Volatility Disagreement and Equilibrium Volatility Trading

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Abstract

We develop a dynamic equilibrium model in which investors disagree on future volatility and trade volatility derivatives to hedge their stock positions and speculate on their beliefs. We find that in equilibrium investors trade less volatility derivatives in more volatile periods, and they may also do so when they disagree more. The variance risk premium is negative on average, but it becomes positive when the market tends to underestimate future volatility. Volatility disagreement generates time-variation in the leverage effect, which gets stronger in more volatile periods, consistent with empirical evidence. Our framework, which can also incorporate an aggregate volatility bias, reconciles other key aspects of volatility derivatives and stock markets in a unified setting.

JEL Classifications: G12, G13.

Keywords: Volatility disagreement, volatility trading, volatility derivatives market, variance swaps, variance risk premium, leverage effect, equilibrium asset prices.

1 Introduction

A key underpinning of financial markets is the ability of investors with different views to trade with each other. Their trading activity allows to achieve a more efficient capital allocations, and their views are ultimately reflected in asset prices. Despite growing survey evidence documenting significant disagreement in investors' volatility expectations (e.g., Graham and Harvey (2001), Amromin and Sharpe (2014), Kaplanski et al. (2016)), this form of disagreement and its implications for financial markets have been largely unexplored. One venue that is naturally affected by volatility disagreement is the volatility derivatives market, which has seen a rapid growth in the last two decades. The prime examples of financial instruments in this market are variance swaps and VIX futures and options, which allow investors to directly hedge the volatility risk in their portfolios and to speculate on their beliefs.[1] The growth of the volatility derivatives market has spurred a vast literature documenting various empirical regularities in this market that, when taken together, are hard to reconcile within existing theoretical studies.[2]

Our goal in this paper is to provide a comprehensive theoretical investigation of the equilibrium effects of volatility disagreement in financial markets. To this end, we develop a tractable dynamic asset pricing model in which investors with different future volatility expectations trade in the stock and, notably, the volatility derivative markets. The presence of volatility derivatives allows investors to hedge their exposure to volatility shocks as well as to speculate on their different volatility expectations. Our model delivers closed-from expressions and generates a rich set of novel predictions for volatility derivatives market quantities, such as the variance risk premium, the variance swap rate and trading activity, as well as for the stock market quantities, such as stock market volatility, stock risk premium, and the leverage effect. To the best of our knowledge, ours is also the first theory work to reconcile several key empirical evidence on the volatility derivatives market in a unified equilibrium setting.

[1]See, for example, Carr and Lee (2009) for a brief survey on the history and the workings of the volatility derivatives market.

[2]In this literature, empirical studies typically focus on key market quantities such as the variance risk premium (e.g., Bakshi and Kapadia (2003), Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), Barras and Malkhozov (2016)), the variance swap rate (e.g., Mixon and Onur (2015), Dew-Becker et al. (2017), Aït-Sahalia, Karaman, and Mancini (2020)), and the VIX futures and options (e.g., Eraker and Wu (2017), Bardgett, Gourier, and Leippold (2019), Cheng (2019, 2020)). Some of these studies also provide evidence on the trading activity in volatility derivatives.

More specifically, in our setting, the fundamental process, determining the risky payoff of the stock representing the aggregate stock market, has stochastic variance, which we model as a mean-reverting square-root process. To allow investors to directly trade assets that are claims on the market return variance, the available securities include volatility derivatives (variance swaps) in zero net supply. The existence of volatility derivatives completes the securities market, and guarantees that asset prices are unique and allocations Pareto optimal. A key aspect of our model is that investors disagree on volatility, as they have different expectations about the future variance of the fundamental process. Specifically, we consider two risk averse investor types: *high-fear* investors, who overestimate the future variance, and *low-fear* investors, who underestimate it.[3] In our baseline economy, investors expectations are symmetric around the true expectation. The valuation of the stock market, the variance swap rate (i.e., the price that volatility buyers pay to volatility sellers), as well as investors' security holdings, are determined endogenously in equilibrium.

In the presence of volatility disagreement, the relative wealth distribution across investors arises as an endogenous state variable. This is because, according with their volatility expectations, investors take different positions in the stock and volatility derivatives, which then leads to wealth transfers in our dynamic economy. Investors whose beliefs are more in line with realized shocks get relatively wealthier in equilibrium and have a stronger impact on asset prices. In particular, high-fear investors become volatility buyers and low-fear investors volatility sellers in the derivatives market. Thus, following positive (negative) variance shocks, high-fear (low-fear) investors get wealthier and more dominant in the economy. We show that the equilibrium wealth-share distribution follows a mean-reverting process, and in the long-run no investor dominates the economy.[4] Moreover, as a novel channel, we also show that the presence of volatility disagreement introduces time variation in the elasticity of the equilibrium state price density with respect to the fundamental variance, which we refer to as "variance elasticity" for short. Fluctuations in the variance elasticity arise because high-fear investors have higher and more persistent variance expectations than low-fear investors. Since high-fear investors' marginal utilities are more sensitive to variance shocks, the equilibrium variance elasticity increases (decreases) when they become more (less) dominant.

[3]Our terminology for investor types is motivated by the financial press and industry commonly referring to the CBOE's volatility index, VIX, as the "fear index."

[4]Equilibrium wealth effects are well-studied in the literature on heterogeneous beliefs (as discussed below). The novel feature of the wealth transfers in our model is that they are due to volatility disagreement and thus driven by shocks to the second-moment of asset returns.

In our model, variance risk is priced and its equilibrium market price is negative on average and becomes more negative when volatility disagreement is higher. This novel result is driven by two separate mechanisms that reinforce each other in equilibrium. First, a higher volatility disagreement spurs volatility trading in the derivatives market, thus exacerbating wealth transfers in the economy. Since risk averse investors dislike this heightened fluctuations in their wealth, they are willing to pay more for assets that are positively exposed to variance shocks. Second, since high-fear investors are more sensitive to risk, the increase in their subjective risk discount due to a larger disagreement is higher in magnitude than the decrease in the risk discount of the low-fear investors. This asymmetry induces the equilibrium variance elasticity to increase when disagreement is higher. These two mechanisms, taken together, leads to a lower market price of variance risk. We also find that when low-fear investors are sufficiently dominant in the economy, the market price of variance risk becomes positive and tend to increase in volatile times. A positive price of risk is somewhat intriguing as it suggests that rather than being willing to pay more for assets that are positively exposed to variance shocks, investors require a premium. This result occurs because, when high-fear investors are small in the economy, they need to be induced by a positive price of variance risk to increase their demand for volatility exposure, in order to meet the large supply provided by low-fear investors.

Investigating the equilibrium behavior of the stock market, we find that higher volatility disagreement leads on average to a lower valuation of the stock market, a higher stock risk premium, and a higher stock return variance. Given the symmetry of investors' beliefs, a higher volatility disagreement leads high-fear investors to discount future stock payoffs more, and low-fear investors to discount them less. Since high-fear investors are more sensitive to risk, the impact of their discounting is stronger in equilibrium, overall decreasing the valuation of the stock market. Moreover, since a higher volatility disagreement leads to a larger wealth transfer and variance elasticity, it exposes investors to more variance shocks. Therefore, investors on average require a higher risk premium to hold the stock. By making the stock more sensitive to variance shocks, higher volatility disagreement also leads to more volatile stock returns.

We also find that the volatility disagreement is a key determinant of the leverage effect (i.e., the negative correlation between stock returns and its variance shocks). In particular, we show that when agents disagree on volatility, the leverage effect gets stronger, and particularly so in more volatile periods, consistent with its documented behavior in the data (e.g., Bandi and Renò (2012), Andersen, Bondarenko, and Gonzalez-Perez (2015)). This novel result arises because the equilibrium variance elasticity becomes higher in more volatile times, reflecting more high-fear investors' risk discounting during those times. Consequently, stocks returns becomes more sensitive to variance shocks per unit of volatility.

A central contribution of our theory is the study of the volatility derivatives market in an equilibrium setting. Although in equilibrium high-fear investors hold long positions in variance swaps (i.e., they are volatility buyers) and low-fear investors hold short positions (i.e., they are volatility sellers), we show that their holdings have a non-monotonic relation with volatility disagreement. In particular, the variance swap holdings of high-fear investors first increase then decrease in disagreement. By market clearing, the opposite holds for the variance swap holdings of low-fear investors. The intuition is that while a higher disagreement about future volatility increases the investors' desire to trade variance swaps, it also increases the riskiness of those derivatives, thus curbing the investors' willingness to hold them in equilibrium. For low levels of disagreement, the former effect dominates, thus generating a positive relationship between variance swap holdings and disagreement. However, for sufficiently high disagreement, the latter effect takes over and the above relationship reverses.

Our model, to the best of our knowledge, is also the first study to reconcile the puzzling empirical evidence that investors on average tend to hold less volatility derivatives in periods of high volatility (e.g., Cheng (2019))). This evidence is puzzling because in volatile times, arguably, there is more volatility risk to be hedged and instruments like variance swaps should be higher in demand. In our setting, this negative relation emerges in equilibrium because the volatility of swap contracts heightens in periods of high volatility, and makes it too risky for investors to hold them.

We next study the equilibrium implications of volatility disagreement for the variance risk premium. Since, as discussed above, higher disagreement tend to amplify both the market price and the quantity of variance risk, the variance risk premium, which is negative on average, becomes more negative with higher disagreement. However, we demonstrate that when low-fear investors are sufficiently large in the economy, the variance risk premium can actually become positive. This intriguing result arises because, when the market is dominated by low-fear investors, who believe that the volatility will be much lower in the near future compared to its current value, the market price of variance risk switches sign and turns positive. Moreover, when this happens, the variance risk premium increases in market volatility. The fact that the variance risk premium is negative on average is well-documented in the empirical literature (e.g., Bakshi and Kapadia (2003), Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009)). However, more recent evidence highlights how it becomes positive especially during market turmoils (Cheng (2019, 2020)). Our model, therefore, offers a plausible explanation for this otherwise puzzling behavior.

In our economy, the endogenous variance swap rate, which volatility buyers pays to volatility sellers, increases in volatility disagreement. This is because a larger disagreement increases both the current stock return variance and the expectations for its future level, triggering an increase in demand for variance swaps by all investors. However, for the variance swap market to clear, the equilibrium swap rate must increase so that the low-fear investors remain willing to provide volatility exposure to the high-fear investors.

We conclude our analysis by studying the implications of an aggregate volatility bias. To this end, we extend our baseline model to allow for asymmetric subjective variance expectations around the true one, introducing a common component in investors' subjective expectations. Therefore, the higher the common component, the higher the bias. The main equilibrium effects of a higher aggregate volatility bias are that it lowers the stock market valuation, the market price of variance risk and the variance risk premium, while increasing the stock market risk premium and its variance, the magnitude of the leverage effect, and the variance swap rate. Counter to our expectations, we find that a higher aggregate volatility bias always reduces the equilibrium trading in volatility derivatives. Indeed, differently from the effect of volatility disagreement, a higher bias increases only the riskiness of the variance swaps, without affecting their perceived expected returns. So, more risky variance swaps lead investors to reduce their holdings of such derivatives. Finally, we show that the presence of a downward aggregate volatility bias, together with a large fraction of wealth held by low-fear investors, can help explain two facts which occur in periods of heightened volatility: a large positive variance risk premium, and a lower volatility trading (Cheng (2019)).

This paper is related to several strands of literature. First, our paper contributes to the large theoretical literature on how investors' belief disagreement affects asset prices. In this literature, the vast majority of works study the effects of disagreement on the fundamental growth rate (first-moment) shocks (e.g., Detemple and Murthy (1994), Zapatero (1998), Basak (2000, 2005), Johnson (2004), David (2008), Yan (2008), Dumas, Kurshev, and Uppal (2009), Cvitanić and Malamud (2011), Banerjee (2011), Bhamra and Uppal (2014), Buraschi, Trojani, and Vedolin (2014), Atmaz and Basak (2018), Andrei, Carlin, and Hasler (2019)). Relatively fewer works study the asset pricing effects of disagreement on volatility (second-moment) shocks as we do (Detemple and Selden (1991), Duchin and Levy (2010), Bakshi, Madan, and Panayotov (2015), Smith (2019)). Differently from us, all these works employ static mean-variance frameworks, thus abstracting from the dynamic trading and wealth transfer mechanisms of our model, which are key to our main results.[5]

Second, our paper is related to the theoretical literature studying the equilibrium effects of investors' volatility expectations. In this literature, Atmaz (2022) and Lochstoer and Muir (2022) consider single agent dynamic extrapolative expectations frameworks to study the effects of biased volatility expectations on asset prices. In this literature, Ghaderi, Kilic, and Seo (2023) consider an incomplete information setting in which a representative agent rationally learns about a hidden state of the economy. Due to incomplete information and learning, they find that variance risk premium can become positive in some states. Since, in contrast to us, all these works employ single-agent economies, they are unable to provide predictions about prices *and* quantities in the volatility derivative market, as well as the effects of volatility disagreement.

Finally, we contribute to the literature studying the effects of trading in nonredundant derivatives (e.g., Detemple and Selden (1991), Brennan and Cao (1996), Franke, Stapleton, and Subrahmanyam (1998), Cao and Ou-Yang (2008), Garleanu, Pedersen, and Poteshman (2008), Bhamra and Uppal (2009), Banerjee and Graveline (2014), Chabakauri, Yuan, and Zachariadis (2022)). The derivatives considered in these works are typically equity options rather than volatility derivatives, with the exception of Chabakauri, Yuan, and Zachariadis (2022), who study the informational role of volatility derivatives in a static asymmetric information framework. They find that volatility derivatives make incomplete markets effectively complete and their prices reflect the shadow value of information. In our model the presence of volatility derivatives completes the markets too. However, differently from them, we consider a dynamic symmetric information framework with a focus on the asset pricing implications of volatility disagreement. Moreover, none of works above are able to generate our equilibrium predictions on the volatility derivatives market.

The remainder of the paper is organized as follows. Section 2 introduces our model with volatility disagreement. Section 3 determines the equilibrium and market prices of risks in our economy. Section 4 presents our results on the stock market, while Section 5 focuses on

[5]For example, Detemple and Selden (1991) and Duchin and Levy (2010) focus only on the stock market and find that a higher volatility disagreement leads to a higher stock price and lower risk premium, the opposite of what we find.

the volatility derivatives market. In Section 6, we extend our baseline model to incorporate an aggregate volatility bias. Section 7 concludes. Appendix A contains all the proofs and Appendix B discusses the parameter values employed in our figures and tables.

2 Model

In this section, we present a simple and tractable pure-exchange economy in which two types of investors disagree about future volatility. The key feature of our model is the presence of volatility derivatives market, which allows investors with different future volatility expectations to trade on their beliefs and speculate against each other.

2.1 Securities Market

We consider a continuous-time economy with horizon T. In this economy, three securities are available for trading: a riskless bond, a risky stock (representing the aggregate stock market), and a volatility derivative. The stock, with its time-t price denoted by S_t, is in positive net supply of one unit and is a claim to the risky payoff D_T at horizon T, so $S_T = D_T$. The stock payoff is the time T realization of the fundamental (cashflow news) process D_t with dynamics

$$\frac{dD_t}{D_t} = \mu dt + \sqrt{V_t} d\omega_{1t}, \tag{1}$$

$$dV_t = \kappa \left(\overline{V} - V_t\right) dt + \sigma \sqrt{V_t} d\omega_{2t}, \tag{2}$$

where μ is the constant mean growth rate and V_t is the stochastic variance of the fundamental process D_t. The positive constants κ, $\overline{V}$, σ, control the mean reversion speed, long-run mean, and the volatility of the fundamental variance process V_t, respectively. Two sources of risk, represented by independent Brownian motions ω_{1t} and ω_{2t}, which are defined on the true probability measure $\mathbb{P}$, represent the *cashflow risk* and *variance risk*, respectively. We assume the initial values $D_0 > 0$ and $V_0 > 0$ and the parameter restriction $2\kappa\overline{V} > \sigma^2$ so that the (square-root) fundamental variance process V_t is positive in finite time.[6] The stock price,

[6] Our framework allows for more general setting by introducing a correlation between the fundamental process D_t and its variance V_t. To keep the model parsimonious we set such correlation to zero. A fundamental process with stochastic variance is consistent with empirical findings in Schorfheide, Song, and

which is determined endogenously in equilibrium, is characterized by following dynamics

$$\frac{dS_t}{S_t} = \mu_{St}dt + \sigma_{S1t}d\omega_{1t} + \sigma_{S2t}d\omega_{2t}, \tag{3}$$

where μ_{St} captures the expected stock return, while the diffusion coefficients σ_{S1t} and σ_{S2t} capture the quantity of cashflow and variance risk for the stock, respectively. Given the definition of the stock return variance $\upsilon_t \equiv \sigma_{S1t}^2 + \sigma_{S2t}^2$, we characterize its dynamics by

$$d\upsilon_t = \mu_{\upsilon t}dt + \sigma_{\upsilon 1t}d\omega_{1t} + \sigma_{\upsilon 2t}d\omega_{2t}, \tag{4}$$

where $\mu_{\upsilon t}$ captures the expected change in return variance, while the diffusion coefficients $\sigma_{\upsilon 1t}$ and $\sigma_{\upsilon 2t}$ capture the quantity of cashflow and variance risk for the return variance, respectively. The riskless (zero-coupon) bond, with its time-t price denoted by Z_t, is in zero net supply with a constant rate of return r, implying $dZ_t = Z_t r dt$ as its dynamics.

To complete the securities market, as a third security, we consider a series of zero net supply volatility derivatives whose payoff depend on the risky stock's future return variance. Toward that, we introduce instantaneous variance swap contracts that are initiated at each time t with maturity over the next instant $t + dt$. Like any swaps, these variance swaps require zero upfront payment at their initiation time t. At their maturity date $t + dt$, an investor who has a long position in this contract receives $\upsilon_t dt + d\upsilon_t$, and in return, pays the variance swap rate $y_t dt$.[7] The variance swap rate y_t is endogenously determined at the contract initiation time t.

Remark 1 (Further discussion on volatility derivatives). The most common financial instruments for getting direct volatility exposure in real world are volatility derivatives such as variance swaps and VIX options/futures. To achieve volatility exposures, investors can also form portfolio of equity options, such as straddles.[8] The exact choice of financial instrument may depend on several factors, which include investors' preferences toward the direct vs. indirect volatility exposure, their complex portfolio management capabilities, and

Yaron (2018) and Pettenuzzo, Sabbatucci, and Timmermann (2020), who find evidence of heteroskedasticity in cashflow growth rates and is also commonly employed in asset pricing models, particularly in the long-run risk models (e.g., Bansal and Yaron (2004)).

[7]The variance swaps typically have notional amounts to scale the payoffs. Since, in our model, the notional amount of the swap does not play any role, we normalize it to 1.

[8]See, for example, Carr and Lee (2009) for a survey on volatility derivatives.

counterparty risk tolerance. In our model, we consider variance swaps as they allow for direct volatility exposures and have simple linear payoffs, which simplifies our analysis. That said, we highlight that the specific choice of volatility derivative is irrelevant for asset prices in our model. Any volatility derivative is sufficient to complete the securities market, leading to a unique state price density in equilibrium, which in turn can be used to recover the prices of other derivative contracts, such as long maturity variance swaps.

2.2 Investors' Beliefs

In this economy, we have two types of investors who observe the fundamental D_t and its fundamental variance V_t at each time t. Investors are assumed to know the fundamental mean growth rate μ and the volatility coefficient of the fundamental variance process σ, but they have different beliefs about the expected future variance. The h-type investors misperceive the expected change in the fundamental variance as

$$\mathrm{E}_t^h\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] + \tfrac{1}{2}\delta V_t dt,$$

whereas the ℓ-type investors misperceive it as

$$\mathrm{E}_t^\ell\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] - \tfrac{1}{2}\delta V_t dt.$$

The positive constant $\delta \geq 0$ controls the *volatility disagreement* in our model since the difference in investors' variance expectations is given by $\mathrm{E}_t^h\left[dV_t\right] - \mathrm{E}_t^\ell\left[dV_t\right] = \delta V_t dt$. We note that the difference between the equally-weighted average of the subjective variance expectations and the true one, which we define as the *aggregate volatility bias* in the economy, is zero in our baseline specification. However, in Section 6, we extend our model to additionally incorporate an aggregate volatility bias by considering asymmetric beliefs around the true one.[9] This specification implies that each i-type investor, $i = h, \ell$, perceives the fundamental variance as

$$dV_t = \kappa_i\left(\overline{V_i} - V_t\right)dt + \sigma\sqrt{V_t}d\omega_{2t}^i,$$

[9]Analogously, the volatility disagreement δ controls the equally-weighted standard deviation of variance expectations, which is equal to $(\delta/2)V_t dt$. Moreover, it is easy to show the parameter δ also controls the wealth-share weighted disagreement in variance expectations using our equilibrium quantities in Section 3.

where $\kappa_h = \kappa - \delta/2$, $\kappa_\ell = \kappa + \delta/2$, and $\overline{V_i} = \overline{V}\kappa/\kappa_i$, are positive constants and ω_2^i is a standard Brownian motion under the i-type investor's subjective probability measure $\mathbb{P}^i$, with the relations $d\omega_{2t}^h = d\omega_{2t} - (1/2\sigma)\delta\sqrt{V_t}dt$, and $d\omega_{2t}^\ell = d\omega_{2t} + (1/2\sigma)\delta\sqrt{V_t}dt$. Given the above beliefs, we interpret h-type investors as *high-fear investors* since they have higher and more persistent variance expectations than ℓ-type investors, who we refer to as *low-fear investors*.[10] To ensure the equilibrium stock price admits a real solution in our model, we impose the parameter restriction of $\kappa_h > \sqrt{2}\sigma$, which also guarantees the fundamental variance being mean-reverting under investors' subjective expectations.

2.2.1 Discussion on Modeling Volatility Expectations

Investors typically do not have uniform expectations on economic variables, and volatility is no exception. Several survey evidence confirm significant differences in volatility expectations. For instance, in a survey of chief financial officers (CFOs) in U.S. corporations, Graham and Harvey (2001) document a cross-sectional average dispersion for the volatility expectations on the next year S&P 500 returns to be 4.6%. Similar findings of volatility disagreement are also present in Amromin and Sharpe (2014) and Kaplanski et al. (2016), who employ different survey data.

Our choice of modeling disagreement proportional to the fundamental variance is to capture that more uncertainty leads to more disagreement. This is economically meaningful and also consistent with the findings in Ben-David, Graham, and Harvey (2013), who show that investors' expectations widen in periods of increased contemporaneous volatility. One could plausibly entertain alternative formulations of volatility expectations in our framework and still be in line with evidence. For instance, a more general non-symmetric beliefs could be modeled as $\mathrm{E}_t^i[dV_t] = \mathrm{E}_t[dV_t] + (\alpha^i + \beta^i V_t)\,dt$ for $i = h, \ell$ such that $\alpha^h \neq \alpha^\ell$, or $\beta^h \neq \beta^\ell$. Alternatively, one could also consider extrapolative beliefs driven by past variance shocks to introduce slow-moving average expectations as in Lochstoer and Muir (2022). These alternative considerations typically lead to additional state variables in equilibrium and complicates the analysis. In this paper, we abstract away from these more complicated

[10] As highlighted in Introduction, our terminology for high- and low-fear is motivated by the CBOE's volatility index, VIX, being commonly referred to as "fear index" in financial press and industry. Our modeling of volatility disagreement with high- and low-fear investors is also akin to the settings with persistently optimistic and pessimistic investors that is commonly employed in growth rate disagreement models (e.g., Detemple and Murthy (1994), Basak (2000), Buraschi and Jiltsov (2006), David (2008)).

settings to focus on the equilibrium implications of volatility trading.

Regarding the sources of volatility disagreement, one plausible economic channel is that investors employ different volatility estimation models. As volatility is latent, investors can rely on different models and methods to estimate volatility dynamics. For instance, one could employ econometric methods such as GARCH or any of its variants, or other stochastic volatility specifications. Moreover, investors could utilize data at different frequencies or from different sources. An alternative source of different volatility expectations could be related to behavioral biases, such as overconfidence and miscalibration. For instance, Ben-David, Graham, and Harvey (2013) show that CFOs in their survey overestimate the precision of their own forecasts and underestimate the variance of risky processes. In particular, they find the average volatility expectation to be around 7%, a much lower value than the historical realized volatility, which suggests a downward volatility bias on average. For this reason, in Section 6, we generalize our model to additionally incorporate an aggregate volatility bias.

2.3 Investors' Preferences and Optimization

Each investor is initially endowed with the same number of stock shares and no bonds, and no variance swap contracts, so that their initial wealth is the same across types, $W_{h0} = W_{\ell 0} = W_0$. At each point in time t, i-type investor, $i = h, \ell$, chooses an admissible dynamic portfolio strategy, defined by the number of bonds α_{it}, the number of shares in the stock ψ_{it}, and the number of variance swap contracts θ_{it} to hold, so as to maximize her logarithmic preferences defined over the value of her wealth at the horizon date T,

$$\max_{\{\alpha_{it}, \psi_{it}, \theta_{it}\}_{t=0}^{T}} \mathrm{E}^i \left[\ln W_{iT} \right],$$

subject to her dynamic budget constraint

$$dW_{it} = \alpha_{it} dZ_t + \psi_{it} dS_t + \theta_{it} \left(\upsilon_t dt + d\upsilon_t - y_t dt \right), \tag{5}$$

where E^i denotes the unconditional expectation under the i-type investor's subjective probability measure $\mathbb{P}^i$.[11] We see in (5) that the ability to trade variance swap contracts allows

[11]In our setting, consumption occurs only at time T (i.e., there is no intermediate consumption, which implies the riskless interest rate can also be taken as exogenous). This setting is suitable for our purposes since it allows variance shocks to be priced even with time-separable logarithmic preferences. This is in

investors to hedge their volatility exposure, as well as to speculate on their different volatility expectations. In particular, a positive (negative) θ_{it} indicates that the i-type investor is long (short) in the variance swap contract at time t, thus, she is a volatility buyer (seller).

3 Equilibrium with Volatility Disagreement

In this section, we determine the equilibrium state price density and market prices of risks in our economy with volatility disagreement. We show that the wealth distribution in our economy affects the equilibrium market price of variance risk but not the market price of cashflow risk. We find that the market price of variance risk is negative on average and its magnitude increases in volatility disagreement. That said, we further show that when the low-fear investors are sufficiently dominant in the economy, the market price of variance risk becomes positive and increases in variance.

Equilibrium in our economy with volatility disagreement is defined in a standard way. The economy is said to be in equilibrium if the stock price S_t, the variance swap rate y_t, and each i-type investor's, $i = h, \ell$, consumption W_{iT} and portfolio strategies $(\alpha_{it}, \psi_{it}, \theta_{it})$ are such that (i) all investors choose their optimal consumption and portfolio strategies given prices, (ii) the goods market clear at time T, $W_{hT} + W_{\ell T} = D_T$, (iii) the bond, the stock, and the variance swap market clear at all times $t \in [0, T]$, $\alpha_{ht} + \alpha_{\ell t} = 0$, $\psi_{ht} + \psi_{\ell t} = 1$, and $\theta_{ht} + \theta_{\ell t} = 0$, respectively. To appreciate the equilibrium implications of volatility disagreement, we will often make comparisons with the equilibrium in an otherwise identical economy where all investors have the same unbiased variance expectations, i.e., $\delta = 0$. We refer to this economy as *benchmark economy without disagreement* and denote economic quantities in this benchmark with an upper bar $(\bar{\ })$.

The presence of a volatility derivative makes financial markets dynamically complete, allowing investors to tailor their exposures to the two sources of risk in the economy. This

contrast to settings with intertemporal consumption in which variance shocks are not priced unless one considers more complex time-inseparable preferences (see, for example, Bansal and Yaron (2004)). As we demonstrate in Sections 3–6, this setting leads to tractable closed-from solutions for all our economic quantities in equilibrium. Other dynamic asset pricing models with no intertemporal consumption and exogenous interest rates include Kogan et al. (2006), Cvitanić and Malamud (2011), Pástor and Veronesi (2012), Basak and Pavlova (2013), Buffa and Hodor (2023).

implies the existence of a unique state price density, denoted by ξ_t, with posited dynamics

$$\frac{d\xi_t}{\xi_t} = -rdt - m_{1t}d\omega_{1t} - m_{2t}d\omega_{2t}, \tag{6}$$

where m_{1t} and m_{2t} denote the endogenous market prices of risks for the cashflow shocks ω_{1t} and variance shocks ω_{2t}, respectively. We employ standard martingale methods (Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989)) to solve for each investor's optimal horizon wealth and portfolio strategies, and apply market clearing conditions above to obtain equilibrium quantities. The equilibrium is characterized by two state variables: the exogenous fundamental variance V_t, and the endogenous wealth-share of the high-fear investors $\mathrm{w}_t \equiv W_{ht}/(W_{ht} + W_{\ell t})$.

Proposition 1 characterizes the equilibrium state price density in closed form.

Proposition 1 (**Equilibrium state price density**). *In the economy with volatility disagreement, the equilibrium state price density follows* (6)*, where the market prices of cashflow and variance risks are given by*

$$m_{1t} = \sqrt{V_t}, \tag{7}$$

$$m_{2t} = -\sigma B_t\sqrt{V_t} - \delta\left(\mathrm{w}_t\Lambda_t - \bar{\mathrm{w}}\right)\frac{1}{\sigma}\sqrt{V_t}, \tag{8}$$

respectively, and the wealth-share of the h-type investor follows

$$d\mathrm{w}_t = \delta^2 \mathrm{w}_t\left(1 - \mathrm{w}_t\right)\left(\bar{\mathrm{w}} - \mathrm{w}_t\right)\frac{1}{\sigma^2}V_t dt + \delta \mathrm{w}_t\left(1 - \mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}, \tag{9}$$

with $\bar{\mathrm{w}} = 1/2$ denoting its long-run mean. The positive processes $\Lambda_t > 1$ and B_t are given by

$$\Lambda_t = \frac{e^{A_h(t)+B_h(t)V_t}}{\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + \left(1 - \mathrm{w}_t\right)e^{A_\ell(t)+B_\ell(t)V_t}}, \qquad B_t = \mathrm{w}_t\Lambda_t B_h\left(t\right) + \left(1 - \mathrm{w}_t\Lambda_t\right)B_\ell\left(t\right), \tag{10}$$

where the positive deterministic functions $A_i(t)$ and $B_i(t)$ for $i = h, \ell$, are provided in Appendix A.

In the benchmark economy without disagreement, the market prices of cashflow and variance risks are given by $\bar{m}_{1t} = \sqrt{V_t}$, and $\bar{m}_{2t} = -\sigma\bar{B}(t)\sqrt{V_t}$, respectively, where the positive deterministic function $\bar{B}(t)$ is provided in Appendix A.

Although in the benchmark economy without disagreement investors agree on the expected fundamental variance, they are exposed to its stochastic fluctuations. This means that, given their investment in the stock, investors' marginal utility of wealth, which determines the equilibrium state price density, is high not only when the current cashflow news D_t is low, but also when the fundamental variance V_t is high. In this economy, the equilibrium market price of cashflow risk $\bar{m}_{1t}$ is positive and equal to the cashflow news volatility $\sqrt{V_t}$, since the marginal utility of risk averse investors increase by that rate following a negative cashflow shock ω_{1t}. Since investors hold a long position in the stock, they dislike positive variance shocks ω_{2t} and would find it desirable to invest in an asset that pays off when V_t is high.[12] Therefore, variance risk is priced in equilibrium, and the corresponding market price of variance risk $\bar{m}_{2t}$ is negative. Its equilibrium magnitude is determined by the volatility of the fundamental variance, as well as by the positive deterministic term $\bar{B}(t)$. We refer to the latter term as *variance-elasticity*, since it captures the rate of increase in the marginal utility following a unit increase in V_t, i.e., $\bar{B}(t) = \partial \ln \bar{\xi}_t / \partial V_t$. A key quantity determining the magnitude of the variance elasticity $\bar{B}(t)$ is the mean-reversion speed, κ, of the fundamental variance. A lower κ implies that V_t reverts to its mean $\overline{V}$ less rapidly, thus making variance shocks more persistent, and consequently investors more sensitive to them. Overall, this leads to a higher variance-elasticity $\bar{B}(t)$. This channel is key for understanding the equilibrium mechanism when investors disagree on volatility.

In the economy with volatility disagreement, the differences in investors' perceptions about future uncertainty leads to different investments in the stock and volatility derivatives. The heterogeneity in their portfolios creates a room for wealth transfers in the economy such that investors whose beliefs are more in line with realized shocks get relatively wealthier in equilibrium, and have a stronger impact on asset prices. Thus, in addition to the fundamental D_t and its variance V_t, in equilibrium, investors' marginal utilities are also driven by the wealth-share distribution in the economy, which is captured by the high-fear investors' wealth-share, $\mathrm{w}_t = W_{ht}/(W_{ht} + W_{\ell t})$. The dynamics of w_t in (9) reveals that, in equilibrium, the wealth-share distribution follows a mean-reverting process where its long-run mean is given by $\bar{\mathrm{w}} = 1/2$. Since beliefs are symmetric around the unbiased variance expectation,

[12] Although investors in the benchmark economy would benefit from trading a volatility derivative, they have zero volatility derivative holdings in equilibrium. This is because they would all want to trade it in the same direction, thus preventing the derivative market to clear.

no investor dominates the economy in the long-run.[13] Changes in the wealth-share distribution crucially depend on the volatility disagreement parameter δ, as this is the source of heterogeneity that leads investors to hold different positions in the stock and volatility derivatives. Since investors agree on cashflow shocks ω_{1t}, but disagree on variance shocks ω_{2t}, in equilibrium, the wealth-share distribution is driven only by the latter. This also implies that the market price of cashflow risk m_{1t} remains as in the benchmark economy without disagreement.

Under volatility disagreement, the equilibrium market price of variance risk m_2 takes a richer form compared to the benchmark economy. The first term in (8), which is due to the fluctuations in the fundamental variance, is negative as in the benchmark economy, since investors' volatility disagreement does not alter their dislike for variance shocks. However, the positive variance-elasticity B_t is no longer deterministic but fluctuates stochastically. Intuitively, it fluctuates between the (deterministic) investor-specific variance-elasticities $B_\ell(t)$ and $B_h(t)$, where the latter is always higher than the former since high-fear investors have higher and more persistent variance expectations than low-fear investors, and hence are more sensitive to variance shocks. Whether B_t is closer to $B_h(t)$ or to $B_\ell(t)$ depends on the (stochastic) weight $\mathrm{w}_t\Lambda_t$, which is driven by the wealth-share w_t and the fundamental variance V_t through the function Λ_t. The function Λ_t, which is always larger than 1, tells us how much high-fear investors discount future cashflows—because of their higher variance expectation—more than the average in the economy. Henceforth, we refer to Λ_t as the *relative risk discount*.

The second term in the equilibrium market price of variance risk (8) reflects the investors' disagreement about volatility, and the ensuing speculation against each other. Indeed, through this term, investors' trading activities in the volatility derivatives make the market price of variance risk higher or lower depending on which of the two agent types "dominates" the economy. Given their respective high and low volatility expectations, high-fear investors are volatility buyers and low-fear investors are volatility sellers (a feature we discuss in more detail in Section 5). Thus, following a positive variance shock, high-fear investors get relatively wealthier and more dominant in the economy. When high-fear investors are sufficiently dominant, i.e., $\mathrm{w}_t > \bar{\mathrm{w}}/\Lambda_t$, this second term is negative, implying that high-fear investors'

[13] As we demonstrate in Section 6, when investors' beliefs are asymmetric around the true variance expectation, the long-run mean of the wealth-share can differ from 1/2 with still no investor completely dominating the economy in the long-run.

demand pressure on the volatility derivative makes the market price of variance risk more negative. When, instead, low-fear investors dominate, this second term is positive, which, if large enough, can turn the market price of variance risk to positive. In this case, a positive price of variance risk is needed to induce high-fear investors to increase their demand for volatility exposure, so that the high supply provided by low-fear investors, holding most of the wealth in the economy, is met in equilibrium.

The overall behavior of the market price of risk m_{2t} depends on the wealth-share distribution in the economy. When the wealth-share is at its long-run mean $\bar{\mathrm{w}}$, m_{2t} is always negative and is lower than its benchmark economy counterpart. This fact suggests that, on average, volatility disagreement amplifies the negative market price of variance risk. However, as discussed above, when low-fear investors' wealth-share is sufficiently high, the market price of variance risk can become positive (see, also, Figure 1, Panel B). A positive market price of risk m_{2t} is somewhat intriguing as it suggests that rather than paying more to hold assets that are positively exposed to variance risk, investors require a premium, a feature that is possible when investors speculate on volatility, and hence never occurs in the benchmark economy without disagreement. A positive market price of variance risk, due to low-fear investors being sufficiently dominant in the economy, can become more positive in high volatility times. We discuss this puzzling result in Corollary 1 below.

We now discuss our model's key implications for the market price of variance risk, which plays a crucial role in the stock and volatility derivative markets as our analysis in Sections 4–5 shows. Here, as well as in our subsequent analysis, we are primarily interested in how economic quantities behave on average so that they can be easily mapped into most empirical evidence. To this end, we refer to the state when the fundamental variance and wealth-share are at their respective long-run means, $V_t = \overline{V}$ and $\mathrm{w}_t = \bar{\mathrm{w}}$, as the *"steady-state"*, capturing the average state in our economy. Corollary 1 presents the effects of the volatility disagreement and fundamental variance on equilibrium market price of variance risk at steady state.[14]

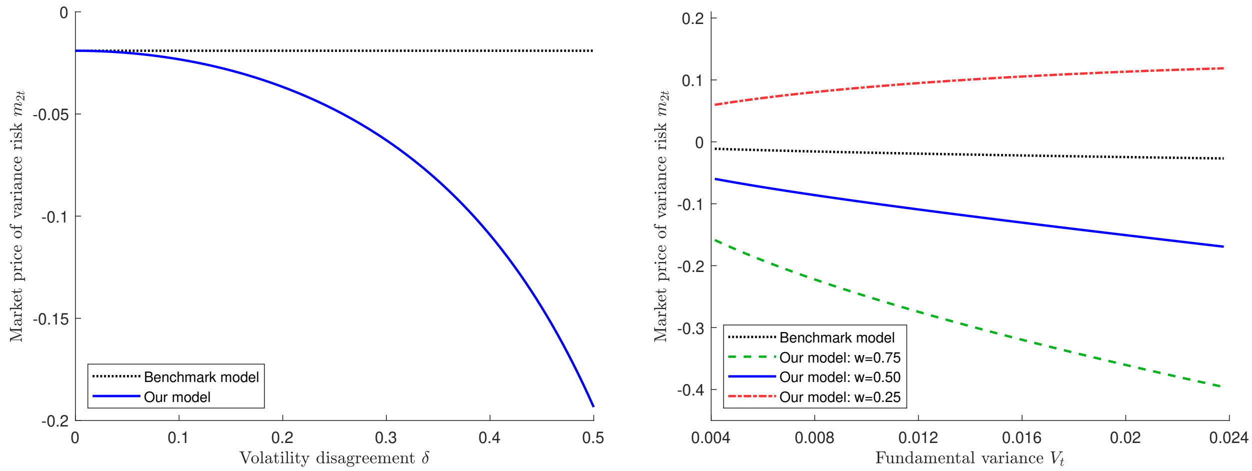
Corollary 1 (**Effects of volatility disagreement and fundamental variance**). *At the steady state of the economy with volatility disagreement, the market price of variance risk m_{2t} is decreasing in both the volatility disagreement δ and the fundamental variance V_t.*

Corollary 1 highlights that, at steady state, the market price of variance risk is negative

[14]We provide the analytic proofs of all our Corollaries in Appendix A.

and its magnitude increases in the volatility disagreement parameter δ. This result is due to a direct and an indirect effect. The direct effect is captured by the scaling of the second term of m_{2t}, which, as discussed earlier, is negative at steady state. Therefore, a higher disagreement leads to a more negative price of variance risk. The indirect effect, instead, works through the investor-specific variance elasticities $B_h(t)$ and $B_\ell(t)$, affecting B_t and Λ_t, and thus the extent to which investors' wealth-shares fluctuate. In particular, a higher volatility disagreement amplifies the stochastic variance-elasticity B_t. This amplification effect occurs because more volatility disagreement leads to high-fear investors to have higher and more persistent variance expectations, while leading to lower and less persistent variance expectations for low-fear investors. However, since high-fear investors are more sensitive to risk, the increase in their elasticity is much higher than the magnitude of the decrease in that of the low-fear investors, $\frac{\partial}{\partial\delta}B_h(t) > |\frac{\partial}{\partial\delta}B_\ell(t)|$, leading to an overall increase in the variance elasticity B_t. Volatility disagreement also controls the additional risk arising from wealth transfers in the economy. This observation can be seen in (9), which shows that at steady state, the expected changes in wealth share is zero but its diffusion term is amplified by the volatility disagreement. Thus, the higher the δ, the higher the risk averse investors' dislike of this additional uncertainty and higher their willingness to pay for assets that are positively exposed to variance shocks. The direct and indirect effects reinforce each other in equilibrium and lead to the market price of variance risk to be decreasing in volatility disagreement as illustrated in Figure 1, Panel A.

As Corollary 1 also highlights that the market price of variance risk is also decreasing in fundamental variance V_t when $\mathrm{w}_t = \bar{\mathrm{w}}$, as also illustrated in Figure 1, Panel B. This result is again due to two, direct and indirect, effects that reinforce each other. The direct effect is also present in the benchmark economy and refers to the amplifying role the fundamental volatility. Given that V_t is a square-root process (see (2)), a higher fundamental variance also means a more volatile process itself, which amplifies the negative market price of variance risk and making it more negative. Whereas, the indirect effect is novel to the volatility disagreement. As discussed above, high-fear investors' marginal utility is more sensitive to the risk, $B_h(t) > B_\ell(t)$. Thus, a higher V_t increases the (high-fear investors') relative risk discount term Λ_t, and consequently the variance-elasticity B_t, which leads to a more negative market price of variance risk m_{2t}. Figure 1, Panel B, also illustrates the behavior of market price of variance risk in other non-steady states. We see that when high-fear investors' have sufficiently low wealth-share, the market price of variance risk can be increasing in



Panel A. Effects of volatility disagreement

Panel B. Effects of fundamental variance

Figure 1. Market price of variance risk. These panels plot the equilibrium market price of variance risk m_{2t} against volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$ (Panel A) and against fundamental variance V_t for different levels of w_t (Panel B). The dotted black lines represent the benchmark economy with no volatility disagreement. The parameter values follow from Table B1 of Appendix B.

V_t. This intriguing positive relation between m_{2t} and V_t occurs because the direct effect now amplifies the positive market price of variance risk arising from high-fear investors' extremely high marginal utility. As we discuss in Section 5, this latter effect have important consequences, as it can help explain the puzzling empirical phenomena that the variance risk premium turning positive and increasing during market stress and extreme volatility (e.g., Cheng (2019)).

4 Stock Market

Having determined the equilibrium, in this section, we investigate how the equilibrium stock price, stock return variance, and the correlation between the stock returns and its variance shocks, i.e., the leverage effect, behave in the economy with volatility disagreement. As novel predictions, we find that on average higher volatility disagreement leads to lower stock price, higher stock risk premium, higher stock return variance, and stronger leverage effect. We also find that under volatility disagreement, the leverage effect is time-varying, with its magnitude increasing in volatility, consistent with empirical evidence.

Proposition 2 reports the equilibrium stock price and its dynamics in our economy with volatility disagreement in closed form.

Proposition 2 (**Equilibrium stock price**). *In the economy with volatility disagreement, the equilibrium stock price is given by*

$$S_t = D_t e^{\mu(T-t)} \frac{1}{e^{r(T-t)}} \frac{1}{\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + (1-\mathrm{w}_t)\, e^{A_\ell(t)+B_\ell(t)V_t}}, \tag{11}$$

where the wealth-share of high-fear investors w_t *is as in Proposition 1, and the positive deterministic functions* $A_i(t)$ *and* $B_i(t)$ *for* $i = h, \ell$*, are provided in Appendix A. The equilibrium quantities of cashflow and variance risks for the stock are given by*

$$\sigma_{S1t} = \sqrt{V_t}, \tag{12}$$

$$\sigma_{S2t} = -\sigma\sqrt{V_t}B_t - \delta \mathrm{w}_t (\Lambda_t - 1)\frac{1}{\sigma}\sqrt{V_t}, \tag{13}$$

respectively, and its equilibrium risk premium by $\mu_{St} - r = m_{1t}\sigma_{S1t} + m_{2t}\sigma_{S2t}$*, where the processes* Λ_t *and* B_t *and the market prices of risks* m_{1t} *and* m_{2t} *are as in Proposition 1.*

In the benchmark economy without disagreement, the equilibrium stock price is given by $\bar{S}_t = D_t e^{\mu(T-t)} \left(1/e^{r(T-t)}\right)\left(1/e^{\bar{A}(t)+\bar{B}(t)V_t}\right)$*, where the positive deterministic functions* $\bar{A}(t)$ *and* $\bar{B}(t)$ *are provided in Appendix A. The equilibrium quantities of cashflow and variance risks for the stock are given by* $\bar{\sigma}_{S1t} = \sqrt{V_t}$*, and* $\bar{\sigma}_{S2t} = -\sigma\bar{B}(t)\sqrt{V_t}$*, respectively, and its equilibrium risk premium by* $\bar{\mu}_{St} - r = \bar{m}_{1t}\bar{\sigma}_{S1t} + \bar{m}_{2t}\bar{\sigma}_{S2t}$*, where the market prices of risks* $\bar{m}_{1t}$ *and* $\bar{m}_{2t}$ *are as in Proposition 1.*

As Proposition 2 shows, in the benchmark economy with no disagreement, the equilibrium stock price can be simply described in three terms. The first term $D_t e^{\mu(T-t)}$ is the expected stock payoff. In the second term, $e^{r(T-t)}$ captures the stock payoff's time-discount, and in the last term, $e^{\bar{A}(t)+\bar{B}(t)V_t}$ captures its *risk discount*. Thus, the fluctuations in the stock price are due to the fluctuations in the fundamental process D and its variance V. In the main economy with volatility disagreement, fluctuations in the equilibrium stock price are also driven by fluctuations in investors' wealth-share distribution w, as (11) illustrates. In particular, the risk discount term in the presence of volatility disagreement takes the form of a (wealth-share) weighted-average of each investor's *subjective risk discount* term $e^{A_i(t)+B_i(t)V_t}$.

The wealth distribution affects the stock price because investors' stock demands are functions of their wealth. As investors get relatively wealthier, their variance expectations affect the stock price more.

Proposition 2 shows that under volatility disagreement, the equilibrium quantity of cashflow risk σ_{S1t} (i.e., the exposure of the stock to cashflow shocks) is positive and equal to the market price of cashflow risk m_{1t}, as in the benchmark economy. In contrast, the equilibrium quantity of variance risk σ_{S2t} (i.e., the exposure of the stock to variance shocks) is negative and different from the market price of variance risk m_{2t}. Such deviation is driven by investors' volatility disagreement and their relative wealth, becomes larger whenever one type of investor is more dominant in the economy. Since investors' relative wealth is driven only by the variance shocks that they disagree on, wealth-share fluctuations affect only the quantity of variance risk but not the quantity of cashflow risk. A negative quantity of variance risk indicates that positive variance shocks leads to lower stock returns. This behavior is consistent with the empirically robust phenomenon called the "leverage effect", which we discuss in detail in Proposition 4.

The equilibrium stock risk premium, denoted buy $\pi_S \equiv \mu_S - r$, admits the standard representation of "market price of risk times quantity of risk." In the benchmark economy without disagreement, the stock risk premium is equal to the stock return variance, $\bar{\pi}_S = \bar{\sigma}_{S1}^2 + \bar{\sigma}_{S2}^2 = \bar{\upsilon}$. In contrast, in our main economy, the stock risk premium can deviate from the stock return variance since as discussed above, σ_{S2t} can be different than m_{2t}. Therefore, deviations are driven by the volatility disagreement and investors' wealth-share distribution in a way that when the high-fear investors are more dominant in the economy, i.e., $\mathrm{w}_t > \bar{\mathrm{w}}$, the stock risk premium is greater than the stock return variance. We defer the discussion on how volatility disagreement and fundamental variance affect the stock price and its risk premium to Corollary 2 towards the end of this section.

Proposition 3 presents the equilibrium stock return variance and its dynamics in closed form.

Proposition 3 (**Equilibrium stock return variance**). *In the economy with volatility disagreement, the equilibrium stock return variance is given by*

$$\upsilon_t = \left[\sigma^2 + \left(\sigma^2 B_t + \delta \mathrm{w}_t \left(\Lambda_t - 1\right)\right)^2\right] \frac{1}{\sigma^2} V_t, \tag{14}$$

where the wealth-share of high-fear investors w_t *and the processes* Λ_t *and* B_t *are as in Pro-*

position 1. The equilibrium quantities of cashflow and variance risks for the stock return variance are given by

$$\sigma_{\upsilon 1t} = 0, \tag{15}$$

$$\begin{aligned}\sigma_{\upsilon 2t} &= \left[\sigma^2 + \left(\sigma^2 B_t + \delta \mathrm{w}_t \left(\Lambda_t - 1\right)\right)^2\right] \frac{1}{\sigma}\sqrt{V_t} \\ &+2\left[\sigma^2\sigma_{B2t}+\delta\mathrm{w}_t\sigma_{\Lambda 2t}+\delta^2\mathrm{w}_t\left(1-\mathrm{w}_t\right)(\Lambda_t-1)\frac{1}{\sigma}\sqrt{V_t}\right]\left[\sigma^2 B_t+\delta\mathrm{w}_t\left(\Lambda_t-1\right)\right]\frac{1}{\sigma^2}V_t,\end{aligned} \tag{16}$$

respectively, and its equilibrium expected change $\mu_{\upsilon t}$, along with $\sigma_{\Lambda 2}$ and σ_{B2}, denoting the diffusion coefficients of the processes Λ_t and B_t, are provided in Appendix A.

In the benchmark economy without disagreement, the equilibrium stock return variance is given by $\bar{\upsilon}_t = \left[\sigma^2 + \left(\sigma^2 \bar{B}\left(t\right)\right)^2\right] V_t/\sigma^2$, where the positive deterministic function $\bar{B}\left(t\right)$ is provided in Appendix A. The equilibrium quantities of cashflow and variance risks for the stock are given by $\bar{\sigma}_{\upsilon 1t} = 0$, and $\bar{\sigma}_{\upsilon 2t} = \left[\sigma^2 + \left(\sigma^2 \bar{B}\left(t\right)\right)^2\right] \sqrt{V_t}/\sigma$, respectively, and its equilibrium expected change $\bar{\mu}_{\upsilon t}$ is provided in Appendix A.

Proposition 3 shows that in the benchmark economy, the equilibrium stock return variance at any given time is proportional to the fundamental variance V_t, and thus follows a square-root process with deterministic coefficients. In the square bracket, the first term is due to the fluctuations in the fundamental process D_t, and the second deterministic term due to the fluctuations in the risk discount term. With volatility disagreement, fluctuations in the stock return variance additionally come from investors' wealth fluctuations, which makes the stock return variance to be no longer proportional to the fundamental variance. As in the benchmark case, the first term in the square bracket of (14) captures the fluctuations in the fundamental process, and the second term fluctuations in the risk discount. However, fluctuations in the risk discount term are not only driven by the variance elasticity B_t, but also by an additional term capturing fluctuations in investors' wealth-shares that are induced by their disagreement.

We also see that both in the benchmark economy and in our main economy, the quantity of cashflow risk for the stock return variance $\sigma_{\upsilon 1t}$ is zero given that the fundamental process D_t and its variance V_t are uncorrelated.[15] However, the equilibrium quantity of variance risk for the return variance $\sigma_{\upsilon 2t}$, which for simplicity, we will refer to as *volatility of variance*,

[15] Our analysis shows that all our main mechanisms and results continue to hold in a more general setting in which the correlation between the fundamental process and its variance is not excessively large.

takes a more complex formulation under volatility disagreement. Relative to the benchmark economy, it is driven not only by the fundamental variance but also by wealth-share fluctuations and investors' disagreement. Our extensive numerical analysis shows that the volatility of variance $\sigma_{\upsilon 2t}$ in (16) is positive for realistic parameter values. A positive $\sigma_{\upsilon 2t}$ is also economically meaningful since it implies that stock returns become more volatile following positive variance shocks (i.e., $dV d\upsilon_t > 0$). This also guarantees that high-fear investors are always volatility buyers since their expectations of the stock return variance is always higher than that of the low-fear investors. Henceforth, our analysis concentrates on this case.

We next investigate how volatility disagreement affects the correlation between the stock returns and changes in stock return variance, $\mathrm{Corr}_t\left[d\ln S_t, d\upsilon_t\right]/dt$, which we denote by ρ_t. Following an established empirical literature, when this correlation is negative, which has been a robust feature of the data, we refer to it as the "leverage effect." This terminology stems from the fact that changes in financial leverage (i.e., a lower stock price leading to a higher debt-to-equity ratio, thus making the stock riskier and more volatile) was offered as a possible explanations for the negative correlation (e.g., Black (1976), Christie (1982)). More recent works on the leverage effect include Bollerslev, Litvinova, and Tauchen (2006), Bandi and Renò (2012), Aït-Sahalia, Fan, and Li (2013), Andersen, Bondarenko, and Gonzalez-Perez (2015), Hu, Jacobs, and Seo (2022).[16]

Proposition 4 presents the equilibrium leverage effect in our economy in closed form.

Proposition 4 (**Equilibrium leverage effect**). *In the economy with volatility disagreement, the leverage effect arises in equilibrium and is given by*

$$\rho_t = -\frac{\sigma^2 B_t + \delta \mathrm{w}_t\left(\Lambda_t - 1\right)}{\sqrt{\sigma^2 + \left(\sigma^2 B_t + \delta \mathrm{w}_t\left(\Lambda_t - 1\right)\right)^2}}, \tag{17}$$

where the wealth-share of high-fear investors w_t *and the processes* Λ_t *and* B_t *are as in Proposition 1.*

In the benchmark economy without disagreement, the equilibrium leverage effect is given by $\bar{\rho}_t = -\sigma^2 \bar{B}(t)/\sqrt{\sigma^2 + (\sigma^2 \bar{B}(t))^2}$, *where the positive deterministic function* $\bar{B}(t)$ *is provided in Appendix A.*

[16] In the literature, the negative correlation between the stock returns and changes in stock return variance is also referred to as the "volatility feedback effect" (e.g., French, Schwert, and Stambaugh (1987), Campbell and Hentschel (1992)).

Proposition 4 reveals that the leverage effect arises in equilibrium and shows its key determinants. While in the benchmark economy the leverage effect is deterministic, it fluctuates with the uncertainty in the economy in the presence of volatility disagreement.[17] These fluctuations are driven by changes in the fundamental variance and the wealth distribution in the economy, both of which affect the variance-elasticity B_t, as well as the price pressure induced by investors disagreeing. This time-varying nature of the leverage effect is consistent with the empirical evidence showing that the leverage effect is affected by market conditions (e.g., Bandi and Renò (2012), Andersen, Bondarenko, and Gonzalez-Perez (2015)). How the leverage effect is affected by fundamental variance in our model is discussed in Corollary 2 below. One of the key contributions of our analysis therefore is to complement the existing literature on the leverage effect by identifying investors' volatility disagreement as a key economic determinant.

We now discuss our model's implications for the stock price, stock risk premium and return variance, as well as the leverage effect. Corollary 2 presents the effects of volatility disagreement and fundamental variance on equilibrium stock market quantities at steady state of our economy.

Corollary 2 (Effects of volatility disagreement and fundamental variance). *At the steady state of the economy with volatility disagreement, a higher volatility disagreement δ or a higher fundamental variance V_t leads to*

i) a lower stock price S_t,

ii) a higher stock risk premium π_{St},

iii) a higher stock return variance υ_t,

iv) a stronger leverage effect ρ_t.

Corollary 2 shows that a higher volatility disagreement leads to a lower stock price, higher stock risk premium, higher stock return variance, and a stronger leverage effect at steady state of our model, as illustrated in Figure 2. These results arise due to the similar effects of volatility disagreement discussed for the market price of variance risk in Section 3. A higher volatility disagreement means that high-fear (low-fear) investors to have higher

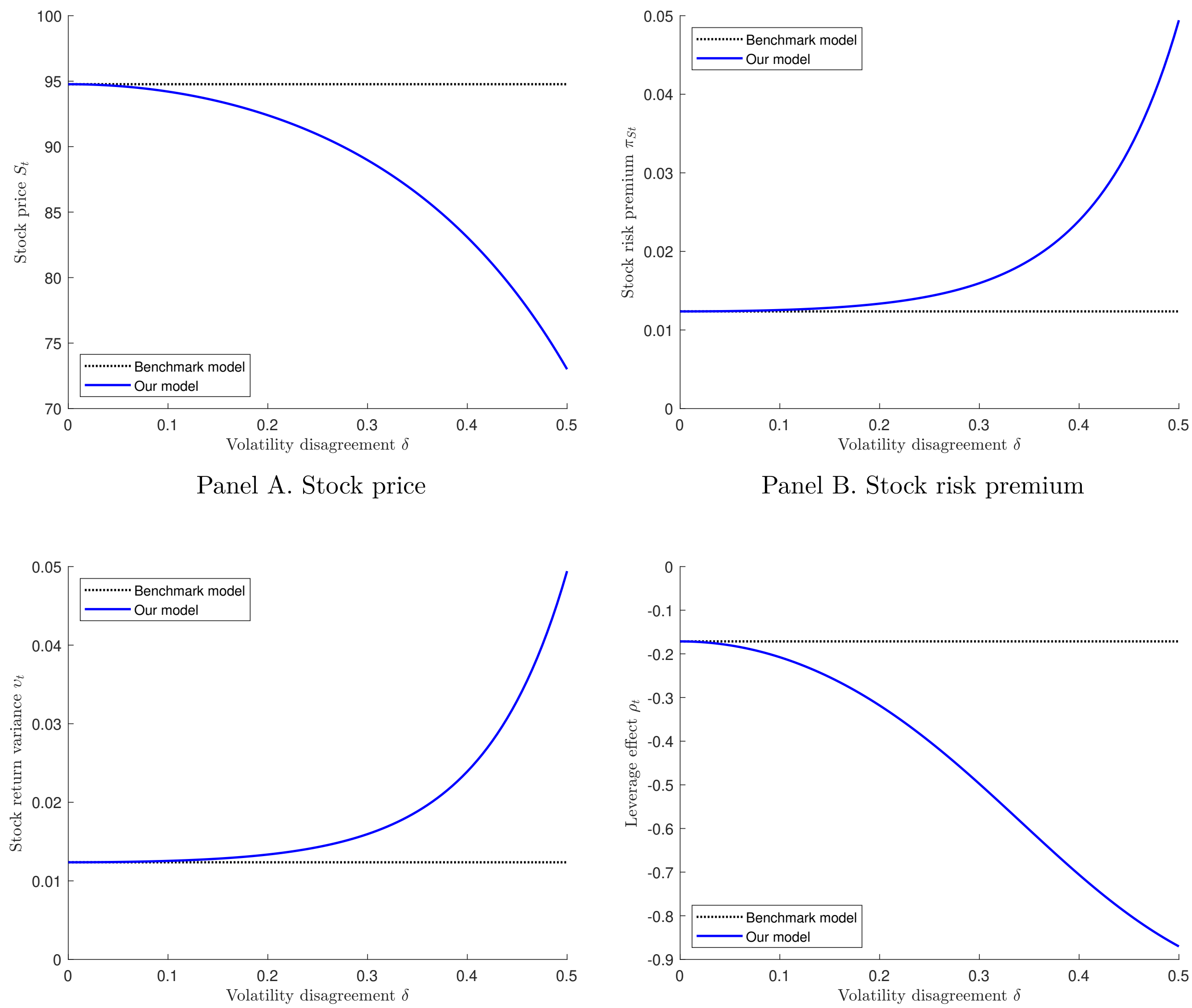
[17] Since in our framework, the stock return variance is only driven by variance shocks, i.e., $\sigma_{\upsilon 1t} = 0$ (see, Proposition 3), the equilibrium leverage effect simplifies to $\rho_t = \sigma_{S2t}/\sqrt{\upsilon_t} < 0$. Therefore, the leverage effect captures how sensitive stock returns are to variance shocks per return volatility.

(lower) and more (less) persistent variance expectations. Since high-fear investors are more sensitive to risk, the increase in their subjective risk discount terms $e^{A_h(t)+B_h(t)\bar{V}}$ is greater than the decrease in that of the low-fear investors, $e^{A_\ell(t)+B_\ell(t)\bar{V}}$, leading to an increase in the average risk discount, which in turn lowers the stock price (Property (i)). Since the volatility disagreement parameter also controls the additional risk arising from wealth distribution in the economy. Under more disagreement, the risk averse investors are exposed to more variance shocks, thus they require higher returns on average to hold the stock (Property (ii)). Moreover, since a higher volatility disagreement also amplifies the variance elasticity B_t and thus making the stock more sensitive to fundamental variance shocks, it leads to a more volatile stock returns that are more negatively correlated with variance shocks (Properties (iii) and (iv)). As discussed in Introduction, existing theoretical works do not generate our model implications regarding the effects of volatility disagreement, thus, to the best of our knowledge, our findings here are all novel.

Corollary 2 shows that a higher fundamental variance V_t also leads to a lower stock price, higher stock risk premium, higher stock return variance, and a stronger leverage effect at steady state, i.e., $\mathrm{w}_t = \bar{\mathrm{w}}$, of our model.[18] These results are illustrated in Figure 3, which additionally shows the effects in states where high-fear or low-fear investors are more dominant in the economy. These results are similar to the effects of volatility disagreement, but the underlying economic mechanisms are different.[19] The mechanism for the stock price result is standard and is also valid for the benchmark economy. Namely, a higher fundamental variance leads to a more uncertain stock payoff, and thus to a greater risk discount terms for the risk averse investors, who are now willing to hold the stock only if its price is lower (Property (i)). When the standard mechanism above coupled with the additional uncertainty due to fluctuations in the wealth distribution, which are also increasing in the fundamental variance, leads to a positive relation between V_t and the stock risk premium (Property (ii)). Figure 3, Panels A and B, also show that when the high-fear investors are more

[18]Our results for the stock price, return variance, and the leverage effect are in fact more general and hold for all wealth-share levels rather than only at steady state. The only exception is the risk premium result, which holds at steady state and other states unless the high-fear investors' wealth-share is particularly low. This occurs because as we discussed in Section 3, when high-fear investors' have sufficiently low wealth-share, the market price of variance risk is increasing in V_t, leading to a similar relation for the risk premium.

[19]For instance, a higher disagreement parameter δ implies an increase in mean-preserving spread about future volatility expectations, which, increases (decreases) the risk discount term for the high-fear (low-fear) investor. Whereas, an increase in fundamental variance V_t leads to an increase in both types of investor's risk discount term $e^{A_i(t)+B_i(t)V}$.



Panel A. Stock price

Panel B. Stock risk premium

Panel C. Stock return variance

Panel D. Leverage effect

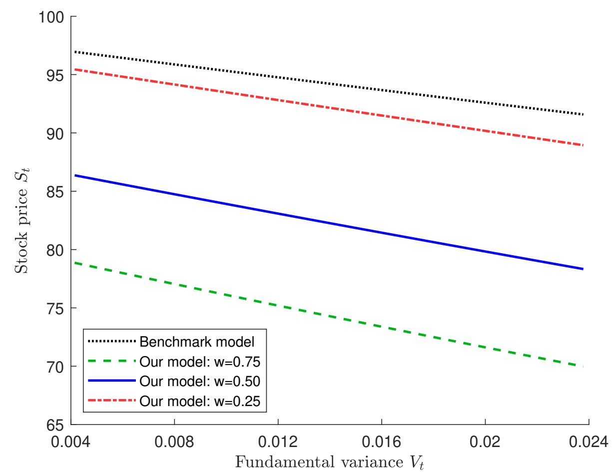
Figure 2. Effects of volatility disagreement in the stock market. These panels plot the effects of volatility disagreement δ on the equilibrium stock price S_t (Panel A), stock risk premium π_{St} (Panel B), stock return variance υ_t (Panel C), and the correlation between stock returns and variance shocks, leverage effect, ρ_t (Panel D) when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$. The dotted black lines represent the benchmark economy with no volatility disagreement. The parameter values follow from Table B1 of Appendix B.

dominant in the economy, $\mathrm{w}_t > \bar{\mathrm{w}}$, the stock price is lower and the risk premium is higher as these investors require higher premium to hold the stock compared to low-fear investors, who expect lower future volatility. The above mechanisms also leads to a positive relation between V_t and the stock return variance (Property (iii)). We also obtain a novel implication of a negative relation between V_t and the leverage effect (Property (iv)). As discussed in Section 3, higher V_t increases both the variance-elasticity B_t and the price pressure induced by investors trading against each other. As Figure 3, Panels C and D, illustrate, these effects are highest at steady state as that is when the variance shocks lead to most wealth transfers, rather than at states in which one type of investor are more dominant. Indeed, when the wealth distribution is extremely skewed towards one investor type, the diffusion term in 9 becomes very small.

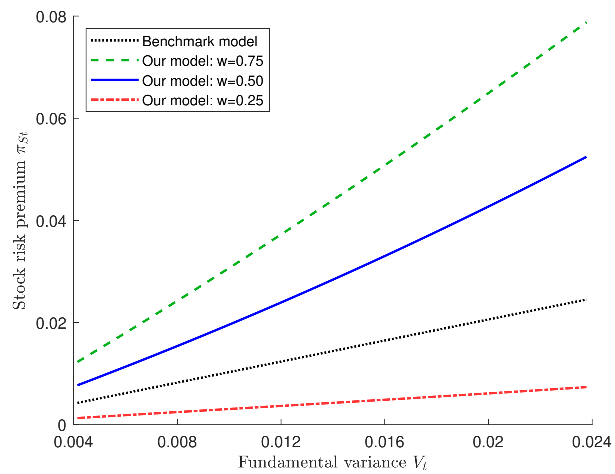
The key implication here is on the leverage effect. As Figure 3, Panel D, illustrates, in the benchmark economy, the leverage effect is relatively small and does not vary with volatility. We see that the presence of volatility disagreement significantly increases the leverage effect compared to the benchmark economy. The magnitude of the leverage effect for our baseline calibration is consistent with the estimates of the leverage effect in the literature, which are typically found to be within the range of -0.50 to -0.90 (e.g., Andersen, Benzoni, and Lund (2002), Aït-Sahalia, Fan, and Li (2013), Andersen, Bondarenko, and Gonzalez-Perez (2015)). Notable, the leverage effect becomes stronger during high volatility periods, consistent with the empirical evidence in Bandi and Renò (2012) and Andersen, Bondarenko, and Gonzalez-Perez (2015).

When we plot the effects of fundamental variance V_t in Figure 3, we kept the other state variable, the wealth-share w_t, fixed. However, as the dynamics of these state variables (2) and (9) show, these state variables are positively correlated, $dV_t d\mathrm{w}_t > 0$. Thus, following a positive variance shock, $d\omega_{2t}$, not only the fundamental variance increases but also the high-fear investors' wealth-share. To better understand the effects of such *pure variance shocks*, in Figure 4 we also plot the equilibrium stock market quantities when the wealth-share varies with the fundamental variance.[20]

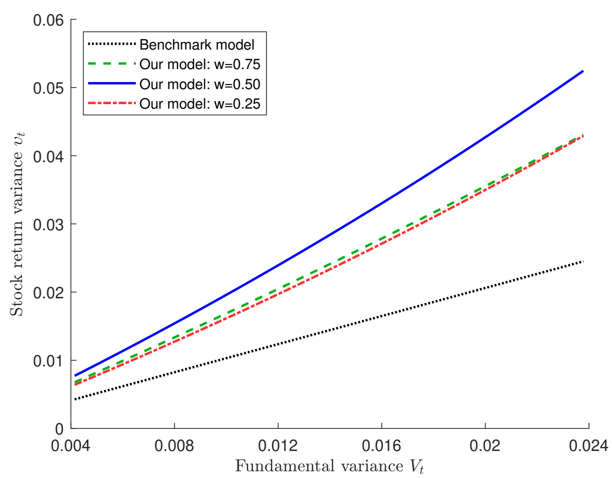
[20] For our illustrations, we measure the effects of pure variance shocks in a straightforward manner. Starting from steady state, $V_t = \overline{V}$ and $\mathrm{w}_t = 1/2$, and using the dynamics (2) and (9), we obtain the changes in the state variables as $dV_t = \sigma\sqrt{\overline{V}} d\omega_{2t}$ and $d\mathrm{w}_t = (1/4)\,\delta\,(1/\sigma)\,\sqrt{\overline{V}} d\omega_{2t}$. These dynamics imply that the relation between these two quantities is $\mathrm{w}_t = (1/2) + (1/4)\,\delta\,\left(1/\sigma^2\right)\left(V_t - \overline{V}\right)$, after integrating and lagging by dt. We note that this simple relation captures the effects of pure variance shocks that whenever the fundamental variance is above (below) its long-run mean, so does the wealth share. By substituting this relation into the economic quantities, and plotting them by varying V_t, we obtain the effects of pure variance shocks that we illustrate in Figure 4.



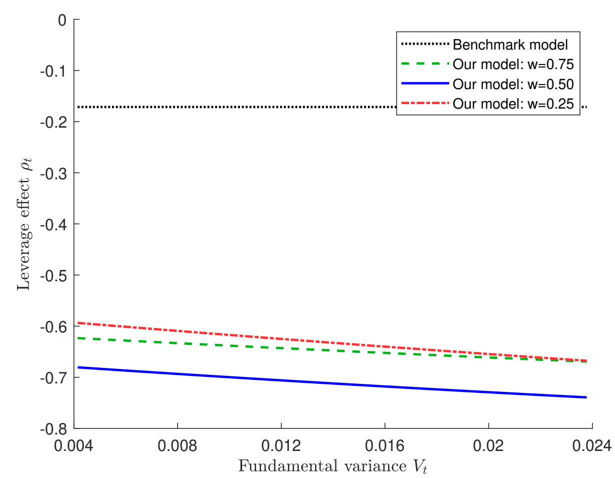
Panel A. Stock price



Panel B. Stock risk premium



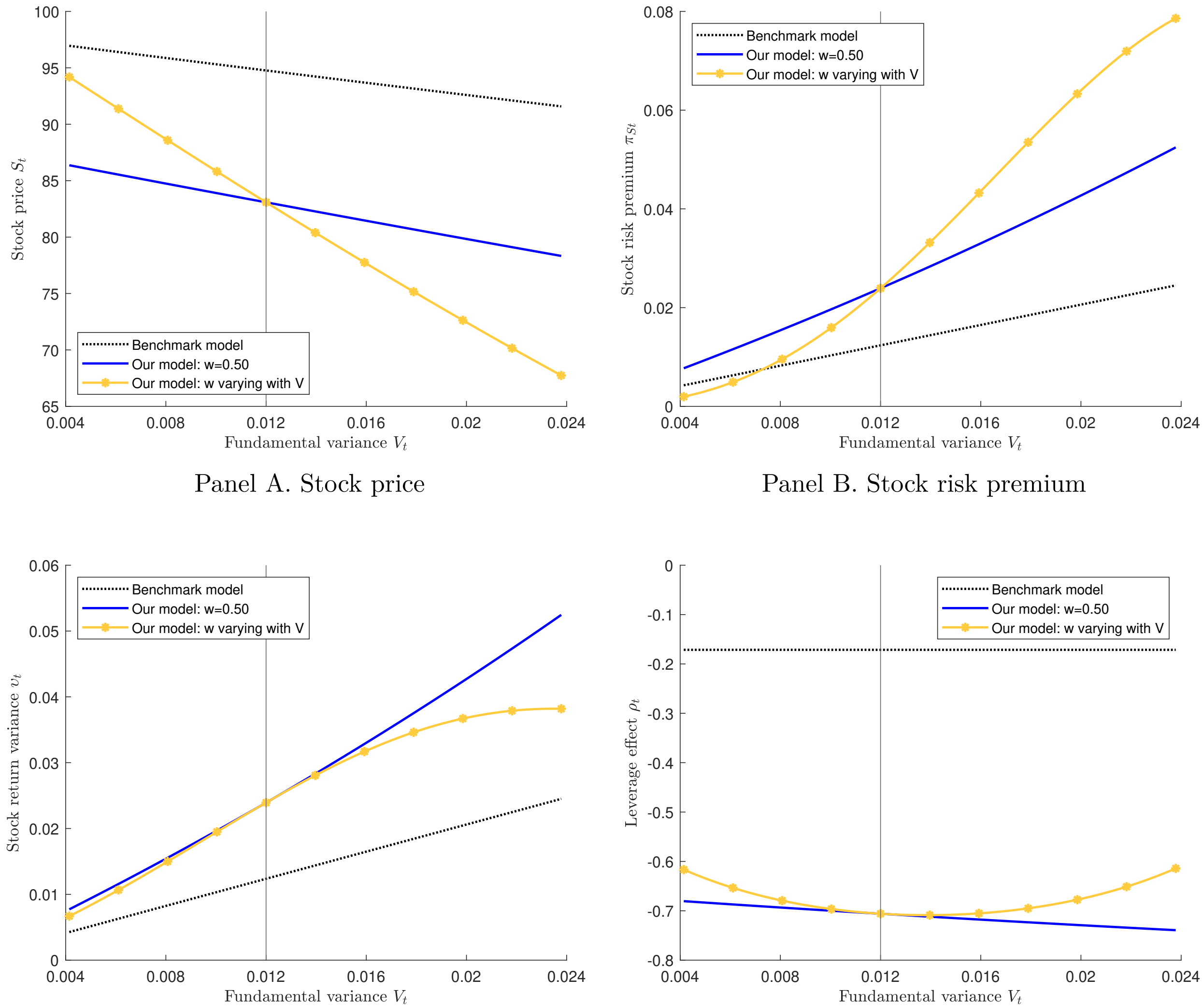
Panel C. Stock return variance



Panel D. Leverage effect

Figure 3. Effects of fundamental variance in the stock market. These panels plot the effects of fundamental variance V_t on the equilibrium stock price S_t (Panel A), stock risk premium π_{St} (Panel B), stock return variance υ_t (Panel C), and the correlation between stock returns and variance shocks, leverage effect, ρ_t (Panel D) for different levels of w_t. The dotted black lines represent the benchmark economy with no volatility disagreement. The parameter values follow from Table B1 of Appendix B.

Figure 4, Panels A and B, illustrate that the stock price and the stock risk premium are more sensitive to the pure variance shocks than to the changes in the fundamental variance.



Panel A. Stock price

Panel B. Stock risk premium

Panel C. Stock return variance

Panel D. Leverage effect

Figure 4. Effects of pure variance shocks in the stock market. These panels plot the effects of pure variance shocks $d\omega_{2t}$ on the equilibrium stock price S_t (Panel A), stock risk premium π_{St} (Panel B), stock return variance υ_t (Panel C), and the correlation between stock returns and variance shocks, leverage effect, ρ_t (Panel D) when V_t and w_t are both varying. The vertical lines represents the long-run mean of the fundamental variance. The solid blue lines represent the effects of fundamental variance when $\mathrm{w}_t = 0.5$. The dotted black lines represent the benchmark economy with no volatility disagreement. The parameter values follow from Table B1 of Appendix B.

By increasing the wealth share of the high-fear investors w_t and the fundamental variance V_t simultaneously, a positive pure variance shock leads to an economy in which high-fear investors are relatively more dominant in more volatile times. Since these investors are more sensitive to risk, they are willing to hold stock only if its price is lower and the risk premium is higher, leading to these amplified effects. On the other hand, Panels C and D show that starting at steady state, moderate pure variance shocks lead to mostly similar behaviors for the stock volatility and the leverage effect as depicted in our earlier Figure 3. However, we also see that a sufficiently large pure variance shock can lead to non-linear relations. These results arise because now a positive (negative) shock is associated with an increase (decrease) in the relative dominance of the high-fear investors in the economy, resulting in a reduction in the "effective disagreement" in the economy, since there is less room for wealth transfers in the future. This reduced wealth transfer risk leads to less volatile returns and less negative correlation between the stock returns and the variance shocks.

5 Volatility Derivatives Market

In this section, we study the equilibrium implications for the volatility derivatives market. We first demonstrate that the relation between the volatility disagreement and the variance swap long holdings is non-monotonic, it is first increasing then decreasing (hump-shaped). Somewhat surprising, but consistent with empirical evidence, we find that investors trade less volatility derivatives in more volatile periods. We then show that higher volatility disagreement gives rise to higher magnitude for the variance risk premium and variance swap rate on average. We further find that variance risk premium is negative on average, but it turns positive and increases in volatility, as occasionally observed in reality, when low-fear investors are more dominant in the economy.

We begin our analysis of the volatility derivatives market by first reporting investors' equilibrium variance swap holdings in closed form in Proposition 5.

Proposition 5 (**Equilibrium variance swap holdings**). *In the economy with volatility disagreement, the equilibrium holdings in the variance swap contracts are given by*

$$\theta_{ht} = \delta\left(1 - \mathrm{w}_t\right) \frac{1}{\sigma} \sqrt{V_t} \frac{1}{\sigma_{v2t}} W_{ht}, \qquad \theta_{\ell t} = -\theta_{ht}, \tag{18}$$

where the wealth-share of high-fear investors w_t *is as in Proposition 1, and the volatility of variance* σ_{v2t} *is as in Proposition 3.*

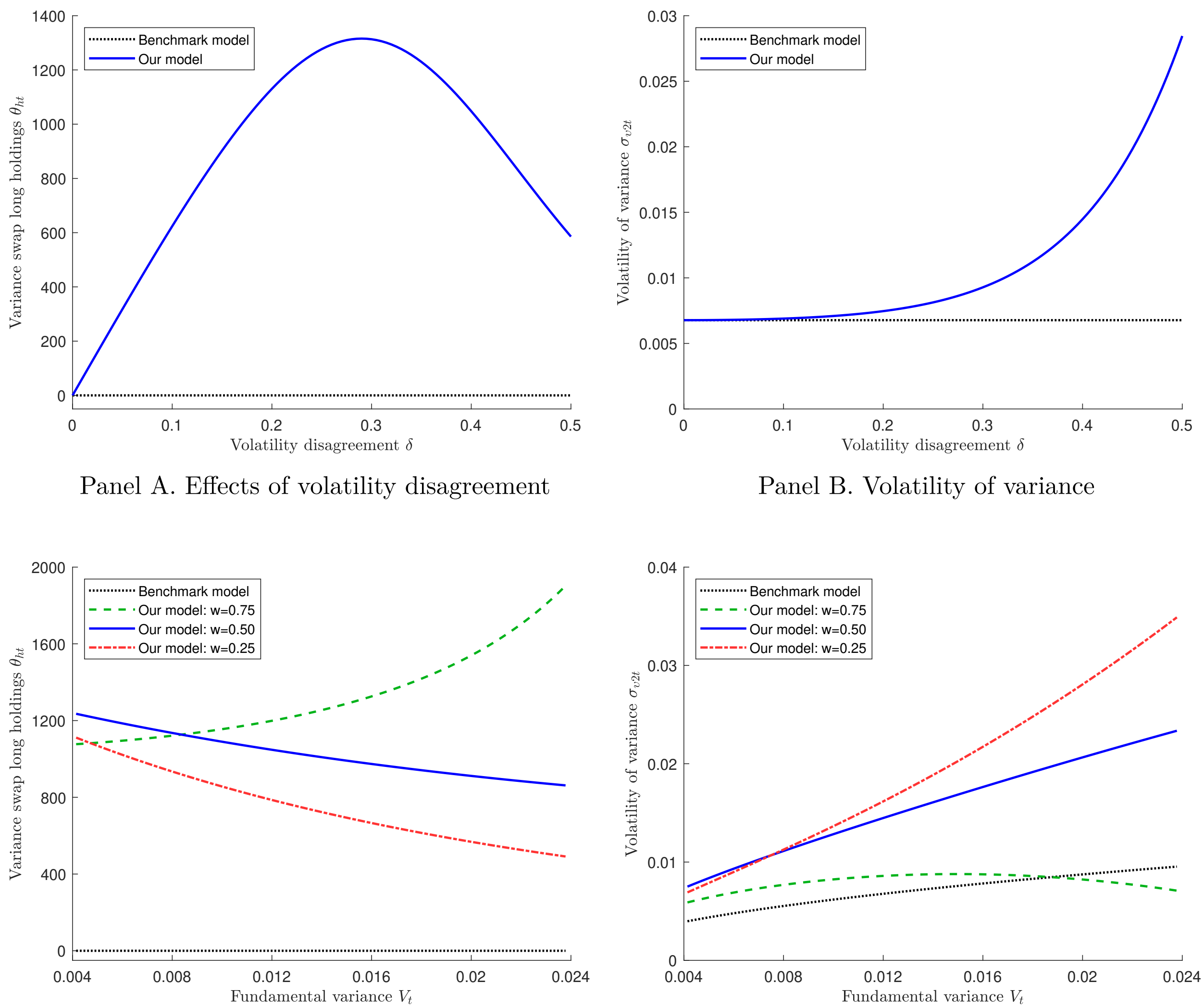
In the benchmark economy without disagreement, the equilibrium holdings in the variance swap contracts are equal to zero.

In the benchmark economy, investors do not hold any variance swap even though they face variance risk. This is because volatility derivatives are in zero-net supply and investor are homogeneous. However, as Proposition 5 demonstrates, under volatility disagreement, investors hedge and speculate on variance risk by taking opposite positions in the variance swaps. In equilibrium, due to their relatively higher future volatility expectations, high-fear investors become volatility buyers by always holding long positions in the volatility derivative, $\theta_{ht} > 0$, while low-fear investors become volatility sellers, $\theta_{\ell t} < 0$.[21] This also implies that in our model, the total number of long positions in the variance swap contract, the open interest, coincides with the high-fear investors' holdings θ_{ht}, which also captures the trading activity in these derivatives. Moreover, due to their logarithmic preferences, investors' variance swap positions in equilibrium has the usual (multivariate) mean-variance efficient portfolio representation adjusted for their subjective expectations.[22] Therefore, not only the variance swap's subjective risk premia, but also its riskiness, which in equilibrium is given by the volatility of variance σ_{v2t}, plays a crucial role in determining investors' variance swap holdings. To better understand the behavior of variance swap holdings under volatility disagreement, in Figure 5, we illustrate how the volatility disagreement and fundamental variance affects it.[23]

[21] Investors being either volatility buyers or sellers all the time in our model is consistent with the evidence in Cheng (2019), who finds a systematic patterns for what type of investors are long or short in the volatility derivative (VIX futures) market. Particularly, Cheng (2019) documents that in recent times, systematically, dealers and asset managers have long positions, while hedge funds have short positions in the volatility derivative market. Thus, given this evidence, one could interpret high-fear investors in our model as mutual fund managers that are worried about the volatility risk inherit in their long stock positions, and low-fear investors as hedge fund managers, who short volatility derivatives to profit from variance risk premium being negative on average.

[22] The mean-variance representation can be seen from the i-type, $i = h, \ell$, investors' portfolio holdings, which can be expressed using the matrix notation as $\boldsymbol{\theta}_{it} = \left(\boldsymbol{\Sigma}^T\right)^{-1} \boldsymbol{m}_{it} = \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1} \boldsymbol{\pi}_{it}$, where $\boldsymbol{\theta}_{it}$ is the i-type investors' portfolio vector in terms of the fraction of her wealth invested in the stock and the number of volatility derivative contracts per wealth, $\boldsymbol{m}_{it}$ and $\boldsymbol{\pi}_{it}$ are vectors of market prices of risks and risk premium, respectively, under her subjective measure, and $\boldsymbol{\Sigma}$ is the volatility matrix of the securities.

[23] Given the complex form of volatility of variance σ_{v2t} in (16), we are unable to obtain analytic comparative statics for volatility derivative quantities that depend on σ_{v2t}. That said, our extensive numerical analysis



Panel A. Effects of volatility disagreement

Panel B. Volatility of variance

Panel C. Effects of fundamental variance

Panel D. Volatility of variance

Figure 5. Variance swap long holdings. These panels plot the equilibrium long holdings in the variance swap contracts θ_{ht} against volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$ (Panel A) and against fundamental variance V_t for different levels of w_t (Panel C). Panels B and D plot the equilibrium volatility of variance σ_{v2t} against volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$ and against fundamental variance V_t for different levels of w_t, respectively. The dotted black lines represent the benchmark economy with no volatility disagreement. The parameter values follow from Table B1 of Appendix B.

shows that the relations depicted by our figures are quite typical and do not vary with alternative plausible parameter values.

Figure 5, Panel A, illustrates a notable non-monotonic relation between the volatility disagreement and the variance swap long holdings; it is first increasing then decreasing. This hump-shaped relation arises because of the two opposing effects of disagreement that can be described in terms of the mean-variance portfolio structure. First, a higher disagreement (linearly) increases the high-fear investors' perceived risk premium in investing in swap contracts by increasing her variance expectation, leading to more swap holdings in equilibrium, ceteris paribus. Second, Panel B of Figure 5 illustrates that a higher disagreement (convexly) increases the riskiness of the swap contract, σ_{v2t}, leading to less swap holdings for risk-averse investors in equilibrium, ceteris paribus. This effect arises because higher disagreement makes the stock more sensitive to fundamental variance shocks, resulting in amplified stock return variance itself as discussed in Section 4. Thus, for low levels of disagreement, the effect of higher perceived risk premium dominates, leading to a positive relation between volatility disagreement and the variance swap holdings. However, for relatively high disagreement levels, the volatility derivative becomes too risky for investors (the effect of the volatility of variance dominates), who now reduce their holdings, leading to a negative relation.[24]

As Figure 5, Panel C, depicts, we find another notable, a negative relation between the variance swap long holdings and the fundamental variance V_t (and also the stock return variance v_t) on average in equilibrium. This negative relation at first seems puzzling as it implies that investors on average hold and trade less volatility derivatives in high volatility periods, during which arguably there is more volatility risk to be hedged. In our model, this negative relation arises primarily because investing in the volatility derivative becomes too risky during high volatility times, since the volatility of the swap contract, σ_{v2t}, heightens in such times, as Figure 5, Panel D, shows.[25] In fact, our analysis confirms that if we shut the volatility of variance channel, we obtain the opposite, a positive relation, which indicates that investors would have traded more volatility derivatives in more volatile times if their riskiness were to remain the same. Our result here is consistent with the empirical evidence in Cheng (2019), who finds increases in market volatility leading to reductions in volatility

[24]We also note that the hump-shape relation is not due to the investors wealth levels in (18). Investors' wealth only have an amplifying role in this relation. As investors get wealthier, they invest more in the stock, which increases their exposure to the variance risk. Therefore, they increase their variance swap holdings without affecting the hump-shaped relation. For this reason, we obtain a similar hump-shaped relation when considering the fraction of wealth invested in the variance swap contracts, θ_{ht}/W_{ht}.

[25]Figure 5, Panel D, also shows that when high-fear investors are more dominant (e.g., $\mathrm{w}_t = 0.75$), the volatility of variance σ_{v2t} can be non-monotonic, which results in a negative relation between the variance swap long holdings and the fundamental variance on these non-steady states, as illustrated in Panel C.

derivatives (VIX futures) positions by long and short holders.[26] To the best of our knowledge, ours is the first work to reconcile this otherwise puzzling evidence.

We next investigate how the risk premium that investors are willing to accept for bearing the variance risk and the price they pay for a claim that hedges the variance risk behave in our economy with volatility disagreement. To that end, Proposition 6 presents the equilibrium variance risk premium and variance swap rate in closed form.

Proposition 6 (Equilibrium variance risk premium and swap rate). *In the economy with volatility disagreement, the equilibrium variance risk premium is given by*

$$\pi_{\upsilon t} = -\left[\sigma^2 B_t + \delta\left(\mathrm{w}_t \Lambda_t - \bar{\mathrm{w}}\right)\right] \frac{\sigma_{\upsilon 2t}}{\sigma} \sqrt{V_t}, \tag{19}$$

and the equilibrium variance swap rate by

$$y_t = \upsilon_t + \mu_{\upsilon t} - \pi_{\upsilon t}, \tag{20}$$

where the wealth-share of high-fear investors w_t and the processes Λ_t and B_t are as in Proposition 1, the stock return variance υ_t and the volatility of variance $\sigma_{\upsilon 2t}$ are as in Proposition 3, and the expected change of the stock return variance $\mu_{\upsilon t}$ is provided in Appendix A.

In the benchmark economy without disagreement, the equilibrium variance risk premium is given by $\bar{\pi}_{\upsilon t} = -\sigma \bar{B}(t)\, \bar{\sigma}_{\upsilon 2t} \sqrt{V_t}$, and the equilibrium variance swap rate by $\bar{y}_t = \bar{\upsilon}_t + \bar{\mu}_{\upsilon t} - \bar{\pi}_{\upsilon t}$, where the stock return variance $\bar{\upsilon}_t$ and the volatility of variance $\bar{\sigma}_{\upsilon 2t}$ are as in Proposition 3, and the expected change of the stock return variance $\bar{\mu}_{\upsilon t}$ and the positive deterministic function $\bar{B}(t)$ are provided in Appendix A.

As for the stock risk premium, the equilibrium variance risk premium (19) admits the standard representation "price of risk times quantity of risk," which arises from the covariance of the variance swap payoff with the state price density. In this case, the quantity of risk is given by the volatility of variance $\sigma_{\upsilon 2t}$. In our model, the equilibrium variance swap rate is determined through volatility buyers' demand and volatility sellers' supply in the volatility derivatives market. However, as (20) illustrates, it still satisfies the standard no-arbitrage

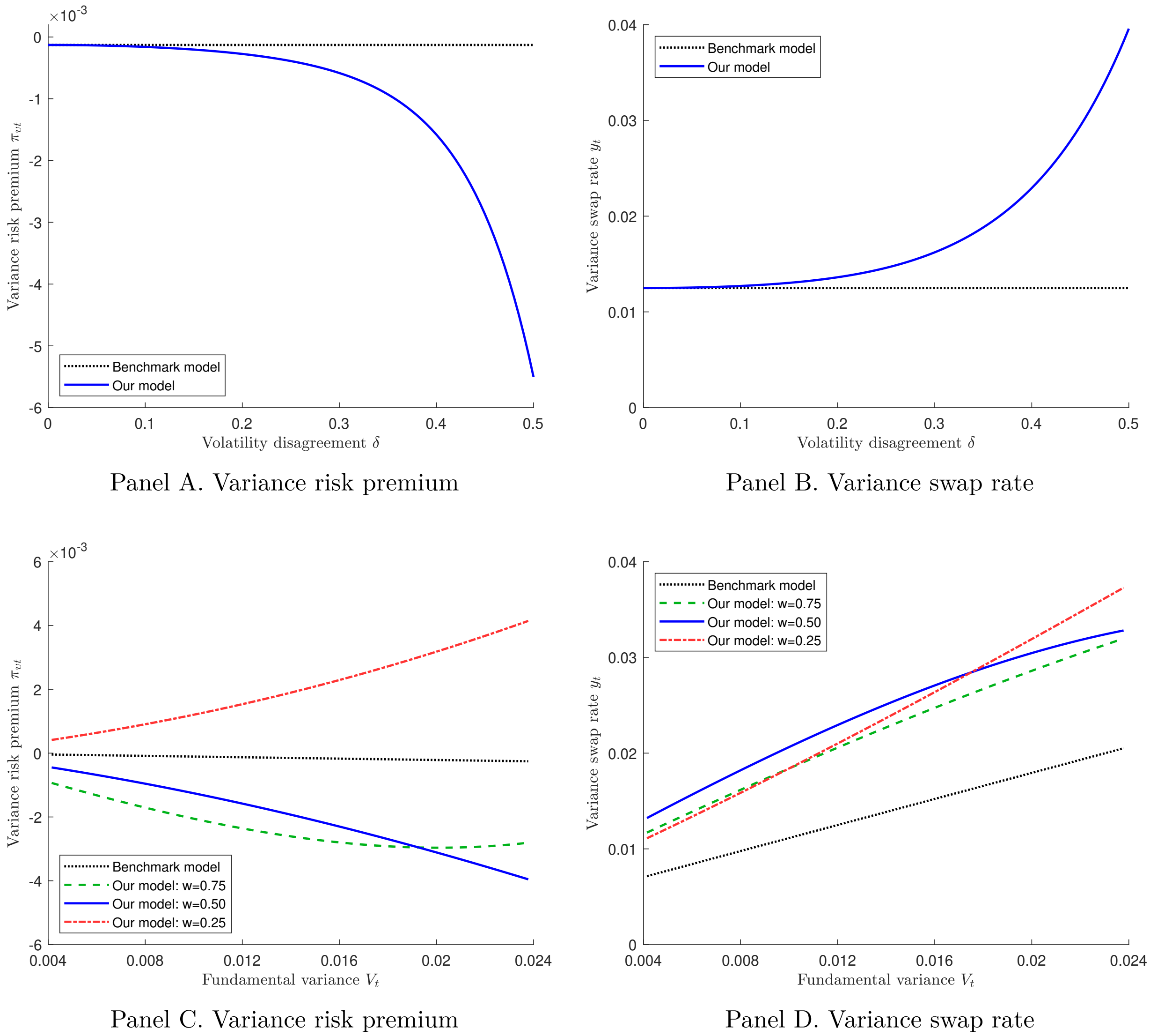
[26] We note that in our model, low-fear investors are long in the stock but short in the variance swap. Thus, a positive variance shock leads to a loss in both security positions for the low-fear types. Whereas, the same shock leads to a loss from the stock position but a gain from variance swap position for the high-fear type, with the overall effect being a gain.

relation that equates it to the risk-neutral expectation of the future stock return variance.[27] In the benchmark economy, the variance risk premium becomes $\bar{\pi}_{vt} = -\sigma^2 \bar{B}(t)\,\bar{v}_t$, which is a negative linear function of the return variance with deterministic coefficients. Under volatility disagreement, these quantities have much richer behavior since they are additionally affected by the wealth-share distribution and the magnitude of the volatility disagreement, over and above the effects of the stock return variance. Again, to better understand the behavior of these quantities under volatility disagreement, we illustrate in Figure 6 how the volatility disagreement and fundamental variance affect them.

Figure 6, Panel A, shows that the variance risk premium is negative and its magnitude increases in volatility disagreement at steady state of our model. This result arises because a higher volatility disagreement increases the magnitude of both the market price of variance risk m_{2t} (which is negative at steady state) and the quantity of variance risk σ_{v2t}. The former effect is discussed in Section 3 in detail and occurs because of a higher volatility disagreement making investors' marginal utilities to be more sensitive to variance shocks on average, i.e., a higher B_t, as well as amplifying the additional risk arising from wealth distribution in the economy. Due to these two channels, a higher volatility disagreement also increases the quantity of risk by making the stock return variance more volatile (see, Figure 5, Panel B). Figure 6, Panel B, shows that the variance swap rate is positively related to volatility disagreement. This result is due to the fact that a higher volatility disagreement leads to an increase not only in current return variance v_t but also in expected future variance $v_t + \mu_{vt}$. Therefore, the trading in the variance swap contract occurs and the derivative market clears only if the variance swap rate becomes higher.

Figure 6, Panel C, highlights that under volatility disagreement, the equilibrium variance risk premium is negative and becomes more negative in more volatile times at steady state of our model. However, we also see that when high-fear investors' have sufficiently low wealth-share, the variance risk premium turns positive and increases in volatility, as occasionally observed in reality. These relations occur because the market price of variance risk m_{2t} is negative on average but turns to positive when the market is dominated by low-fear investors, who believe the return volatility will be much lower in the near future, as discussed in Section

[27] To see this, note that the sum of the first two terms in (20), $v_t + \mu_{vt}$, is the true expectation of the future conditional return variance. Subtracting the variance risk premium from that sum leads to the risk-neutral expectation. We also highlight that since volatility is not a traded asset $\mu_{vt} - \pi_{vt}$ is not equal to the riskless rate r.



Panel A. Variance risk premium

Panel B. Variance swap rate

Panel C. Variance risk premium

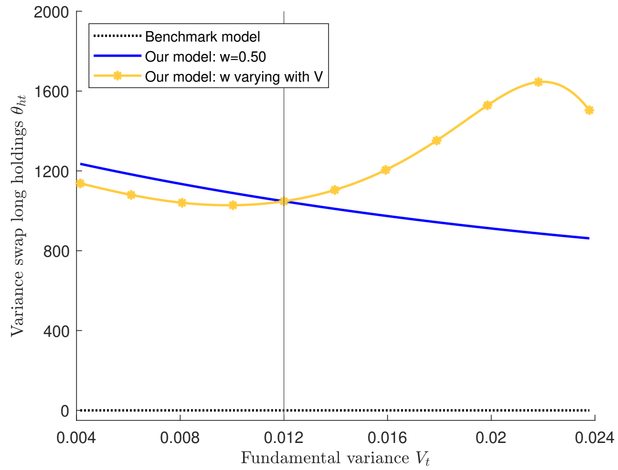
Panel D. Variance swap rate

Figure 6. Variance risk premium and variance swap rate. These panels plot the equilibrium variance risk premium π_{vt} against volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$ (Panel A) and against fundamental variance V_t for different levels of w_t (Panel C). Panel B and D plot the equilibrium variance swap rate y_t against volatility disagreement δ when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$ and against fundamental variance V_t for different levels of w_t, respectively. The dotted black lines represent the benchmark economy with no volatility disagreement. The parameter values follow from Table B1 of Appendix B.

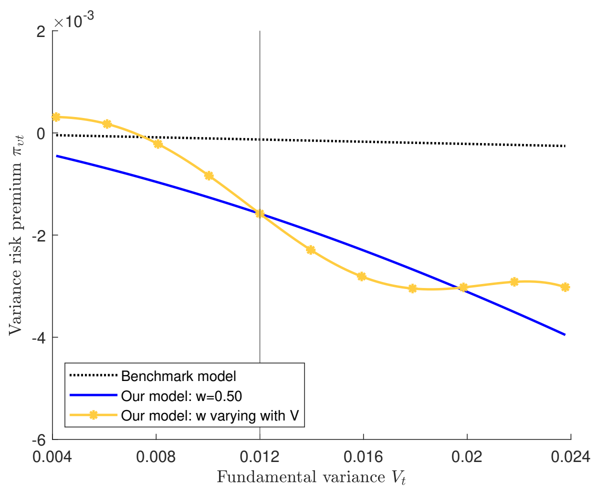
3. We also see in Figure 6, Panel D, that the variance swap rate is increasing in volatility even when low-fear investors are more dominant in the economy. The last observation may sound counter intuitive at first given the behavior of the variance risk premium. The reason we still obtain a positive relation between y_t and V_t in such times is that those times are precisely when the expected future variance $v_t + \mu_{vt}$ is higher, offsetting the effects of the variance risk premium.

The variance risk premium being negative on average is well-documented in the empirical literature (e.g., Bakshi and Kapadia (2003), Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009)). For our model, this finding also implies that long volatility derivative positions lose money on average, which is supported by the direct evidence in Eraker and Wu (2017), who find that 1-month VIX futures lose about 30% per year on average. Our finding of the magnitude of the variance risk premium increasing in volatility on average is also consistent with its behavior documented in Barras and Malkhozov (2016). More notably, there is a puzzling behavior in the data that during market turmoils, due to sharp increases in realized variance, variance risk premium switches sign and increases in volatility (Cheng (2019, 2020)). Our model offers one plausible explanation for this otherwise puzzling behavior by demonstrating that such a pattern arises when the market is dominated by investors who believe the currently high volatility will quickly revert back to its lower level in the near future. We elaborate more on this evidence in Section 6.2, which also illustrates the puzzling evidence in Figure 9.

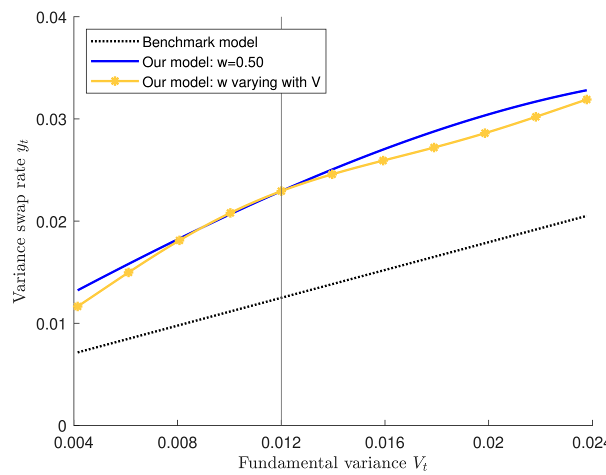
Finally, similar to our analysis in Section 4, we now look at the effects of pure variance shocks for the volatility derivatives market quantities. To that end, Figure 7 plots the behavior of these quantities when they experience a such pure variance shock by varying the wealth-share w_t along with the fundamental variance V_t. We see that, pure variance shocks lead to non-linear but mostly similar economic behaviors for the variance risk premium and variance swap rate as depicted in our earlier Figure 6. That said, we also see that positive pure variance shocks can lead to an increase in the variance swap long holdings. This result occurs because such a positive shock is associated with an increase in the wealth share of the high-fear investors. This accompanying wealth transfer makes them relatively more dominant and reduces the "effective disagreement" in the economy, since there is less room for such transfers in the future. This diminished wealth transfer effect reduces the riskiness of the swap contract, σ_{v2t}, leading to more swap holdings for risk-averse investors in equilibrium.



Panel A. Variance swap long holdings



Panel B. Variance risk premium



Panel C. Variance swap rate

Figure 7. Effects of pure variance shocks in the volatility derivatives market. These panels plot the effects of pure variance shocks $d\omega_{2t}$ on the equilibrium long holdings in the variance swap contracts θ_{ht} (Panel A), variance risk premium π_{vt} (Panel B), variance swap rate y_t (Panel C) when V_t and w_t are both varying. The vertical lines represents the long-run mean of the fundamental variance. The solid blue lines represent the effects of fundamental variance when $\mathrm{w}_t = 0.5$. The dotted black lines represent the benchmark economy with no volatility disagreement. The parameter values follow from Table B1 of Appendix B.

6 Aggregate Volatility Bias

Thus far we have studied an economy in which investors' subjective variance expectations are symmetric around the true one. Our baseline economy was sufficient to demonstrate our key insights on the effects of volatility disagreement. In this section, we extend our model to study the implications of an aggregate volatility bias by considering *asymmetric* subjective variance expectations around the true one. A novel finding is that a higher aggregate volatility bias leads to lower equilibrium trading in volatility derivatives. Moreover, we show that the presence of a downward aggregate volatility bias can help explain the observed pattern of the variance risk premium becoming positive while investors trading less volatility derivatives during high volatility periods.

6.1 Equilibrium with Volatility Disagreement and Aggregate Bias

We incorporate an aggregate volatility bias into our framework in a straightforward manner. That is, we only modify investors' volatility expectations in Section 2.2, while keeping all the other features of our baseline model the same. Now, high-fear investors misperceive the expected change in the fundamental variance as

$$\mathrm{E}_t^h\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] + \left(\beta + \tfrac{1}{2}\delta\right)V_t dt,$$

and the ℓ-type investors misperceive it as

$$\mathrm{E}_t^\ell\left[dV_t\right] = \mathrm{E}_t\left[dV_t\right] + \left(\beta - \tfrac{1}{2}\delta\right)V_t dt.$$

Under this specification, the constant β controls the *aggregate volatility bias* in the economy. When the bias is absent ($\beta = 0$) we revert to our baseline economy characterized by only volatility disagreement. To ensure that high-fear (low-fear) investors' volatility expectations are greater (less) than the true one as in our main economy, we impose a parameter restriction of $-\delta/2 \leq \beta \leq \delta/2$.[28] With this generalization, in addition to the volatility disagreement, our model can also address the documented average biases in variance expectations in surveys

[28]Moreover, the earlier parameter restriction that ensures the equilibrium stock price admits a real solution now becomes $\kappa - (\beta + \delta/2) > \sqrt{2}\sigma$.

(e.g., Graham and Harvey (2001), Ben-David, Graham, and Harvey (2013), Amromin and Sharpe (2014)) (see also our discussion in Remark **??** of Section 2).

We proceed by first determining the equilibrium in our extended economy with volatility disagreement and bias, and present our findings in Proposition 7.

Proposition 7 (**Equilibrium with aggregate volatility bias**). *The equilibrium in the economy with volatility disagreement and aggregate bias is characterized as in our baseline economy where the wealth-share's long-run mean* $\bar{\mathrm{w}} = 1/2$ *is replaced by* $\bar{\mathrm{w}} = 1/2 - \beta/\delta$*.*

Proposition 7 confirms that in the economy with volatility disagreement and bias, the economic quantities have similar structures to those in our baseline economy with only disagreement. The only difference is that the long-run mean of the high-fear investors' wealth-share now becomes $\bar{\mathrm{w}} = 1/2 - \beta/\delta$. Therefore, in the polar case of $\beta = \delta/2$ ($\beta = -\delta/2$), low-fear (high-fear) investors eventually dominate the economy (i.e., their wealth-shares converges to unity in the long run), since they have correct beliefs. This is consistent with the survival effects demonstrated in the literature (e.g., Kogan et al. (2006), Yan (2008), Cvitanić and Malamud (2011)). In all other relevant cases, $-\delta/2 < \beta < \delta/2$, both types of investors have incorrect volatility expectations, hence no investor type completely dominates the economy in the long run. Therefore, as in our baseline economy, both types of investors have a price impact, even in the long-run. We also see that the aggregate volatility bias is negatively related to the high-fear investors' long-run dominance. This observation implies that, under a downward volatility bias, $\beta < 0$, their wealth will increase relative to that of low-fear investors' in the long-run. This finding is intuitive since under such downward bias, high-fear investors' beliefs are relatively more accurate. Thus, they accumulate wealth on average over time through their investments in the stock and volatility derivatives.

We now investigate how the aggregate volatility bias affects equilibrium market price of variance risk and stock market quantities, and present our findings in Corollary 3.

Corollary 3 (**Effects of aggregate volatility bias**). *In the economy with volatility disagreement and aggregate bias, a higher bias* β *leads to*

i) a lower market price of variance risk m_{2t}*,*

ii) a lower stock price S_t*,*

iii) a higher stock risk premium π_{St}*,* *iv) a higher stock return variance v_t,*

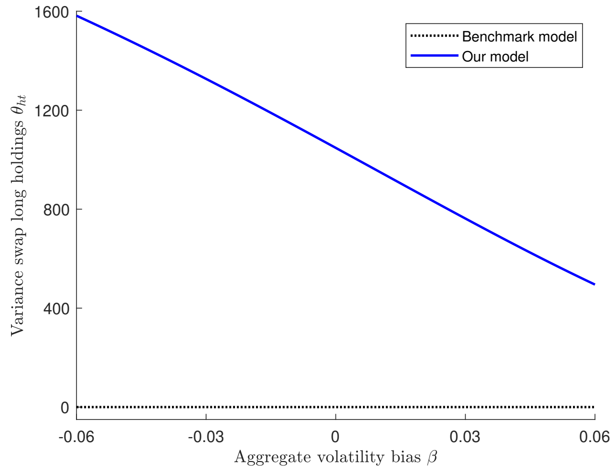
v) a stronger leverage effect ρ_t.

Corollary 3 reveals that, similar to the effects of volatility disagreement, a higher aggregate volatility bias leads to a lower market price of variance risk, lower stock price, higher stock risk premium, higher stock return variance, and a stronger leverage effect. However, the underlying mechanisms of volatility disagreement and bias are notably different. For instance, in the case of a higher volatility disagreement, which implies a higher mean-preserving spread for expectations, high-fear investors' variance elasticity increases but that of the low-fear investors' decreases (see, Section 3). In the case of a higher volatility bias, instead, all investors perceive the variance to be relatively higher and more persistent, resulting in an increase in both investor-specific variance elasticities. This increased sensitivity to variance shocks in investors' marginal utilities translates into a higher demand for volatility exposure. However, for the volatility derivatives market to clear, the market price of variance risk needs to go down in equilibrium, leading to more forgone returns on assets that are positively exposed to variance shocks (Property (i)). Moreover, the higher variance elasticities along with the higher volatility expectations on average reduces the willingness of investors to hold the stock, resulting in lower stock price and higher stock returns on average (Properties (ii) and (iii)). Furthermore, since the stock price is more sensitive to variance shocks, we also find more volatile stock returns and stronger leverage effect under higher aggregate volatility bias (Properties (iv) and (v)).[29]

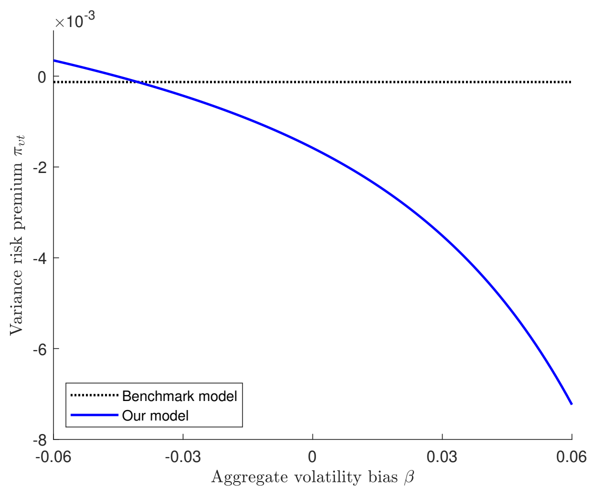
Next, we look at the effects of the aggregate volatility bias for the volatility derivatives market quantities, and present our findings in Figure 8.

Figure 8, Panel A, shows our novel and somewhat surprising result that a higher aggregate volatility bias leading to a lower volatility derivative holdings (and trade) in equilibrium. This monotonic behavior arises because as we discussed above, under higher volatility bias, the stock price becomes more sensitive to variance shocks, which not only increases its volatility but also the volatility of variance, σ_{v2t}, and thus the volatility of the swap contract. Because of this increase in riskiness of the swap contract, the risk-averse investors hold fewer volatility derivatives in equilibrium. Our finding here also highlights that even though the aggregate volatility bias and disagreement have similar effects on the stock market quantities, their

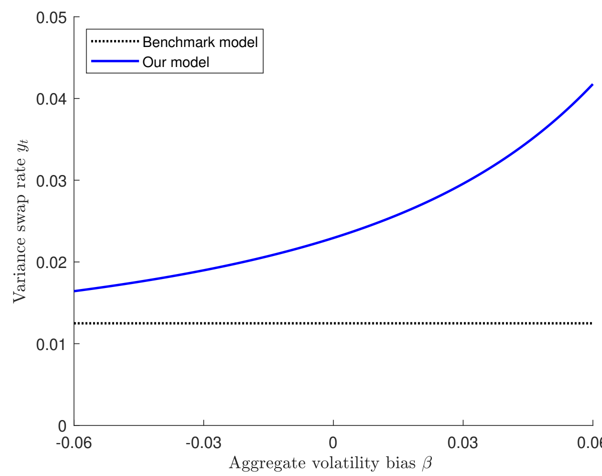
[29]Since the changes in bias β go in the same direction in investors' expectations, not so surprisingly, we find the results presented in Corollary 3 not only hold at steady state but to hold for all states.



Panel A. Variance swap long holdings



Panel B. Variance risk premium



Panel C. Variance swap rate

Figure 8. Effects of aggregate volatility bias in the volatility derivatives market. These panels plot the effects of aggregate volatility bias β on the equilibrium long holdings in the variance swap contracts θ_{ht} (Panel A), variance risk premium π_{vt} (Panel B), variance swap rate y_t (Panel C) when $V_t = \overline{V}$ and $\mathrm{w}_t = 0.5$. The dotted black lines represent the benchmark economy with no volatility disagreement and aggregate bias. The parameter values follow from Table B1 of Appendix B.

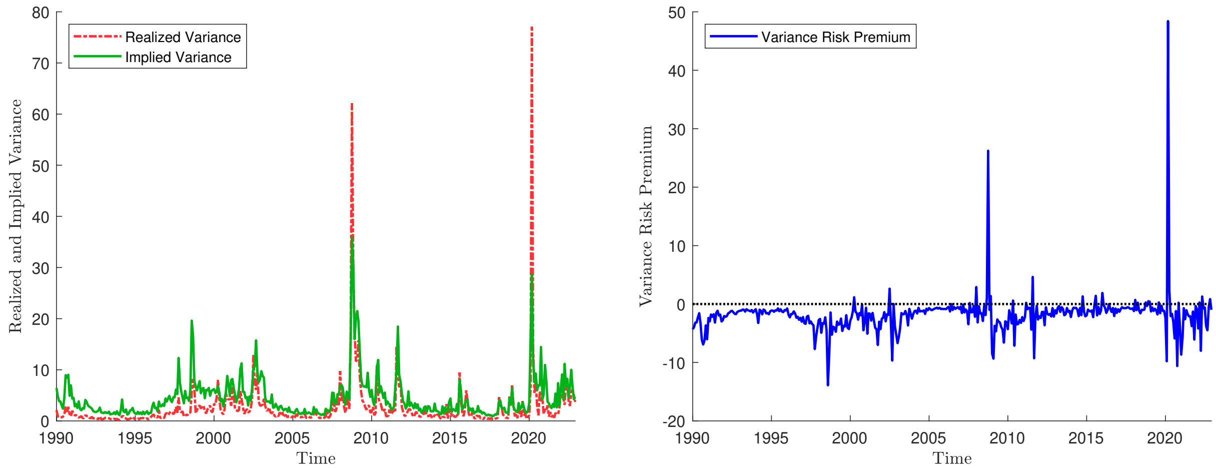
effects on the volatility derivatives are distinctly different. This difference can be clearly summarized in terms of the mean-variance portfolio structure. A higher aggregate volatility bias increases only the riskiness of the volatility derivative without affecting the perceived mean returns. However, a higher disagreement increases both the perceived mean returns and the riskiness of the volatility derivative, resulting in the hump-shaped relation in Figure 5.

Figure 8, Panels B and C, show that the variance risk premium decreases, while the variance swap rate increases with the aggregate volatility bias in equilibrium. The latter result arises because a higher volatility bias not only leads to a lower variance risk premium but also a higher conditional variance and higher expected future variance. The former result occurs because, as highlighted above, a higher bias reduces the market price of variance risk m_{2t} while increasing the quantity of variance risk σ_{v2t}. When $m_{2t} < 0$, both effects reinforce each other and lead to a more negative variance risk premium. We also observe that for a sufficiently negative (downward) volatility bias, the variance risk premium switches sign and becomes positive. Therefore, in addition to a high wealth-share of the low-fear investors (i.e., low w_t), a downward bias (i.e., $\beta < 0$) can also generates a positive variance risk premium. Both channels are related since a downward aggregate volatility bias means investors on average believe the return volatility is less persistent and is more likely to be lower in the near future, akin to the effects of low w_t. In Section 6.2, we demonstrate how such a downward bias can also help explain the behavior of the variance risk premium during heightened volatility.

6.2 Aggregate Volatility Bias and Heightened Volatility

In this section, we highlight the role of the downward aggregate volatility bias in explaining the large positive variance risk premium observed during periods of heightened volatility. In Figure 9, we first illustrate the time-series behavior of the realized and the implied variance in Panel A, and their difference, the variance risk premium, in Panel B. We see that most of the times, the implied variance, that is, the variance swap rate, is higher that the realized variance in the data, resulting in a negative variance risk premium on average. However, during high volatility times, such as during the 2008 financial crisis or 2020 COVID period, the implied variance is much lower that the realized variance, leading to large positive spikes in the variance risk premium.

To shed light on the behavior of the variance risk premium during heightened volatility periods, Table 1 presents the relevant economic quantities in our extended model under a



Panel A. Realized and implied variance

Panel B. Variance risk premium

Figure 9. Time-series behavior of variance risk premium. These panels plot the time series of the realized and implied variance (Panel A) and their difference, the variance risk premium, (Panel B). The sample period is from January 1990 to December 2022. Data is obtained from Hao Zhou's website.

downward aggregate volatility bias ($\beta < 0$), as well as under no bias ($\beta = 0$). To link our model to the empirical literature, in Table 1 we refer to the (expected) realized variance $v_t + \mu_{vt}$ as the realized variance (RV), and the variance swap rate y_t as the implied variance (IV). The variance risk premium (VRP) π_{vt} is given by the difference between RV and IV. In addition to these quantities, we also report the model implied open interest (OI) in the variance swap contracts, which corresponds to the total (long) holdings of the high-fear investors, θ_{ht}.

Table 1, Panel A, presents our findings for our baseline economy with volatility disagreement but without an aggregate volatility bias ($\beta = 0$). As we discuss in Section 5, when the stock market return volatility is relatively high, a positive VRP can arise when the market is dominated by investors with lower future variance expectations (e.g, $\mathrm{w}_t = 0.25$). Panel B presents our findings for the economy with a downward aggregate volatility bias ($\beta = -0.10$). We see that during high volatility periods, a positive VRP still arises when the market is dominated by investors with lower future variance expectations. As Table 1 illustrates the presence of a downward aggregate volatility bias, over and above volatility disagreement, increases the wedge between RV and IV, thus leading to a larger positive VRP.

Table 1. Downward aggregate volatility bias during heightened volatility. This table reports the volatility derivatives market quantities during heightened volatility in our model without aggregate volatility bias (Panel A) and with a downward volatility bias (Panel B). To generate comparable high stock return volatility around 27%, we set the fundamental variance level in Panel A to $V_t = 0.03$ and to $V_t = 0.05$ in Panel B. In the table, w_t refers to the wealth-share of the h-type investors, volatility refers to the conditional stock return volatility $\sqrt{\upsilon_t}$, RV refers to the (expected) realized variance $\upsilon_t + \mu_{\upsilon t}$, IV refers to the implied variance (variance swap rate) y_t, VRP refers to the variance risk premium $\pi_{\upsilon t}$, and the OI refers to open interest, the total long holdings in the variance swap contract, θ_{ht}. All other parameter values are as in Table B1.

Panel A. Economy with no aggregate volatility bias ($\beta = 0$)					
w_t	Volatility	RV	IV	VRP	OI
0.25	23.87	5.22	4.62	**0.60**	392
0.50	26.40	2.91	3.45	-0.55	798
0.75	23.68	3.51	3.68	-0.17	3,801
Panel B. Economy with a downward volatility bias ($\beta = -0.10$)					
w_t	Volatility	RV	IV	VRP	OI
0.25	26.04	6.33	3.78	**2.55**	622
0.50	27.80	2.80	2.31	**0.50**	1,323
0.75	26.10	3.60	3.64	-0.05	6,531

Moreover, as the last column in both panels indicates, periods with relatively larger positive VRP times also coincide with periods of low volatility derivative holdings, consistent with the findings in Cheng (2019).[30]

An economy characterized by a downward aggregate volatility bias during heightened volatility periods is realistic. For instance, since volatility is a mean-reverting process, when investors observe an extremely high volatility, excessively deviating from its long-run mean, they may all adjust their future volatility expectations downward, while still disagreeing about its future level. To the best of our knowledge, ours is the first theoretical work to simultaneously reconcile the puzzling behavior of positive VRP, as well as the reduction in

[30]In contrast, in the economy without volatility disagreement ($\delta = 0$), irrespective of whether there is aggregate volatility bias or not, we do not observe a positive VRP even when the stock market return volatility is relatively high. In the literature, it is shown that a positive VRP can arise in biased volatility expectations models of Atmaz (2022) and Lochstoer and Muir (2022), and in rational learning model of Ghaderi, Kilic, and Seo (2023). However, since all these works employ single-agent economies, they are unable to speak to the quantity side of the volatility derivative market.

the quantity of volatility derivatives traded in the market during high volatility times.

7 Conclusion

In this paper, we develop a tractable dynamic complete-market model in which investors with different volatility expectations trade a riskless asset, a risky stock, and a volatility derivative. Presence of volatility derivatives allows investors to hedge their volatility exposure as well as to speculate on their differing beliefs. The model delivers closed-from expressions and generates novel implications for the volatility derivatives and stock market quantities.

As novel predictions, we find that on average higher volatility disagreement leads to more negative market price of variance risk, lower stock price, higher stock risk premium, higher stock return variance, stronger leverage effect, higher variance swap rate, and more negative variance risk premium. We also find that the relation between the volatility disagreement and the volatility derivative (variance swap) trades is non-monotonic, it is first increasing then decreasing. Consistent with empirical evidence, we also find that under volatility disagreement the leverage effect is time-varying with its magnitude increasing in volatility, and investors trade less volatility derivatives in more volatile periods. We further show that variance risk premium can turn positive and increase in volatility when investors with lower volatility expectations are more dominant in the economy. Finally, in an extension of our model, we show that a higher aggregate volatility bias leads to lower equilibrium trading in volatility derivatives, and the presence of a downward aggregate volatility bias can generate the occasionally observed behavior of a positive variance risk premium along with a lower volatility derivatives trading during high volatility periods. To the best of our knowledge, our results summarized here are all new and have not been demonstrated in the extant theoretical literature.

To demonstrate our insights on volatility disagreement as clearly as possible, in this paper we employ a fairly simple dynamic framework in the sense that there are only two sources of (Brownian) risk and two types of heterogeneous-belief investors with standard logarithmic preferences. Our framework can be extended along different dimensions to study other potentially important features such as non-Brownian jump risks, more general investor type space with beliefs, and more general constant relative risk aversion (CRRA) preferences. We leave these considerations to future research.

Appendix A: Proofs

In this appendix, we first provide the proofs of Propositions 1–7, then the proofs of Corollaries 1–3. At the end of this section, we also provide Lemma A1 that is used in our proofs. For brevity, we solve the more general version of our model presented in Section 6 that additionally features the aggregate volatility bias parameter β. The relevant economic quantities in our baseline economy arise as the special case of $\beta = 0$ in the following proofs.

Proof of Proposition 1. To determine the equilibrium in our complete market economy, we first solve investors' optimization problem using standard martingale methods. Each i-type investor's static optimization problem, $i = h, \ell$, becomes

$$\max_{W_{iT}} \mathrm{E}^i \left[\ln W_{iT}\right], \qquad \text{subject to} \qquad \mathrm{E}^i \left[\xi_{iT} W_{iT}\right] \leq \xi_{i0} W_{i0}, \tag{A.1}$$

where, ξ_{it} is the i-type investor's subjective state price density. The consistency relation across investors' subjective beliefs implies $\xi_{ht} = L_t \xi_{\ell t}$, where L_t is the likelihood ratio process defined as

$$L_T \equiv \frac{d\mathbb{P}^\ell}{d\mathbb{P}^h} = e^{-\int_0^T \delta \frac{1}{\sigma}\sqrt{V_u} d\omega_{2u}^h - \frac{1}{2}\int_0^T \left(\delta \frac{1}{\sigma}\sqrt{V_u}\right)^2 du},$$

with dynamics

$$dL_t = -\delta L_t \frac{1}{\sigma}\sqrt{V_t} d\omega_{2t}^h. \tag{A.2}$$

The first order conditions of the static optimization problem (A.1) gives the optimal terminal wealth of each i-type as $W_{iT} = \lambda_i^{-1} \xi_{iT}^{-1}$, where λ_i is the Lagrange multiplier which solves the static budget constraint with equality, leading to $\lambda_i^{-1} = \xi_{i0} W_{i0}$.

Next, we impose the goods market clear condition $W_{hT} + W_{\ell T} = D_T$ along with the consistency relation $\xi_{ht} = L_t \xi_{\ell t}$, and obtain the h-type investor's subjective state price density at time-T as

$$\xi_{hT} = D_T^{-1} \left(\lambda_h^{-1} + \lambda_\ell^{-1} L_T\right). \tag{A.3}$$

The subjective state price density at an earlier time $t < T$ is determined through the relation $\xi_{ht} = e^{r(T-t)} \mathrm{E}_t^h \left[\xi_{hT}\right]$, which implies

$$\xi_{ht} = e^{r(T-t)} \left(\lambda_h^{-1} \mathrm{E}_t^h \left[D_T^{-1}\right] + \lambda_\ell^{-1} \mathrm{E}_t^h \left[D_T^{-1} L_T\right]\right).$$

To compute above expectations, we use Lemma A1 at the end of this section. By taking

$a = -1$ and $b = 0$ in Lemma A1, we obtain the first expectation as

$$M_{ht} \equiv \mathrm{E}_t^h \left[D_T^{-1} \right] = D_t^{-1} e^{-\mu(T-t)} e^{A_h(t) + B_h(t) V_t},$$

with its dynamics by

$$\frac{dM_{ht}}{M_{ht}} = -\sqrt{V_t} d\omega_{1t} + \sigma B_h(t) \sqrt{V_t} d\omega_{2t}^h, \tag{A.4}$$

and by taking $a = -1$ and $b = 1$ in Lemma A1, we obtain the second expectation as

$$M_{\ell t} \equiv \mathrm{E}_t^h \left[D_T^{-1} L_T \right] = D_t^{-1} L_t e^{-\mu(T-t)} e^{A_\ell(t) + B_\ell(t) V_t},$$

with its dynamics by

$$\frac{dM_{\ell t}}{M_{\ell t}} = -\sqrt{V_t} d\omega_{1t} + \left(\sigma^2 B_\ell(t) - \delta \right) \frac{1}{\sigma} \sqrt{V_t} d\omega_{2t}^h, \tag{A.5}$$

where the positive deterministic functions $A_i(t)$ and $B_i(t)$ for $i = h, \ell$, are given by

$$A_i(t) = \frac{2\kappa\overline{V}}{\sigma^2} \ln \frac{2\eta_i e^{\frac{1}{2}(\kappa_i + \eta_i)(T-t)}}{(\kappa_i + \eta_i)\left(e^{\eta_i(T-t)} - 1\right) + 2\eta_i}, \qquad B_i(t) = 2 \frac{e^{\eta_i(T-t)} - 1}{(\kappa_i + \eta_i)\left(e^{\eta_i(T-t)} - 1\right) + 2\eta_i}, \tag{A.6}$$

along with the positive constants $\kappa_h = \kappa - \left(\beta + \frac{1}{2}\delta\right)$, $\eta_h = \sqrt{\kappa_h^2 - 2\sigma^2}$, $\kappa_\ell = \kappa - \left(\beta - \frac{1}{2}\delta\right)$, and $\eta_\ell = \sqrt{\kappa_\ell^2 - 2\sigma^2}$.

Using (A.4) and (A.5), we next apply Itô's Lemma to $\xi_{ht} = e^{r(T-t)} \left(\lambda_h^{-1} M_{ht} + \lambda_\ell^{-1} M_{\ell t} \right)$, to obtain

$$\frac{d\xi_{ht}}{\xi_{ht}} = -rdt - \sqrt{V_t} d\omega_{1t} + \left[\sigma^2 \frac{\lambda_h^{-1} M_{ht} B_h(t) + \lambda_\ell^{-1} M_{\ell t} B_\ell(t)}{\lambda_h^{-1} M_{ht} + \lambda_\ell^{-1} M_{\ell t}} - \frac{\delta \lambda_\ell^{-1} M_{\ell t}}{\lambda_h^{-1} M_{ht} + \lambda_\ell^{-1} M_{\ell t}} \right] \frac{1}{\sigma} \sqrt{V_t} d\omega_{2t}^h. \tag{A.7}$$

To express this in terms of investors' wealth share, we also compute their time-t wealth using no arbitrage relations as

$$W_{ht} = \frac{1}{\xi_{ht}} \mathrm{E}_t^h \left[\xi_{hT} W_{hT} \right] = \frac{1}{\xi_{ht}} \lambda_h^{-1}, \qquad W_{\ell t} = \frac{1}{\xi_{\ell t}} \mathrm{E}_t^\ell \left[\xi_{\ell T} W_{\ell T} \right] = \frac{1}{\xi_{\ell t}} \lambda_\ell^{-1},$$

which along with with the consistency relation $\xi_{ht} = L_t \xi_{\ell t}$, gives the wealth share of the

h-type investor as

$$\mathrm{w}_t = \frac{W_{ht}}{W_{ht} + W_{\ell t}} = \frac{\lambda_h^{-1}}{\lambda_h^{-1} + \lambda_\ell^{-1} L_t}, \tag{A.8}$$

which in turn implies

$$\frac{\lambda_h^{-1} M_{ht}}{\lambda_h^{-1} M_{ht} + \lambda_\ell^{-1} M_{\ell t}} = \frac{\lambda_h^{-1} e^{A_h(t)+B_h(t)V_t}}{\lambda_h^{-1} e^{A_h(t)+B_h(t)V_t} + \lambda_\ell^{-1} L_t e^{A_\ell(t)+B_\ell(t)V_t}} = \mathrm{w}_t \Lambda_t,$$

where the high-fear investors' relative risk discount term Λ_t is as in (10). Therefore, we can simply rewrite the dynamics in (A.7) as

$$\frac{d\xi_{ht}}{\xi_{ht}} = -rdt - \sqrt{V_t} d\omega_{1t} + \left[\sigma^2 B_t - \delta\left(1 - \mathrm{w}_t \Lambda_t\right)\right] \frac{1}{\sigma} \sqrt{V_t} d\omega_{2t}^h,$$

where the stochastic variance elasticity B_t is as in (10).

Matching the above dynamics to the h-type investor's subjective dynamics $d\xi_{ht}/\xi_{ht} = -rdt - m_{1t}^h d\omega_{1t} - m_{2t}^h d\omega_{2t}^h$, gives her perceived market prices of risks as $m_{1t}^h = \sqrt{V_t}$, and $m_{2t}^h = -\left[\sigma^2 B_t - \delta\left(1 - \mathrm{w}_t \Lambda_t\right)\right] \sqrt{V_t}/\sigma$. Lastly, using the consistency relations between the objective measure and the h-type investor's subjective measure, which imply $m_{1t} = m_{1t}^h$ and $m_{2t} = m_{2t}^h - (\beta + \delta/2) \sqrt{V_t}/\sigma$, we obtain the market prices of risks for the shocks ω_1 and ω_2 are as in (7) and (8), respectively.

Moreover, applying Itô's Lemma to the wealth share (A.8) using the dynamics of L in (A.2) gives the subjective dynamics

$$d\mathrm{w}_t = \delta^2 \mathrm{w}_t \left(1 - \mathrm{w}_t\right)^2 \frac{1}{\sigma^2} V_t dt + \delta \mathrm{w}_t \left(1 - \mathrm{w}_t\right) \frac{1}{\sigma} \sqrt{V_t} d\omega_{2t}^h,$$

which after $d\omega_{2t}^h = d\omega_{2t} - \left(\beta + \frac{1}{2}\delta\right) \frac{1}{\sigma} \sqrt{V_t} dt$ substituted in becomes the objective dynamics as in (9) with the constant long-run mean $\bar{\mathrm{w}} = 1/2 - \beta/\delta$.

The benchmark economy quantities are simply obtained by substituting $\delta = \beta = 0$, which yields $\Lambda_t = 1$ and the deterministic functions $\bar{B}(t)$ and $\bar{A}(t)$ such that

$$\bar{A}(t) = \frac{2\kappa \overline{V}}{\sigma^2} \ln \frac{2\bar{\eta} e^{\frac{1}{2}(\bar{\kappa}+\bar{\eta})(T-t)}}{(\bar{\kappa}+\bar{\eta})\left(e^{\bar{\eta}(T-t)} - 1\right) + 2\bar{\eta}}, \qquad \bar{B}(t) = 2 \frac{e^{\bar{\eta}(T-t)} - 1}{(\bar{\kappa}+\bar{\eta})\left(e^{\bar{\eta}(T-t)} - 1\right) + 2\bar{\eta}},$$

along with the positive constants $\bar{\kappa} = \kappa$ and $\bar{\eta} = \sqrt{\bar{\kappa}^2 - 2\sigma^2}$. $\square$

Proof of Proposition 2. By no arbitrage, the stock price satisfies $\xi_{ht}S_t = \mathrm{E}_t^h\left[\xi_{hT}D_T\right]$. Using (A.3) and the martingale property of L under $\mathbb{P}^h$ gives the expectation as $\mathrm{E}_t^h\left[\xi_{hT}D_T\right] = \lambda_h^{-1} + \lambda_\ell^{-1}L_t$. Substituting h-type investor's subjective state price density in terms of the wealth share

$$\xi_{ht} = D_t^{-1}e^{-(\mu-r)(T-t)}\left(\lambda_h^{-1} + \lambda_\ell^{-1}L_t\right)\left[\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + \left(1-\mathrm{w}_t\right)e^{A_\ell(t)+B_\ell(t)V_t}\right],$$

immediately gives (11).

Applying Itô's Lemma to the stock price (11) using the dynamics of wealth share (9) yields the diffusion coefficients as in (12) and (13).

The benchmark economy stock price and dynamics are simply obtained by substituting $\delta = \beta = 0$ into (11)–(13). □

Proof of Proposition 3. The equilibrium stock return variance (14) is readily given by the stock diffusion terms (12) and (13) through the relation $\upsilon_t = \sigma_{S1t}^2 + \sigma_{S2t}^2$.

To determine the dynamics of the the stock return variance, we first apply Itô's Lemma to Λ_t and B_t using their representations in (10). After straightforward but lengthy algebra, we obtain the dynamics for Λ_t as $d\Lambda_t = \mu_{\Lambda t}dt + \sigma_{\Lambda 1t}d\omega_{1t} + \sigma_{\Lambda 2t}d\omega_{2t}$, where $\sigma_{\Lambda 1t} = 0$,

$$\sigma_{\Lambda 2t} = \Lambda_t\left[\sigma^2\left(1-\mathrm{w}_t\Lambda_t\right)\left(B_h\left(t\right)-B_\ell\left(t\right)\right) - \delta\mathrm{w}_t\left(\Lambda_t-1\right)\right]\frac{1}{\sigma}\sqrt{V_t}, \tag{A.9}$$

and the drift term

$$\begin{aligned}\mu_{\Lambda t} = {} & \Lambda_t\left(1-\mathrm{w}_t\Lambda_t\right)\left[\dot{B}_h\left(t\right)-\dot{B}_\ell\left(t\right)-\kappa\left(B_h\left(t\right)-B_\ell\left(t\right)\right)+\left(\frac{1}{2}-\mathrm{w}_t\Lambda_t\right)\sigma^2\left(B_h\left(t\right)-B_\ell\left(t\right)\right)^2\right]V_t \\ & - \delta\mathrm{w}_t\Lambda_t\left(1-\mathrm{w}_t\Lambda_t\right)\left(2\Lambda_t-1\right)\left(B_h\left(t\right)-B_\ell\left(t\right)\right)V_t - \delta^2\mathrm{w}_t\Lambda_t\left(\Lambda_t-1\right)\left(\bar{\mathrm{w}}-\mathrm{w}_t\Lambda_t\right)\frac{1}{\sigma^2}V_t,\end{aligned} \tag{A.10}$$

with $\dot{B}_i\left(t\right) = dB_i\left(t\right)/dt$ as in the proof of Lemma A1.

Similarly, we obtain the dynamics $dB_t = \mu_{Bt}dt + \sigma_{B1t}d\omega_{1t} + \sigma_{B2t}d\omega_{2t}$, where $\sigma_{B1t} = 0$,

$$\sigma_{B2t} = \left(B_h\left(t\right)-B_\ell\left(t\right)\right)\left(\mathrm{w}_t\sigma_{\Lambda 2t} + \delta\mathrm{w}_t\left(1-\mathrm{w}_t\right)\Lambda_t\frac{1}{\sigma}\sqrt{V_t}\right), \tag{A.11}$$

and the drift term

$$\mu_{Bt} = \mathrm{w}_t \left(B_h(t) - B_\ell(t)\right) \left[\mu_{\Lambda t} + \delta\sigma_{\Lambda 2t}\left(1-\mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t} + \delta^2\Lambda_t\left(1-\mathrm{w}_t\right)\left(\bar{\mathrm{w}}-\mathrm{w}_t\right)\frac{1}{\sigma^2}V_t\right]$$
$$+ \mathrm{w}_t\Lambda_t\dot{B}_h(t) + \left(1-\mathrm{w}_t\Lambda_t\right)\dot{B}_\ell(t). \tag{A.12}$$

Finally, we apply Itô's Lemma to stock return variance (14) using the above dynamics to obtain the diffusion terms as in (15) and (16), and its drift term as

$$\begin{aligned}
\mu_{\upsilon t} &= \left[\sigma^2 + \left(\sigma^2 B_t + \delta\mathrm{w}_t\left(\Lambda_t - 1\right)\right)^2\right]\kappa\left(\overline{V} - V_t\right)\frac{1}{\sigma^2} \\
&+ \left[\sigma^2\sigma_{B2t} + \delta\mathrm{w}_t\sigma_{\Lambda 2t} + \delta^2\left(\Lambda_t - 1\right)\mathrm{w}_t\left(1-\mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t}\right]^2\frac{1}{\sigma^2}V_t \\
&+ 2\left[\sigma^2\mu_{Bt} + \delta\mathrm{w}_t\mu_{\Lambda t} + \delta^2\sigma_{\Lambda 2t}\mathrm{w}_t\left(1-\mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t}\right]\frac{\sigma^2 B_t + \delta\mathrm{w}_t\left(\Lambda_t - 1\right)}{\sigma^2}V_t \\
&+ 2\delta^3\left(\Lambda_t - 1\right)\mathrm{w}_t\left(1-\mathrm{w}_t\right)\left(\bar{\mathrm{w}}-\mathrm{w}_t\right)\frac{\sigma^2 B_t + \delta\mathrm{w}_t\left(\Lambda_t - 1\right)}{\sigma^4}V_t^2 \\
&+ 2\left[\sigma^2\sigma_{B2t} + \delta\mathrm{w}_t\sigma_{\Lambda 2t} + \delta^2\left(\Lambda_t - 1\right)\mathrm{w}_t\left(1-\mathrm{w}_t\right)\frac{1}{\sigma}\sqrt{V_t}\right]\frac{\sigma^2 B_t + \delta\mathrm{w}_t\left(\Lambda_t - 1\right)}{\sigma}\sqrt{V_t}.
\end{aligned} \tag{A.13}$$

The benchmark economy stock return variance and dynamics are simply obtained by substituting $\delta = \beta = 0$ into the above dynamics, which leads to $\bar{\sigma}_{\Lambda 1t} = \bar{\sigma}_{\Lambda 2t} = \bar{\mu}_{\Lambda t} = 0$, $\bar{\sigma}_{B1t} = \bar{\sigma}_{B2t} = 0$, and $\bar{\mu}_{Bt} = \dot{\bar{B}}(t) = -1 + \bar{\kappa}\bar{B}(t) - \frac{1}{2}\sigma^2\bar{B}^2(t)$. These imply the diffusion coefficients of the stock return variance as $\bar{\sigma}_{\upsilon 1t} = 0$ and $\bar{\sigma}_{\upsilon 2t} = \left[\sigma^2 + \left(\sigma^2\bar{B}(t)\right)^2\right]\sqrt{V_t}/\sigma$, and

$$\bar{\mu}_{\upsilon t} = \left[\sigma^2 + \left(\sigma^2\bar{B}(t)\right)^2\right]\kappa\left(\overline{V} - V_t\right)\frac{1}{\sigma^2} + 2\sigma^2\bar{B}(t)\left(-1 + \bar{\kappa}\bar{B}(t) - \frac{1}{2}\sigma^2\bar{B}^2(t)\right)V_t,$$

as its expected change. □

Proof of Proposition 4. The equilibrium correlation between stock returns and variance shocks is given by

$$\rho_t dt = \frac{\mathrm{Cov}_t\left[d\ln S_t, d\upsilon_t\right]}{\sqrt{\mathrm{Var}_t\left[d\ln S_t\right]}\sqrt{\mathrm{Var}_t\left[d\upsilon_t\right]}} = \frac{\sigma_{S1t}\sigma_{\upsilon 1t} + \sigma_{S2t}\sigma_{\upsilon 2t}}{\sqrt{\sigma_{S1t}^2 + \sigma_{S2t}^2}\sqrt{\sigma_{\upsilon 1t}^2 + \sigma_{\upsilon 2t}^2}}dt,$$

which immediately yields (17) when $\sigma_{\upsilon 2t} > 0$ after substituting (12)–(13) and $\sigma_{\upsilon 1t} = 0$.

The benchmark economy correlation is obtained by substituting $\beta = \delta = 0$ into (17). □

Proof of Proposition 5. To determine the equilibrium holdings in variance swaps, we begin with the observation that for $i = h, \ell$, i-type investor's discounted wealth process satisfies $\xi_{it} W_{it} = \mathrm{E}_t^i [\xi_{iT} W_{iT}] = \lambda_i^{-1}$, which implies the dynamics $d(\xi_{it} W_{it}) = 0$ under their subjective measure $\mathbb{P}^i$. Matching this dynamics to their discounted budget constraint

$$d(\xi_{it} W_{it}) = \xi_{it} \left[\psi_{it} S_t \sigma_{S1t} + \theta_{it} \sigma_{\upsilon 1t} - W_{it} m_{1t}^i\right] d\omega_{1t} + \xi_{it} \left[\psi_{it} S_t \sigma_{S2t} + \theta_{it} \sigma_{\upsilon 2t} - W_{it} m_{2t}^i\right] d\omega_{2t}^i,$$

yields the system of two equations in two unknowns, ψ_{it} and θ_{it},

$$\begin{aligned} \left[\psi_{it} S_t \sigma_{S1t} + \theta_{it} \sigma_{\upsilon 1t} - W_{it} m_{1t}^i\right] &= 0, \\ \left[\psi_{it} S_t \sigma_{S2t} + \theta_{it} \sigma_{\upsilon 2t} - W_{it} m_{2t}^i\right] &= 0, \end{aligned}$$

where m_{jt}^i is i-type investor's perceived market price of risk for the Brownian motion ω_j^i, which satisfy $d\xi_{it}/\xi_{it} = -r dt - m_{1t}^i d\omega_{1t} - m_{2t}^i d\omega_{2t}^i$. Solving the above system of equations along with the facts that $m_{1t}^h = m_{1t}$ and $m_{2t}^h = m_{2t} + (\beta + \delta/2)\sqrt{V_t}/\sigma$, yields the variance swap contract holdings $\theta_{ht} = W_{ht} \left[\sigma_{S1t} \left(m_{2t} + (\beta + \delta/2)\sqrt{V_t}/\sigma\right) - \sigma_{S2t} m_{1t}\right] / [\sigma_{S1t}\sigma_{\upsilon 2t} - \sigma_{S2t}\sigma_{\upsilon 1t}]$. Substituting (7)–(8), (12)–(13), and $\sigma_{\upsilon 1t} = 0$ yields θ_{it} as in (18). □

Proof of Proposition 6. The equilibrium variance risk premium (19) is given by the conditional covariance of the state price density growth with the variance swap payoff

$$\pi_{\upsilon t} dt = -\frac{d\xi_t}{\xi_t} (\upsilon_t dt + d\upsilon_t - y_t dt) = -\frac{d\xi_t}{\xi_t} d\upsilon_t = (m_{1t}\sigma_{\upsilon 1t} + m_{2t}\sigma_{\upsilon 2t}) dt,$$

which becomes (19) after substituting (8) and $\sigma_{\upsilon 1t} = 0$. Note that the variance risk premium is also equivalent to the difference $\mathrm{E}_t[d\upsilon_t] - \mathrm{E}_t^*[d\upsilon_t]$, where E_t^* denotes the expectation under the risk-neutral measure, under which ω_1^* and ω_2^* are standard Brownian motions such that $d\omega_{1t}^* = d\omega_{1t} + m_{1t} dt$ and $d\omega_{2t}^* = d\omega_{2t} + m_{2t} dt$.

By no-arbitrage, the equilibrium variance swap rate is given by equating the risk-neutral expectation of the variance swap payoff to zero. That is, $\mathrm{E}_t^*[\upsilon_t dt + d\upsilon_t - y_t dt] = 0$, which immediately gives (20) after substituting $\mathrm{E}_t^*[d\upsilon_t] = \mathrm{E}_t[d\upsilon_t] - \pi_{\upsilon t} dt = (\mu_{\upsilon t} - \pi_{\upsilon t}) dt$.

The benchmark economy variance risk premium and variance swap rate are obtained by substituting $\delta = \beta = 0$ into the above expressions. □

Proof of Proposition 7. The quantities in the economy with volatility disagreement and aggregate bias are already obtained in the proofs of Propositions 1–6. □

Proof of Corollary 1. The property that the market price of variance risk is decreasing in volatility disagreement at steady state, follows from the partial derivative of (8) with respect to δ. This property holds if and only if $\delta \frac{\partial}{\partial \delta} \Lambda_t + \sigma^2 \frac{\partial}{\partial \delta} B_t + (\mathrm{w}_t \Lambda_t - \bar{\mathrm{w}}) > 0$. To that end, we first show that $\frac{\partial}{\partial \delta} \Lambda_t > 0$ for any given V_t and w_t, and $\frac{\partial}{\partial \delta} B_t > 0$ for any given V_t and $\mathrm{w}_t \geq \bar{\mathrm{w}}$, which are sufficient to prove $\frac{\partial}{\partial \delta} m_{2t} < 0$ when $\mathrm{w}_t = \bar{\mathrm{w}}$.

The partial derivative of Λ_t with respect to δ using (10) is given by

$$\frac{\partial}{\partial \delta} \Lambda_t = \frac{-\left(1 - \mathrm{w}_t\right) \frac{\partial}{\partial \delta} e^{-[A_h(t) - A_\ell(t)] - [B_h(t) - B_\ell(t)] V_t}}{\left[\mathrm{w}_t + \left(1 - \mathrm{w}_t\right) e^{-[A_h(t) - A_\ell(t)] - [B_h(t) - B_\ell(t)] V_t}\right]^2}.$$

Thus, $\frac{\partial}{\partial \delta} \Lambda_t > 0$ if and only if $\frac{\partial}{\partial \delta} \left[A_h\left(t\right) - A_\ell\left(t\right)\right] + \frac{\partial}{\partial \delta} \left[B_h\left(t\right) - B_\ell\left(t\right)\right] V_t > 0$. Now we use the facts that the deterministic functions $A_i\left(t\right)$ and $B_i\left(t\right)$ in (A.6) are both convexly decrease in κ. Since $\kappa_h = \kappa - \beta - \frac{1}{2}\delta$ and $\kappa_\ell = \kappa - \beta + \frac{1}{2}\delta$, a mean-preserving spread around $\kappa - \beta$ leads to more increases in the $A_h\left(t\right)$ and $B_h\left(t\right)$ than the decreases in $A_\ell\left(t\right)$ and $B_\ell\left(t\right)$, respectively. That is, $\frac{\partial}{\partial \delta} \left[A_h\left(t\right) - A_\ell\left(t\right)\right] > 0$ and $\frac{\partial}{\partial \delta} \left[B_h\left(t\right) - B_\ell\left(t\right)\right] > 0$, which in turn shows that the above inequality holds for any given V_t and w_t.

Similarly, the partial derivative of B_t with respect to δ using (10) is given by

$$\frac{\partial}{\partial \delta} B_t = \mathrm{w}_t \Lambda_t \frac{\partial}{\partial \delta} B_h\left(t\right) + \left(1 - \mathrm{w}_t \Lambda_t\right) \frac{\partial}{\partial \delta} B_\ell\left(t\right) + \mathrm{w}_t \left[B_h\left(t\right) - B_\ell\left(t\right)\right] \frac{\partial}{\partial \delta} \Lambda_t.$$

Since $\frac{\partial}{\partial \delta} \Lambda_t > 0$ and $B_h\left(t\right) > B_\ell\left(t\right)$, the last term above is always positive. A sufficient condition for the sum of first two terms to be positive is $\mathrm{w}_t \geq \bar{\mathrm{w}}$, since in that case $\mathrm{w}_t \Lambda_t > 1 - \mathrm{w}_t \Lambda_t$. Thus, we conclude that $\frac{\partial}{\partial \delta} B_t > 0$ for any given V_t and $\mathrm{w}_t \geq \bar{\mathrm{w}}$.

The property that the market price of variance risk is decreasing in fundamental variance at the steady-state, $\mathrm{w}_t = \bar{\mathrm{w}}$, follows from the partial derivative of (8) with respect to V_t. This property holds if and only if $\left(\frac{\partial}{\partial V_t} \Lambda_t + \frac{\partial}{\partial V_t} B_t\right) \sigma^2 V_t + \frac{1}{2}\left(\sigma^2 B_t + \delta\left(\mathrm{w}_t \Lambda_t - \bar{\mathrm{w}}\right)\right) > 0$. To that end, we first show that $\frac{\partial}{\partial V_t} \Lambda_t > 0$ and $\frac{\partial}{\partial V_t} B_t > 0$ for any given V_t and w_t, which are sufficient to prove $\frac{\partial}{\partial V_t} m_{2t} < 0$, since when $\mathrm{w}_t = \bar{\mathrm{w}}$, we also have the last term positive.

The partial derivative of Λ_t with respect to V_t using (10) immediately gives the condition that $\frac{\partial}{\partial V_t} \Lambda_t > 0$ if and only if $\left(B_h\left(t\right) - B_\ell\left(t\right)\right) e^{-[A_h(t) - A_\ell(t)] - [B_h(t) - B_\ell(t)] V_t} > 0$, which holds for any V_t and w_t since $B_h\left(t\right) > B_\ell\left(t\right)$. Similarly, the partial derivative of B_t with respect to V_t using (10) immediately gives the condition that $\frac{\partial}{\partial V_t} B_t > 0$ if and only if $\left(B_h\left(t\right) - B_\ell\left(t\right)\right) \mathrm{w}_t \frac{\partial}{\partial V_t} \Lambda_t > 0$, which holds for any V_t and w_t. $\square$

Proof of Corollary 2. Property (i), which states that the stock price is decreasing in volatility disagreement at steady state, follows from the partial derivative of (11) with respect to δ. This property holds if and only if $\frac{\partial}{\partial\delta}\left[\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + (1-\mathrm{w}_t)\, e^{A_\ell(t)+B_\ell(t)V_t}\right] > 0$. Since due to the convexity of the deterministic functions $A_i(t)$ and $B_i(t)$, we have $\frac{\partial}{\partial\delta}\left[A_h(t) + B_h(t)V_t\right] > \left|\frac{\partial}{\partial\delta}\left[A_\ell(t) + B_\ell(t)V_t\right]\right|$, and thus, the above inequality is satisfied at steady state. The property that the stock price is decreasing in fundamental variance, follows from the partial derivative of (11) with respect to V_t. This property holds if and only if $\frac{\partial}{\partial V_t}\left[\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + (1-\mathrm{w}_t)\, e^{A_\ell(t)+B_\ell(t)V_t}\right] > 0$, which holds since $B_i(t) > 0$.

Property (ii), which states that the stock risk premium is increasing in volatility disagreement at steady state, follows from the partial derivative of $\pi_{St} = \sigma_{S1t}m_{1t} + \sigma_{S2t}m_{2t} = V_t + \sigma_{S2t}m_{2t}$, with respect to δ. This property holds if and only if $\sigma_{S2t}\frac{\partial}{\partial\delta}m_{2t} + m_{2t}\frac{\partial}{\partial\delta}\sigma_{S2t} > 0$. Knowing that $\frac{\partial}{\partial\delta}m_{2t} < 0$ at $\mathrm{w}_t = \bar{\mathrm{w}}$, and $\sigma_{S2t} < 0$ implies the positivity of the first term. Similarly, knowing that $\frac{\partial}{\partial\delta}\sigma_{S2t} < 0$ at $\mathrm{w}_t = \bar{\mathrm{w}}$, and $m_{2t} < 0$ at $\mathrm{w}_t = \bar{\mathrm{w}}$, implies the positivity of the second term. The property that the stock risk premium is increasing in fundamental variance at steady state, follows from the partial derivative of $\pi_{St} = V_t + \sigma_{S2t}m_{2t}$, with respect to V_t. This property holds if and only if $1 + \sigma_{S2t}\frac{\partial}{\partial V_t}m_{2t} + m_{2t}\frac{\partial}{\partial V_t}\sigma_{S2t} > 0$. Knowing that $\frac{\partial}{\partial V_t}m_{2t} < 0$ at $\mathrm{w}_t = \bar{\mathrm{w}}$, and $\sigma_{S2t} < 0$ implies the positivity of the second term. Similarly, knowing that $\sigma_{S2t} = m_{2t}$ at $\mathrm{w}_t = \bar{\mathrm{w}}$, implies the positivity of the last term.

Property (iii), which states that the stock return variance is increasing in volatility disagreement at steady state, follows from the partial derivative of (14) with respect to δ. This property holds if and only if $\sigma_{S1t}\frac{\partial}{\partial\delta}\sigma_{S1t} + \sigma_{S2t}\frac{\partial}{\partial\delta}\sigma_{S2t} > 0$. Knowing that $\sigma_{S1t} = \sqrt{V_t}$ implies the first term is zero. Knowing that $\frac{\partial}{\partial\delta}\sigma_{S2t} < 0$ at $\mathrm{w}_t = \bar{\mathrm{w}}$, and $\sigma_{S2t} < 0$ implies the positivity of the second term. The property that the stock return variance is increasing in fundamental variance, follows from the partial derivative of (14) with respect to V_t. This property holds if and only if $\sigma_{S1t}\frac{\partial}{\partial V_t}\sigma_{S1t} + \sigma_{S2t}\frac{\partial}{\partial V_t}\sigma_{S2t} > 0$. Knowing that $\sigma_{S1t} = \sqrt{V_t}$ and $\frac{\partial}{\partial V_t}\sigma_{S2t} < 0$ at $\mathrm{w}_t = \bar{\mathrm{w}}$ with $\sigma_{S2t} < 0$ implies the positivity of the both terms.

Property (iv), which states that the leverage effect gets stronger in volatility disagreement at steady state, follows from the partial derivative of (17) with respect to δ. This property holds if and only if $\delta\mathrm{w}_t\frac{\partial}{\partial\delta}\Lambda_t + \sigma^2\frac{\partial}{\partial\delta}B_t + \mathrm{w}_t(\Lambda_t - 1) > 0$. Knowing that $\frac{\partial}{\partial\delta}\Lambda_t > 0$ and $\frac{\partial}{\partial\delta}B_t > 0$ at $\mathrm{w}_t = \bar{\mathrm{w}}$, imply that this inequality holds. The property that the leverage effect gets stronger in fundamental variance, follows from the partial derivative of (17) with respect to V_t. This property holds if and only if $\delta\mathrm{w}_t\frac{\partial}{\partial V_t}\Lambda_t + \sigma^2\frac{\partial}{\partial V_t}B_t > 0$. Knowing that $\frac{\partial}{\partial V_t}\Lambda_t > 0$ and $\frac{\partial}{\partial V_t}B_t > 0$, imply that this inequality holds. $\square$

Proof of Corollary 3. Property (i), which states that the market price of variance risk is decreasing in aggregate volatility bias, follows from the partial derivative of (8) with respect to β. This property holds if and only if $\delta \mathrm{w}_t \frac{\partial}{\partial\beta}\Lambda_t + \sigma^2 \frac{\partial}{\partial\beta} B_t + 1 > 0$. To that end, we show that $\frac{\partial}{\partial\beta}\Lambda_t > 0$ and $\frac{\partial}{\partial\beta} B_t > 0$ for any given V_t and w_t, which are sufficient to prove this property. Using (10), we have $\frac{\partial}{\partial\beta}\Lambda_t > 0$ if and only if $\frac{\partial}{\partial\beta}\left[A_h(t) - A_\ell(t)\right] + \frac{\partial}{\partial\beta}\left[B_h(t) - B_\ell(t)\right] V_t > 0$. Now we use the facts that the deterministic functions $A_i(t)$ and $B_i(t)$ in (A.6) are both convexly decrease in κ. Since an increase in β is equivalent to a decrease in κ, we obtain $\frac{\partial}{\partial\beta} A_h(t) > \frac{\partial}{\partial\beta} A_\ell(t)$ and $\frac{\partial}{\partial\beta} B_h(t) > \frac{\partial}{\partial\beta} B_\ell(t)$, since $\kappa_h = \kappa - \beta - \frac{1}{2}\delta < \kappa - \beta + \frac{1}{2}\delta = \kappa_\ell$. Similarly, using (10), we have $\frac{\partial}{\partial\beta} B_t > 0$ if and only if $\frac{\partial}{\partial\beta}\left[\mathrm{w}_t \Lambda_t B_h(t) + (1 - \mathrm{w}_t \Lambda_t) B_\ell(t)\right] > 0$, which holds since $\frac{\partial}{\partial\beta}\Lambda_t > 0$, $\frac{\partial}{\partial\beta} B_i(t) > 0$, and $B_h(t) > B_\ell(t)$.

Property (ii), which states that the stock price is decreasing in aggregate volatility bias, follows from the partial derivative of (11) with respect to β. This property holds if and only if $\frac{\partial}{\partial\beta}\left[\mathrm{w}_t e^{A_h(t)+B_h(t)V_t} + (1-\mathrm{w}_t)\, e^{A_\ell(t)+B_\ell(t)V_t}\right] > 0$. Since $\frac{\partial}{\partial\beta}\left[A_h(t) + B_h(t) V_t\right] > 0$ and $\frac{\partial}{\partial\beta}\left[A_\ell(t) + B_\ell(t) V_t\right] > 0$, the above inequality holds.

Property (iii), which states that the stock risk premium is increasing in aggregate volatility bias, follows from the partial derivative of $\pi_{St} = m_{1t}\sigma_{S1t} + m_{2t}\sigma_{S2t}$, with respect to β. This property holds since $\frac{\partial}{\partial\beta}\Lambda_t > 0$, $\frac{\partial}{\partial\beta} B_t > 0$, $\frac{\partial}{\partial\beta}\bar{\mathrm{w}} < 0$.

Property (iv), which states that the stock return variance is increasing in aggregate volatility volatility bias, follows from the partial derivative of (14) with respect to β. This property holds if and only if $\sigma_{S1t}\frac{\partial}{\partial\beta}\sigma_{S1t} + \sigma_{S2t}\frac{\partial}{\partial\beta}\sigma_{S2t} > 0$. Knowing that $\sigma_{S1t} = \sqrt{V_t}$ implies the first term is zero. Knowing that $\frac{\partial}{\partial\beta}\sigma_{S2t} < 0$ and $\sigma_{S2t} < 0$ implies the positivity of the second term.

Property (v), which states that the leverage effect gets stronger in volatility bias, follows from the partial derivative of (17) with respect to β. This property holds if and only if $\delta \mathrm{w}_t \frac{\partial}{\partial\beta}\Lambda_t + \sigma^2 \frac{\partial}{\partial\beta} B_t > 0$. Knowing that $\frac{\partial}{\partial\beta}\Lambda_t > 0$ and $\frac{\partial}{\partial\beta} B_t > 0$, imply that this inequality holds. □

Lemma A1. *Let the processes D and L be as in* (1) *and* (A.2)*. Then for all numbers a and b, we have the conditional joint moment generating function of* $\ln D_T$ *and* $\ln L_T$ *under $\mathbb{P}^h$, denoted by $M_t(a,b)$ is given by*

$$M_t(a,b) = \mathrm{E}_t^h\left[D_T^a L_T^b\right] = D_t^a L_t^b e^{a\mu(T-t)} e^{A(t;a,b)+B(t;a,b)V_t}, \tag{A.14}$$

with its dynamics given by

$$\frac{dM_t\left(a,b\right)}{M_t\left(a,b\right)} = a\sqrt{V_t}d\omega_{1t} + \left(\sigma^2 B\left(t;a,b\right) - b\delta\right)\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h, \tag{A.15}$$

where the deterministic functions are by

$$A\left(t;a,b\right) = \frac{2\kappa\overline{V}}{\sigma^2}\ln\frac{2\eta e^{\frac{1}{2}\left(\tilde{\kappa}+\eta\right)\left(T-t\right)}}{\left(\tilde{\kappa}+\eta\right)\left(e^{\eta\left(T-t\right)}-1\right)+2\eta}, \tag{A.16}$$

$$B\left(t;a,b\right) = \left(a\left(a-1\right)+b\left(b-1\right)\delta^2\frac{1}{\sigma^2}\right)\frac{\left(e^{\eta\left(T-t\right)}-1\right)}{\left(\tilde{\kappa}+\eta\right)\left(e^{\eta\left(T-t\right)}-1\right)+2\eta}, \tag{A.17}$$

with the constants

$$\tilde{\kappa} = \kappa + \left(b-\frac{1}{2}\right)\delta - \beta, \qquad \eta = \sqrt{\tilde{\kappa}^2 - a\left(a-1\right)\sigma^2 - b\left(b-1\right)\delta^2}. \tag{A.18}$$

Proof of Lemma A1. We use the standard transform analysis to compute the conditional joint moment generating function (A.14).

To that end, using (1) and (A.2), we first apply Itô's Lemma to obtain the dynamics

$$\frac{dD_t^a L_t^b}{D_t^a L_t^b} = \left[a\mu + \frac{1}{2}\left(a\left(a-1\right)+b\left(b-1\right)\delta^2\frac{1}{\sigma^2}\right)V_t\right]dt + a\sqrt{V_t}d\omega_{1t} - b\delta\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h,$$

for all numbers a and b. Next, we rewrite the process $D_t^a L_t^b$ in terms of the martingale Q and the finite variation process N by defining $D_t^a L_t^b = Q_t N_t$ with

$$\begin{aligned}\frac{dQ_t}{Q_t} &= a\sqrt{V_t}d\omega_{1t} - b\delta\frac{1}{\sigma}\sqrt{V_t}d\omega_{2t}^h,\\ \frac{dN_t}{N_t} &= \left[a\mu + \frac{1}{2}\left(a\left(a-1\right)+b\left(b-1\right)\delta^2\frac{1}{\sigma^2}\right)V_t\right]dt,\end{aligned}$$

which implies $\mathrm{E}_t^h\left[D_T^a L_T^b\right] = \mathrm{E}_t^h\left[Q_T N_T\right]$. Next, we define $d\omega_{1t}^Q = d\omega_{1t} - a\sqrt{V_t}dt$ and $d\omega_{2t}^Q = d\omega_{2t}^h + b\delta\frac{1}{\sigma}\sqrt{V_t}dt$ with the likelihood ratio process

$$\frac{d\mathbb{P}^Q}{d\mathbb{P}^h} = Q_T = e^{\int_0^T a\sqrt{V_u}d\omega_{1u} - \int_0^T b\delta\sigma^{-1}\sqrt{V_u}d\omega_{2u}^h - \frac{1}{2}\int_0^T\left(a^2+b^2\delta^2\sigma^{-2}\right)V_u du},$$

and by changing the measure to $\mathbb{P}^Q$, we obtain the required expectation as $\mathrm{E}_t^h\left[D_T^a L_T^b\right] = D_t^a L_t^b \mathrm{E}_t^Q\left[\frac{N_T}{N_t}\right]$, where $\frac{N_T}{N_t} = e^{\int_t^T (a\mu + cV_u)du}$ with the constant $c \equiv \frac{1}{2}\left[a(a-1) + b(b-1)\delta^2\sigma^{-2}\right]$. Since the dynamics of V under measure $\mathbb{P}^Q$ becomes a standard square-root process $dV_t = \left(\kappa\overline{V} - \tilde{\kappa}V_t\right)dt + \sigma\sqrt{V_t}d\omega_{2t}^Q$, where $\tilde{\kappa} \equiv \kappa + \left(b - \frac{1}{2}\right)\delta - \beta$, using the standard moment generating function of the square-root process, we obtain (A.14) where the deterministic functions solve the ODEs

$$\begin{aligned}
\frac{d}{dt}A(t;a,b) &= -\kappa\overline{V}B(t;a,b), \\
\frac{d}{dt}B(t;a,b) &= -c + \tilde{\kappa}B(t;a,b) - \frac{1}{2}\sigma^2 B^2(t;a,b),
\end{aligned}$$

with $A(T;a,b) = B(T;a,b) = 0$, whose solutions are as in (A.16) and (A.17), with the restriction $\tilde{\kappa}^2 > 2c\sigma^2$.

The subjective dynamics of the $M_t(a,b)$ in (A.15) is obtained by applying Itô's Lemma to (A.14) while employing the dynamics in (1), (2), (A.2), and the ODEs above. $\square$

Appendix B: Parameter Values

In this Appendix, we discuss the parameter values employed in our Figures. We note that the behaviors of the equilibrium quantities depicted in our Figures are typical and do not vary much with alternative plausible parameter values.

The interest rate and the fundamental mean growth rate do not affect any of our key quantities, apart from the stock price level, so we simply them to $r = 1\%$ and to $\mu = 2\%$, consistently with data and other works in the literature. We set the long-run mean of the fundamental variance to $\overline{V} = 1.2\%$, implying the average volatility of $\sqrt{\overline{V}} = 11\%$, which is consistent with the time-series average of the aggregate dividend volatility as reported in Beeler and Campbell (2012). We set the fundamental variance mean reversion speed, which is also the mean reversion speed of the realized variance of our benchmark model in the limit, $T \to \infty$, to $\kappa = 0.35$, since it roughly corresponds to the reported first-order auto-correlation of 0.70 for realized variance in Bollerslev, Tauchen, and Zhou (2009). We set the fundamental variance volatility parameter to the corresponding volatility value in the stochastic volatility model estimation of Andersen, Benzoni, and Lund (2002, Table 6), $\sigma = 6\%$. We set the disagreement parameter $\delta = 0.40$ so that the stock return volatility

Table B1. Parameter values. This table reports the parameter values used in our numerical illustrations.

Parameter	Symbol	Value
Interest rate	r	0.01
Fundamental mean growth rate	μ	0.02
Fundamental variance long-run mean	$\overline{V}$	0.012
Fundamental variance mean reversion speed	κ	0.35
Fundamental variance volatility	σ	0.06
Volatility disagreement	δ	0.40
Payoff time	T	50
Current time	t	25

at the steady state of our main model is comparable to the average volatility of the S&P 500, 15.5%. Finally, we set the time to maturity $T - t$ to 25 years so that model horizon is comparable to the duration of the aggregate stock market.[31] To that end, we set $T = 50$ years and take the model evaluation time to be $t = 25$ years. This procedure yields the parameter values in Table B1.

Table B2 reports the quantitative effects of volatility disagreement in the steady-state of our main model $\left(\mathrm{w}_t = 1/2,\ V_t = \overline{V}\right)$ relative to the benchmark economy, along with corresponding empirical evidence. The stock return volatility and risk premium evidence is from Beeler and Campbell (2012), but many other studies find similar magnitudes for these quantities. As compared to the benchmark economy, we see that the presence of volatility disagreement increases both the volatility and risk premium. However, the stock risk premium 2.39% is lower than the evidence. This finding is also unsurprising since investors have low risk aversion (logarithmic preferences), they require a relatively low risk premium to hold the stock in our model. We see that the presence of volatility disagreement significantly increases the leverage effect compared to the benchmark economy, raising its magnitude roughly from -17% to -70%. The reported evidence is from Aït-Sahalia and Kimmel (2007), which is broadly consistent with other estimates of this quantity in the literature, which are typically found to within the range of -0.50 to -0.90 (e.g., Andersen, Benzoni, and Lund (2002),

[31] Most researchers find the stock market duration to be around 20-30 years using the classic dividend growth model, which implies the stock duration as the average price-dividend ratio. See, for example, a recent work of Van Binsbergen (2020) who finds that the aggregate stock market duration to lie somewhere between 20 and 50 years.

Table B2. Quantitative effects of volatility disagreement. This table reports the effects of volatility disagreement on our key economic quantities using the parameter values in Table B1, as well as the corresponding reported empirical evidence. The benchmark economy quantities are obtained by setting the fundamental variance to its long-run mean $V_t = \overline{V}$, the main economy quantities are obtained by additionally setting the wealth-share to its long-run mean, $\mathrm{w}_t = 1/2$. All quantities are reported in annualized percentage form. The empirical evidence for the stock return volatility and risk premium are from Beeler and Campbell (2012), the leverage effect is from Aït-Sahalia and Kimmel (2007), the variance risk premium and the swap rate are from Bollerslev, Tauchen, and Zhou (2009), who report the monthly mean for those quantities as -18.30 and 33.23 (IV) in percentage squares. We multiply these reported quantities by 12 and divide by 100 to obtain annualized quantities in percentage form.

		Benchmark model	Our model	Evidence
Stock return volatility	$\sqrt{v}$	11.12	**15.47**	16.52
Stock risk premium	π_S	1.24	**2.39**	6.36
Leverage effect	ρ	-17.14	**-70.6**	-75.0
Variance risk premium	π_v	-0.01	**-0.16**	-2.20
Variance swap rate	y	1.25	**2.29**	3.99

Aït-Sahalia, Fan, and Li (2013), Andersen, Bondarenko, and Gonzalez-Perez (2015)). Not only for the equity risk, but investors also require a lower premia in magnitude for the variance risk, resulting with the variance risk premium being smaller than the reported evidence in Bollerslev, Tauchen, and Zhou (2009), even though compared to the benchmark model it is amplified from -0.01% to -0.16%. In our baseline calibration, the variance swap rate is 2.29%, which is somewhat smaller than the evidence in Bollerslev, Tauchen, and Zhou (2009).

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