RANDOM DYNAMICAL SYSTEMS WITH MULTIPLICATIVE NOISE

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Abstract. The paper considers random economic systems generating nonlinear time series on the positive half-ray $\mathbb{R}_+$. Using Liapunov techniques, new conditions for existence, uniqueness and stability of stationary equilibria are obtained. The conditions generalize earlier results from the mathematical literature, and extend to models outside the scope of existing economic methodology. An application to the stochastic growth problem with increasing returns is given.

1. Introduction

Increasingly, modern economics is implemented within the framework of stochastic dynamic systems. Physical laws, equilibrium constraints and restrictions on the behavior of agents jointly determine evolution of endogenous state variable $x \in X$ according to some transition rule

$$x_{t+1} = h(x_t, z_t, \varepsilon_t), \quad t = 0, 1, \ldots, \tag{1}$$

where h is an arbitrary function, (z_t) is a sequence of exogenous forcing variables and (ε_t) is uncorrelated noise.

For some models, either z_t is constant or the the endogenous variables can be redefined such that the system is autonomous:

$$x_{t+1} = h(x_t, \varepsilon_t), \quad t = 0, 1, \ldots \tag{2}$$

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Assume that this is the case. Of primary concern is whether the autonomous system (2) is in some sense stationary, in which case one can anticipate convergence of the sequence of distributions (φ_t) associated with the sequence of random variables (x_t) to some unique limiting distribution φ^*. The latter is then interpreted as the long-run equilibrium of the economy (2). Typically, comparative statics or policy simulation will be performed by analyzing the relationship between its moments and the underlying structural parameters contained in the function h and the distribution of the shock ε.

When h is linear on real vector space, (2) is the standard autoregression (AR) model. Conditions for stationarity are familiar from elementary time series analysis (Hamilton, 1994). When the map is nonlinear, dynamic behavior is potentially more complicated. General conditions for existence of unique and stable equilibria are not known.

In this case, a common approach is to linearize (2) using a first order Taylor expansion or similar technique, and then examine the stability properties of the resulting AR model. However, it is by no means clear that stability properties obtained for the AR model have *any* homeomorphic implications for the behavior of the true model (2). In other words, it is not in general legitimate to infer stability of (2) from stability of the corresponding linear form. Moreover, linearization may eliminate important features of the model.[1]

A more correct method is to examine the Markov chain generated by (2), and determine whether appropriate conditions for stability of Markovian systems are satisfied. A well-known survey of these conditions is provided by Futia (1982). Stokey, Lucas and Prescott (1989, Chapter 13) outline ways to verify these and related conditions for common economic models. Prescott and Hopenhayn (1992) study weak-star stability using a monotonicity requirement. Bhattacharya and Majumdar (2001) obtain exponential convergence in the Kolmogorov metric for systems that satisfy a "splitting" condition.

In general, a Markov process is characterized by its transition kernel, which generalizes the notion of Markov transition matrix to the case

[1]For example, Durlauf and Quah (1999) find evidence to the effect that the standard linearization procedure applied to Solow-Ramsey growth models (e.g., Mankiw, Romer and Weil, 1992) fails to extract nonlinear local increasing returns dynamics that are crucial to understanding the evolution of the cross-country income distribution.

where the state space is arbitrary. In the papers listed above, stability conditions for Markov processes are stated in terms of properties of the kernel. For the system (2), the kernel is defined implicitly, and the conditions must be verified on the basis of restrictions on the function h and the distribution of the shock ε.

Kernel-based methods are important for their generality. At the same time, when the original model is stated as a stochastic difference equation such as (2), translating it into the Markov kernel formula involves some loss of information. Conversely, working directly with the formulation (2) provides structure not available for generic Markov processes. This structure can be exploited when deriving equilibrium existence results and stability conditions. Moreover, such an approach leads naturally to conditions stated directly in terms of the primitives h and ε rather than the implied transition kernel, making them easy to verify in applications.

Adopting this approach, the paper provides new conditions for existence, uniqueness and stability of equilibrium in a class of models defined by (2). In particular, we consider the special case where the shock ε is multiplicative and the state space for the endogenous variable x_t is the positive half-ray $\mathbb{R}_+ = [0, \infty)$. That is,

$$x_{t+1} = g(x_t)\varepsilon_t, \quad t = 0, 1, \dots, \tag{3}$$

where $g\colon \mathbb{R}_+ \to \mathbb{R}_+$, and $\varepsilon_t \in \mathbb{R}_+$. Our motivation for treating this case is that models evolving in $\mathbb{R}_+$ are common in economics, where state variables typically denote physical quantities or prices. Stochastic one-sector accumulation models are an important example.

The above problem has also been studied in the mathematical literature. In particular, there exists for (3) a well-known set of stability conditions due to K. Horbacz (1989, Theorem 1). The results obtained here provide a general principle which yields the conditions of Horbacz as a special case.

Our arguments are based on the framework for studying integral Markov semigroups in the function space L_1 proposed by Lasota (1994). Previously, Stachurski (2002) has applied Lasota's method to the stochastic neoclassical growth problem.

The paper proceeds as follows. Section 2 defines random systems and equilibria in the space $\mathbb{R}_+$. Section 3 states our results. Section

4 discusses applications. The proofs of the main results are given in Section 5.

2. Formulation of the problem

A perturbed dynamical system on the positive reals can be defined as follows. Let $\mathbb{R}$ be the real numbers, let $\mathcal{B}$ be the Borel sets of $\mathbb{R}$, let $\mathbb{R}_+ = [0, \infty)$, and let $\mathcal{B}_+ = \mathcal{B} \cap \mathbb{R}_+$. Lebesgue measure is denoted by μ, and $L_1(\mu)$ is the space of μ-integrable real functions on $\mathbb{R}_+$. Integration where the measure is not made explicit refers to integration with respect to μ. The symbol $\int$ without subscript refers to integration over the whole space $\mathbb{R}_+$.

As usual, $L_1(\mu)$ is interpreted as a Banach lattice of equivalence classes; functions equal off a μ-null set are identified. A *density function* on $\mathbb{R}_+$ is an element $\varphi \in L_1(\mu)$ such that $\varphi \geq 0$ and $\int \varphi = \|\varphi\| = 1$. The set of all density functions is denoted $D(\mu)$.

Let $(\Omega, \mathcal{F})$ be a measurable space, where $\mathcal{F}$ is a σ-algebra on the set Ω, and let $\mathbf{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Random outcomes are implemented as follows. A state of nature is selected from Ω according to $\mathbf{P}$, and mapped into the real line by random variable $\varepsilon \colon \Omega \to \mathbb{R}$. As usual, the random variable defines a probability distribution associating event $B \in \mathcal{B}$ with the real number $\mathbf{P}[\varepsilon^{-1}(B)] \in [0, 1]$. We assume throughout that ε is nonnegative and represented by a density function:

Assumption 2.1. The distribution $\mathcal{B} \ni B \mapsto \mathbf{P}[\varepsilon^{-1}(B)] \in [0, 1]$ satisfies $\mathbf{P}[\varepsilon^{-1}(\mathbb{R}_+)] = 1$ and $\mathbf{P}[\varepsilon^{-1}(B)] = 0$ whenever $\mu(B) = 0$.

Given Assumption 2.1, there exists a unique representative density $\psi \in D(\mu)$ satisfying $\int_B \psi = \mathbf{P}[\varepsilon^{-1}(B)]$ for all $B \in \mathcal{B}_+$; ψ is called the Radon-Nikodým (RN) derivative of $B \mapsto \mathbf{P}[\varepsilon^{-1}(B)]$ with respect to μ.

Definition 2.1. Let $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. In this paper, a *perturbed dynamical system* on $\mathbb{R}_+$ refers to the pair (g, ψ), where, given current state value $x_t \in \mathbb{R}_+$, a shock $\varepsilon_t \in \mathbb{R}_+$ is selected independently from density ψ, and the next period state is realized as in (3).

Let $\mathbf{1}_B \colon \mathbb{R}_+ \to \{0, 1\}$ be the characteristic function for $B \in \mathcal{B}_+$. The pair (g, ψ) determines a Markov process on $\mathbb{R}_+$ with transition kernel

$$\mathbf{N} \colon \mathbb{R}_+ \times \mathcal{B}_+ \ni (x, B) \mapsto \int \mathbf{1}_B[g(x)z]\psi(z)dz \in [0, 1]. \tag{4}$$

(See, e.g., Futia, 1982, Definition 1.1.) The value $\mathbf{N}(x, B)$ should be interpreted as the conditional probability that the next period state is in Borel set B, given that the current state is equal to x. A Markov process is fully characterized by its transition kernel.

Let $\mathcal{M}$ be the vector space of finite signed measures on $(\mathbb{R}_+, \mathcal{B}_+)$. Let $\mathcal{M}_1$ be the elements $\nu \in \mathcal{M}$ such that $\nu \geq 0$ and $\nu(\mathbb{R}_+) = 1$. The subset $\mathcal{M}_1$ will be called the *distributions* on $\mathbb{R}_+$. Finally, let B be any Borel set, and let $\nu_t \in \mathcal{M}_1$ be the marginal distribution for the random variable x_t.[2] By the law of total probability, if ν_{t+1} is the distribution for x_{t+1}, then

$$\nu_{t+1}(B) = \int \mathbf{N}(x, B)\nu_t(dx). \tag{5}$$

Intuitively, the probability that the state variable is in B next period is the sum of the probabilities that it travels to B from x across all $x \in \mathbb{R}_+$, weighted by the probability $\nu_t(dx)$ that x occurs as the current state.

Suppose we now define an operator $P \colon \mathcal{M} \ni \nu \mapsto P\nu \in \mathcal{M}$ by

$$P\nu(B) = \int \mathbf{N}(x, B)\nu(dx). \tag{6}$$

Evidently $P\mathcal{M}_1 \subset \mathcal{M}_1$. A linear self-mapping on $\mathcal{M}$ satisfying $P\mathcal{M}_1 \subset \mathcal{M}_1$ is called a *Markov operator*.[3] It follows from (5) and (6) that if ν_t is the distribution for the current state x_t, then $\nu_{t+1} = P\nu_t$ is the distribution for the next period state x_{t+1}.

Repeated iteration of P on a fixed distribution ν is equivalent to moving forward in time. If P^t is defined by $P^t = P \circ P^{t-1}$ and $P^1 = P$, and if ν is the current marginal distribution for the state variable, then $P^t\nu$ is the distribution t periods hence.

2.1. **The L_1 method.** In this paper we study the Markov process generated by (g, ψ) using L_1 techniques (Hopf, 1954). Embedding the Markov problem in the function space $L_1(\mu)$ requires that the transition

[2]The distribution for the entire stochastic process $(x_t)_{t\geq 0}$ can be constructed uniquely from the transition kernel and an initial value x_0 (see, e.g., Shiryaev, 1996, Theorem II.9.2). The real number $\nu_t(B)$ is the probability that this distribution assigns to the event $x_t \in B$ and $x_s \in \mathbb{R}_+$ for all other $s \neq t$.

[3]The operator P closely corresponds to T^* in Futia (1982, p. 380). Markov operators are called *stochastic operators* by some authors. Our terminology follows the literature on Markov processes in L_1.

probabilities can be represented by density functions. We suppose in particular that

Assumption 2.2. The map g is strictly positive almost everywhere on $\mathbb{R}_+$.

It can be verified under Assumption 2.2 that for almost all x, the distribution $B \mapsto \mathbf{N}(x, B)$ is absolutely continuous with respect to μ, and can therefore be represented by density $y \mapsto p(x, y)$. For such x,

$$p(x,y) = \psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}, \tag{7}$$

because then

$$\int_B p(x,y)dy = \int \mathbf{1}_B[g(x)z]\psi(z)dz = \mathbf{N}(x,B).$$

For other x set $p(x, \cdot)$ equal to any density.

Heuristically, the number $p(x, y)dy$ is the probability of traveling from state x to state y in one step. In this paper, p is called the *stochastic kernel* corresponding to (g, ψ).[4]

The Markov operator P corresponding to (g, ψ) can now be reinterpreted as a linear self-mapping on the function space $L_1(\mu)$, where if $h \in L_1(\mu)$, then

$$Ph(y) = \int p(x,y)h(x)dx. \tag{8}$$

It can be verified that the two definitions of P are equivalent for the absolutely continuous measures in $\mathcal{M}$ when these measures and their RN derivatives in $L_1(\mu)$ are identified. That is, if $h \in L_1(\mu)$ is the RN derivative of $\lambda \in \mathcal{M}$, then Ph defined by (8) is the RN derivative of $P\lambda$ defined by (6).[5]

Note that $PD(\mu) \subset D(\mu)$, as can be shown using Fubini's theorem. As before, if φ is the current marginal density for the state variable, then $P^t\varphi$ is that of the state t periods hence.

[4]It will become clear below that stochastic kernels need be defined only up to the complement of a null set—systems with kernels equal $\mu \times \mu$-a.e. have identical dynamics and we do not distinguish between them in what follows.

[5]Formally, the semidynamical systems defined by $(\mathcal{M}_\mu, P)$, where $\mathcal{M}_\mu$ is the absolutely continuous measures and P is the Markov operator on measures; and $(L_1(\mu), P)$, where P is the Markov operator on functions; are topologically conjugate, in that they commute with the homeomorphism defined by Radon-Nikodým differentiation—an isometric isomorphism from $\mathcal{M}_\mu$ to $L_1(\mu)$. Topologically conjugate dynamical systems have identical dynamic properties.

Definition 2.2. Let (g, ψ) be a perturbed dynamical system satisfying Assumptions 2.1–2.2. Let P be the corresponding Markov operator. An *equilibrium* or *steady state* for (g, ψ) is a density φ^* on $\mathbb{R}_+$ such that $P\varphi^* = \varphi^*$. An equilibrium φ^* is called *unique* if there exists no other fixed point of P in the space $D(\mu)$, and *globally stable* if $P^t\varphi \to \varphi^*$ in the $L_1(\mu)$ metric as $t \to \infty$ for every $\varphi \in D(\mu)$.

These definitions are consistent with standard definitions used in Markovian economic models (c.f., e.g., Stokey, Lucas and Prescott, 1989, pp. 317–8).

One advantage of the techniques used here is that stability is defined in the strong topology on $L_1(\mu)$. The distributions corresponding to the density functions in Definition 2.2 converge in the strong (total variation) topology on $\mathcal{M}$. Existing techniques typically obtain only weak or weak-star stability.[6]

3. Results

In this section we give two stability results for the perturbed dynamical system (g, ψ) on $\mathbb{R}_+$ defined by (3), with ψ the density of the shock ε. Central to our conditions is the notion of a *Liapunov function on* $\mathbb{R}_+$, which we define to be a continuous, nonnegative function V from $\mathbb{R}_+$ into $\mathbb{R}_+ \cup \{\infty\}$ such that $V(0) = \infty$, $V(x) < \infty$ for $x \neq 0$ and $\lim_{x\to\infty} V(x) = \infty$.

Condition 3.1. Corresponding to (g, ψ), there exists a Liapunov function V on $\mathbb{R}_+$ and constants $\alpha, C \geq 0$, $\alpha < 1$, such that

$$\int V[g(x)z]\psi(z)dz \leq \alpha V(x) + C, \quad \forall x \in \mathbb{R}_+.$$

The function V in Condition 3.1 is large at 0 and $+\infty$. The condition restricts the probability that the state variable moves toward these limits without bound.

Condition 3.2. The density ψ is strictly positive on $\mathbb{R}_+$.[7]

[6] Horbacz (1992, Example 1) exhibits a simple perturbed dynamical system with multiplicative shock which is weakly globally stable but not strongly globally stable.

[7] More precisely, there exists a strictly positive function in the equivalence class $\psi \in L_1(\mu)$. When ψ is treated as a function it is to this element of the equivalence class that we refer.

Most "named" densities on $\mathbb{R}_+$ have this property, such as the lognormal, exponential, chi-squared, gamma, and Weibull densities.

Condition 3.3. The density ψ satisfies $\psi(z)z \leq M$ on $\mathbb{R}_+$.

Condition 3.3 also holds for the lognormal, exponential, chi-squared, gamma and Weibull distributions. The condition is used here to bound the probability that ψ assigns to closed intervals in $\mathbb{R}_+ \setminus \{0\}$.

The first theorem can now be stated.

Theorem 3.1. *Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 2.1 and 2.2. If g and ψ also satisfy Conditions 3.1, 3.2 and 3.3, then (g, ψ) has a unique, globally stable equilibrium.*

Alternatively, suppose that

Condition 3.4. The map g is weakly monotone increasing on the nonempty interval $[0, r)$, and $g(x) \geq b > 0$ on $[r, \infty)$.

Theorem 3.2. *Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 2.1 and 2.2. If g and ψ also satisfy Conditions 3.1, 3.2 and 3.4, then (g, ψ) has a unique, globally stable equilibrium.*

Corollary 3.1. *Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 2.1 and 2.2. If g is weakly monotone increasing and, in addition, g and ψ together satisfy Conditions 3.1 and 3.2, then (g, ψ) has a unique, globally stable equilibrium.*

Proof. Evidently Condition 3.4 is also satisfied if Assumption 2.2 holds and g is weakly monotone increasing on $\mathbb{R}_+$. $\square$

4. Applications

Applications of the results are presented in this section.

4.1. **Stability in a model with externalities.** Consider the following growth model with increasing returns. The framework is overlapping generations. Agents live for two periods, working in the first and living off savings in the second. Savings in the first period forms capital stock, which in the following period will be combined with the labor of a new generation of young agents for production under the technology

$$y_{t+1} = A(k_t)k_t^{\alpha}\ell_t^{1-\alpha}\varepsilon_t.$$

where y is output, k is capital and ℓ is labor input. The function $k \mapsto A(k)$ signifies the existence of increasing social returns resulting from sensitivity of "technology" to economy-wide capital aggregates. This dependence is external to individual agents, and A is treated as constant with respect to private investment. The capital share α satisfies $0 < \alpha < 1$.

Regarding the nature of the function $k \mapsto A(k)$, we assume only that increasing returns are bounded:

Assumption 4.1. The function $A\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ is measurable and takes values in a closed and bounded subset of $\mathbb{R}_+ \setminus \{0\}$.

Well-known macroeconomic models with external effects satisfying Assumption 4.1 include Azariadis and Drazen (1990), Galor and Zeira (1993), and Quah (1996).

For convenience, labor supply is normalized to unity. The productivity shocks ε_t are uncorrelated and identically distributed by the lognormal density, denoted here by ψ. In other words, $\ln \varepsilon$ has the (standard) normal distribution.

Let c (respectively, c') denote consumption while young (respectively, old). Agents maximize utility

$$U(c_t, c'_{t+1}) = \ln c_t + \beta \mathbb{E}(\ln c'_{t+1})$$

subject to the budget constraint $c'_{t+1} = (w_t - c_t)(1 + r_{t+1})$, where w_t and r_t are the wage and interest rates at t respectively. In this case optimization implies a savings rate from wage income of $\beta/(1+\beta)$, and hence $k_{t+1} = (\beta/(1+\beta))w_t$.

Assuming that labor is paid its marginal factor product yields the law of motion

$$k_{t+1} = DA(k_t)k_t^{\alpha}\varepsilon_t = g(k)\varepsilon_t, \tag{9}$$

where $D = \frac{\beta}{1+\beta}(1-\alpha)$.

Considerable technical difficulties are presented by the increasing returns model (9), which is potentially highly nonlinear. In the absence of further assumptions, the transition kernel generated by this model is in general neither Feller stable (Stokey, Lucas and Prescott, 1989, p. 220) nor increasing (Hopenhayn and Prescott, 1992, p. 1392). The state space cannot be taken to be compact. The splitting condition of Bhattacharya and Majumdar (2001, p. 212) is not satisfied. Finally, we note that in the deterministic case (when ε_t is held constant), the model (9) may have a countable infinity of local attractors. Never the less,

Proposition 4.1. *The economy (9) has a unique, globally stable stochastic equilibrium.*

Proof. We verify the conditions of Theorem 3.1. Set $V(0) = \infty$ and $V(k) = |\ln k|$ for $k > 0$. The function V so constructed is a Liapunov function on $\mathbb{R}_+$. Moreover,

$$\begin{aligned}\int V[g(k)z]\psi(z)dz &= \int |\ln D + \ln A(k) + \alpha \ln k + \ln z|\psi(z)dz \\ &\leq \alpha|\ln k| + C \\ &= \alpha V(k) + C,\end{aligned}$$

where $C = |\ln D| + \sup_k |\ln A(k)| + \mathbb{E}|\ln \varepsilon|$. Since $\alpha < 1$ and $C < \infty$, Condition 3.1 holds. Since Conditions 3.2 and 3.3 are also satisfied, existence, uniqueness and global stability now follow from Theorem 3.1. $\square$

4.2. **Existing conditions.** Previously a set of conditions for obtaining stability of the model (3) was obtained by K. Horbacz (1989, Theorem 1). We now derive her results as a special case of Theorem 3.2.

Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 2.1 and 2.2. Horbacz (1989, Theorem 1) proves that (g, ψ) has a unique and globally stable equilibrium whenever

(i) The map g is weakly monotone increasing and continuously differentiable on $[0, r) \neq \emptyset$, and $g(x) \geq b > 0$ on $[r, \infty)$;
(ii) the map g satisfies $g(0) = 0$ and $g'(0) > 0$;
(iii) there exist a, $B \geq 0$ such that $g(x) \leq ax + B$ for all $x \in \mathbb{R}_+$;
(iv) the mean $\mathbb{E}(\varepsilon) = \int z\psi(z)dz$ is finite and, moreover, $\mathbb{E}(\varepsilon)a < 1$;
(v) there exists a $\lambda > 0$ such that $\mathbb{E}[(g'(0)\varepsilon)^{-\lambda}] < 1$; and
(vi) the density ψ is everywhere positive on $\mathbb{R}_+$.

We show that (i)–(vi) imply the conditions of Theorem 3.2. Evidently Conditions 3.2 and 3.4 of the theorem are satisfied. It remains to verify Condition 3.1. To this end, let λ be as in (v). If we set $V(0) = \infty$ and $V(x) = x^{-\lambda} + x$ for $x > 0$, then V is a Liapunov function on $\mathbb{R}_+$, and

$$\int V[g(x)z]\psi(z)dz = \int [g(x)z]^{-\lambda}\psi(z)dz + \int g(x)z\psi(z)dz. \tag{10}$$

Consider the first term in the sum (10). By (v), there exists a positive number σ so small that

$$\int [(g'(0) - \sigma)z]^{-\lambda}\psi(z)dz < 1. \tag{11}$$

By (i) and (ii), there exists a $\delta > 0$ such that

$$g(x) \geq (g'(0) - \sigma)x \quad \text{whenever } x \in [0, \delta). \tag{12}$$

Combining (11) and (12) yields a $\gamma < 1$ such that

$$\int [g(x)z]^{-\lambda}\psi(z)dz \leq \gamma x^{-\lambda}, \quad \forall\, x \in [0, \delta).$$

Moreover, (i) implies the existence of a $c > 0$ such that

$$g(x) \geq c \quad \text{whenever } x \in [\delta, \infty).$$

Thus, for all $x \in \mathbb{R}_+$, we have the bound

$$\int [g(x)z]^{-\lambda}\psi(z)dz \leq \gamma x^{-\lambda} + C_0, \tag{13}$$

where $\gamma < 1$ and C_0 is a finite constant.

Regarding the second term in the sum (10), (iii) implies that

$$\int g(x)z\psi(z)dz \leq \mathbb{E}(\varepsilon)ax + C_1, \quad x \in \mathbb{R}_+, \tag{14}$$

where C_1 is a finite constant.

Combining (13) and (14) gives

$$\int V[g(x)z]\psi(z)dz \leq \alpha V(x) + C, \tag{15}$$

where $\alpha = \max[\mathbb{E}(\varepsilon)a, \gamma] < 1$ and $C = C_0 + C_1 < \infty$. This confirms Condition 3.1. Hence all of the conditions of Theorem 3.2 are satisfied.

5. Proofs

Verification of Theorems 3.1 and 3.2 proceeds by outlining a framework for obtaining existence, uniqueness and stability of equilibria, and then establishing the required lemmas. The framework for studying integral Markov operators used here is due to Lasota (1994). Our exposition of Lasota's method is based on Stachurski (2002).

By the definition of equilibrium, the proof requires a fixed point argument for a mapping $T\colon U \to U$ on a metric space (U, ϱ), where in the present case T corresponds to the Markov operator P defined in (6), U is the space of density functions $D(\mu)$, and ϱ is the distance in $D(\mu)$ induced by the L_1 norm.

A standard result which gives existence, uniqueness and stability of equilibrium in the form desired here is the Banach contraction theorem. (Note that the underlying space is indeed complete.) However, the contraction condition of Banach is not always satisfied under Conditions 3.1–3.4. Here we pursue an alternative contraction-based argument, using a slightly weaker condition.

Definition 5.1. Let U be a metric space, and let $T\colon U \to U$. The map T is called *contracting* on U if

$$\varrho(Tx, Tx') < \varrho(x, x'), \quad \forall x, x' \in U, \quad x \neq x'. \tag{16}$$

The contraction condition (16) immediately implies uniqueness of fixed points for T in U, because if x and x' are any two fixed points in U, then $\varrho(Tx, Tx') = \varrho(x, x')$, from which it follows that $x = x'$.

Lemma 5.1. *Let (g, ψ) be a perturbed dynamical system satisfying Assumptions 2.1 and 2.2. If Condition 3.2 holds, then the associated Markov operator P is contracting on $D(\mu)$ with respect to the metric induced by the $L_1(\mu)$ norm.*

Proof. Note that under Condition 3.2, the stochastic kernel $y \mapsto p(x, y)$ is strictly positive for almost all x, as can be verified from the representation (7). Pick any two densities $\varphi \neq \varphi'$. Evidently the function $\varphi - \varphi'$ is both strictly positive on a set of positive measure and strictly negative on a set of positive measure. Pick any $y \in \mathbb{R}_+$. Since $p(x, y) > 0$ for almost all x, it follows that $x \mapsto p(x, y)[\varphi(x) - \varphi'(x)]$ is also strictly positive on a set of positive measure and strictly negative on a set of positive measure. Therefore, by the strict triangle inequality,

$$\begin{aligned}
\|P\varphi - P\varphi'\| &= \|P(\varphi - \varphi')\| \\
&= \int \left| \int p(x, y)[\varphi(x) - \varphi'(x)]dx \right| dy \\
&< \int \int \big| p(x, y)[\varphi(x) - \varphi'(x)] \big| \, dx \, dy \\
&= \int \int p(x, y)|\varphi(x) - \varphi'(x)| dx \, dy \\
&= \int \int p(x, y) dy |\varphi(x) - \varphi'(x)| dx \\
&= \|\varphi - \varphi'\|,
\end{aligned}$$

as was to be proved. □

By the comment preceding the lemma, this result establishes uniqueness of equilibrium in the sense of Definition 2.2.

Consider now the problem of existence and stability. It is known that when $T\colon U \to U$ is contracting on a *compact* metric space (U, ϱ), then T has a unique fixed point $x^* \in U$. Uniqueness holds for all contractions, as discussed above. To obtain existence, define $r\colon U \to \mathbb{R}$ by $r(x) = \varrho(Tx, x)$. Evidently r is continuous. Since U is compact, r has a minimizer x^*. But then $Tx^* = x^*$, because otherwise Tx^* minimizes r on U.[8]

It is less well-known but also true that under these conditions all points in the space are convergent to x^* under iteration of T. To prove this, pick any $x \in U$, and define $\alpha_n = \varrho(T^n x, x^*)$. Since (α_n) is monotone decreasing and nonnegative it has a limit α. If $\alpha = 0$ then we are done. Suppose otherwise. By compactness, $(T^n x)$ has a convergent subsequence $T^{n_k} x \to \bar{x} \in U$. Evidently $\varrho(\bar{x}, x^*) = \alpha > 0$, so $\bar{x}$ and x^* are distinct. But then

$$\begin{aligned}
\varrho(T\bar{x}, Tx^*) &= \varrho(T \lim_k T^{n_k} x, x^*) \\
&= \lim_k \varrho(TT^{n_k} x, x^*) \\
&= \lim_k \alpha_{n_k+1} = \alpha,
\end{aligned}$$

which contradicts (16). This argument proves convergence to the fixed point.

We have proved that contractiveness of the operator and compactness of the space together imply existence, uniqueness and global stability of equilibrium. In the case of the perturbed dynamical system (g, ψ), while P is strongly contracting the metric space on $D(\mu)$ with L_1 distance by Lemma 5.1, $D(\mu)$ is not compact in the same topology. Some weakening of the compactness condition is required. Consider the following approach. Suppose that, in addition to strong contractiveness of P on $D(\mu)$, the set of iterates $\{P^t\varphi : t \geq 0\}$ is precompact for any initial distribution $\varphi \in D(\mu)$.[9] Such a property is called *Lagrange stability.* Let $\Gamma(\varphi)$ denote the closure of this collection, that is, $\Gamma(\varphi) = \mathrm{cl}\{P^t\varphi : t \geq 0\}$. It is straightforward to check that

[8]Strictness of the inequality in (16) is necessary for both uniqueness and existence. For example, existence fails if U is the boundary of the unit sphere in $\mathbb{R}^2$, and $Tx = -x$.

[9]A subset of a topological space is precompact if it has compact closure.

$P\Gamma(\varphi) \subset \Gamma(\varphi)$.[10] In this case, P is a contracting self-mapping on the compact set $\Gamma(\varphi)$. Therefore P has a fixed point φ^* in $\Gamma(\varphi) \subset D(\mu)$. Moreover, the stability result discussed above implies that $P^t\varphi \to \varphi^*$ in L_1 norm. Finally, since P is a contraction on the whole space, the fixed point φ^* is unique and does not depend on φ.

Thus it remains only to establish Lagrange stability of the Markov operator P associated with (g, ψ) on the density space $D(\mu)$. Lasota (1994, Theorem 4.1) has made the important insight that in the case of integral Markov operators such as (8), it is sufficient to prove that $\{P^t\varphi : t \geq 0\}$ is *weakly* precompact for every $\varphi \in D(\mu)$. The reason is that integral Markov operators map weakly precompact subsets of $L_1(\mu)$ into strongly precompact subsets. Therefore, if $\{P^t\varphi : t \geq 0\}$ is weakly precompact, then $\{P^t\varphi : t \geq 1\}$ is strongly precompact. But then $\{P^t\varphi : t \geq 0\}$ is also strongly precompact.

In fact Lasota (1994, Proposition 3.4) has used a Cantor diagonal argument to show that weak precompactness of $\{P^t\varphi : t \geq 0\}$ need only be established for a collection of φ such that the closure of the collection contains $D(\mu)$. In summary, then, both Theorem 3.1 and Theorem 3.2 will be verified if we are able to show that under Condition 3.1 and either one of Condition 3.3 or 3.4, there exists a set $\mathcal{D}$ such that $\mathcal{D}$ is dense in $D(\mu)$ and $\{P^t\varphi : t \geq 0\}$ is weakly precompact for each $\varphi \in \mathcal{D}$.

Lemma 5.2. *Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 2.1 and 2.2, and let P be the associated Markov operator. If Condition 3.1 and either one of Condition 3.3 or 3.4 holds, then there exists a set $\mathcal{D}$ such that $\mathcal{D}$ is dense in $D(\mu)$ and $\{P^t\varphi : t \geq 0\}$ is weakly precompact for each $\varphi \in \mathcal{D}$.*

Proof of Lemma 5.2. Let V be the Liapunov function in Condition 3.1. Let $\mathcal{D}$ be the set of all density functions φ in $L_1(\mu)$ such that

$$\int V(x)\varphi(x)dx < \infty. \tag{17}$$

We claim that $\mathcal{D}$ has the desired properties.

Pick any density φ. To see that there exists a $(\varphi_k) \subset \mathcal{D}$ with $\varphi_k \to \varphi$, define first $\varphi_k^0 = \mathbf{1}_{[1/k,k]}\varphi$. By the monotone convergence theorem, $\|\varphi_k^0\| \to 1$. Hence $\|\varphi_k^0\| > 0$ for all k greater than some constant K.

[10]We are using the fact that P is continuous, which is true of any positive linear self-mapping on the Banach lattice $L_1(\mu)$.

For all such k define $\varphi_k = \|\varphi_k^0\|^{-1}\varphi_k^0$. Then $\varphi_k \in D(\mu)$ for all $k \geq K$ by construction. Moreover, $\varphi_k \to \varphi$ pointwise, and hence in L_1 norm by Scheffé's lemma. Finally, $\varphi_k \in \mathcal{D}$ for all $k \geq K$, because

$$\int V(x)\varphi_k(x)dx = \frac{1}{\|\varphi_k^0\|}\int \mathbf{1}_{[1/k,k]}(x)V(x)\varphi(x)dx,$$

and V is bounded on compact subsets of $\mathbb{R}_+ \setminus \{0\}$ by continuity.

It remains to show that if $\varphi \in \mathcal{D}$, then $\{P^t\varphi : t \geq 0\}$ is weakly precompact. Note first that $\{P^t\varphi\}$ is nonnegative and norm-bounded, because $PD(\mu) \subset D(\mu)$. By the Dunford-Pettis theorem (1940, Theorem 3.2.1), a norm-bounded collection of nonnegative functions $\{P^t\varphi\}$ in $L_1(\mu)$ is weakly precompact whenever

(i) $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that if $A \in \mathcal{B}$ and $\mu(A) < \delta$, then

$$\int_A P^t\varphi < \varepsilon, \quad \forall t \geq 0; \text{ and}$$

(ii) $\forall \varepsilon > 0$, $\exists\, G \in \mathcal{B}_+$ such that $\mu(G) < \infty$ and

$$\int_{\mathbb{R}_+\setminus G} P^t\varphi < \varepsilon, \quad \forall t \geq 0.$$

Evidently it is sufficient to verify that these conditions are satisfied for all but a finite (i.e., all but $t < N$) number of the collection $\{P^t\varphi : t \geq 0\}$.

Regarding (i), pick any $\varepsilon > 0$. We exhibit a $\delta > 0$ and an $N \in \mathbb{N}$ such that

$$\mu(A) < \delta \implies \int_A P^t f(x)dx < \varepsilon, \quad \forall\, t \geq N.$$

Define $E(V|g) = \int Vg$. By Fubini's theorem,

$$\begin{aligned} E(V|P^t\varphi) &= \int V(y)P^t\varphi(y)dy \\ &= \int V(y)\left[\int p(x,y)P^{t-1}\varphi(x)dx\right]dy \\ &= \int \left[\int V(y)p(x,y)dy\right]P^{t-1}\varphi(x)dx. \end{aligned}$$

But

$$\int V(y)p(x,y)dy = \int V[g(x)z]\psi(z)dz \leq \alpha V(x) + C$$

for all x by Condition 3.1. Therefore,

$$E(V|P^t\varphi) \leq \int [\alpha V(x) + C] P^{t-1}\varphi(x)dx = \alpha E(V|P^{t-1}\varphi) + C.$$

Repeating this argument obtains

$$E(V|P^t\varphi) \leq \alpha^n E(V|\varphi) + \frac{C}{1-\alpha}.$$

Since $E(V|\varphi)$ is finite by (17), it follows that

$$E(V|P^t\varphi) \leq 1 + \frac{C}{1-\alpha}, \quad t \geq N,$$

for some $N \in \mathbb{N}$.

On the other hand, it can be verified that for arbitrary positive a,

$$a \int_{\mathbb{R}_+ \setminus G_a} P^t\varphi \leq E(V|P^t\varphi)$$

when G_a is defined as the set of $x \in \mathbb{R}_+$ with $V(x) \leq a$.

$$\therefore \quad \int_{\mathbb{R}_+ \setminus G_a} P^t\varphi \leq \frac{1}{a}\left(1 + \frac{C}{1-\alpha}\right), \quad \forall\, t \geq N, \quad \forall\, a > 0. \tag{18}$$

Choose a so large that

$$\frac{1}{a}\left(1 + \frac{C}{1-\alpha}\right) \leq \frac{\varepsilon}{2}. \tag{19}$$

Consider now the decomposition

$$\int_A P^t\varphi = \int_{A\cap G_a} P^t\varphi + \int_{A\cap[\mathbb{R}_+ \setminus G_a]} P^t\varphi.$$

Using (18) and (19) gives

$$\int_A P^t\varphi \leq \int_{A\cap G_a} P^t\varphi + \frac{\varepsilon}{2}, \tag{20}$$

whenever $t \geq N$. Here a is the constant determined in (19).

The next step is to bound the first term in the sum on the right hand side of (20), taking the constant a as given, and assuming that at least one of Condition 3.3 or Condition 3.4 holds.

Assume first that Condition 3.3 holds. Using the expression for the stochastic kernel given in (7), we have

$$\begin{aligned}
P^t\varphi(y) &= \int p(x,y)P^{t-1}\varphi(x)dx \\
&= \int \psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}P^{t-1}\varphi(x)dx \\
&= \int \psi\left(\frac{y}{g(x)}\right)\frac{y}{g(x)}\frac{1}{y}P^{t-1}\varphi(x)dx \\
&\leq \frac{M}{y}.
\end{aligned}$$

$$\therefore \quad \int_{A\cap G_a} P^t\varphi(y)dy \leq \int_{A\cap G_a}\frac{M}{y}dy \leq \int_A J(a)dy = J(a)\mu(A),$$

where the finite number $J(a)$ is the maximum of M/y over the closed and bounded interval $G_a \subset \mathbb{R}_+ \setminus \{0\}$.

Now pick any positive δ satisfying $\delta \leq \varepsilon/(J(a)2)$. For such a δ we have

$$\mu(A) < \delta \implies \int_{A\cap G_a} P^t f(x)dx < \frac{\varepsilon}{2}.$$

Combining this with (20) proves (i) of the Dunford-Pettis characterization for the collection $\{P^t\varphi : t \geq N\}$ when Condition 3.3 holds.

We now establish the same when Condition 3.4 holds, again by bounding the first term in the sum (20). Suppose first that there exists a c with $g(x) \geq c > 0$ for *all* x in $\mathbb{R}_+$. In this case, because

$$\begin{aligned}
\int_{A\cap G_a} P^t\varphi(y)dy &= \int_{A\cap G_a}\int p(x,y)P^{t-1}\varphi(x)dxdy \\
&= \int\left[\int_{A\cap G_a} p(x,y)dy\right]P^{t-1}\varphi(x)dx,
\end{aligned}$$

and because

$$\begin{aligned}
\int_{A\cap G_a} p(x,y)dy &= \int_{A\cap G_a}\psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}dy \\
&= \int_{\frac{A\cap G_a}{g(x)}}\psi(z)dz \\
&\leq \int_{\frac{A\cap G_a}{c}}\psi(z)dz \\
&\leq \int_{\frac{A}{c}}\psi(z)dz
\end{aligned}$$

for all $x \in \mathbb{R}_+$, it follows that if $\delta' > 0$ is chosen such that

$$\mu(A) < \delta' \implies \int_A \psi(z)dz < \frac{\varepsilon}{2}$$

(existence of such a δ' is by absolute continuity of $A \mapsto \int_A \psi$ with respect to μ), then

$$\int_{A\cap G_a} p(x,y)dy \leq \int_{\frac{A}{c}} \psi(z)dz < \frac{\varepsilon}{2}$$

whenever $\mu(A) < \delta$, $\delta = \delta' c$, and, therefore,

$$\mu(A) < \delta \implies \int_{A\cap G_a} P^t\varphi < \frac{\varepsilon}{2}.$$

Again, combining this with (20) yields (i) of the Dunford-Pettis characterization.

Finally, suppose to the contrary that while Condition 3.4 is satisfied, there exists no c with $g(x) \geq c > 0$ for all $x \in \mathbb{R}_+$. In this case Condition 3.4 implies that $g(x) \downarrow 0$ as $x \downarrow 0$, and hence there exists a $d > 0$ such that

$$\int_{A\cap G_a} p(x,y)dy = \int_{\frac{A\cap G_a}{g(x)}} \psi(z)dz \leq \frac{\varepsilon}{2} \quad \text{for almost all } x \in [0,d), \tag{21}$$

owing to the fact that $A \cap G_a$ is bounded away from 0. For $x \geq d$, $g(x) \geq c' = \min[g(d), b] > 0$, where b is the positive constant in Condition 3.4.[11] In this case, an argument similar to that given above for the case $g(x) \geq c > 0$ implies that

$$\int_{A\cap G_a} p(x,y)dy \leq \int_{\frac{A}{c'}} \psi(z)dz < \frac{\varepsilon}{2} \tag{22}$$

whenever $x \in [d, \infty)$ and $\mu(A) < \delta$, $\delta = \delta' c'$. Combining (21) and (22) yields

$$\mu(A) < \delta \implies \int_{A\cap G_a} P^t\varphi < \frac{\varepsilon}{2}.$$

Once again, (i) of the Dunford-Pettis characterization holds.

It remains to establish that the Dunford-Pettis condition (ii) also holds for the same collection. We have already shown that

$$\int_{\mathbb{R}_+\setminus G_a} P^t\varphi \leq \frac{1}{a}\left(1 + \frac{C}{1-\alpha}\right)$$

[11]Here $g(d) > 0$ by the almost everywhere positivity of g.

for all positive a, all $t \geq N$. But this inequality is sufficient, because G_a is always bounded. Hence condition (ii) is also satisfied for $\{P^t\varphi : t \geq N\}$. This completes the proof of the lemma. $\square$

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