

ISSN 0819-2642
ISBN 0 7340 2499 1



THE UNIVERSITY OF
MELBOURNE

THE UNIVERSITY OF MELBOURNE
DEPARTMENT OF ECONOMICS

RESEARCH PAPER NUMBER 844

APRIL 2002

**PARTY FORMATION IN
COLLECTIVE DECISION-MAKING**

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Party formation in collective decision-making

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April 9, 2002

Abstract

We study party formation in a general model of collective decision-making, modeling parties as agglomerations of policy positions championed by decision-makers. We show that if there are economies of party size and the policy chosen is not beaten by another policy in pairwise voting, then players agglomerate into exactly two parties. This result does not depend on the magnitude of the economies of party size or sensitively on the nature of the individuals' preferences. Our analysis encompasses a wide range of models, including decision-making in committees with costly participation and representative democracy in which the legislature is elected by citizens.

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1. Introduction

Many societies make collective decisions in legislative assemblies in which the policies chosen are compromises that depend on the policies championed by the legislators. In these assemblies, legislators tend to be grouped into “parties”. Under what circumstances do parties form? What determines the number of parties? What positions do the parties take?

We model political parties as agglomerations of positions championed by decision makers. A group of legislators constitutes a party when all members support and vote for the same position (“party discipline”.) We focus on a classical reason for agglomeration: economies of scale. In societies in which an electoral process generates a set of decision-makers (legislators), the operating cost of a party derives in part from an extra-parliamentary organization whose main role is to rally support in election times. Economies of scale may exist because some costs have a fixed component (e.g. advertising, registration), organizational economies exist (e.g. the average cost of ensuring that supporters of potential legislators are registered voters may decline with party membership), and large parties get disproportionate public subsidies.

We assume that the policy enacted by the legislature is the outcome of voting. Each legislator champions a position, and the policy enacted is one that does not lose to any other under majority rule pairwise voting. We assume that political issues can be arrayed from left to right on a one-dimensional spectrum, so the policy enacted by the legislature is the median of the positions championed by legislators.¹

We formulate a general model of collective decision-making, rather than starting with a detailed model tailored to a specific setting. A key finding is that exactly two parties form, and these parties are of equal or almost equal size. This agglomeration does not depend on the magnitude of the economies of party size or sensitively on the nature of the individuals’

¹The median is, more generally, the only subgame perfect equilibrium outcome of any “binary agenda” (a procedure in which the outcome is the result of a sequence of pairwise votes) in which the players use weakly undominated strategies (see, for example, Miller 1995, Section 6.3).

preferences. Our model has several applications, including collective decision-making in committees with costly participation and legislative decision-making when legislators are elected by citizens. The latter model applies to a large class of electoral processes, including proportional representation and majority rule.

Some history

The evolution of Britain's two-party system motivates our model well.² Prior to the *Great Reform Act* of 1832, the English parliament was partitioned into two loosely knit groups, the reformist Whigs and the conservative Tories; little extra-parliamentary party machinery existed. Further, most bills were local or personal, and party discipline was minimal.

In the unreformed electoral system of Hanoverian England, a member of the élite faced an essentially fixed personal cost to joining parliament. The cost structure changed with the reform acts of 1832, 1867, and 1884, which lead to the evolution of a solidly two-party system—initially Conservative and Liberal, later Conservative and Labour. These reform acts afforded changes that played an important part in the development of modern political parties. The reforms were, broadly, three-fold: the expansion of the franchise;³ the annual compilation of a voter registry; and the adoption of simple plurality rule within the electoral process.⁴

For our purposes, the most relevant consequence of these reforms is that they introduced significant economies of scale and meant that average participation costs for parliamentarians could be reduced by supporting a party. First, the large increase in the size of the electorate and the introduction of plurality rule meant that a candidate had to rally support to get into parliament.

²Our historical account draws on O'Gorman (1989) and Ball (1987).

³The 1832 Act increased the electorate by almost 50%, that of 1867 increased it by 38% in counties and 13% in boroughs, that of 1884 increased it by 66%, and that of 1918 (which enfranchised all women over 30) increased it by around 50% (Ball 1987, pp. 18, 24).

⁴In 1866, prior to the reforms of 1867, half the members of the House of Commons were elected by one fifth of the electorate (Ball 1987, p. 21), so that these members had to do little work in terms of rallying support to get into parliament.

Second, the introduction of a voter registry introduced clear administrative costs to rallying support—costs that could be shared within a party. Indeed, by the 1880's "most of the energies of the party agents and the bulk of party election funds were devoted to filling the electoral register with one set of supporters and stripping the same register of the opposing voters through the Registration Courts" (Ball 1987, p. 20; see also p. 26).⁵

This evolution of a political party system was intra-parliamentary in the sense that existing members of parliament grouped themselves into parties. By contrast, the later emergence of the British Labour Party was extra-parliamentary: it resulted mainly from the work of grass-root activists from the working class who were enfranchised by the 1918 Reform Act. Our model contributes to the explanation of both types of party formation.

Framework

We study a model of collective decision-making in which each of a finite number of players chooses whether or not to participate in the decision-making process, and if so which policy to champion. We assume that the policy space is completely ordered (e.g. there is a single issue, or many issues ordered lexicographically), and that the outcome is the median of the policies championed by the participants.

We make two basic assumptions about the players' payoff functions, the first of which generalizes the idea that participation is costly.

C (costly participation) If a participating player's switching to nonparticipation does not change the outcome, then her payoff increases.

This condition is satisfied in a model of committee voting in which players incur a cost if they choose to participate and care about the policy chosen, because a player whose participation does not affect the outcome can profitably

⁵The cost of registering supporters and opposing the registration of opponents' supporters was associated with the complex eligibility requirements. Political parties litigated against the inclusion of their opponents' supporters and in defense of the inclusion of their own supporters.

withdraw and save the cost of participation. It is satisfied also in a richer two-stage model in which citizens elect legislators to an assembly.

Our second basic assumption is that there are economies of party size. We consider three variants of this assumption.

E (*economies of party size*) *If a participant's switching to a larger party does not change the median championed policy, then her payoff increases.*

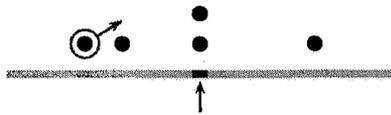


Figure 1. Moving to a larger party increases payoff if outcome does not change.

This condition holds in the example of committee voting when the cost of belonging to a party decreases with the party's size. It is illustrated in Figure 1, in which the horizontal line is the policy space, each dot is the policy championed by a player, and the arrow indicates the median policy championed.

The two-stage electoral games that we study may not satisfy condition **E**: when a candidate's position changes, voters' strategic incentives may change, causing a change in the voting equilibrium in the subgame which may affect the candidate's probability of winning.

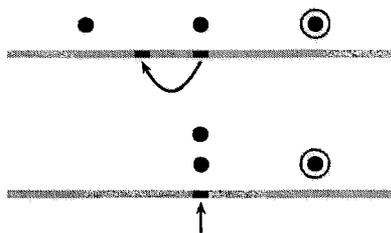


Figure 2. When a candidate's position changes, voters' strategic incentives may change.

In Figure 2, before the left candidate moves to the center to form a two-member party (top), citizens with favorite policies on the right have an incentive to vote for the right candidate, because were they to abstain the right candidate would not be elected and the policy outcome would be further to the left (as indicated by the arrow). If the left candidate moves to the middle (bottom), the outcome does not change if citizens vote as they did before. However, after

the left candidate moves the citizens no longer have an incentive to vote for the right candidate, because their switching to abstention has no effect on the policy.

The following variant of condition **E** accommodates this problem and fits well with models in which the players are elected. It says that there are

economies of party size if a move to a larger party does not change the incentives to participate.

PE (*participation dependent economies of party size*) If a participant's switching to a larger party (a) does not change the median championed policy and (b) would not change the median championed policy if the action of any given other player were to be fixed at nonparticipation, then her payoff increases.

The left party's move to the center in the example in Figure 2 satisfies (a) but not (b) (because the move would change the outcome were the rightist not to participate).

Condition **PE** is appropriate for a two-stage model in which each candidate is committed to the policy she champions. However, in a model in which a legislator can renege on a commitment to champion a policy, we need a version of the condition that applies only if the legislators' incentives to maintain their positions remain unchanged.

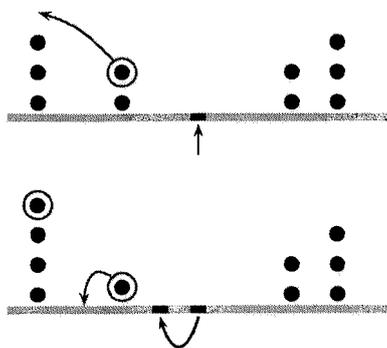


Figure 3. Incentives change when a member of the center-left party moves to the extreme left.

Consider Figure 3. A move to the left party by a member (say i) of the center-left party satisfies both (a) and (b) of **PE**. However, after i 's move, the remaining member of the center-left party (say j) can, by changing her position, affect the median position in ways that she could not originally (assuming she remains elected). Specifically, after i 's move, j 's moving to the left moves the median to the left (assuming j remains elected), whereas prior to i 's move, no move of j results in such a change in the median. Thus, depending on j 's preferences, i 's

move may give j an incentive to renege on j 's commitment to champion the position of the center-left party. Consequently, i should be concerned that her move to the left will precipitate an undesirable outcome-changing move to the left by her ex-comrade. This consideration leads us to study the following condition, which is weaker than **PE**.

SE (*strategy dependent economies of party size*) *If a participant's switching to a larger party (a) does not change the median championed policy and (b) does not allow any other player, by changing her action, to induce outcomes that she could not induce before the participant's move, then her payoff increases.*

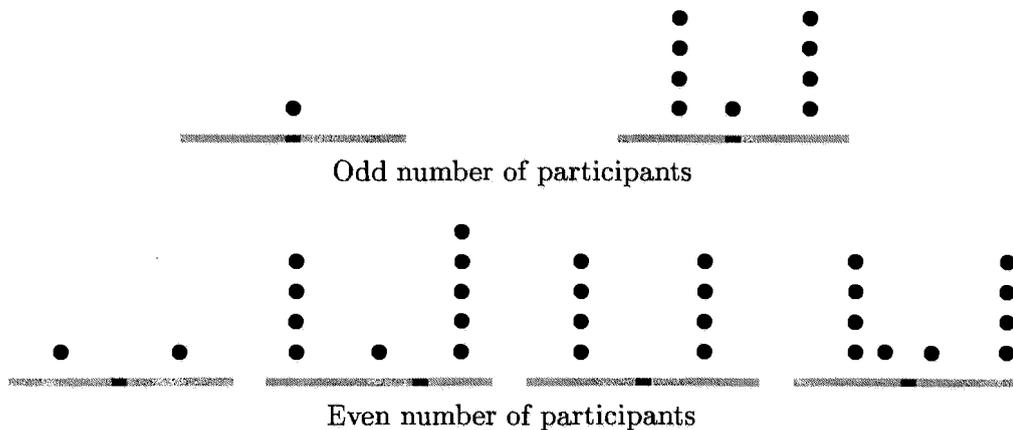
The move from the center-left party to the left party by player i that we considered in Figure 3 does not satisfy (b) because after the move, the other member of the center-left party can, by moving her position to the left, move the median championed policy to the left, a change she cannot induce before i 's move.

Results

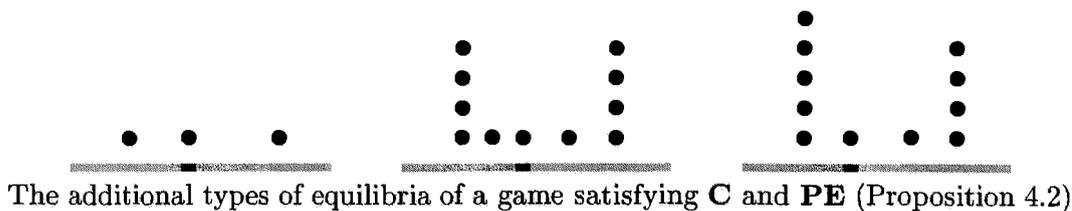
A key finding is that if participation is costly (**C**) and the game has participation dependent economies of party size (**PE**), then exactly *two* parties form, and these parties are of equal or almost equal size. In addition, up to three "independents" participate. If the game satisfies the stronger condition of economies of party size (**E**), then at most two "independents" participate. When participation is costly (**C**) and the game has strategy dependent economies of party size (**SE**), then exactly two parties with more than two members form, and these parties are of equal or almost equal size. In addition, up to three small parties or independents participate (see Figure 4).

Our two-party result is very general. It does not depend on the magnitude of the economies of party size or sensitively on the nature of the individuals' payoffs beyond conditions **C** and **E**, **PE**, or **SE**. Thus the results in our applications do not depend sensitively on the nature of the individuals' preferences over policies; for example, we do not assume these preferences are single-peaked.

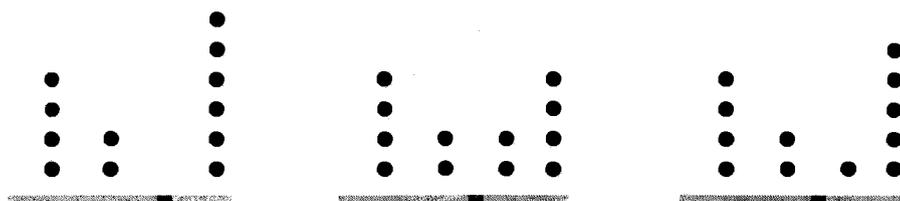
The idea behind the result is straightforward. In deciding whether to participate in a party or to champion a position as an individual, a player potentially faces a tradeoff. Joining a party saves her some cost, but may force her to compromise her position. However, if the policy outcome is the median



The types of equilibria of a game satisfying **C** and **E** (Proposition 4.1)



The additional types of equilibria of a game satisfying **C** and **PE** (Proposition 4.2)



The additional types of equilibria of a game satisfying **C** and **SE** (Proposition 4.3)

Figure 4. The equilibria of a game satisfying **C** and **E**, **PE**, or **SE**. Agglomerations of four or more players are parties, which may contain any number of players.

of the legislators' proposed policies, then joining a party whose position is on the same side of the median as is her favorite position is just as effective in determining the outcome as is championing her own favorite position. Thus in fact the individual faces no tradeoff. A left-leaning legislator is better off joining a left-leaning party than acting as an independent, regardless of the size of the cost saving, and a right-leaning legislator is better off joining a right-leaning party.

Relation with literature

The applications of our model to committee voting and the electoral process in a representative democracy are related to models in the literature.

Both our model of committee voting and Feddersen's (1992) voting model seek to explain two-party competition. The models differ in their explanatory variables. Feddersen's model emphasizes the role of voting decisions, while our model emphasizes the role of legislators. Feddersen assumes that the outcome is the policy that obtains the most votes, an assumption that captures well a single-district plurality rule election. By contrast, the median rule in our model is an appropriate model of legislative compromise.

Feddersen's first-past-the-post outcome function leads immediately to the conclusion that parties form in equilibrium. (Only a party can win an election.) His two-party result hinges also on this outcome function; it depends in addition on an assumption about the distribution of voters' preferences (see his Proposition 4). In our model, by contrast, party-formation is driven by our assumption of economies of party size. Furthermore, our two-party result does not depend on *any* assumption about the nature of preferences.

Gerber and Ortuño-Ortín (1998) study a model related to Feddersen's, with a continuum of voters. They assume a continuous outcome function that weights parties by their sizes and gives proportionally larger weight to large parties. (Such a function is not consistent with a winner-takes-all electoral rule.) They show that a unique strong Nash equilibrium exists, in which there are two parties.

The model of Osborne, Rosenthal, and Turner (2000) is related to our example of committee decision-making with costly participation. The key respect in which their model differs from our application to committee voting lies in their assumption that each player's participation cost is independent of the other participants' actions. Under this assumption, a player's strategy of proposing her favorite policy weakly dominates her other strategies, so that there is no cost-based incentive for individuals with different preferences to form parties. In the equilibria of their model, moderates do not participate and participation in large populations is low. The equilibria of our model, where

the participation cost declines with the number of individuals proposing a given policy, do not necessarily have these characteristics. In particular, while equilibria may exist in which some moderates do not participate, there also exist equilibria in which all individuals participate, even in large populations. Further, in equilibrium two parties form.

Several other models generate parties from principles different from those that drive the application of our model to the electoral process. Morelli (2001), for example, studies a model of party formation as an extensive game in which two potential candidates with extreme preferences propose compromise positions to a moderate potential candidate; subsequently each potential candidate chooses whether to stand, an election is held, and the outcome is the median of the elected politicians' positions. He finds that the number of parties depends on the electoral system and the distribution of preferences. The primary role of parties in his model is that they coordinate citizens' votes in an election.

Baron (1993) studies a model of proportional representation within the Hotelling–Downs framework. Citizens are not strategic, party formation is not costly, and the number of parties is fixed. Party size is determined by the fact that a large party has a diverse, and thus harder to please, membership, whereas such a party is more likely to be part of the government and be able to implement a policy appealing to its members.

Duverger (1954, Book II, Ch. I) argues that the evidence suggests that “the simple-majority single-ballot system favours the two-party system” (p. 217), while “the simple-majority system with second ballot and proportional representation favour multi-partism” (p. 239). (This claim is sometimes called “Duverger’s law”.) The first part of the claim is given theoretical support by Cox (1987) and Palfrey (1989) in a model in which strategic voters elect a single representative. They formalize the idea that votes for candidates with little chance of winning are wasted, resulting in equilibria in which there are two candidates. Feddersen’s model, discussed above, also lends the first part of the claim theoretical support, as do the “citizen-candidate” models of Osborne and Slivinski (1996) and Besley and Coate (1997). Our result provides an alternative strong rationale for the emergence of two parties under ma-

majority rule; it further suggests that when there is a single issue, two parties also emerge under proportional representation, contrary to the second part of Duverger's claim.

2. Model

Each member of a group $I = \{1, 2, \dots, n\}$ of n people chooses whether to champion a *policy*, and if so, which one. A policy is a member of the nonempty set X ; the action of non-participation is denoted θ . A person who champions a policy is called a *participant*. Each person i 's payoff function is $u_i: (X \cup \{\theta\})^n \rightarrow \mathbb{R}$.

If two or more people champion the same policy $x \in X$, we say that x is a *party*; if a single person champions x , we say that x is an *independent*. We call a party with more than two members a *large* party and one with exactly two members a *small* party.

The policy outcome is a compromise among the policies championed by participants: a *compromise function* $M: (X \cup \{\theta\})^n \rightarrow X$ assigns to each action profile an *outcome*, with $M(\theta, \dots, \theta)$ (the outcome if no one participates) equal to some given default policy. We discuss the compromise function in Section 3.

In summary, we study a strategic game in which the set of players is $I = \{1, \dots, n\}$, the set of actions of each player is $X \cup \{\theta\}$, and the payoff function of each player i is $u_i: (X \cup \{\theta\})^n \rightarrow \mathbb{R}$. The relation between the actions and payoffs depends on the compromise function M , and we denote the game $\Gamma(I, (X \cup \{\theta\})_{i \in I}, (u_i)_{i \in I}, M)$; when there is no possibility for confusion we use the abbreviation $\Gamma(M)$.

We say that participation in the game $\Gamma(M)$ is costly if a player prefers to withdraw if her withdrawal does not change the outcome. This condition captures the idea that participation is costly.

Definition 2.1. For a strategy profile $a \in (X \cup \{\theta\})^n$ the game $\Gamma(M)$ has

- C *costly participation at a* if $u_i(\theta, a_{-i}) > u_i(a)$ whenever $a_i \in X$ and $M(a) = M(\theta, a_{-i})$ for any player i .

We say that a game that has costly participation for every action profile has *costly participation*.

Our assumptions about economies of party size are explained in the *Framework* section of the Introduction. Here, and subsequently, we use $\#Z$ to denote the number of members of the set Z .

Definition 2.2. For a strategy profile $a \in (X \cup \{\theta\})^n$ the game $\Gamma(M)$ has

E *economies of party size at a* if $u_i(x, a_{-i}) > u_i(a)$ whenever $a_i \in X$, $M(a) = M(x, a_{-i})$, and $\#\{j \in I : a_j = a_i\} \leq \#\{j \in I : a_j = x\}$

PE *participation dependent economies of party size at a* if $u_i(x, a_{-i}) > u_i(a)$ whenever $a_i \in X$, $M(a) = M(x, a_{-i})$, $M(\theta, a_{-h}) = M(\theta, (x, a_{-i})_{-h})$ for each player h , and $\#\{j \in I : a_j = a_i\} \leq \#\{j \in I : a_j = x\}$

SE *strategy dependent economies of party size at a* if $u_i(x, a_{-i}) > u_i(a)$ whenever $a_i \in X$, $M(a) = M(x, a_{-i})$, for any participating player h and any $y \in X \cup \{\theta\}$ there exists $z \in X \cup \{\theta\}$ such that $M(z, a_{-h}) = M(y, (x, a_{-i})_{-h})$, and $\#\{j \in I : a_j = a_i\} \leq \#\{j \in I : a_j = x\}$.

We say that a game that satisfies **E** for every action profile has *economies of party size*, one that satisfies **PE** for every action profile has *participation dependent economies of party size*, and one that satisfies **SE** for every action profile has *strategy dependent economies of party size*.

Examples

Example 2.1 (Committee voting). Each member of a committee of n people chooses whether to champion a *policy* in X or not to participate. Each person i obtains the payoff $v_i(x)$ from the policy x , where $v_i : X \rightarrow \mathbb{R}$; we refer to v_i as i 's *valuation function*. If person i participates and chooses x she bears the cost $c_i(k(x))$, where $k(x)$ is the number of people proposing x and $c_i : \mathbb{N} \rightarrow \mathbb{R}$ is positive and decreasing.

We may model this situation as a game in which the set of players is the set of n people, the set of actions of each player is $X \cup \{\theta\}$, and player i 's

payoff function is given by

$$u_i(a) = \begin{cases} v_i(M(a)) & \text{if } a_i = \theta \\ v_i(M(a)) - c_i(\#\{j \in I : a_j = a_i\}) & \text{if } a_i \in X. \end{cases}$$

This payoff function satisfies conditions **C** and **E** at every action profile: participation is costly and the game has economies of party size.

This example may be generalized so that each person's cost of championing a policy depend on the policies championed by others, not simply on the number of others championing it, which captures the idea that arguing the case for a policy may be more difficult if it is extreme, or if some other particularly attractive position is being championed. The resulting game also satisfies **C** and **E**.

Example 2.2 (Elections without commitment). Consider a two-stage game in which the players consist of both citizens and candidates for a legislature. In the first stage each citizen votes for a candidate; these decisions are made simultaneously. The set of winners of the election is determined by an arbitrary rule. (For example, the set of winners may consist of the top k vote-getters, with ties broken randomly.) In the second stage, each legislator chooses whether to actively participate, incurring an effort cost, and if so which policy to champion. (A politician cannot commit to a policy prior to the election.) We assume that the effort cost displays economies of party size. As before, the policy outcome is the median of the policies championed by the legislators. Both and citizens and politicians care about this policy outcome. (In particular, the politicians are "ideological".)

More precisely, the players in the subgame following any first-period history are the elected legislators, and the subgame has the form of the game in Example 2.1. Thus each subgame satisfies **C** and **E**. Implicit in the game is the assumption that the electorate knows the preferences of each politician and thus can anticipate the behavior of the elected legislators in each subgame.

Proposition 4.2 below implies that in every subgame the politicians form two parties. (Thus under plurality rule with ties broken randomly, every real-

ization of the legislature contains two parties.)

Example 2.3 (Elections with commitment). We now consider a model that allows a richer interaction between voters and candidates, and permits candidates who are not ideological.

The electoral process has two stages.

- First, each potential legislator chooses whether to become a candidate, and if so the policy she *commits* to champion.
- Second, each citizen chooses whether to vote for a candidate, and if so, which one.

An electoral rule determines the members of the legislature. The policy outcome is the median of the policies championed by the elected candidates.

Each citizen cares about the policy outcome and incurs a fixed cost if she votes. Each potential legislator incurs a cost if she participates as a candidate and may or may not care about the policy outcome. If elected she obtains a “prize” that compensates her for the costs; this prize depends on the number of elected legislators who propose the same policy (i.e. on the size of her party after the election.) This prize could reflect the fact that the cost of running is lower for a member of a larger party or public subsidies to parties.

We study this game in detail in Section 6. Fix a subgame perfect equilibrium and consider the strategic game in which the players are the potential legislators and the payoff of each legislator to the action profile a of policy commitments is her equilibrium in the subgame following a . We show in Proposition 6.1 that this strategic game satisfies **C** (costly participation) and **PE** (participation dependent economies of party size) under a natural refinement of subgame perfection that we call *subgame persistence*.

We consider a modification of the game in which any elected legislator may change her position after the election—i.e. may renege on her policy commitment. We propose an alternative refinement of subgame perfection, which takes into account *incentive compatibility* requirements in subgames, and show that the associated strategic game between the potential legislators satisfies **C** and **SE** (Proposition 6.3).

Example 2.4 (Proportional representation with party lists). The previous example can be specialized to proportional representation based on party lists (the electoral system in Austria, Belgium, Denmark, Finland, the Netherlands, Norway, Sweden, and Switzerland). Candidates proposing the same position are ordered into a party-list and each citizen votes for one of these lists. Each list is awarded a number of seats in proportion to the votes received. Each action profile in the two-stage game of the previous example corresponds to an action profile in this party-list model, with the same payoffs, and vice versa. Thus the results for the previous example apply also to this example.

3. The compromise

We assume that the set X of policies is “completely ordered”. That is, the members of X are related by an ordering, denoted \leq ,⁶ which we interpret as embodying the policies’ positions on a left–right political spectrum. That is, if $x < y$ then all players agree that x is *more left than* y , and if $x \leq y$ then they agree that x is *at least as left as* y . The members of X may be numbers, in which case \leq may take its usual meaning, but they may alternatively be points in a higher-dimensional space, as long as they may be ordered.

We assume that the compromise is the median of the policies championed by participants. The median is the middle proposed policy if the number of participants is odd, a policy between the two middle positions if the number of participants is positive and even, and a *default outcome* $d \in X$ if no player participates. Precisely, order the participants (the players whose actions are in X) so that $a_1 \leq \dots \leq a_k$. If k is odd, the *median* of a is $m(a) = a_{(k+1)/2}$. If k is positive and even, the *left median* of a is $m^l(a) = a_{k/2}$ and the *right median* of a is $m^r(a) = a_{k/2+1}$, and we assume that the compromise is $M(a) = S(m^l(a), m^r(a))$, where $S : X \times X \rightarrow X$ with $S(x, y) = S(y, x)$ if $x \in X$ and $y \in X$, and $x \leq S(x, y) \leq y$ if $x \leq y$.

⁶That is, \leq is transitive, complete (either $x \leq y$ or $y \leq x$ for any two policies $x \in X$ and $y \in X$), reflexive ($x \leq x$), and antisymmetric ($x \leq y$ and $y \leq x$, implies $x = y$).

In summary, for any action profile a the outcome is

$$M(a) = \begin{cases} d & \text{if } a = (\theta, \dots, \theta) \\ m(a) & \text{if } \#\{i \in I : a_i \in X\} \text{ is odd} \\ S(m^l(a), m^r(a)) & \text{if } \#\{i \in I : a_i \in X\} \text{ is positive and even.} \end{cases}$$

The median championed policy is not beaten by any policy in pairwise voting when each participant's preferences are single-peaked (relative to the ordering) with a peak at the policy she champions. The understanding underlying this outcome function is that a player's championing a policy entails her committing to vote according to some single-peaked preference that peaks at that policy.

4. Properties of equilibrium

We study the properties of a (pure strategy) Nash equilibrium of a game satisfying our conditions. The following result is illustrated in Figure 4 (page 7).

Proposition 4.1. *At any Nash equilibrium in which participation is positive and **C** and **E** are satisfied, one of the following conditions holds.*

- a. *The number of participants is odd and there is either a single independent or two equal-sized parties between which there is an independent.*
- b. *The number of participants is even, and there are (i) two equal-sized parties and no independents, (ii) two independents, (iii) two equal-sized parties between which there are two independents, or (iv) two parties, one larger by one member than the other, between which there is a single independent.*

This result and all subsequent ones are proved in the appendix. The proof rests on two main ideas. First, if there are two parties on one side of the median, then any member of the smaller party (or of either party, if the sizes are the same) can switch to the other party without affecting the outcome. Such a move is profitable by condition **E**. Second, if a single position is proposed

and more than one player proposes it, then no player's withdrawal affects the outcome. Therefore, by condition **C** some participating player can profitably withdraw.

When the weaker condition **PE** is satisfied three further types of equilibria may occur (again, see Figure 4).

Proposition 4.2. *Any Nash equilibrium in which participation is positive and **C** and **PE** are satisfied either takes one of the forms given in Proposition 4.1 or has an odd number of participants and (i) three independents, (ii) two equal-sized parties and three independents holding positions between the parties, or (iii) two parties, one larger by one member, and two independents holding positions between the parties.*

Under the even weaker condition **SE** we get only an additional three new types of equilibria.

Proposition 4.3. *Any Nash equilibrium in which participation is positive and **C** and **SE** are satisfied either takes one of the forms given in Proposition 4.2 or has an even number of participants and (i) two large parties of equal size and two small parties holding positions between the large parties, (ii) two large parties, one larger than the other by two members, and one small party holding a position between the large parties, or (iii) two large parties, one larger than the other by one member, and a small party and an independent holding positions between the large parties with the independent holding a position closer to the largest party.*

5. Equilibrium of committee voting

The game of committee voting in Example 2.1 satisfies **C** and **E**, so by Proposition 4.1 any Nash equilibrium has one of the forms shown in Figure 4. We now fully characterize the equilibria in which there are two parties and no independents, which we refer to here as *two-party equilibria*. Notice that if the cost of being the sole proponent of a position is too high to justify any possible benefit, then every equilibrium with positive participation is of this type.

We use the following assumption.

- A** The policy space X is a non-trivial interval of the real line \mathbb{R} . The valuation function of each player i is given by $v_i(x) = v(|x - x_i|)$, where $v : \mathbb{R} \rightarrow \mathbb{R}$ is strictly concave, each player's cost function c_i is the same, equal to c , and $S(x, y) = \frac{1}{2}(x + y)$ for all $x \in X$ and $y \in X$.

Under this assumption, for any two-party equilibrium in which each party has k members and the distance between the parties' positions is z we find a number τ , which depends on k and z (and may be positive or negative), such that all players whose favorite positions are to the left of $x - \tau$ belong to the left party, the favorite position of every member of the left party is to the left of $\frac{1}{2}(x + y) - \tau$, all players whose favorite positions are to the right of $y + \tau$ belong to the right party, and the favorite position of every member of the right party is to the right of $\frac{1}{2}(x + y) + \tau$. Further, every action profile that satisfies these conditions is an equilibrium. In particular, the result gives conditions under which a two-party equilibrium with parties of a given size and separation exists. It is illustrated in Figure 6.

We can state the result precisely with the aid of a function t defined as follows. For any $z > 0$ define

$$\bar{C}(z) = \max_t [v(t) - v(t + \frac{1}{2}z)]$$

(which may be infinite). Now, for any C with $0 \leq C < \bar{C}(z)$ there is a unique point $t(z, C)$ such that $v(t(z, C)) - v(t(z, C) + \frac{1}{2}z) = C$ (refer to Figure 5). Note that $t(z, 0) = -\frac{1}{4}z$ for all $z > 0$, and t is decreasing in z for any given value of C and increasing in C for any given value of z .

Proposition 5.1. *Assume A. The action profile a is a two-party Nash equilibrium if and only if every player either belongs to one of two parties, x and $y > x$, or does not participate, the parties have the same number $k \geq 2$ of*

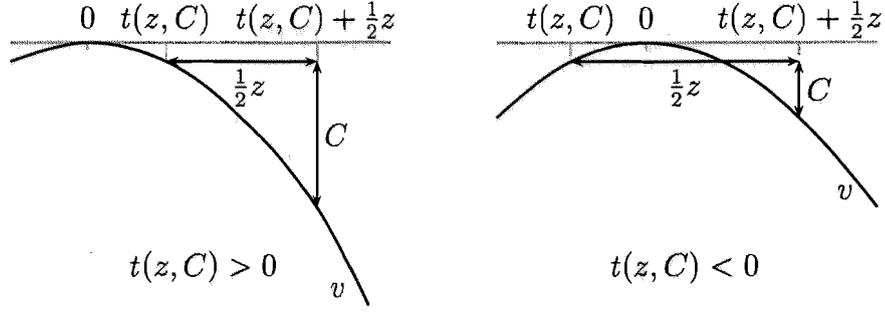


Figure 5. Two examples of the determination of $t(z, C)$.

members, $c(k) \leq \bar{C}(y - x)$, and

$$x_i < x - t(y - x, c(k)) \Rightarrow a_i = x$$

$$x_i > y + t(y - x, c(k)) \Rightarrow a_i = y$$

$$a_i = x \Rightarrow x_i \leq \frac{1}{2}(x + y) - t(y - x, c(k))$$

$$a_i = y \Rightarrow x_i \geq \frac{1}{2}(x + y) + t(y - x, c(k)).$$

The proof of this result involves finding players who are indifferent between belonging to one of the parties or withdrawing. It then uses the concavity of the valuation function to conclude that all players with more extreme, or less extreme, favorite positions prefer one action or the other.

6. Elections with commitment

In Example 2.3, each of a finite number n of *potential legislators* has the opportunity to vie for a place in a legislative assembly. Candidates are elected to the assembly by a population of *citizens*, modelled as a finite positive atomless measure space $(\Omega, \mathcal{F}, \mu)$. (We adopt this formulation to avoid integer problems.) The electoral process has two stages. First, each potential legislator chooses whether to become a candidate, and if so which policy (member of X) to champion. Second, each citizen chooses whether to vote for a candidate, and if so, which one.

In the subgame following the potential legislators' action profile $a \in (X \cup$

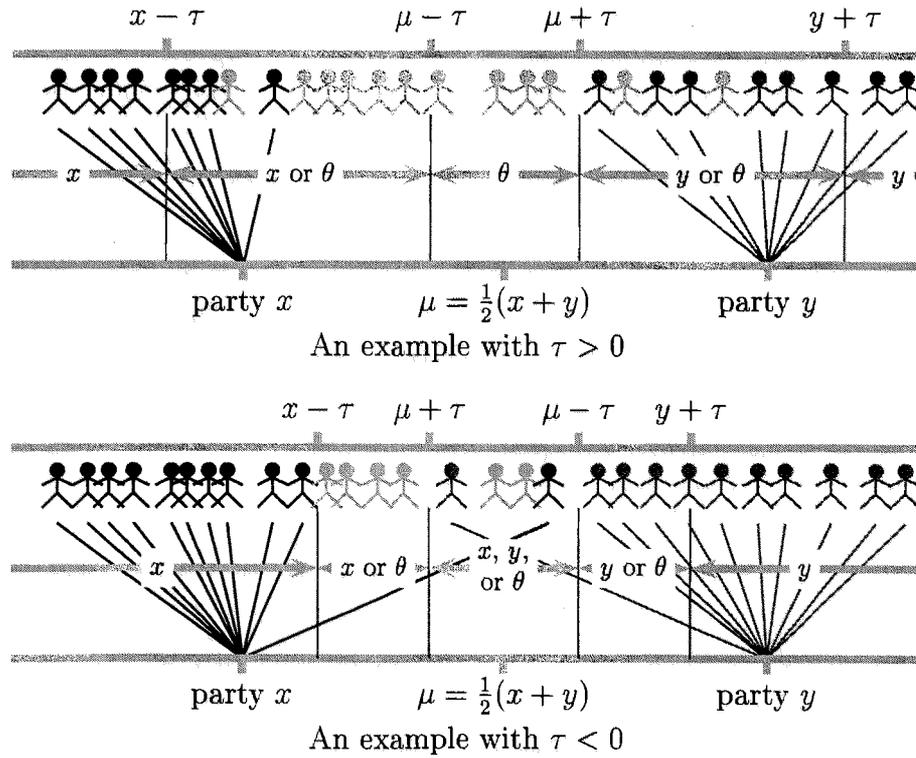


Figure 6. The equilibria under assumption A (Proposition 5.1). The parties' positions are x and y , $\mu = \frac{1}{2}(x + y)$, and $\tau = t(y - x, c(k))$, where k is the size of each party.

$\{\theta\}^n$, a voting profile for the citizens is a measurable function

$$b^a : \Omega \rightarrow \{i \in I : a_i \in X\} \cup \{\theta\},$$

where $b^a(\omega) = \theta$ means that citizen ω does not vote and $b^a(\omega) = i$ means she votes for candidate $i \in I$ (who proposes the policy $a_i \in X$).

Electoral rule An electoral rule translates the voting profile into a subset of the candidates (those who are elected). We work here with rules that elect a potential legislator if and only if the number of votes she obtains is at least equal to some "quota" of votes. This quota is given by a quota function $Q : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ that is continuous, nondecreasing ($x \geq y$ implies $Q(x) \geq Q(y)$), anonymous in the sense that for any one-to-one function $\lambda : I \rightarrow I$ we

have $Q(\alpha_1, \dots, \alpha_n) = Q(\alpha_{\lambda(1)}, \dots, \alpha_{\lambda(n)})$, and vanishes, if at all, only at zero ($Q(x) = 0$ implies $x = 0$). Given a profile a of policies championed by the potential legislators and a voting profile b^a , $(b^a)^{-1}(j)$ is the set of citizens who vote for candidate j and $\mu((b^a)^{-1}(j))$ is the size of this set. Thus legislator i is *elected* if

$$\mu((b^a)^{-1}(i)) \geq Q(\mu((b^a)^{-1}(1)), \dots, \mu((b^a)^{-1}(n))).$$

The following three examples of quota functions are continuous, nondecreasing, and anonymous. A potential legislator is elected if and only if she gets at least as many votes as

- i. (*first-past-the-post*) every other candidate: $Q(\alpha_1, \dots, \alpha_n) = \max_{i \in I} \alpha_i$
- ii. (*Hare Quota*) the total number of votes divided by \bar{k} , the number of seats in the legislature: $Q(\alpha_1, \dots, \alpha_n) = (\sum_{i=1}^n \alpha_i) / \bar{k}$
- iii. (*fixed quota*) a fixed number (which might be related to the size of the population): $Q(\alpha) = \delta$ for all $\alpha \in \mathbb{R}^n$.

Payoffs The policy championed by legislator i is

$$A_i(a, b^a) = \begin{cases} a_i & \text{if } i \text{ is elected} \\ \theta & \text{otherwise.} \end{cases}$$

Each citizen $\omega \in \Omega$ attaches the value $v(\omega, x)$ to the policy $x \in X$, where $v : \Omega \times X \rightarrow \mathbb{R}$. We refer to $v(\omega, \cdot)$ as ω 's *valuation function*. We assume that the function $v(\cdot, x)$ is integrable for any x .

A citizen who votes incurs the cost $C_z > 0$. The payoff $u(\omega, (a, b^a))$ of citizen ω depends on the outcome and whether she votes:

$$u(\omega, (a, b^a)) = \begin{cases} v(\omega, M(A(a, b^a))) - C_z & \text{if } b^a(\omega) \neq \theta \\ v(\omega, M(A(a, b^a))) & \text{if } b^a(\omega) = \theta. \end{cases}$$

Each potential legislator i incurs the cost $C_\ell > 0$ if she participates as a candidate and has a *valuation function* $v_i : X \rightarrow \mathbb{R}$ over policies. If elected she obtains a “prize” that depends on the number of elected legislators that propose the same policy (i.e. on the size of her party after the election). Specifically, the payoff of potential legislator i is

$$u_i(a, b^a) = \begin{cases} v_i(M(A(a, b^a))) & \text{if } a_i = \theta \\ v_i(M(A(a, b^a))) - C_\ell & \text{if } a_i \neq \theta \text{ \& } i \text{ not elected} \\ v_i(M(A(a, b^a))) - C_\ell + P_i(a, b^a) & \text{if } a_i \neq \theta \text{ \& } i \text{ is elected,} \end{cases}$$

where

$$P_i(a, b^a) = p(\#\{j \in I : A_j(a, b^a) = a_i\})$$

and $p : \mathbb{N} \rightarrow \mathbb{R}_{++}$ is a positive increasing function, called the *prize function*.

Equilibrium Given the continuum of voters, no single voter affects the outcome of the election. To obtain meaningful equilibria, we consider strategy profiles from which no arbitrarily small group of citizens has an incentive to deviate.

Each action profile $a \in (X \cup \{\theta\})^n$ of the potential legislators leads to a subgame Γ^a in which the citizens vote. Given a voting profile b in the subgame Γ^a and $\epsilon > 0$, we say that a measurable set $S \subset \Omega$ is an ϵ -club if $S \subset b^{-1}(j)$ for some $j \in \{i \in I : a_i \in X\} \cup \{\theta\}$ (either all members of S vote for the same candidate, or none votes) and $0 < \mu(S) \leq \epsilon$.

Definition 6.1. The voting profile b in the subgame Γ^a is a *small clubs Nash equilibrium* (or simply an *equilibrium*) of the subgame if there exists $\epsilon > 0$ such that for every ϵ -club $S \subset \Omega$ and every $j \in \{i \in I : a_i \in X\} \cup \{\theta\}$

$$\int_S u(\omega, (a, b)) d\mu(\omega) \geq \int_S u(\omega, (a, [j, b|_{\Omega \setminus S}])) d\mu(\omega).$$

Notice that when Ω is finite and μ is the counting measure, the notion of a small clubs Nash equilibrium coincides with the notion of a pure strategy Nash equilibrium.

A strategy profile in the whole game is a pair (a, B) where $a \in (X \cup \{\theta\})^n$ and B is a function that associates each $a^* \in (X \cup \{\theta\})^n$ with a voting profile $B(a^*) : \Omega \rightarrow \{i \in I : a_i^* \in X\} \cup \{\theta\}$ in the subgame Γ^{a^*} .⁷ For a strategy profile (a, B) we say that there is *positive voter turnout* if $B(a)$ does not vanish almost everywhere.

For each strategy profile B of the citizens define the strategic game for the legislators

$$G^B(I, (X \cup \{\theta\})_{i \in I}, (w_i)_{i \in I}, M)$$

by letting $w_i(a) = u_i(a, B(a))$ for each $a \in (X \cup \{\theta\})^n$ and each i , where u_i is the payoff of the potential legislator.

Each voting subgame Γ^a may have multiple equilibria, with a variety of outcomes. We restrict attention to equilibria of the whole game in which each legislator expects a deviation in the first stage to have no effect on the citizens' voting behavior *if this behavior remains an equilibrium of the subgame reached after the deviation*. We say that the subgames Γ^{a^*} and Γ^a are *adjacent* if there exists some player i and some $x \in X \cup \{\theta\}$ such that $a^* = (x, a_{-i})$ (i.e. the histories a and a^* differ only in the action of a single candidate).

Definition 6.2. A strategy profile (a, B) is an *equilibrium* if a is a Nash equilibrium of the game G^B and the voting profile $B(a)$ is a small clubs Nash equilibrium of the subgame following the history a . An equilibrium strategy profile (a, B) is a *subgame persistent equilibrium* if for every a^* adjacent to a we have $B(a^*) = B(a)$ whenever $B(a)$ is a small clubs Nash equilibrium of the subgame following a^* .

Results

Proposition 6.1. *If (a, B) is a subgame persistent equilibrium with positive voter turnout, then the game G^B satisfies **C** and **PE** at a .*

The next result is an immediate consequence of Propositions 4.2 and 6.1.

⁷We use the notation $B(a^*)$ instead of b^{a^*} to emphasize the fixed strategy profile B for the second stage of the game.

Corollary 6.2. *In a subgame persistent equilibrium with positive voter turnout all participating candidates are elected and their positions are configured according to Proposition 4.2.*

To see the logic behind the result, note first that at equilibrium, the continuity of the quota function and the fact that it is nondecreasing imply that all elected candidates get exactly the quota of votes and candidates that are not elected get no votes. Therefore, by subgame persistence any unelected candidate can profitably drop out without affecting the policy outcome, so that all candidates at equilibrium get elected. To show that condition **C** is satisfied at a , we note that if withdrawal does not change the outcome, a small club of citizens voting for that candidate can profitably drop out. To show that condition **PE** is satisfied at a , we argue that if a move by candidate i to a larger party x satisfies the conditions of **PE**, then the equilibrium voting profile of the subgame following the history a is also an equilibrium voting profile of the adjacent subgame following the history (x, a_{-i}) . The reason is that if the members of a small club of voters find it profitable to deviate in the adjacent subgame, then they also find it profitable to deviate in the original subgame, contradicting equilibrium. So by subgame persistence candidate i finds it profitable to move to the larger party x .

Incentive compatibility We now study the implications of allowing the candidates to renege on their policy commitments. We look at equilibria that satisfy two conditions. First, no elected legislator can profitably change her policy after she has incurred the cost of participation and obtained the “prize” that depends on the size of her party. Second, each potential legislator expects a deviation in the first stage to have no effect on the citizens’ voting behavior if this behavior remains an equilibrium of the subgame reached after the deviation *and* no elected legislator wishes to renege on her policy commitment in this subgame. By weakening the restriction on deviants’ beliefs about the resulting voting equilibrium, we potentially allow beliefs for which deviations that were profitable under subgame persistence are no longer profitable, and thus expand the set of equilibria.

Let b be a voting profile for citizens in the subgame following a . We say that (a, b) is *incentive compatible* if for any policy $x \in X$ and for any elected legislator i in that subgame we have

$$v_i(M(A(a, b))) \geq v_i(M(x, A_{-i}(a, b))),$$

where v_i is i 's valuation function. That is, a voting profile is incentive compatible for a given strategy profile for potential legislators if no elected legislator can profitably change her policy after the election.

Definition 6.3. An equilibrium strategy profile (a, B) is a *subgame IC-persistent equilibrium* if $(a, B(a))$ is incentive compatible and for every a^* adjacent to a we have $B(a^*) = B(a)$ whenever $B(a)$ is a small clubs Nash equilibrium of the subgame following a^* and $(a^*, B(a))$ is incentive compatible.

A subgame persistent equilibrium (a, B) may not be a subgame IC-persistent equilibrium, because $(a, B(a))$ may not be incentive compatible. However,

$$\begin{array}{l} (a, B) \text{ is subgame persistent and} \\ (a, B(a)) \text{ is incentive compatible} \end{array} \Rightarrow (a, B) \text{ is subgame IC-persistent.}$$

Further, a subgame IC-persistent equilibrium (a, B) may not be a subgame persistent equilibrium, because at a^* adjacent to a we may have $B(a^*) \neq B(a)$ even though $B(a)$ is a small clubs Nash equilibrium of the subgame following a^* (i.e. a voting profile may remain an equilibrium for an adjacent subgame but may no longer be incentive compatible.)

The next result follows easily from the proof of Proposition 6.1.

Proposition 6.3. *If (a, B) is a subgame IC-persistent equilibrium with positive voter turnout, then the game G^B satisfies **C** and **SE** at a .*

Corollary 6.4. *In a subgame IC-persistent equilibrium with positive voter turnout, all participating candidates get elected and their positions are configured according to Proposition 4.3.*

The Hare Quota Consider briefly the specific election rule given by the Hare Quota, whereby a potential legislator is elected if and only if the number of votes she obtains is at least the total number of votes divided by \bar{k} , the number of seats in the legislature.

In this case, if candidates are not ideological then the conclusion of Proposition 6.1 holds for subgame perfect equilibria. This result follows from the fact that for the Hare Quota, in any equilibrium of a voting subgame with positive voter turnout exactly \bar{k} legislators are elected, each receiving exactly the Hare Quota of votes.

Another implication of this property is that the Hare Quota can be considered in the class of the Single Transferable Vote (STV) procedures.⁸ At an equilibrium of our model each elected legislator receives exactly the Hare Quota of votes. Therefore, this procedure provides transferability implicitly—if a candidate has enough votes to get elected, then in equilibrium excess supporters allocate their votes to other candidates or drop out.⁹

Existence of equilibrium A subgame persistent equilibrium of the two-stage game exists under weak conditions. If we assume, for example, that potential legislators are not ideological, citizens' preferences are single-peaked, and the distribution of favorite positions is nonatomic, then we can construct an equilibrium. For a small enough cost of participation, we find a two-party equilibrium in which all citizens vote, citizens whose favorite positions are to the left of the population median vote for the left party, and those whose positions are to the right of the population vote for the right party.

⁸In the canonical STV procedure if a candidate receives more than the quota of votes, then the excess votes are transferred to another candidate, usually using the voters' secondary preferences.

⁹For a history of the STV procedure see Tideman (1995) and Richardson and Tideman (2000). One of the earliest versions of STV voting was in a local election in Tasmania, Australia. In that election a voter recorded his name on his preferred candidate's list. A candidate secured election by receiving the required number of votes. As in our model, Richardson and Tideman point out that this procedure provides transferability implicitly, in that when a candidate had enough votes to be elected, supporters see this and allocate their votes to other candidates.

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Appendix: Proofs

Proof of Proposition Proposition 4.3 (SE). Let a be a Nash equilibrium with at least two participants.

Number of participants is even: Ignore nonparticipating players and re-index the participants from $-k$ to -1 and from 1 to k so that $a_i \leq a_j$ if $i \leq j$, so that the left median is a_{-1} and the right median is a_1 . In this proof we use $a_{(I \setminus i)}$ to denote the list of actions of the players other than i and a_{-i} to denote the action of player $-i$.

We prove the result by establishing four properties of an equilibrium. (By the symmetry of the situation we need only look at players with positive indexes.)

(i) $a_1 \neq a_{-1}$: If $a_1 = a_{-1}$ and either player 1 or player -1 withdraws then the outcome does not change, so by **C** the player’s withdrawal is profitable, which is inconsistent with equilibrium.

(ii) If $i \geq 3$ and $j \geq 3$, then $a_i = a_j$: Suppose by way of contradiction that $a_i \neq a_j$. Without loss of generality, assume that a_j is held by at least as many players as is a_i . If player i moves to a_j , then the policy outcome does not change and she joins a larger party. (I.e. the second and fourth conditions in **SE** are satisfied for $x = a_j$.)

Now take a participating player h . If we fix that player’s action at nonparticipation, the new outcome becomes a_1 or a_{-1} irrespective of player i ’s move to the position of j . Further, if we fix player h ’s action to $y \in X$ then the new outcome is the same before and after i moves to a_j , because the positions of player 2 and all players to the left of her remain unchanged with i ’s move. That is, $M(y, (a_j, a_{(I \setminus i)})(I \setminus h)) = M(y, a_{(I \setminus h)})$. Thus by **SE**, player i ’s move to a_j is profitable, which is inconsistent with equilibrium.

(iii) *Players 2 is not an independent*: Suppose by way of contradiction that player 2 is an independent. Then by (i) player 1 is also an independent. Now if 2 shifts her position to 1, then she joins a two member party and does not change the outcome. Take some participating player h and fix her position to $y \in X \cup \{\theta\}$. Notice that when $M(y, a_{(I \setminus h)}) \neq M(y, (a_1, a_{(I \setminus 2)})_{(I \setminus h)})$, we have $M(y, (a_1, a_{(I \setminus 2)})_{(I \setminus h)}) = a_1$. Therefore, letting $z = a_1$ we have $M(z, a_{(I \setminus h)}) = M(y, (a_1, a_{(I \setminus 2)})_{(I \setminus h)})$, so that by **SE** player 2's move to a_1 is profitable, which is inconsistent with equilibrium.

(iv) *If $a_2 \neq a_3$ then player 3 is a member of a party with three or more members*: Suppose that $a_2 \neq a_3$ and that a_3 is held by two members or an independent. By (iii), players 1 and 2 are members of one party. Now by the same argument as for (ii), the conditions of **SE** are satisfied for $i = 3$ and $x = a_2$, so that a move by player 3 to a_2 is profitable, which is inconsistent with equilibrium.

Number of participants is odd: Re-index the participants from $-k$ to k with $a_i \leq a_j$ if $i \leq j$, so that the median position is a_0 .

(i) *Player 0 is an independent*: If not, the withdrawal of any player whose position is a_0 does not change the outcome, so that by **C** it is profitable, which is inconsistent with equilibrium.

(ii) *If $i \geq 2$ and $j \geq 2$, then $a_i = a_j$* : Suppose by way of contradiction that $a_i \neq a_j$. Without loss of generality, assume that a_j is held by at least as many players as is a_i . If player i moves to a_j then the policy outcome does not change and she joins a larger party. (I.e. the second and fourth conditions in **SE** are satisfied for $x = a_j$.) Now take some participating player h and fix her position to $x \in X \cup \{\theta\}$. Then $M(x, (a_i, a_{(I \setminus j)})_{(I \setminus h)}) = M(x, a_{(I \setminus h)})$, so by **SE** the move to a_j is profitable, which is inconsistent with equilibrium.

(iii) *If $i \geq 2$, then i is not an independent*: If i is an independent, then by (i) player 1 is an independent and by the same argument as in (ii) player i can profitably move to a_1 . \square

Proof of Proposition 4.2 (PE). Let a be a Nash equilibrium. **PE** implies **SE**, so we need only exclude as an equilibrium an action profile with an even

number of participants in which a two member party holds the right median position and there is a three or more member party to its right.

Once again re-index the participants from $-k$ to k , excluding 0, with $a_i \leq a_j$ if $i \leq j$. Reconsider case (iv) of the proof of Proposition 4.3.

(iv') *Players 2 and 3 cannot hold different positions:* If $a_2 \neq a_3$ and player 2 moves to a_3 then she joins a larger party and does not change the outcome. Fixing the action of some player h to be nonparticipation, we know that $M(\theta, a_{(I \setminus h)})$ is a_1 if $h < 0$ and a_{-1} if $h > 0$. Therefore $M(\theta, (a_3, a_{(I \setminus 2)})_{(I \setminus h)}) = M(\theta, a_{(I \setminus h)})$, so that **PE** implies that player 2's move to a_3 is profitable. \square

Proof of Proposition 4.1 (E). Let a be a Nash equilibrium. **E** implies **PE**, which implies **SE**, so we need only exclude as an equilibrium an action profile with an odd number of participants in which an independent holds a position other than the median. Such an action profile is excluded as an equilibrium because the median position is held by an independent in any equilibrium with an odd number of participants, so that any other independent can profitably move to the median, given **E**. \square

Proof of Proposition 5.1. First suppose that the parties' positions and players' actions satisfy the conditions in the result.

Consider deviations by a player i for whom $x_i \leq \mu - t(y - x, c(k))$ who supports party x . If she becomes an independent, the outcome either does not change or moves away from her favorite position, and her cost does not decrease; thus she is no better off. If she either withdraws or switches to party y the outcome changes from μ to y . By the definition of $t(y - x, c(k))$ and the concavity of u , neither deviation increases her payoff. Similarly a player i for whom $x_i \geq \mu + t(y - x, c(k))$ does not increase her payoff by deviating from supporting party y .

Now consider a player i for whom $x - t(y - x, c(k)) \leq x_i \leq \mu$ who does not participate. If she deviates to supporting party x the outcome changes from μ to x ; by the definition of $t(y - x, c(k))$ her payoff does not increase. Given that the $c(1) \geq c(k)$, the same argument implies that she is not better off becoming an independent at any position. Similarly a player i for whom

$\mu \leq x_i \leq y + t(y - x, c(k))$ who does not participate does not increase her payoff by deviating to support party y or becoming an independent.

We conclude that every action profile that satisfies the conditions in the result is a Nash equilibrium.

Now consider a two-party Nash equilibrium. By Proposition 4.1, the parties have the same number of members, say k . Denote the parties' positions by x and $y > x$. If $c(k) > \bar{C}(y - x)$ then every member of each party is better off withdrawing, which moves the outcome from $\frac{1}{2}(x + y)$ to the position of the other party. Thus $c(k) \leq \bar{C}(y - x)$.

Now consider a player i for whom $x_i < \mu + t(y - x, c(k))$. If she supports party y then by withdrawing she changes the outcome from μ to x and saves the cost $c(k)$. By the definition of $t(y - x, c(k))$ her withdrawal increases her payoff, so that in no equilibrium does she support party y . Similarly a player i for whom $x_i > \mu - t(y - x, c(k))$ does not support party x in any equilibrium.

Now consider a player i for whom $x_i < x - t(y - x, c(k))$. If she does not participate then by switching to support party x she changes the outcome from μ to x and incurs the cost $c(k)$. By the definition of $t(y - x, c(k))$ this deviation is profitable, so that in no equilibrium does she not participate.

We conclude that in any Nash equilibrium the action profile satisfies the conditions in the result. \square

Proof of Proposition 6.1. Given positive voter turnout, the quota q is positive.

We first show that no unelected candidate receives votes. Suppose to the contrary that candidate i is not elected and obtains a positive measure of votes. Let $\delta \in (0, q)$ be the largest measure of votes received by any unelected candidate. Because the quota function is continuous and nondecreasing and Ω is atomless, there exists $\epsilon > 0$ such that the quota remains in $(\delta, q]$ if an ϵ -club of citizens voting for i switches to not voting. Thus these citizens' withdrawal does not affect the set of elected candidates and hence the policy outcome. The members of the ϵ -club decrease their costs, however, contradicting the fact that $B(a)$ is an equilibrium of the subgame following a . Thus no unelected candidate receives votes.

We now show that every candidate is elected. We argue that an unelected

candidate who drops out does not affect the outcome, and hence increases her payoff. Suppose that candidate i is not elected. Then by the previous paragraph she receives no votes. We show that the voting equilibrium $B(a)$ of the subgame Γ^a in which i is a candidate is also an equilibrium of the adjacent subgame $\Gamma^{(\theta, a_{-i})}$ in which i is not a candidate. Because $B(a)$ is an equilibrium of Γ^a we know that there exists $\epsilon > 0$ such that no ϵ -club has an incentive to deviate from $B(a)$ in this subgame. Choose $\nu \in (0, \min\{\epsilon, q\})$. For ν small enough, when a ν -club changes its action from $B(a)$ in the subgame Γ^a , candidate i remains unelected, because the quota remains positive. Therefore, if the ν -club changes its action from $B(a)$ in the subgame $\Gamma^{(\theta, a_{-i})}$, the change in the policy outcome is the same as it is in the subgame Γ^a . Consequently, if some ν -club can profitably change its vote in the subgame $\Gamma^{(\theta, a_{-i})}$, then it can also profitably change its vote in the subgame Γ^a . Thus $B(a)$ is an equilibrium of the subgame $\Gamma^{(\theta, a_{-i})}$; subgame persistence implies that $B(a) = B(\theta, a_{-i})$. Therefore, candidate i 's dropping out reduces her costs, and does not change the policy outcome, contradicting the fact that (a, B) is an equilibrium. Hence all candidates are elected.

*The game G^B satisfies **C** at a .* Condition **C** is satisfied if we show that there exist no candidate i for whom $M(\theta, a_{-i}) = M(a)$. Assume the contrary. We know that the quota is positive and that candidate i , who is elected, gets some votes. However, given $M(\theta, a_{-i}) = M(a)$, any small club voting for i can profitably withdraw and reduce its cost without changing the policy outcome: after its withdrawal the quota does not go up and all candidates (with the possible exception of i) remain elected. This contradicts the fact that $B(a)$ is an equilibrium of the subgame Γ^a .

*The game G^B satisfies **PE** at a .* First notice that in an equilibrium all candidates get exactly the quota of votes, because otherwise some ϵ -club voting for a candidate can profitably drop out without changing the election outcome. Thus the policy outcome after the election is $M(a)$.

Now, **PE** is satisfied if there exists no candidate i and policy $x \in X$ with $x \neq a_i$ such that

- $M(a) = M(x, a_{-i})$
- $M(\theta, a_{-j}) = M(\theta, (x, a_{-i})_{-j})$ for each player j
- $\#\{j \in I \setminus \{i\} : a_j = a_i\} < \#\{j \in I \setminus \{i\} : a_j = x\}$.

Assume that the conditions hold for elected candidate i . We show that i can profitably move to x , contradicting the fact that (a, B) is an equilibrium.

For any history a^* , denote the set of small clubs Nash equilibria of Γ^{a^*} by $\beta(a^*)$.

Our strategy is to show that $B(a) \in \beta(x, a_{-i})$, which by subgame persistence implies that $B(x, a_{-i}) = B(a)$. This is because candidate i finds it profitable to switch to the policy x , increase her prize, and not change the electoral or policy outcomes.

So to complete the proof we prove that $B(a) \in \beta(x, a_{-i})$. Suppose that the citizens use the strategy $B(a)$ in the subgame following (x, a_{-i}) . We argue that no ϵ -club can profitably deviate from $B(a)$.

If an ϵ -club voting for some legislator j stops voting, then j is no longer elected whereas all other participating legislators are still elected. But we know that $M(\theta, a_{-j}) = M(\theta, (x, a_{-i})_{-j})$, so if this ϵ -club can profitably drop out when the strategy profile is $B(a)$ in the subgame following (x, a_{-i}) , then it can profitably drop out when the strategy profile is $B(a)$ in the subgame following a , contradicting $B(a) \in \beta(a)$.

If an ϵ -club voting for some legislator j switches to voting for legislator j' , then because all candidates get exactly the quota of votes one of the following cases occurs in the subgames following a and (x, a_{-i}) : 1) only candidate j' remains elected, 2) all candidates except j are elected, 3) no candidate remains elected, 4) all candidates remain elected.

In no case is the deviation profitable for the ϵ -club in the subgame following a , because (a, B) is an equilibrium. Therefore case 1 for $j' \neq i$, and cases 2, 3, and 4 are not profitable in the subgame following (x, a_{-i}) , because they result in the same change in the policy outcome for both subgames. For case 1 with $j' = i$, there exists another candidate h holding position x . If in the subgame $\Gamma^{(x, a_{-i})}$ the ϵ -club can profitably switch its vote from j to i , then (given the

anonymity of the quota function) in the subgame Γ^a it can profitably switch from j to h , which contradicts equilibrium.

A similar argument shows that an ϵ -club that does not vote cannot profitably vote for any candidate j in the $\Gamma^{(x, a_{-i})}$, because such a change does not decrease the quota and either only candidate j remains elected or all candidates remain elected.

The previous paragraphs imply that for some $\epsilon > 0$ the members of no ϵ -club can profitably change their vote from $B(a)$ in the subgame following (x, a_{-i}) , completing the proof. \square

Proof of Proposition 6.3. We know from the proof of Proposition 6.1 that all elected candidates get the quota of votes and candidates that are not elected get no votes.

We need to show that all candidates are elected. Suppose by way of contradiction that candidate i is not elected. We know from the proof of Proposition 6.1 that $B(a) \in \beta(\theta, a_{-i})$. Furthermore, (a, B) is incentive compatible, so $((\theta, a_{-i}), B)$ is incentive compatible because i is not elected. Therefore by subgame IC-persistence we have $B(\theta, a_{-i}) = B(a)$ and i finds it profitable to withdraw. Thus all candidates are elected.

The rest of the proof follows with little modification from the proof of Proposition 6.1. \square

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