# The distribution of total dividend payments in a Sparre Andersen Model

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**Abstract**

We study the distribution of the total dividend payments in a Sparre Andersen model with phase-type inter-claim times in the presence of a constant dividend barrier. This paper shows that the distribution of the total dividend payments prior to the time of ruin is a mixture of a degenerate distribution at zero and a phase-type distribution. Further, the total dividends prior to ruin can be expressed as a compound geometric sum.

**Keywords:** Sparre Andersen model; Time of ruin; Dividend; Constant dividend barrier; Phase-type distribution

## 1 Introduction

Consider a continuous-time Sparre Andersen model in which the surplus process is given by

$$U(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i \,, \quad t \geq 0 \,, \tag{1}$$

where $u \geq 0$ is the initial surplus, $c \geq 0$ is the premium income rate, while the random sum represents aggregate claims. The $X_i$'s are i.i.d. random variables with common distribution function (d.f.) $P(x) = \mathbb{P}(X \leq x)$ ($P(0) = 0$) and

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density function $p(x) = P'(x)$. Denote the Laplace transform (LT) of $p$ by $\hat{p}(s) = \int_0^\infty e^{-sx}\, p(x)\, dx$. The renewal process $\{N(t)\}_{t\geq 0}$ denotes the number of claims up to time $t$ and is defined as follows.

For $k = 1, 2, \ldots$, let $V_k$ denote the time when the $k$-th claim occurs. Let $W_1 = V_1$ and $W_i = V_i - V_{i-1}$ for $i = 2, 3, \ldots$. We assume that $W_1, W_2, \ldots$ are independent and the inter-claim times $W_2, W_3, \ldots$ have a common distribution function $A(x) = \mathbb{P}(W \leq x)$ and density function $a(x) = A'(x)$. Denote by $\hat{a}(s) = \int_0^\infty e^{-sx}\, a(x)\, dx$ the LT of $a$. Then $N(t) = \max\{n \in \mathbb{N}^+ : W_1 + W_2 + \cdots + W_n \leq t\}$.

Further assume that $\{W_i\}_{i\geq 1}$ and $\{X_i\}_{i\geq 1}$ are independent and that $c\,\mathbb{E}(W) > \mathbb{E}(X)$, providing a positive safety loading factor.

In this paper, we assume that the distribution of the inter-claim times $A$ is phase-type with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{B})$, where $\mathbf{B} = (b_{i,j})_{i,j=1}^m$ is an $m \times m$ matrix with $b_{i,i} < 0$, $b_{i,j} \geq 0$ for $i \neq j$, and $\sum_{j=1}^m b_{i,j} \leq 0$ for any $i = 1, 2, \ldots, m$, $\vec{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $\sum_{i=1}^m \alpha_i = 1$. Then

$$
\begin{aligned}
A(t) &= 1 - \vec{\boldsymbol{\alpha}} e^{t\mathbf{B}} \vec{\mathbf{e}}^\top, \qquad t \geq 0, \\
a(t) &= \vec{\boldsymbol{\alpha}} e^{t\mathbf{B}} \vec{\mathbf{b}}^\top, \qquad t \geq 0, \\
\hat{a}(s) &= \vec{\boldsymbol{\alpha}} (s\mathbf{I} - \mathbf{B})^{-1} \vec{\mathbf{b}}^\top,
\end{aligned}
$$

where $\vec{\mathbf{e}}$ is a row vector of length $m$ with each element being 1, $\mathbf{I}$ is the $m \times m$ identity matrix, and $\vec{\mathbf{b}}^\top = -\mathbf{B}\vec{\mathbf{e}}^\top$.

It follows that $W$ corresponds to the time to absorption in a terminating continuous-time Markov chain $\{J(t)\}_{t\geq 0}$ with $m+1$ states, one of which is absorbing. The state space of $\{J(t)\}_{t\geq 0}$ is $\{1, 2, 3, \ldots, m, 0\} = E \cup \{0\}$ and the initial distribution is $(\alpha_1, \alpha_2, \ldots, \alpha_m, 0)$. The generator of $\{J(t)\}_{t\geq 0}$ is

$$
\begin{pmatrix} \mathbf{B} & \vec{\mathbf{b}}^\top \\ \mathbf{0} & 0 \end{pmatrix}.
$$

A detailed introduction to phase-type distributions and their properties can be found in Neuts (1981), Asmussen (1992), and references therein.

Note that $W_1$ may not follow the same distribution as the inter-claim times. In the case of an ordinary renewal risk process, $W_1$ follows the distribution $A$, that is, a claim has taken place at time 0 and $u$ is the surplus after the claim is paid. For $i \in E$, if $J(0) = i$, then $W_1$ follows a distribution with density function $\vec{\mathbf{e}}_i e^{t\mathbf{B}} \vec{\mathbf{b}}^\top$, where $\vec{\mathbf{e}}_i$ is a $1 \times m$ row vector with the $i$-th element being 1 and all other elements being 0. In the rest of the paper, we employ $\mathbb{P}$ to denote the probability measure when the surplus process is an ordinary renewal risk process and employ $\mathbb{P}_i$ to denote the probability measure given $J(0) = i$, i.e., the density function of $W_1$ is $\vec{\mathbf{e}}_i e^{t\mathbf{B}}\vec{\mathbf{b}}^\top$. $\mathbb{E}$ and $\mathbb{E}_i$ represent the expectation operators under $\mathbb{P}$ and $\mathbb{P}_i$, respectively.

Let $T$ denote the time of ruin,

$$T = \inf\{t \geq 0\,;\, U(t) < 0\}$$

($T = \infty$ if ruin does not occur). Define

$$\Psi(u) = \mathbb{P}(T < \infty \,|\, U(0) = u)\ , \qquad u \geq 0\ ,$$

as the infinite-horizon ruin probability in the corresponding ordinary renewal risk model. Further, define

$$\Psi_i(u) = \mathbb{P}_i(T < \infty \,|\, U(0) = u)\ , \qquad i \in E, u \geq 0\ ,$$

to be the ruin probability given that the initial state is $i$. Then

$$\Psi(u) = \vec{\boldsymbol{\alpha}}\vec{\boldsymbol{\Psi}}(u), \tag{2}$$

where $\vec{\boldsymbol{\Psi}}(u) = (\Psi_1(u), \Psi_1(u), \ldots, \Psi_m(u))^\top$. In Section 2, each $\Psi_i(u)$ will be further decomposed into $m$ components according to the state at the time of recovery after ruin.

The Sparre Andersen model with Erlang inter-claim times has been studied by Li and Garrido (2004), Gerber and Shiu (2005), and references therein. The Sparre Andersen model with phase-type inter-claim times has been studied by Avram and Usábel (2004), Schmidli (2005), Albrecher and Boxma (2005), Ren (2007) and Li (2008a, b). Li and Garrido (2005) studies the discounted penalty (Gerber-Shiu) function in a general renewal process with $K_n$-family distribution as the inter-claim distribution, which includes the phase-type distribution as a special case, and Albrecher and Boxma (2005) analyze the Gerber-Shiu function through the Laplace-Stieltjes transforms for a semi-Markovian risk model which also includes the Sparre Andersen model with phase-type inter-claim times as a special case.

The barrier strategy was initially proposed by de Finetti (1957) for a binomial model. More general barrier strategies have been studied in a number of papers and books. See Lin et al. (2003), Dickson and Waters (2004), Li and Lu (2007), Lu and Li (2008), and references therein for details. The main focus of the mentioned papers is on optimal dividend payouts and problems associated with time of ruin, under various barrier strategies and other economic conditions. Cheung (2007) studies the moments of the discounted dividend payments prior to ruin in a Sparre Andersen model with phase-type claim inter-arrival times in the presence of a constant dividend barrier. The purposes of this note is to extend some results in Dickson and Waters (2004) and Li and Dickson (2006) and to find the distribution of total dividend payments prior to ruin.

# 2 Preliminaries

## 2.1 The decomposition of ruin probabilities

In this paper, we allow the surplus process to continue if ruin occurs and define

$$T' = \inf\{t : t > T, U(t) = 0\}$$

to be the time of the first upcrossing of the surplus process through level zero after ruin. This time is also called the time of recovery in literature, see Gerber and Shiu (1998). Since we assume that the loading factor is positive, it is certain that the surplus process will attain this level. Define

$$\Psi_{i,j}(u) = \mathbb{P}_i\left(T < \infty, J(T') = j \middle| U(0) = u\right), \quad i, j \in E, u \geq 0,$$

to be the probability of ruin with the state of recovery being $j$ given that the initial surplus is $u$ and the initial state is $i$. Then $\Psi_i(u)$ can be decomposed as

$$\Psi_i(u) = \sum_{j=1}^{m} \Psi_{i,j}(u), \qquad i \in E.$$

In matrix form,

$$\vec{\boldsymbol{\Psi}}(u) = \boldsymbol{\Psi}(u)\vec{\mathbf{e}}^\top, \quad \Psi(u) = \vec{\boldsymbol{\alpha}}\boldsymbol{\Psi}(u)\vec{\mathbf{e}}^\top,$$

where $\boldsymbol{\Psi}(u)$ is an $m \times m$ matrix with the $(i, j)$ element being $\Psi_{i,j}(u)$.

It follows from Li (2008b) that

$$\boldsymbol{\Psi}(u) = \mathbb{E}\left[e^{\mathbf{K}U(T)} I(T < \infty) \middle| U(0) = u\right],$$

where $I(\cdot)$ is the indicator function and matrix $\mathbf{K}$ satisfies the following matrix equation:

$$c\mathbf{K} = -\mathbf{B} - \vec{\mathbf{b}}^\top \vec{\boldsymbol{\alpha}} \int_0^\infty e^{-\mathbf{K}\,x} p(x)dx. \tag{3}$$

The solution of the matrix equation (3) can be obtained as the limit of $\mathbf{K}_n$ as $n$ goes to infinity, where

$$\mathbf{K}_{n+1} = -\mathbf{B} - \vec{\mathbf{b}}^\top \vec{\boldsymbol{\alpha}} \int_0^\infty e^{-\mathbf{K}_n\,x} p(x)dx, \qquad n \geq 0$$

with $\mathbf{K}_0 = \mathbf{0}$. See Li (2008b) for the details on computation of $\boldsymbol{\Psi}(u)$ for some special claim amount distributions.

We remark that when $m = 1$, the model simplifies to the classical risk model, $\mathbf{K}$ simplifies to zero, and $\boldsymbol{\Psi}(u)$ is simply the ruin probability $\Psi(u)$ for the classical risk model.

## 2.2 The probability of the surplus process attaining a certain level before ruin

For $0 \le u \le b$ and $i, j \in E$, define $\chi_{i,j}(u;b)$ to be the probability that the surplus process attains level $b$ at state $j$ from initial state $i$ and initial surplus $u$ without first falling below zero. Clearly, $\chi_{i,j}(b;b) = I(i = j)$ for $i, j \in E$, where $I(\cdot)$ is the indicator function. Then

$$\chi_{i,\cdot}(u;b) = \sum_{j=1}^{m} \chi_{i,j}(u;b) , \qquad i \in E ,$$

is the probability that the surplus process attains level $b$ from initial state $i$ and initial surplus $u$ without first falling below zero,

$$\chi_{\cdot,j}(u;b) = \sum_{j=1}^{m} \alpha_i \chi_{i,j}(u;b) , \qquad j \in E ,$$

is the probability that the surplus process attains level $b$ at state $j$ from initial surplus $u$ without first falling below zero, and

$$\chi(u;b) = \sum_{i=1}^{m} \alpha_i \chi_{i,\cdot}(u;b) = \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \chi_{i,j}(u;b), \qquad 0 \le u \le b,$$

is the probability that the surplus process attains level $b$ without first falling below zero in the ordinary renewal risk model. In matrix form,

$$\chi(u;b) = \vec{\boldsymbol{\alpha}} \boldsymbol{\chi}(u;b) \vec{\mathbf{e}}^{\top}, \quad 0 \le u \le b, \tag{4}$$

where $\boldsymbol{\chi}(u,b) = \big(\chi_{i,j}(u;b)\big)_{i,j=1}^{m}$ with $\boldsymbol{\chi}(b;b) = \mathbf{I}$.

Using the same arguments as in Ko (2007), we can show that $\boldsymbol{\chi}(u,b)$ satisfies the following matrix form integral-differential equation:

$$c\boldsymbol{\chi}'(u;b) = -\mathbf{B}\boldsymbol{\chi}(u;b) - \vec{\mathbf{b}}^{\top}\vec{\boldsymbol{\alpha}} \int_0^u \boldsymbol{\chi}(u-x;b)p(x)dx, \tag{5}$$

where $\boldsymbol{\chi}'(u;b) = \partial\boldsymbol{\chi}(u;b)/\partial u$. Further, Li (2008b) shows that

$$\boldsymbol{\chi}(u;b) = [e^{\mathbf{K}u} - \boldsymbol{\Psi}(u)][e^{\mathbf{K}b} - \boldsymbol{\Psi}(b)]^{-1}, \quad 0 \le u \le b. \tag{6}$$

In particular, when $m = 1$, $\mathbf{K} = 0$ and $\boldsymbol{\Psi}(u) = \Psi(u)$, then

$$\chi(u;b) = \frac{1 - \Psi(u)}{1 - \Psi(b)},$$

which can be found in Dickson and Gray (1984).

## 3 Main results

Now we consider the surplus process (1) modified by the payment of dividends. When the surplus exceeds a constant barrier $b\,(\geq u)$, dividends are paid continuously so the surplus stays at the level $b$ until a new claim occurs. Let $\{U_b(t)\}_{t\geq 0}$ be the surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy, i.e.,

$$U_b(t) = u + c\,t - \sum_{i=1}^{N(t)} X_i - D(t),$$

where $D(t)$ is the aggregate dividends paid by time $t$.

Define $\bar{T} = \inf\{t \geq 0 : U_b(t) < 0\}$ to be the time of ruin. Then $D(\bar{T})$ is the total dividend payments prior to the time of ruin.

Let $V_{i,n}(u;b)$ denote the $n$-th moment of $D(\bar{T})$ given that the initial state is $i$ for $i \in E$ and $n \in \mathbb{N}$, and denote $\vec{\mathbf{V}}_n(u;b) = (V_{1,n}(u;b), V_{2,n}(u;b), \ldots, V_{m,n}(u;b))^\top$. Then

$$V_n(u;b) = \vec{\boldsymbol{\alpha}}\vec{\mathbf{V}}_n(u;b)$$

is the $n$-th moment of $D(\bar{T})$ in the ordinary renewal risk model. Further define

$$M_i(u,y;b) = \mathbb{E}_i\left[e^{yD(\bar{T})}\big|U(0) = u\right], \quad i \in E,$$

to be the moment generating function of $D(\bar{T})$ given that the initial state is $i$. Then

$$M(u,y;b) = \vec{\boldsymbol{\alpha}}\vec{\mathbf{M}}(u,y;b)$$

is the moment generating function of $D(\bar{T})$ in the ordinary renewal risk model, where $\vec{\mathbf{M}}(u,y;b) = (M_1(u,y;b), M_2(u,y;b), \ldots, M_m(u,y;b))^\top$.

Since the dividends are only payable if the surplus attains level $b$ prior to ruin, then

$$V_{i,n}(u;b) = \sum_{j=1}^{m} \chi_{i,j}(u;b)V_{j,n}(b;b), \qquad 0 \leq u \leq b,$$

or in matrix form,

$$\vec{\mathbf{V}}_n(u;b) = \boldsymbol{\chi}(u;b)\vec{\mathbf{V}}_n(b;b).$$

The vector $\vec{\mathbf{V}}_n(b;b)$ can be evaluated by the following boundary condition (see Cheung (2007))

$$\vec{\mathbf{V}}'_n(b;b) = n\vec{\mathbf{V}}_{n-1}(b;b), \qquad n \in \mathbb{N}^+, \tag{7}$$

where $\vec{\mathbf{V}}_0(u;b) = \vec{\mathbf{e}}^\top$. Then $\vec{\mathbf{V}}_n(b;b) = n[\boldsymbol{\chi}'(b;b)]^{-1}\vec{\mathbf{V}}_{n-1}(b;b)$ and

$$\begin{aligned}
\vec{\mathbf{V}}_n(u;b) &= n\boldsymbol{\chi}(u;b)[\boldsymbol{\chi}'(b;b)]^{-1}\vec{\mathbf{V}}_{n-1}(b;b) \\
&= n!\boldsymbol{\chi}(u;b)[\boldsymbol{\chi}'(b;b)]^{-1}\{\boldsymbol{\chi}(b;b)[\boldsymbol{\chi}'(b;b)]^{-1}\}^{n-1}\vec{\mathbf{e}}^\top \\
&= n!\boldsymbol{\chi}(u;b)[\boldsymbol{\chi}(b;b)]^{-1}\{\boldsymbol{\chi}(b;b)[\boldsymbol{\chi}'(b;b)]^{-1}\}^{n}\vec{\mathbf{e}}^\top \\
&= n!\boldsymbol{\chi}(u;b)\{\boldsymbol{\chi}(b;b)[\boldsymbol{\chi}'(b;b)]^{-1}\}^{n}\vec{\mathbf{e}}^\top. \qquad (8)
\end{aligned}$$

Denote $\mathbf{W}(u;b) = \boldsymbol{\chi}(u;b)[\boldsymbol{\chi}'(b;b)]^{-1}$. It follows from (6) that

$$\mathbf{W}(u;b) = \left[e^{\mathbf{K}u} - \boldsymbol{\Psi}(u)\right]\left[\mathbf{K}e^{\mathbf{K}b} - \boldsymbol{\Psi}'(b)\right]^{-1}.$$

Then (8) can be rewritten as

$$\vec{\mathbf{V}}_n(u;b) = n!\boldsymbol{\chi}(u;b)[\mathbf{W}(b;b)]^n\vec{\mathbf{e}}^\top. \qquad (9)$$

Taylor expansion gives

$$\begin{aligned}
\vec{\mathbf{M}}(u,y;b) &= \sum_{n=0}^{\infty}\frac{y^n}{n!}\vec{\mathbf{V}}_n(u;b) \\
&= \left\{\mathbf{I} + \boldsymbol{\chi}(u;b)\sum_{n=1}^{\infty} y^n[\mathbf{W}(b;b)]^n\right\}\vec{\mathbf{e}}^\top \\
&= \left\{\mathbf{I} - \boldsymbol{\chi}(u;b) + \boldsymbol{\chi}(u;b)[\mathbf{I} - y\mathbf{W}(b;b)]^{-1}\right\}\vec{\mathbf{e}}^\top \\
&= [\mathbf{I} - \boldsymbol{\chi}(u;b)]\vec{\mathbf{e}}^\top \\
&\quad + \boldsymbol{\chi}(u;b)\left[[\mathbf{W}(b;b)]^{-1} - y\mathbf{I}\right]^{-1}[\mathbf{W}(b;b)]^{-1}\vec{\mathbf{e}}^\top. \qquad (10)
\end{aligned}$$

Then

$$\begin{aligned}
M(u,y;b) &= \mathbb{E}\left[e^{yD(\bar{T})}\right] = \vec{\boldsymbol{\alpha}}\vec{\mathbf{M}}(u,y;b) \\
&= 1 - \vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\vec{\mathbf{e}}^\top + \vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\left[[\mathbf{W}(b;b)]^{-1} - y\mathbf{I}\right]^{-1}[\mathbf{W}(b;b)]^{-1}\vec{\mathbf{e}}^\top \\
&= 1 - \chi(u;b) + \chi(u;b)\frac{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)}{\chi(u;b)}\left[[\mathbf{W}(b;b)]^{-1} - y\mathbf{I}\right]^{-1}[\mathbf{W}(b;b)]^{-1}\vec{\mathbf{e}}^\top.
\end{aligned}$$

This shows that the distribution of $D(\bar{T})$ is a mixture of the degenerate distribution at zero with weight $q = 1 - \chi(u;b)$ and a continuous distribution with weight $1 - q = \chi(u;b)$ and phase-type pdf

$$f(x) = \vec{\boldsymbol{\gamma}}e^{\mathbf{T}x}\vec{\mathbf{t}}^\top,$$

where

$$
\begin{aligned}
\vec{\gamma} &= \frac{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)}{\chi(u;b)} = \frac{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)}{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\vec{\mathbf{e}}^{\top}}, \\
\mathbf{T} &= -[\mathbf{W}(b;b)]^{-1} = -\left[\mathbf{K}e^{\mathbf{K}b} - \mathbf{\Psi}'(b)\right]\left[e^{\mathbf{K}b} - \mathbf{\Psi}(b)\right]^{-1}, \\
\vec{\mathbf{t}}^{\top} &= -\mathbf{T}\vec{\mathbf{e}}^{\top}.
\end{aligned}
$$

Next we show that the total dividend payments prior to ruin, $D(\bar{T})$, can be decomposed as

$$D(\bar{T}) = \sum_{i=1}^{N} \Delta_i,$$

where $N$ denotes the number of streams of dividend payments, and $\Delta_i$ denotes the amount of dividends in the $i$-th dividend stream. Then $N$ has a zero-modified geometric distribution, with $\mathbb{P}(N=0) = 1-\chi(u;b)$, and

$$P(N=k) = \chi(u;b)[1-a(b)]a(b)^{k-1}, \qquad k=1,2,\ldots, \tag{11}$$

where $a(b) = \int_0^b \chi(b-x;b)p(x)dx$. Further $\Delta_1$ has a mixture of $m$ phase-type distributions with density function

$$
\begin{aligned}
f_1(x) &= \sum_{j=1}^{m} \frac{\chi_{\cdot,j}(u;b)}{\chi(u;b)}\,\vec{\mathbf{e}}_j e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c} \\
&= \sum_{j=1}^{m} \frac{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\vec{\mathbf{e}}_j^{\top}}{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\vec{\mathbf{e}}^{\top}}\,\vec{\mathbf{e}}_j e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c} \\
&= \frac{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\left(\sum_{j=1}^{m}\vec{\mathbf{e}}_j^{\top}\vec{\mathbf{e}}_j\right)e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c}}{\vec{\boldsymbol{\alpha}}\boldsymbol{\chi}(u;b)\vec{\mathbf{e}}^{\top}} = \vec{\boldsymbol{\gamma}}e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c}.
\end{aligned}
\tag{12}
$$

This is to say, $\Delta_1$ follows a phase-type distribution with representation $(\vec{\boldsymbol{\gamma}}, \mathbf{B}/c)$. Denote $a_j(b) = \int_0^b \chi_{\cdot,j}(b-x;b)p(x)dx$. Then $\Delta_2, \Delta_3, \ldots$ are i.i.d. with density function

$$
\begin{aligned}
f_2(x) &= \sum_{j=1}^{m} \frac{a_j(b)}{a(b)}\,\vec{\mathbf{e}}_j e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c} \\
&= \sum_{j=1}^{m} \frac{\vec{\boldsymbol{\alpha}}\int_0^b \boldsymbol{\chi}(b-x;b)p(x)dx\,\vec{\mathbf{e}}_j^{\top}}{\vec{\boldsymbol{\alpha}}\int_0^b \boldsymbol{\chi}(b-x;b)p(x)dx\vec{\mathbf{e}}^{\top}}\,\vec{\mathbf{e}}_j e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c} \\
&= \frac{\vec{\boldsymbol{\alpha}}\int_0^b \boldsymbol{\chi}(b-x;b)p(x)dx}{\vec{\boldsymbol{\alpha}}\int_0^b \boldsymbol{\chi}(b-x;b)p(x)dx\,\vec{\mathbf{e}}^{\top}} e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c} = \vec{\boldsymbol{\beta}}e^{x\frac{\mathbf{B}}{c}}\frac{\vec{\mathbf{b}}^{\top}}{c},
\end{aligned}
\tag{13}
$$

where

$$\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{\alpha}} \int_0^b \boldsymbol{\chi}(b-x;b)p(x)dx}{\vec{\boldsymbol{\alpha}} \int_0^b \boldsymbol{\chi}(b-x;b)p(x)dx\vec{\mathbf{e}}^\top}$$

is a $1 \times n$ row vector with $\vec{\boldsymbol{\beta}}\vec{\mathbf{e}}^\top = 1$. Then $f_2(x)$ is a phase-type density function with representation $\left(\vec{\boldsymbol{\beta}}, \mathbf{B}/c\right)$.

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