studied a more general Markovian regime-switching risk model in which the premium, the claim intensity, the claim amount, the dividend payment rate and the dividend threshold level were influenced by an external Markovian environment process. Some ruin related functions were investigated and closed-form solutions to systems of integro-differential equations were obtained when the underlying Markovian environment process has only two states and the claim amounts are exponentially distributed. However, in this paper we assume that only the Poisson claim arrival rates and the claim amount distributions vary in time depending on the states of the environment process. Under these particular settings, we adopt an approach which is akin to the one used in Lin and Pavlova (2006) to study the Gerber-Shiu discounted penalty function and the moments of the discounted dividend payments.

Lin and Sendova (2008) further considered a multi-threshold compound Poisson risk model. A piecewise integro-differential equation was derived for the Gerber-Shiu discounted penalty function, and an elegant recursive approach for obtaining general solutions to the equation was presented. By the similar approach, the results in this paper for the Markovian regime-switching risk model with a fixed threshold level can be mathematically extended to the one under a multi-threshold dividend strategy. Piecewise integro-differential equations in matrix form would be derived and corresponding solutions would be obtained recursively in terms of analytical matrix expressions. The risk model with multi-layer (multi-threshold) dividend strategy was also studied in Albrecher and Hartinger (2007), Zhou (2007), and Yang and Zhang (2008), while the risk model discussed in Lin and Pavlova (2006) can be considered as a two-layer model. See also Badescu et al. (2007) for an analysis of a threshold dividend strategy for a risk model with the Markovian arrival process which includes the Markovian regime-switching risk model investigated in this paper as a special case. However a different approach is used.

Now denote by $\{J(t); t \geq 0\}$ the external environment process, and suppose that it is a homogeneous, irreducible and recurrent Markov process with a finite state space $E = \{1, 2, \ldots, m\}$ and intensity matrix $\mathbf{\Lambda} = (\alpha_{i,j})_{i,j=1}^{m}$, where $\alpha_{i,i} := -\alpha_i$ for $i \in E$. Let $N(t)$ be the number of claims occuring in $(0, t]$. If $J(s) = i$ for all s in a small interval $(t, t+h]$, then the number of claims occuring in that interval, $N(t+h) - N(t)$, is assumed to follow a Poisson distribution with parameter $\lambda_i\,(> 0)$, and the corresponding claim amounts have distribution F_i with density function f_i and finite mean μ_i $(i \in E)$. Moreover, we assume that premiums are received continuously at a positive constant rate c_1. The corresponding surplus process $\{U(t);\, t \geq 0\}$ is then given by

$$U(t) = u + c_1\, t - \sum_{n=1}^{N(t)} X_n\,, \qquad t \geq 0\,, \tag{1.1}$$

where $u \geq 0$ is the initial surplus and X_n is the amount of the n-th claim.

In this paper we consider the surplus process (1.1) modified by the payment of dividends. Let d $(0 \leq d \leq c_1)$ be the dividend rate. When the surplus exceeds the constant barrier $b\,(\geq u)$, dividends are paid continuously at rate d so that the net premium rate after dividend payments is $c_1 - d = c_2$. Let $\{U_b(t);\, t \geq 0\}$ be the surplus process with initial surplus $U_b(0) = u$ under the threshold dividend strategy above; then it is defined by

$$U_b(t) = u + \int_0^t c[U_b(s)]ds - \sum_{n=1}^{N(t)} X_n\,, \qquad t \geq 0\,, \tag{1.2}$$

where $c(y)$ is c_1 for $0 \leq y < b$, and is c_2 for $y > b$. Further, we assume that $\sum_{i=1}^m \pi_i(c_2 - \lambda_i\, \mu_i) > 0$ so that the loading is positive and the ruin is not certain, where $\vec{\boldsymbol{\pi}} = (\pi_1, \ldots, \pi_m)$ is the stationary distribution of $\{J(t);\, t \geq 0\}$.

Define $\tau_b = \inf\{t \geq 0 : U_b(t) < 0\}$ to be the time of ruin and let $w(x, y)$, for $x, y \geq 0$, be a non-negative penalty function. For notational convenience, let $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|J(0) = i)$. Let $\delta \geq 0$ be the force of interest for valuation. For $i \in E$, define

$$\mathbb{E}_i\left[e^{-\delta\, \tau_b}\, w(U_b(\tau_b-),\, |U_b(\tau_b)|)I(\tau_b < \infty) \;\Big|\; U_b(0) = u\right] = \begin{cases} \phi_{i1}(u; b), & 0 \leq u < b \\ \phi_{i2}(u; b), & b < u < \infty \end{cases}, \tag{1.3}$$

to be the expected discounted penalty function at ruin, given the initial surplus u and the initial environment $i \in E$, for the surplus $U_b(\tau_b-)$ before ruin and the deficit $|U_b(\tau_b)|$ at ruin, where $I(\cdot)$ is the indicator function. This so-called Gerber-Shiu function was introduced originally in their influential paper by Gerber and Shiu (1998). In particular, when $\delta = 0$ and $w(x, y) = 1$, (1.3) simplifies to $\Psi_i(u; b)$, the conditional ruin probability

$$\Psi_i(u; b) = \mathbb{P}_i\{\tau_b < \infty \,|\, U_b(0) = u\} = \begin{cases} \Psi_{i1}(u; b), & 0 \leq u < b \\ \Psi_{i2}(u; b), & b < u < \infty \end{cases}, \qquad i \in E\,.$$

The rest of the paper is organized as follows. In Section 2 we review the main results for the Gerber-Shiu discounted penalty function at ruin for a Markovian regime-switching risk model without dividends involved. Systems of integro-differential equations in matrix form for the discounted penalty functions under the threshold dividend strategy are presented in Section 3. Then in Section 4, the analytical formulas for the discounted penalty functions when the initial surplus is below and above the dividend threshold b are derived, respectively. A constant vector which is crucial for completing the results in Section 4 is determined in Section 5. The moment of the dividend payments for the model is considered in Section 6. Matrix forms of the integro-differential equations are derived and their analytical solutions are presented in Theorem 2. Finally, numerical examples for a two-state model are illustrated in Section 7 for the ruin probability and the expected total dividend payments until ruin when claim amounts are exponentially distributed.

2. Preliminaries

In this section, we first review some results for surplus process (1.1) where no dividends are involved, i.e., $b = \infty$. As we will see in the next section, the discounted penalty functions $\phi_{i1}(u;b)$ and $\phi_{i2}(u;b)$ defined in (1.3) under the threshold strategy are associated with the discounted penalty function for the process without such a strategy. Define $\tau = \inf\{t \ge 0 : U(t) < 0\}$ to be the time of ruin, and for $\delta \ge 0$

$$\phi_i(u) = \mathbb{E}_i\left[e^{-\delta\,\tau}\, w(U(\tau-), |U(\tau)|) I(\tau < \infty) \mid U(0) = u\right], \qquad u \ge 0,\, i \in E\,,$$

to be the expected discounted penalty (Gerber-Shiu) function at ruin, given the initial surplus u and the initial state i. Function $\phi_i(u)$ has been investigated by Ng and Yang (2006) and Li and Lu (2008).

Let $\vec{\boldsymbol{\phi}}(u) = (\phi_1(u), \ldots, \phi_m(u))^\top$, "$\top$" denoting transpose. An integro-differential equation in matrix form for $\vec{\boldsymbol{\phi}}(u)$ is given by

$$\vec{\boldsymbol{\phi}}'(u) = \mathbf{P}_{c_1}\, \vec{\boldsymbol{\phi}}(u) + \int_0^u \mathbf{G}_{c_1}(t) \vec{\boldsymbol{\phi}}(u-t)dt + \vec{\boldsymbol{\zeta}}_1(u)\,, \qquad 0 \le u < \infty\,, \tag{2.1}$$

where both $\mathbf{P}_{c_1} = [\mathrm{diag}(\lambda_1+\delta, \ldots, \lambda_m+\delta) - \boldsymbol{\Lambda}]/c_1$ and $\mathbf{G}_{c_1}(t) = -\mathrm{diag}(\lambda_1 f_1(t), \ldots, \lambda_m f_m(t))/c_1$ are $m \times m$ matrices, and $\vec{\boldsymbol{\zeta}}_1(u)$ defined by

$$\vec{\boldsymbol{\zeta}}_1(u) = \int_u^\infty w(u, t-u)\, \mathbf{G}_{c_1}(t) \vec{\mathbf{1}}\, dt\,, \qquad 0 \le u < \infty\,,$$

is an m-dimensional vector, in which $\vec{\mathbf{1}} = (1, 1, \ldots, 1)^\top$ is an $m \times 1$ column vector. The corresponding homogeneous integro-differential equation of (2.1) is

$$\vec{\boldsymbol{\phi}}'(u) = \mathbf{P}_{c_1}\, \vec{\boldsymbol{\phi}}(u) + \int_0^u \mathbf{G}_{c_1}(t) \vec{\boldsymbol{\phi}}(u-t)dt\,, \qquad 0 \le u < \infty\,. \tag{2.2}$$

By referring to Section 2.3 in Burton (2005), we present the analytical expression for $\vec{\boldsymbol{\phi}}(u)$ in the following lemma.

Lemma 1 Let $\mathbf{v}(u) = (v_{i,j}(u))_{i,j=1}^m, 0 \le u < \infty$, be the $m \times m$ matrix whose columns are particular solutions to (2.2) with $\mathbf{v}(0) = \mathbf{I}$, where $\mathbf{I}$ is the $m \times m$ identity matrix. The solution to Eq. (2.1) is

$$\vec{\boldsymbol{\phi}}(u) = \mathbf{v}(u)\, \vec{\boldsymbol{\phi}}(0) + \int_0^u \mathbf{v}(u-t) \vec{\boldsymbol{\zeta}}_1(t)dt\,, \qquad 0 \le u < \infty\,. \tag{2.3}$$

Proof. See Theorem 2.3.1 in Burton (2005). The explicit expression for $\vec{\boldsymbol{\phi}}(0)$ is given by (2.7) in Li and Lu (2008), and will also be given by (5.6). □

As in Li and Lu (2007), we apply Laplace transforms to find the particular solution $\mathbf{v}(u)$ to Eq. (2.2), satisfying

$$\mathbf{v}'(u) = \mathbf{P}_{c_1}\, \mathbf{v}(u) + \int_0^u \mathbf{G}_{c_1}(t) \mathbf{v}(u-t)dt\,, \qquad 0 \le u < \infty\,, \tag{2.4}$$

with $\mathbf{v}(0) = \mathbf{I}$. Let $\hat{\mathbf{v}}(s) = (\hat{v}_{i,j}(s))_{i,j=1}^m$ where $\hat{v}_{i,j}(s) = \int_0^\infty e^{-su} v_{i,j}(u)du$ is the Laplace transform of function $v_{i,j}$ for $i, j \in E$. By taking Laplace transforms of both sides of Eq. (2.4) and noting that $\mathbf{v}(0) = \mathbf{I}$, we obtain

$$\hat{\mathbf{v}}(s) = [s\,\mathbf{I} - \mathbf{P}_{c_1} - \hat{\mathbf{G}}_{c_1}(s)]^{-1} ,$$

where $\hat{\mathbf{G}}_{c_1}(s) = -\mathrm{diag}(\lambda_1 \hat{f}_1(s), \ldots, \lambda_m \hat{f}_m(s))/c_1$ and $\hat{f}_i(s) = \int_0^\infty e^{-sx} f_i(x)dx$. Hence the particular solution to (2.2) is the Laplace inversion of inverse matrix of $\mathbf{A}_{c_1}(s) = s\,\mathbf{I} - \mathbf{P}_{c_1} - \hat{\mathbf{G}}_{c_1}(s)$, that is,

$$\mathbf{v}(u) = \mathcal{L}^{-1}\left\{[\mathbf{A}_{c_1}(s)]^{-1}\right\} , \qquad 0 \le u < \infty . \tag{2.5}$$

Equation $\det[\mathbf{A}_{c_1}(s)] = 0$ is called the characteristic equation or a generalized Lundberg's equation for risk process (1.1). Using the same arguments as in Albrecher and Boxma (2005), we can show that Eq. (2.5) has exactly m roots with positive real parts, which play an important role in determining the initial vector $\vec{\boldsymbol{\phi}}(0)$ [see Li and Lu (2008)]. We remark that when the claim amounts are rationally distributed, each element of $\mathbf{A}_{c_1}(s)$ is a rational function, so is each element of $[\mathbf{A}_{c_1}(s)]^{-1}$; therefore an explicit expression for $v_{i,j}(u)$ can be obtained by inverting $\hat{v}_{i,j}(s)$ through partial fractions. This is illustrated by an example in Section 7.

In the following sections, two operations on matrices are used. We firstly extend the definition of operator T_r for a real-valued integrable function to a matrix function with respect to a complex number r ($\Re(r) \ge 0$). For matrix function $\mathbf{B}(y)$ with each element being a real-valued integrable function of y, define

$$T_r\mathbf{B}(y) = \int_y^\infty e^{-r(x-y)}\mathbf{B}(x)dx , \qquad r \in \mathbb{C} ,\ y \ge 0 .$$

The composition of operators can be obtained recursively, for example,

$$T_{r_1}T_{r_2}\mathbf{B}(y) = T_{r_2}T_{r_1}\mathbf{B}(y) = \frac{T_{r_1}\mathbf{B}(y) - T_{r_2}\mathbf{B}(y)}{r_2 - r_1} , \qquad r_1 \ne r_2 \in \mathbb{C} ,\ y \ge 0 .$$

Operator T_r has been introduced for the real-valued integrable function by Dickson and Hipp (2001). For properties of this operator for functions, see Li and Garrido (2004) and Gerber and Shiu (2005).

Next, we also extend the definition of divided differences for functions to matrices as in Li and Lu (2008) and Lu and Tsai (2007). For matrix function $\mathbf{B}(s)$, define its divided differences with respect to distinct numbers $r_1, r_2, \ldots$, recursively, as follows:

$$\mathbf{B}[r_1, s] = \frac{\mathbf{B}(s) - \mathbf{B}(r_1)}{s - r_1} , \qquad \mathbf{B}[r_1, r_2, s] = \frac{\mathbf{B}[r_1, s] - \mathbf{B}[r_1, r_2]}{s - r_2} .$$

An alternative formula for the $(k-1)$-th divided difference [see Gerber and Shiu (2005)] is also given by

$$\mathbf{B}[r_1, r_2, \ldots, r_k] = \sum_{j=1}^{k} \frac{\mathbf{B}(r_j)}{\prod_{l=1, l\ne j}^{k}(r_j - r_l)} . \tag{2.6}$$

3. Systems of integro-differential equations

In this section, we derive integro-differential equations for the discounted penalty functions defined by (1.3) when the initial surplus below or above the barrier b. By similar arguments used in Zhu and Yang (2007) (see Theorem 4.1), we have for $0 \le u < b$,

$$(\lambda_i + \delta)\phi_{i1}(u; b) = c_1\, \phi'_{i1}(u; b) + \sum_{k=1}^{m} \alpha_{i,k}\, \phi_{k1}(u; b) + \lambda_i \left[\int_0^u \phi_{i1}(u - x; b)\, dF_i(x) + \int_u^\infty w(u, x - u)\, dF_i(x) \right], \qquad i \in E, \tag{3.1}$$

and for $b < u < \infty$,

$$(\lambda_i + \delta)\phi_{i2}(u; b) = c_2\, \phi'_{i2}(u; b) + \sum_{k=1}^{m} \alpha_{i,k}\, \phi_{k2}(u; b) + \lambda_i \int_0^{u-b} \phi_{i2}(u - x; b)\, dF_i(x) + \lambda_i \left[\int_{u-b}^{u} \phi_{i1}(u - x; b)\, dF_i(x) + \int_u^\infty w(u, x - u)\, dF_i(x) \right], \qquad i \in E. \tag{3.2}$$

Integro-differential Eqs. (3.1) and (3.2) can easily be rewritten in matrix forms. Let $\vec{\boldsymbol{\phi}}_j(u; b) = (\phi_{1j}(u; b), \ldots, \phi_{mj}(u; b))^\top$ for $j = 1, 2$. Then the vectors of the discounted penalty function $\vec{\boldsymbol{\phi}}_1(u; b)$ and $\vec{\boldsymbol{\phi}}_2(u; b)$ satisfy the integro-differential equations

$$\begin{cases} \vec{\boldsymbol{\phi}}'_1(u;\, b) = \mathbf{P}_{c_1}\, \vec{\boldsymbol{\phi}}_1(u;\, b) + \int_0^u \mathbf{G}_{c_1}(t) \vec{\boldsymbol{\phi}}_1(u - t;\, b) dt + \vec{\boldsymbol{\zeta}}_1(u)\,, & 0 \le u < b\,, \\ \vec{\boldsymbol{\phi}}'_2(u;\, b) = \mathbf{P}_{c_2}\, \vec{\boldsymbol{\phi}}_2(u;\, b) + \int_0^{u-b} \mathbf{G}_{c_2}(t) \vec{\boldsymbol{\phi}}_2(u - t;\, b) dt & \\ \qquad\qquad + \int_{u-b}^{u} \mathbf{G}_{c_2}(t) \vec{\boldsymbol{\phi}}_1(u - t;\, b) dt + \vec{\boldsymbol{\zeta}}_2(u)\,, & b < u < \infty\,, \end{cases} \tag{3.3}$$

where $m \times m$ matrices $\mathbf{P}_{c_2}$ and $\mathbf{G}_{c_2}(t)$, and vector $\vec{\boldsymbol{\zeta}}_2(u)$ have the same formate as $\mathbf{P}_{c_1}$, $\mathbf{G}_{c_1}(t)$ and $\vec{\boldsymbol{\zeta}}_1(u)$ with replacing c_1 by c_2. The continuity condition for $\vec{\boldsymbol{\phi}}_1(u; b)$ and $\vec{\boldsymbol{\phi}}_2(u; b)$ is $\vec{\boldsymbol{\phi}}_1(b-; b) = \vec{\boldsymbol{\phi}}_2(b+; b)$. Note that both matrix equations in (3.3) are non-homogeneous integro-differential equations and the solutions to them will be discussed in the next section.

Remarks:

1. If we set in Zhu and Yang (2007) that all the premium rates, the dividend rates and the barrier levels are equal at any states of the underlying Markov chain in their model, the function $L(x; i)$ is a special case of our discounted penalty function when $\delta = 0$ and $w(x, y) = e^{-\delta^* y}$. Then Eqs. (3.1) and (3.2) reduce to (4.2) and (4.3) in their paper, respectively.

2. When $m = 1$ (i.e., there is no Markovian environment involved in the surplus process), implying $\alpha_1 = 0$, Eqs. (3.1) and (3.2) reduce to (3.1) in Lin and Pavlova (2006).

4. Analytical expressions for $\vec{\boldsymbol{\phi}}_1(u; b)$ and $\vec{\boldsymbol{\phi}}_2(u; b)$

In this section, using the result in Lemma 1, we derive firstly an analytical expression for the discounted penalty function $\vec{\boldsymbol{\phi}}_1(u; b)$, then obtain an equation which shows the relationship between discounted penalty functions $\vec{\boldsymbol{\phi}}(u;b)$ and $\vec{\boldsymbol{\phi}}_1(u;b)$. The latter equation allows us to determine a constant vector and then derive the analytical expression for $\vec{\boldsymbol{\phi}}_2(u;b)$.

Obviously, vector function $\vec{\boldsymbol{\phi}}_1(u;b)$ $(0 \le u < b)$ in (3.3) satisfies a non-homogeneous integro-differential equation. By the similar arguments as in Lemma 1, we immediately get

$$\vec{\boldsymbol{\phi}}_1(u;b) = \mathbf{v}(u)\,\vec{\boldsymbol{\phi}}_1(0;b) + \int_0^u \mathbf{v}(u-t)\vec{\boldsymbol{\zeta}}_1(t)dt, \qquad 0 \le u < b\,, \tag{4.1}$$

where matrix $\mathbf{v}(u)$ for $0 \le u < b$ is given by (2.5). Now restricting $\vec{\boldsymbol{\phi}}(u;b)$ in Eq. (2.3) to $0 \le u < b$, it is observed that vector $\vec{\boldsymbol{\phi}}_1(u;b)$ in (4.1) can be rewritten as

$$\vec{\boldsymbol{\phi}}_1(u;b) = \vec{\boldsymbol{\phi}}(u) + \mathbf{v}(u)\left[\vec{\boldsymbol{\phi}}_1(0;b) - \vec{\boldsymbol{\phi}}(0)\right] = \vec{\boldsymbol{\phi}}(u) + \mathbf{v}(u)\vec{\boldsymbol{\kappa}}(b)\,, \qquad 0 \le u < b\,, \tag{4.2}$$

where the unknown vector $\vec{\boldsymbol{\kappa}}(b)$ is to be determined in the next section. As it will be seen in the following, Eq. (4.2) plays an important role in deriving the analytical expressions for $\vec{\boldsymbol{\phi}}_1(u;b)$ and $\vec{\boldsymbol{\phi}}_2(u;b)$.

Note that in (4.2), the expected discounted penalty function $\vec{\boldsymbol{\phi}}_1(u;b)$ for modified Markovian regime-switching surplus processes with a threshold dividend strategy can be expressed as the summation of the expected discounted penalty function $\vec{\boldsymbol{\phi}}(u)$ for the corresponding process without dividend strategy applied and a vector which is the product of $\mathbf{v}(u)$, a matrix function of u, and $\vec{\boldsymbol{\kappa}}(b)$, a vector function of b.

Remark: When $m = 1$, the model reduces to the classical compound Poisson risk model; the expected discounted penalty function $\vec{\boldsymbol{\phi}}_1(u;b)$ in (4.2) simplifies to

$$\phi_1(u;b) = \phi(u) + v(u)\kappa(b), \qquad 0 \le u < b\,,$$

which is Eq. (5.1) in Lin and Pavlova (2006). Here $\phi(u)$ ($m_\infty(u)$ in their paper) is the expected discounted penalty function under the classical risk process with premium rate c, and the function v satisfies reduced integro-differential equation (2.4) and the constant $\kappa(b)$ is determined in Lin and Pavlova (2006).

To obtain the analytical expression for $\vec{\boldsymbol{\phi}}_2(u;b)$, we need to derive an equivalent equation for it. When $b < u < \infty$, letting $y = u - b$, $\varphi_i(y;b) = \phi_{i2}(y+b;b)$, and $\vec{\boldsymbol{\varphi}}(y;b) = (\varphi_1(y;b), \ldots, \varphi_m(y;b))^\top$ for $y > 0$, we can rewrite equation for $\vec{\boldsymbol{\phi}}_2(u;b)$ in (3.3) as

$$\vec{\boldsymbol{\varphi}}'(y;\,b) = \mathbf{P}_{c_2}\,\vec{\boldsymbol{\varphi}}(y;\,b) + \int_0^y \mathbf{G}_{c_2}(t)\vec{\boldsymbol{\varphi}}(y-t;\,b)dt + \vec{\boldsymbol{\eta}}(y;b)\,, \qquad y > 0\,, \tag{4.3}$$

with initial condition $\vec{\boldsymbol{\varphi}}(0;\,b) = \vec{\boldsymbol{\phi}}_2(b+;b) = \vec{\boldsymbol{\phi}}_1(b-;b)$, where $\vec{\boldsymbol{\eta}}(y;b)$ is an m-dimensional vector, given by

$$\vec{\boldsymbol{\eta}}(y;b) = \int_y^{y+b} \mathbf{G}_{c_2}(t)\vec{\boldsymbol{\phi}}_1(y+b-t;\,b)dt + \vec{\boldsymbol{\zeta}}_2(y+b)\,, \qquad y > 0\,.$$

Employing an analogy to Lemma 1, we obtain that the solution to (4.3) is

$$\vec{\boldsymbol{\varphi}}(y;b) = \mathbf{w}(y)\vec{\boldsymbol{\varphi}}(0;\,b) + \int_0^y \mathbf{w}(y-t)\vec{\boldsymbol{\eta}}(t;b)dt\,, \qquad y > 0\,, \tag{4.4}$$

or

$$\vec{\boldsymbol{\phi}}_2(u;b) = \mathbf{w}(u-b)\vec{\boldsymbol{\phi}}_2(b;b) + \int_0^{u-b} \mathbf{w}(u-b-t)\vec{\boldsymbol{\eta}}(t;b)dt\,, \qquad u > b\,,$$

where $\mathbf{w}(y) = (w_{i,j}(y))_{i,j=1}^m$ is an $m \times m$ matrix satisfying

$$\mathbf{w}'(y) = \mathbf{P}_{c_2}\,\mathbf{w}(y) + \int_0^y \mathbf{G}_{c_2}(t)\mathbf{w}(y-t)dt\,, \qquad y \geq 0\,,$$

with $\mathbf{w}(0) = \mathbf{I}$, and similar to (2.5) it has the expression

$$\mathbf{w}(y) = \mathcal{L}^{-1}\left\{[s\,\mathbf{I} - \mathbf{P}_{c_2} - \hat{\mathbf{G}}_{c_2}(s)]^{-1}\right\}\,, \qquad y \geq 0\,. \tag{4.5}$$

Finally, we conclude our result for $\vec{\boldsymbol{\phi}}_1(u;b)$ and $\vec{\boldsymbol{\phi}}_2(u;b)$ in the theorem below.

Theorem 1 The analytical expression for discounted penalty functions $\vec{\boldsymbol{\phi}}_1(u;b)$ and $\vec{\boldsymbol{\phi}}_2(u;b)$ are obtained as follows:

(i) the discounted penalty function $\vec{\boldsymbol{\phi}}(u)$ for surplus process (1.1) (without dividend involved) is

$$\vec{\boldsymbol{\phi}}(u) = \mathbf{v}(u)\,\vec{\boldsymbol{\phi}}(0) + \int_0^u \mathbf{v}(u-t)\vec{\boldsymbol{\zeta}}_1(t)dt\,, \qquad 0 \leq u < \infty\,,$$

where matrix function $\mathbf{v}(u)$ is given by (2.5) and constant vector $\vec{\boldsymbol{\phi}}(0)$ is given by (2.7) in Li and Lu (2008) (see also (5.6));

(ii) the discounted penalty function $\vec{\boldsymbol{\phi}}_1(u;b)$ when $0 \leq u \leq b$ for the surplus process under the threshold dividend strategy is

$$\vec{\boldsymbol{\phi}}_1(u;b) = \mathbf{v}(u)\,\vec{\boldsymbol{\phi}}_1(0;b) + \int_0^u \mathbf{v}(u-t)\vec{\boldsymbol{\zeta}}_1(t)dt = \vec{\boldsymbol{\phi}}(u) + \mathbf{v}(u)\vec{\boldsymbol{\kappa}}(b)\,, \qquad 0 \leq u \leq b\,,$$

where constant vector $\vec{\boldsymbol{\kappa}}(b)$ is determined in the next section;

(iii) the discounted penalty function $\vec{\boldsymbol{\phi}}_2(u;b)$ when $b < u < \infty$ for the surplus process under the threshold dividend strategy is

$$\vec{\boldsymbol{\phi}}_2(u;b) = \mathbf{w}(u-b)\vec{\boldsymbol{\phi}}_2(b;b) + \int_0^{u-b} \mathbf{w}(u-b-t)\vec{\boldsymbol{\eta}}(t;b)dt\,, \qquad u > b\,,$$

where matrix function $\mathbf{w}(u)$ is given by (4.5) and $\vec{\boldsymbol{\phi}}_2(b;b) = \vec{\boldsymbol{\phi}}_1(b;b)$.

5. Determine the constant vector $\vec{\boldsymbol{\kappa}}(b)$

For simplicity, we write $\vec{\boldsymbol{\phi}}_i(u;\,b) = \vec{\boldsymbol{\phi}}_i(u)$ for $i = 1, 2$ in this section. We adopt an approach which is similar to the one used in Lin and Pavlova (2006) for the compound Poisson risk model under the threshold dividend strategy. By multiplying both sides of Eq. (3.2) by

$e^{-s(u-b)}$ and integrating with respect to u from b to ∞, and after some manipulations, we can write Eq. (3.2) as

$$\left[s\,\mathbf{I} - \mathbf{P}_{c_2} - \hat{\mathbf{G}}_{c_2}(s)\right] T_s\vec{\boldsymbol{\phi}}_2(b) = \vec{\boldsymbol{\phi}}_2(b) + \int_0^b T_s\mathbf{G}_{c_2}(b-t)\vec{\boldsymbol{\phi}}_1(t)dt + T_s\vec{\boldsymbol{\zeta}}_2(b)\,. \tag{5.1}$$

Let $\mathbf{A}_{c_2}(s) = s\,\mathbf{I} - \mathbf{P}_{c_2} - \hat{\mathbf{G}}_{c_2}(s)$. Similarly, we can show that the characteristic equation $\det[\mathbf{A}_{c_2}(s)] = 0$ has exactly m roots with positive real parts, say, $\rho_1, \ldots, \rho_m$, which play an important role in determining the constant vector $\vec{\boldsymbol{\phi}}_2(b)$.

It follows from (5.1) that

$$\begin{aligned} T_s\vec{\boldsymbol{\phi}}_2(b) &= [\mathbf{A}_{c_2}(s)]^{-1}\left[\vec{\boldsymbol{\phi}}_2(b) + \int_0^b T_s\mathbf{G}_{c_2}(b-t)\vec{\boldsymbol{\phi}}_1(t)dt + T_s\vec{\boldsymbol{\zeta}}_2(b)\right] \\ &= \frac{\mathbf{A}^*_{c_2}(s)\left[\vec{\boldsymbol{\phi}}_2(b) + T_s\mathbf{G}_{c_2} * \vec{\boldsymbol{\phi}}_1(b) + T_s\vec{\boldsymbol{\zeta}}_2(b)\right]}{\det[\mathbf{A}_{c_2}(s)]}\,, \end{aligned} \tag{5.2}$$

where $\mathbf{A}^*_{c_2}(s)$ is the adjoint matrix of $\mathbf{A}_{c_2}(s)$. By a similar approach used in Li and Lu (2008), we can find the explicit expression for $\vec{\boldsymbol{\phi}}_2(b)$ in terms of vector $\vec{\boldsymbol{\phi}}_1(b)$. Then we are able to obtain the constant vector $\vec{\boldsymbol{\kappa}}(b)$. Details are as follows.

Note for distinct $\rho_1, \ldots, \rho_m$ each element of $T_s\vec{\boldsymbol{\phi}}_2(b)$ in (5.2) is finite for $\Re(s) \geq 0$, then

$$\mathbf{A}^*_{c_2}(\rho_i)\vec{\boldsymbol{\phi}}_2(b) = -\mathbf{A}^*_{c_2}(\rho_i)\left[T_{\rho_i}\mathbf{G}_{c_2} * \vec{\boldsymbol{\phi}}_1(b) + T_{\rho_i}\vec{\boldsymbol{\zeta}}_2(b)\right], \qquad i \in E\,.$$

Let $\vec{\boldsymbol{\omega}}(\rho_i) = T_{\rho_i}\mathbf{G}_{c_2} * \vec{\boldsymbol{\phi}}_1(b) + T_{\rho_i}\vec{\boldsymbol{\zeta}}_2(b)$. We further have

$$\mathbf{A}^*_{c_2}[\rho_1, \rho_2]\vec{\boldsymbol{\phi}}_2(b) = -(\mathbf{A}^*_{c_2}\,\vec{\boldsymbol{\omega}})[\rho_1, \rho_2]\,,$$

where $(\mathbf{A}^*_{c_2}\,\vec{\boldsymbol{\omega}})[\rho_1, \rho_2]$ is the divided difference of the product of matrices $\mathbf{A}^*_{c_2}(s)$ and $\vec{\boldsymbol{\omega}}(s)$ with respect to ρ_1 and ρ_2, given by

$$(\mathbf{A}^*_{c_2}\,\vec{\boldsymbol{\omega}})[\rho_1, \rho_2] = \mathbf{A}^*_{c_2}(\rho_1)\,\vec{\boldsymbol{\omega}}[\rho_1, \rho_2] + \mathbf{A}^*_{c_2}[\rho_1, \rho_2]\,\vec{\boldsymbol{\omega}}(\rho_2)\,.$$

Recursively, we have for $i = 2, 3, \ldots, m$, that

$$\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_i]\vec{\boldsymbol{\phi}}_2(b) = -(\mathbf{A}^*_{c_2}\vec{\boldsymbol{\omega}})[\rho_1, \ldots, \rho_i] = -\sum_{l=1}^{m}\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_l]\,\vec{\boldsymbol{\omega}}[\rho_l, \ldots, \rho_i]\,. \tag{5.3}$$

In particular, when $i = m$, by making use of formula (2.6) and for $i, j \in E$, putting $\Delta_{i,j} = \prod_{l=i, l\neq j}^{m}(\rho_j - \rho_l)$, it follows from (5.3) that

$$\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_m]\vec{\boldsymbol{\phi}}_2(b) = -\sum_{i=1}^{m}\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_i]\sum_{j=i}^{m}\frac{\vec{\boldsymbol{\omega}}(\rho_j)}{\Delta_{i,j}}\,,$$

which then gives the following result for $\vec{\boldsymbol{\phi}}_2(b)$:

$$\vec{\boldsymbol{\phi}}_2(b) = -\left\{\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_m]\right\}^{-1}\left[\sum_{i=1}^{m}\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_i]\sum_{j=i}^{m}\frac{T_{\rho_j}\mathbf{G}_{c_2} * \vec{\boldsymbol{\phi}}_1(b) + T_{\rho_j}\vec{\boldsymbol{\zeta}}_2(b)}{\Delta_{i,j}}\right].$$

Since $\vec{\boldsymbol{\phi}}_1(u)$ and $\vec{\boldsymbol{\phi}}_2(u)$ is continuous at $u = b$, it follows that

$$-\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_m]\vec{\boldsymbol{\phi}}_1(b) = \sum_{i=1}^{m} \mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_i] \sum_{j=i}^{m} \frac{T_{\rho_j}\mathbf{G}_{c_2} * \vec{\boldsymbol{\phi}}_1(b) + T_{\rho_j}\vec{\boldsymbol{\zeta}}_2(b)}{\Delta_{i,j}}\,.$$

By Eq. (4.2), we finally have

$$\vec{\boldsymbol{\kappa}}(b) = -\left[\mathbf{U}(b)\right]^{-1}\left[\mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_m]\vec{\boldsymbol{\phi}}(b) + \sum_{i=1}^{m} \mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_i] \sum_{j=i}^{m} \frac{T_{\rho_j}\mathbf{G}_{c_2} * \vec{\boldsymbol{\phi}}(b) + T_{\rho_j}\vec{\boldsymbol{\zeta}}_2(b)}{\Delta_{i,j}}\right], \tag{5.4}$$

in which $\mathbf{U}(b)$ is defined by

$$\mathbf{U}(b) = \sum_{i=1}^{m} \mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_i] \sum_{j=i}^{m} \frac{T_{\rho_j}\mathbf{G}_{c_2} * \mathbf{v}(b)}{\Delta_{i,j}} + \mathbf{A}^*_{c_2}[\rho_1, \ldots, \rho_m]\mathbf{v}(b)\,. \tag{5.5}$$

The derivation of the constant vector $\vec{\boldsymbol{\kappa}}(b)$ in this section thus completes the result presented in Theorem 1.

Remarks:

1. In fact, by applying the exact same method for finding $\vec{\boldsymbol{\phi}}_2(b)$ to (2.1), we can get the expression for $\vec{\boldsymbol{\phi}}(0)$ in (2.3) as

$$\vec{\boldsymbol{\phi}}(0) = -\{\mathbf{A}^*_{c_1}[\theta_1, \ldots, \theta_m]\}^{-1}\left[\sum_{i=1}^{m} \mathbf{A}^*_{c_1}[\theta_1, \ldots, \theta_i] \sum_{j=i}^{m} \frac{T_{\theta_j}\vec{\boldsymbol{\zeta}}_1(b)}{\Theta_{i,j}}\right], \tag{5.6}$$

   where $\theta_1, \ldots, \theta_m$ are m roots with positive real parts to the characteristic equation $\det[\mathbf{A}_{c_1}(s)] = \det[s\,\mathbf{I} - \mathbf{P}_{c_1} - \hat{\mathbf{G}}_{c_1}(s)] = 0$, and $\Theta_{i,j} = \prod_{l=i,l\neq j}^{m}(\theta_j - \theta_l)$, for $i, j \in E$.

2. When $\delta = 0$ and $w(x, y) = 1$, the expected discounted penalty functions simplifies to the ruin probabilities. In this case, Li and Lu (2008) shows that $\mathbf{v}(u)$ and $\mathbf{w}(u)$ can be expressed in terms of the ruin probability matrices for the risk model in (1.1) with premium rates c_1 and c_2, respectively.

6. Moments for the dividend payments

For modified surplus process (1.2) with $U_b(0) = u$ and $\delta > 0$ define

$$D_{u,b} = \int_0^{\tau_b} e^{-\delta t} dD(t) = d\int_0^{\tau_b} e^{-\delta t} 1(U_b(t) > b) dD(t)$$

to be the total discounted dividends until time of ruin τ_b, where $D(t)$ is the aggregate dividends paid by time t and $d = c_1 - c_2$. Define the moment-generating function of $D_{u,b}$, given that the initial environment state is i, by

$$M_i(u, y;\, b) = \mathbb{E}_i\left[e^{y\,D_{u,b}}\right] = \begin{cases} M_{i1}(u, y;\, b), & 0 \le u < b \\ M_{i2}(u, y;\, b), & b < u < \infty \end{cases}, \qquad i \in E,$$

where y is such that $M_i(u,y;\,b)$ exists. Further for $0 \le u < \infty$ define

$$V_i^{[n]}(u;\,b) = \mathbb{E}_i\left[D_{u,b}^n\right] = \begin{cases} V_{i1}^{[n]}(u;b), & 0 \le u < b \\ V_{i2}^{[n]}(u;b), & b < u < \infty \end{cases}, \qquad i \in E,\ n \in \mathbb{N},$$

to be the n-th moment of $D_{u,b}$, with $V_i^{[0]}(u;\,b) = 1$ and $V_i^{[1]}(u;\,b) = V_i(u;\,b)$, the expected value of the total dividend payments until ruin.

Similar to the derivation in Li and Lu (2007), we can obtain systems of integro-differential equations for $M_{i1}(u,y;\,b)$ and $M_{i2}(u,y;\,b)$, $i \in E$, respectively. With the help of the representation $M_i(u,y;\,b) = 1 + \sum_{n=1}^{\infty}(y^n/n!)V_i^{[n]}(u;\,b)$, we get the following systems of integro-differential equations for $0 \le u < b$ and $i \in E$,

$$c_1\,\frac{dV_{i1}^{[n]}(u;\,b)}{du} - (\lambda_i + n\,\delta)V_{i1}^{[n]}(u;\,b) + \lambda_i \int_0^u V_{i1}^{[n]}(u-x;\,b)\,dF_i(x) + \sum_{k=1}^{m}\alpha_{i,k}V_{k1}^{[n]}(u;\,b) = 0\,, \tag{7.1}$$

and for $b < u < \infty$ and $i \in E$,

$$\begin{aligned} c_2\,\frac{dV_{i2}^{[n]}(u;\,b)}{du} &- (\lambda_i + n\,\delta)V_{i2}^{[n]}(u;\,b) + n(c_1 - c_2)V_{i2}^{[n-1]}(u;\,b) + \sum_{k=1}^{m}\alpha_{i,k}V_{k2}^{[n]}(u;\,b) \\ &+ \lambda_i\left[\int_0^{u-b} V_{i2}^{[n]}(u-x;\,b)\,dF_i(x) + \int_{u-b}^{u} V_{i1}^{[n]}(u-x;\,b)\,dF_i(x)\right] = 0\,, \end{aligned} \tag{7.2}$$

with boundary conditions $\lim_{u\to\infty} V_{i2}^{[n]}(u;\,b) = (d/\delta)^n$, and the continuity condition for $V_i^{[n]}(u;\,b)$ as a function of u at $u = b$, that is,

$$V_{i1}^{[n]}(b-;b) = V_{i2}^{[n]}(b+;b)\,. \tag{7.3}$$

Remarks:

1. When $m = 1$, that is, there is no Markovian environment involved in the surplus process, the surplus (1.1) reduces to the classical compound Poisson model with multilayer dividend strategy (2-layer case) studied by Albrecher and Hartinger (2007). Integro-differential equations (7.1) and (7.2) simplified to (2.18) (when $k = 2$, $\alpha_1 = 0$ and $\alpha_2 = c_1 - c_2$) in their paper.

2. Furthermore when $m = 1$ and $n = 1$, Eqs. (7.1) and (7.2) reduce to (5.1) and (5.2) in Gerber and Shiu (2006) for the compound Poisson model under threshold dividend strategy.

3. It is worth noting that the derivative of $V_i^{[n]}(u;\,b)$ with respect to u is not continuous at $u = b$. In fact, it follows from Eqs. (7.1) and (7.2) that

$$c_1\,\frac{dV_{i1}^{[n]}(u;b)}{du}\Big|_{u=b-} = c_2\,\frac{dV_{i2}^{[n]}(u;b)}{du}\Big|_{u=b+} + n(c_1 - c_2)V_{i2}^{[n-1]}(b+;b)\,.$$

Let $\vec{\mathbf{V}}_j^{[n]}(u;b) = (V_{1j}^{[n]}(u;b), \ldots, V_{mj}^{[n]}(u;b))^\top$ for $j = 1, 2$, then (7.1) and (7.2) are of matrix forms

$$\frac{d\vec{\mathbf{V}}_1^{[n]}(u;\, b)}{du} = \mathbf{P}_{c_1,n}\vec{\mathbf{V}}_1^{[n]}(u;\, b) + \int_0^u \mathbf{G}_{c_1}(t)\vec{\mathbf{V}}_1^{[n]}(u-t;\, b)dt\,, \qquad 0 \le u \le b\,, \tag{7.4}$$

$$\begin{aligned}\frac{d\vec{\mathbf{V}}_2^{[n]}(u;\, b)}{du} &= \mathbf{P}_{c_2,n}\vec{\mathbf{V}}_2^{[n]}(u;\, b) + \int_0^{u-b} \mathbf{G}_{c_2}(t)\vec{\mathbf{V}}_2^{[n]}(u-t;\, b)dt - \frac{n(c_1-c_2)}{c_2}\vec{\mathbf{V}}_2^{[n-1]}(u;\, b)\\ &\quad + \int_{u-b}^{u} \mathbf{G}_{c_2}(t)\vec{\mathbf{V}}_1^{[n]}(u-t;\, b)dt\,, \qquad b < u < \infty\,,\end{aligned} \tag{7.5}$$

where $m \times m$ matrix $\mathbf{P}_{c_i,n}$ is defined as $\mathbf{P}_{c_i,1} = \mathbf{P}_{c_i}$, and for $n \ge 2$ and $i = 1, 2$, $\mathbf{P}_{c_i,n} = [\mathrm{diag}(\lambda_1 + n\,\delta, \ldots, \lambda_m + n\,\delta) - \mathbf{\Lambda}]/c_i$. The continuity condition (7.3) becomes

$$\vec{\mathbf{V}}_1^{[n]}(b-;b) = \vec{\mathbf{V}}_2^{[n]}(b+;b)\,. \tag{7.6}$$

It is observed that (7.4) is a homogeneous integro-differential equation while (7.4) is non-homogeneous and hence by Lemma 1 and similar arguments in Section 3, we have

$$\begin{cases}\vec{\mathbf{V}}_1^{[n]}(u;b) = \mathbf{v}_n(u)\,\vec{\mathbf{V}}_1^{[n]}(0;b)\,, & 0 \le u \le b\,,\\ \vec{\mathbf{V}}_2^{[n]}(u;b) = \mathbf{w}_n(u-b)\vec{\mathbf{V}}_2^{[n]}(b;b) + \int_0^{u-b}\mathbf{w}_n(u-b-t)\vec{\boldsymbol{\xi}}_n(t;b)dt\,, & b < u < \infty\,,\end{cases} \tag{7.7}$$

where $\mathbf{v}_n(u)$ and $\mathbf{w}_n(u)$ are given by the following Laplace inversions

$$\begin{cases}\mathbf{v}_n(u) = \mathcal{L}^{-1}\left\{[s\,\mathbf{I} - \mathbf{P}_{c_1,n} - \hat{\mathbf{G}}_{c_1}(s)]^{-1}\right\}\,, & u > 0\,,\\ \mathbf{w}_n(u) = \mathcal{L}^{-1}\left\{[s\,\mathbf{I} - \mathbf{P}_{c_2,n} - \hat{\mathbf{G}}_{c_2}(s)]^{-1}\right\}\,, & u > 0\,,\end{cases} \tag{7.8}$$

with $\mathbf{v}_n(0) = \mathbf{w}_n(0) = \mathbf{I}$, and $\vec{\boldsymbol{\xi}}_n(y;b)$ $(n \ge 1)$, an m-dimensional vector, is defined as

$$\vec{\boldsymbol{\xi}}_n(y;b) = -\frac{n(c_1-c_2)}{c_2}\vec{\mathbf{V}}_2^{[n-1]}(y+b;\, b) + \int_y^{y+b}\mathbf{G}_{c_2}(t)\vec{\mathbf{V}}_1^{[n]}(y+b-t;\, b)dt\,.$$

By the exact same approach used to determine $\vec{\boldsymbol{\phi}}_2(b)$, an analytical expression for vector $\vec{\mathbf{V}}_2^{[n]}(b;b)$ $(n \ge 1)$ in (7.7) can be derived recursively as follows:

$$\begin{aligned}\vec{\mathbf{V}}_2^{[n]}(b;b) = &-\left\{\mathbf{A}^*_{c_2,n}[\rho_1^{[n]}, \ldots, \rho_m^{[n]}]\right\}^{-1}\\ &\cdot\left[\sum_{i=1}^m \mathbf{A}^*_{c_2,n}[\rho_1^{[n]}, \ldots, \rho_i^{[n]}]\sum_{j=i}^m \frac{T_{\rho_j^{[n]}}\mathbf{G}_{c_2} * \vec{\mathbf{V}}_1^{[n]}(b;b) - \frac{n(c_1-c_2)}{c_2}T_{\rho_j^{[n]}}\vec{\mathbf{V}}_2^{[n-1]}(b;\, b)}{\Delta_{i,j}^{[n]}}\right],\end{aligned} \tag{7.9}$$

with $\vec{\mathbf{V}}_2^{[0]}(b;\, b) = \vec{\mathbf{1}}$, $\mathbf{A}_{c_2,n}(s) = s\,\mathbf{I} - \mathbf{P}_{c_2,n} - \hat{\mathbf{G}}_{c_2}(s)$, and $\rho_1^{[n]}, \ldots, \rho_m^{[n]}$ are m roots with positive real parts to equation $\det[\mathbf{A}_{c_2,n}(s)] = 0$, and $\Delta_{i,j}^{[n]} = \prod_{l=i,l\neq j}^m(\rho_j^{[n]} - \rho_l^{[n]})$ for $i, j \in E$.

Next we derive the expression for vector $\vec{\mathbf{V}}_1^{[n]}(0;b)$ in (7.7). In fact, it follows from the continuity condition (7.6) and Eq. (7.9) that

$$
\begin{aligned}
&\mathbf{A}^*_{c_2,n}[\rho_1^{[n]},\ldots,\rho_m^{[n]}]\,\mathbf{v}_n(b)\vec{\mathbf{V}}_1^{[n]}(0;b) = \mathbf{A}^*_{c_2,n}[\rho_1^{[n]},\ldots,\rho_m^{[n]}]\,\vec{\mathbf{V}}_2^{[n]}(b;b)\\
&\quad = -\sum_{i=1}^{m}\mathbf{A}^*_{c_2,n}[\rho_1^{[n]},\ldots,\rho_i^{[n]}]\sum_{j=i}^{m}\frac{\Big(T_{\rho_j^{[n]}}\mathbf{G}_{c_2}*\mathbf{v}_n(b)\Big)\vec{\mathbf{V}}_1^{[n]}(0;b)-\frac{n(c_1-c_2)}{c_2}T_{\rho_j^{[n]}}\vec{\mathbf{V}}_2^{[n-1]}(b;\,b)}{\Delta_{i,j}^{[n]}}
\end{aligned}
$$

then $\vec{\mathbf{V}}_1^{[n]}(0;b)$ can be solved as

$$
\vec{\mathbf{V}}_1^{[n]}(0;b) = [\mathbf{U}_n(b)]^{-1}\,\frac{n(c_1-c_2)}{c_2}\sum_{i=1}^{m}\mathbf{A}^*_{c_2,n}[\rho_1^{[n]},\ldots,\rho_i^{[n]}]\sum_{j=i}^{m}\frac{T_{\rho_j^{[n]}}\vec{\mathbf{V}}_2^{[n-1]}(b;\,b)}{\Delta_{i,j}^{[n]}}\,, \tag{7.10}
$$

where $\mathbf{U}_1(b)=\mathbf{U}(b)$, and for $n\geq 2$,

$$
\mathbf{U}_n(b) = \mathbf{A}^*_{c_2,n}[\rho_1^{[n]},\ldots,\rho_m^{[n]}]\,\mathbf{v}_n(b) + \sum_{i=1}^{m}\mathbf{A}^*_{c_2,n}[\rho_1^{[n]},\ldots,\rho_i^{[n]}]\sum_{j=i}^{m}\frac{T_{\rho_j^{[n]}}\mathbf{G}_{c_2}*\mathbf{v}_n(b)}{\Delta_{i,j}^{[n]}}\,.
$$

The above results for $\vec{\mathbf{V}}_1^{[n]}(u;b)$ and $\vec{\mathbf{V}}_2^{[n]}(u;b)$ are summarized in the theorem below.

Theorem 2 The analytical expression for n-th moment of the total dividend payments until ruin, $\vec{\mathbf{V}}_1^{[n]}(u;b)$ and $\vec{\mathbf{V}}_2^{[n]}(u;b)$, with initial values $\vec{\mathbf{V}}_2^{[0]}(u;b)=\vec{\mathbf{1}}$, can be calculated recursively as follows:

(i) the n-th moment vector $\vec{\mathbf{V}}_1^{[n]}(u;b)$ is

$$
\vec{\mathbf{V}}_1^{[n]}(u;b) = \mathbf{v}_n(u)\,\vec{\mathbf{V}}_1^{[n]}(0;b)\,, \qquad 0\leq u\leq b\,,
$$

where matrix function $\mathbf{v}_n(u)$ and constant vector $\vec{\mathbf{V}}_1^{[n]}(0;b)$ are given by (7.8) and (7.10), respectively;

(ii) the interim vector $\vec{\boldsymbol{\xi}}_n(y;b)$ for $n\geq 1$ is

$$
\vec{\boldsymbol{\xi}}_n(y;b) = -\frac{n(c_1-c_2)}{c_2}\vec{\mathbf{V}}_2^{[n-1]}(y+b;\,b) + \int_y^{y+b}\mathbf{G}_{c_2}(t)\vec{\mathbf{V}}_1^{[n]}(y+b-t;\,b)dt\,;
$$

(iii) the n-th moment vector $\vec{\mathbf{V}}_2^{[n]}(u;b)$ is

$$
\vec{\mathbf{V}}_2^{[n]}(u;b) = \mathbf{w}_n(u-b)\vec{\mathbf{V}}_2^{[n]}(b;b) + \int_0^{u-b}\mathbf{w}_n(u-b-t)\vec{\boldsymbol{\xi}}_n(t;b)dt\,, \qquad b<u<\infty\,,
$$

where matrix function $\mathbf{w}_n(u)$ is given in (7.8) and $\vec{\mathbf{V}}_2^{[n]}(b;b)=\vec{\mathbf{V}}_1^{[n]}(b;b)$.

As an illustration, we give the expressions for $\vec{\mathbf{V}}_1(u;b)$ and $\vec{\mathbf{V}}_2(u;b)$, the expected present values of the total dividend payments until ruin, in the corollary below.

Corollary 1 The valuation formulas for $\vec{\mathbf{V}}_1(u;b)$ and $\vec{\mathbf{V}}_2(u;b)$ are:

(i) the expected present value of the total dividend payments until ruin when $0 \le u \le b$

$$\vec{\mathbf{V}}_1(u;b) = \mathbf{v}(u)\,\vec{\mathbf{V}}_1(0;b)\,, \qquad 0 \le u \le b\,,$$

where matrix function $\mathbf{v}(u)$ is given by (2.5) and constant vector $\vec{\mathbf{V}}_1(0;b)$ becomes

$$\vec{\mathbf{V}}_1(0;b) = [\mathbf{U}(b)]^{-1}\,\frac{c_1-c_2}{c_2}\sum_{i=1}^{m}\mathbf{A}^*_{c_2}[\rho_1,\ldots,\rho_i]\sum_{j=i}^{m}\frac{1}{\rho_j\,\Delta_{i,j}}\,\vec{\mathbf{1}}\,,$$

in which $\mathbf{U}(b)$ is given by (5.5);

(ii) the expected present value of the total dividend payments until ruin when $b < u < \infty$

$$\vec{\mathbf{V}}_2(u;b) = \mathbf{w}(u-b)\vec{\mathbf{V}}_2(b;b) + \int_0^{u-b}\mathbf{w}(u-b-t)\vec{\boldsymbol{\xi}}(t;b)dt\,, \qquad b < u < \infty\,,$$

where matrix function $\mathbf{w}(u)$ is given in (4.5), vector $\vec{\boldsymbol{\xi}}(y;b)$ is of the form

$$\vec{\boldsymbol{\xi}}(y;b) = -\frac{c_1-c_2}{c_2}\,\vec{\mathbf{1}} + \int_y^{y+b}\mathbf{G}_{c_2}(t)\vec{\mathbf{V}}_1(y+b-t;\,b)dt\,,$$

and $\vec{\mathbf{V}}_2(b;b) = \vec{\mathbf{V}}_1(b;b)$.

7. Numerical illustrations for a two-state model

In this section, we illustrate some results numerically. Consider a two-state regime-switching risk model under a threshold dividend strategy, that is, $\{J(t);\, t \ge 0\}$ is a two-state Markov process, which reflects the random environmental effects due to, probably, normal risk or abnormal risk conditions.

In the case where the claim amount distributions f_1 and f_2 are exponentially distributed with Laplace transformations $\hat{f}_i(s) = \beta_i/(s+\beta_i)$, $\beta_i > 0$ and $i = 1, 2$, Li and Lu (2007) obtained the explicit expression for $\mathbf{v}(u)$ given by (2.5) as

$$v_{i,j}(u) = \sum_{k=1}^{4} r_{i,j,k}\, e^{R_k\, u}\,, \qquad i, j = 1, 2\,,$$

where R_k, for $k = 1, 2, 3, 4$, are four roots of equation $\det[\mathbf{A}_{c_1}(s)] = 0$, and the coefficients, $r_{i,j,k}$, are given by

$$\begin{pmatrix} r_{1,1,k} & r_{1,2,k} \\ r_{2,1,k} & r_{2,2,k} \end{pmatrix} = \frac{(R_k+\beta_1)(R_k+\beta_2)}{\prod_{l=1,l\neq k}^{4}(R_k-R_l)}\mathbf{A}^*_{c_1}(R_k)\,, \qquad k = 1, 2, 3, 4\,.$$

The expression for $\mathbf{w}(u)$ can also be obtained analogously.

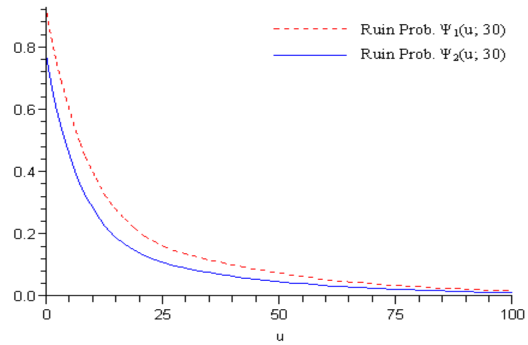

Figure 1: Ruin probabilities under the threshold dividend strategy

To illustrate the results numerically, set $c_1 = 110$, $d = 10$ (accordingly, $c_2 = 100$), $\lambda_1 = 100$, $\lambda_2 = 40$, $\alpha_1 = 1/4$, $\alpha_2 = 3/4$, $\beta_1 = 1$, and $\beta_2 = 0.5$. We also set the constant barrier $b = 30$ for the threshold dividend strategy. We first consider the ruin probabilities for the underlying risk process which is the case when $\delta = 0$ and $w(x, y) = 1$. Then we get four roots for $\det[\mathbf{A}_{c_1}(s)] = 0$, $0,\ 0.03911,\ -.10215,\ -.15514$, and four roots for $\det[\mathbf{A}_{c_2}(s)] = 0$, $0,\ 0.06597,\ -0.03011,\ -.12586$, respectively. We also get $\vec{\boldsymbol{\phi}}(0) = (0.90262, 0.74669)^\top$, and the elements of the matrix $\mathbf{v}(u)$ are obtained, for $u \geq 0$, as follows:

$$\begin{pmatrix} v_{1,1}(u) \\ v_{1,2}(u) \\ v_{2,1}(u) \\ v_{2,2}(u) \end{pmatrix} = \begin{pmatrix} 5.50000 & -.29699 & -7.28890 & 3.08589 \\ 1.83333 & 0.41473 & -1.06177 & -1.18630 \\ 5.50000 & 1.24419 & -3.18531 & -3.55889 \\ 1.83333 & -1.73746 & -0.46400 & 1.36813 \end{pmatrix} \begin{pmatrix} 1 \\ e^{-.15514u} \\ e^{-.10215u} \\ e^{0.39111u} \end{pmatrix}.$$

Following the results from Theorem 1, we can further obtain expressions for $\vec{\boldsymbol{\Psi}}(u)$, $\vec{\boldsymbol{\Psi}}_1(u; 30)$ and $\vec{\boldsymbol{\Psi}}_2(u; 30)$ in terms of linear combinations of exponential functions, given by

$$\vec{\boldsymbol{\Psi}}(u) = \begin{pmatrix} -0.07614 & 0.97876 \\ 0.31896 & 0.42772 \end{pmatrix} \begin{pmatrix} e^{-0.15514u} \\ e^{-0.10215u} \end{pmatrix}, \qquad u \geq 0,$$

$$\vec{\boldsymbol{\Psi}}_1(u; 30) = \begin{pmatrix} 0.08345 & -0.07066 & 0.89474 & 0.00378 \\ 0.08345 & 0.29602 & 0.39101 & -0.00436 \end{pmatrix} \begin{pmatrix} 1 \\ e^{-0.15514u} \\ e^{-0.10215u} \\ e^{0.03911u} \end{pmatrix}, \quad 0 \leq u \leq 30,$$

$$\vec{\boldsymbol{\Psi}}_2(u; 30) = \begin{pmatrix} 0.33921 & -0.03035 \\ 0.21239 & 0.18962 \end{pmatrix} \begin{pmatrix} e^{-0.03011u} \\ e^{-0.12586u} \end{pmatrix}, \qquad u > 30,$$

with $\vec{\boldsymbol{\kappa}}(b)$ given by (5.4) as $\vec{\boldsymbol{\kappa}}(30) = (0.00870, 0.01943)^\top$.

Figure 1 shows the ruin probabilities for the risk model under the threshold dividend strategy with $b = 30$, where the dotted (solid) line represents the ruin probability $\Psi_1(u; 30)$ ($\Psi_2(u; 30)$) with starting state 1 (2) taking values $\Psi_{11}(u; 30)$ ($\Psi_{21}(u; 30)$) when $0 \leq u \leq 30$ and $\Psi_{12}(u; 30)$ ($\Psi_{22}(u; 30)$) when $u > 30$. For comparison, we also show the ruin probabilities with and without the threshold dividend strategy in Figure 2, where dotted lines are the ones (starting from state 1 and 2 respectively) under the threshold dividend strategy,

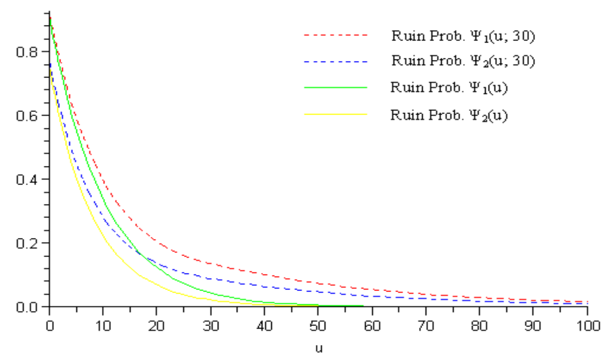

Figure 2: Comparison of ruin probabilities with and without the threshold dividend strategy

while solid lines are corresponding ones without the dividend strategy applied. As expected, $\Psi_i(u;30)$ is uniformly greater than $\Psi_i(u)$ for $i = 1$ and 2. Moreover, the ruin probabilities for the model without dividend payments drop more quickly as u goes larger than those for the model under the threshold strategy.

Further look at the numerical results for the expected present value of the total dividend payments until ruin. We set $\delta = 0.1$ and again $b = 30$. In this case we get four roots for $\det[\mathbf{A}_{c_1}(s)] = 0$, 0.00617, 0.04318, −0.10874, −0.15697, and four roots for $\det[\mathbf{A}_{c_2}(s)] = 0$, 0.01304, 0.07252, −0.04537, −0.12820. Following the results from Corollary 1, we get the following expressions for $\vec{\mathbf{V}}_1(u;30)$ and $\vec{\mathbf{V}}_2(u;30)$:

$$\vec{\mathbf{V}}_1(u;30) = \begin{pmatrix} -67.029 & 6.386 & 67.459 & 0.612 \\ -29.669 & -25.620 & 76.780 & -.637 \end{pmatrix} \begin{pmatrix} e^{-0.10874u} \\ e^{-0.15697u} \\ e^{0.00617u} \\ e^{0.04318u} \end{pmatrix}, \quad 0 \le u \le 30\,,$$

$$\vec{\mathbf{V}}_2(u;30) = \begin{pmatrix} 100 \\ 100 \end{pmatrix} + \begin{pmatrix} -75.124 & 7.273 \\ -40.384 & -44.659 \end{pmatrix} \begin{pmatrix} e^{-0.04537u} \\ e^{-0.12820u} \end{pmatrix}, \qquad u > 30\,.$$

Table 1 shows some values of these functions. The upper (lower) rows in Table 1 give expected present values of the total dividend payments until ruin with initial state 1(2).

Table 1: $\vec{\mathbf{V}}_1(u;30)$ and $\vec{\mathbf{V}}_2(u;30)$ for $u = 10, 20, \ldots, 80$

u	10	20	30	40	50	60	70	80
$\vec{\mathbf{V}}_1(u;30)$	51.427	70.426	80.894					
	65.350	80.870	88.692					
$\vec{\mathbf{V}}_2(u;30)$				87.807	92.238	95.065	96.864	98.007
				93.157	95.748	97.325	98.308	98.927

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