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# RESERVING CONSECUTIVE LAYERS OF INWARDS EXCESS-OF-LOSS REINSURANCE

by

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# RESERVING CONSECUTIVE LAYERS OF INWARDS EXCESS-OF-LOSS REINSURANCE

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# Summary

Consider an excess of loss reinsurance arranged in a number of layers. A loss reserve is required for each layer.

There are two major reasons why the independent application of some conventional loss reserving technique to each layer is inappropriate. First, the experiences in different layers in respect of a particular treaty year will be linked; favourable or adverse experience in one layer is likely to be reflected in favourable or adverse experience in the next. Second, experience data will typically become sparse in the higher layers, rendering analysis in isolation from other layers relatively uninformative.

The purpose of the present paper is to analyse the linkages between the loss experiences of different layers, and apply these to obtain linked loss reserves. A numerical example is provided.

# 1 Introduction

Consider the case of a reinsurer of two or more consecutive layers of an excess-of-loss (XoL) treaty. Suppose that the usual triangulation of incurred losses is available separately in respect of each layer, but no other claims information is available.

It would be possible, in reserving, to apply some particular type of analysis, e.g. chain ladder or some variant thereof, to each layer. To do this independently for each layer would, however, fail to recognise the coupling of consecutive layers' experiences.

For example, heavy experience in one layer will increase the frequency and/or the severity of hits on the next layer above. It is preferable, therefore, for each layer's experience to be modelled in a manner which takes due account of the modelled experience in the layer immediately below.

The following sections make some suggestions as to a very basic form of such modelling. Undoubtedly, more sophisticated techniques could be developed, as is briefly discussed in Section 8.

# 2 Statement of the problem

The purpose of the present section is to describe in precise terms the problem discussed in Section 1.

Consider $K + 1$ layers of insurance, labelled layers $0, 1, 2, \ldots, K$. The labelling begins at $0$ to accommodate a ground-up layer which may not be reinsured, but for which experience is available. The use of a ground-up layer is not essential to the following analysis, however. The $0\text{-}th$ layer may just as well be taken as a reinsured layer.

The layer number will be represented throughout by a bracketed superscript.

Let

$L_{ij}^{(k)}$ = the incurred loss in layer $k$ deriving from treaty year $i$ and as recorded at the end of development year $j$ (=1, 2, etc.).

The data assumed available comprise an incurred loss triangulation for each layer, i.e. $L_{ij}^{(k)},\ i=1,...,I;\ j=1,2,...,I-j+1;\ k=0,1,...,K.$

Let

$d_i^{(k)}$ = upper limit of layer $k$,

$W_i^{(k)}$ = $d_{ij}^{(k)},\ k=0;$

= $d_i^{(k)}-d_i^{(k-1)},\ k=1,2,...,K;$

= height of layer $k$.

With this notation, layer $k(>0)$ reinsures $W_i^{(k)}$ excess of $d_i^{(k-1)}$.

Incurred losses in a layer may be zero. The zero and non-zero cases are considered separately in each layer:

$$P_{ij}^{(k)} = Prob[L_{ij}^{(k)}=0], \tag{2.1}$$

$$A_{ij}^{(k)} = E[L_{ij}^{(k)} \mid L_{ij}^{(k)} \neq 0], \tag{2.2}$$

for values of $i,j$ representing the future, i.e. $i+j>I+1$.

It will be convenient to represent the loss $L_{ij}^{(k)}$ as a multiple of the relevant layer height. Thus, define

$$Q_{ij}^{(k)} = L_{ij}^{(k)} / W_i^{(k)}, \tag{2.3}$$

and

$$R_{ij}^{(k)} = A_{ij}^{(k)} / W_i^{(k)}. \tag{2.4}$$

Further, define

$$\begin{aligned} \Delta Q_{ij}^{(k)} &= Q_{i,j+1}^{(k)} - Q_{ij}^{(k)} \\ &= \Delta L_{ij}^{(k)} \ / \ W_i^{(k)}, \end{aligned} \tag{2.5}$$

and $\Delta R_{ij}^{(k)}$ similarly. It is assumed that $\Delta L_{ij}^{(k)} \geq 0$.

A version of the chain ladder will be applied to the bottom layer. Age-to-age ratios will therefore be required. In fact, it will be useful to define

$$Z_{ij} = \log[L_{i,j+1}^{(0)} / L_{ij}^{(0)}]. \tag{2.6}$$

# 3 General structure of the model

Since layer 0 is a ground-up layer, it is assumed here that

$$P_{ij}^{(0)} = 0. \tag{3.1}$$

It is possible, of course, for the contrary to occur, i.e. for the line of business under consideration to have a non-zero probability of generating no claims at the primary level. It would be unusual to protect risks of this type with a sequence of XoL layers.

In any event, the possibility is ignored here. Its inclusion would not require any major structural change to the paper. It would be necessary simply to model the function $P_{ij}^{(0)}$.

The functions $P_{ij}^{(k)}$ are modelled for $k > 0$. In fact, the modelling is generally divided into three cases:

$$L_{ij}^{(k)} = 0; \tag{3.2}$$

$$L_{ij}^{(k)} \neq 0,\ \Delta L_{i,j-1}^{(k)} = 0; \tag{3.3}$$

$$\Delta L_{i,j-1}^{(k)} \neq 0, \tag{3.4}$$

where

$$\Delta L_{i,j-1}^{(k)} = L_{ij}^{(k)} - L_{i,j-1}^{(k)}. \tag{3.5}$$

The model excludes the possibility of $L_{ij}^{(k)} = 0$ when $L_{i,j-1}^{(k)} > 0$, and so the first and third of the enumerated cases are in fact mutually exclusive, as are other pairs of cases.

The relevance of the case $\Delta L^{(k)} = 0$ is that it implies $\Delta L^{(k+1)} = 0$. The distribution of $\Delta L^{(k+1)}$ will be continuous for strictly positive values but will have a discrete mass at zero. The separation of cases (3.3) and (3.4) recognises this in the modelling of $\Delta L^{(k+1)}$. Case (3.2) is a special case of $\Delta L^{(k)} = 0$.

# 4 Modelling the upper layers

## 4.1 Probability of nil incurred

To calculate the array of values $P_{ij}^{(k)}$, one must consider the evolution of each underwriting year, in particular transitions of $L_{i,j-1}^{(k)} = \alpha$ to $L_{ij}^{(k)} = \beta$.

It is assumed that the different underwriting years develop independently, and so any particular underwriting year may be considered in isolation from the others. In this case, the suffix $i$ is constant, and so is suppressed through the remainder of the present sub-section unless the contrary is explicitly stated.

The event $L_{ij}^{(k)} = \alpha$ is abbreviated to $\{j,k,\alpha\}$. Similarly, the twofold event $L_{ij}^{(k-1)} = \alpha$, $L_{ij}^{(k)} = \beta$ is denoted by $\{j,k,\alpha,\beta\}$. When $A$ appears in place of $\alpha$ in this last expression, it denotes $A_{ij}^{(k-1)}$.

The expression $\{j,k,\alpha(>0),\beta\}$ will denote $\{j,k,\alpha,\beta\}$, i.e. relating to a specific value of $\alpha$, but subject to the restriction that $\alpha>0$. In distinction from this, $\{j,k,\alpha>0,\beta\}$ will denote the event $L_{ij}^{(k-1)} > 0$, $L_{ij}^{(k)} = \beta$, i.e. unrelated to a specific value of $\alpha$.

Transitions of the type mentioned above will be influenced by experience in the next layer below, specifically by $L_{i,j-1}^{(k-1)}$ and $L_{ij}^{(k-1)}$ (suffix $i$ shown for consistency with the earlier notation).

It will be necessary, therefore, to consider transitions of pairs

$$[L_{i,j-1}^{(k-1)},\ L_{i,j-1}^{(k)}] \rightarrow [L_{ij}^{(k-1)},\ L_{ij}^{(k)}], \tag{4.1}$$

or, in the abbreviated notation,

$$\{j-1,k,\alpha,\beta\} \rightarrow \{j,k,\gamma,\delta\}. \tag{4.2}$$

There are three distinct types of state to be recognised in such transitions. On the left side, for example, of (4.2) they are:

$\alpha = \beta = 0;$

$\alpha \neq 0,\ \beta = 0;$

$\alpha \neq 0,\ \beta \neq 0;$

respectively.

The matrix of transitions between these states, written in the unabbreviated notation, appears as follows.

| | $L_{ij}^{(k-1)}=0,\ L_{ij}^{(k)}=0$ | $L_{ij}^{(k-1)}=\gamma\neq 0,\ L_{ij}^{(k)}=0$ | $L_{ij}^{(k-1)}=\gamma\neq 0\ L_{ij}^{(k)}=\delta\neq 0$ |
|---|---|---|---|
| $L_{i,j-1}^{(k-1)}=0,\ L_{i,j-1}^{(k)}=0$ | $\bar{*}$ | $*$ | $*$ |
| $L_{i,j-1}^{(k-1)}=\alpha\neq 0,\ L_{i,j-1}^{(k)}=0$ | 0 | $\bar{*}$ | $*$ |
| $L_{i,j-1}^{(k-1)}=\alpha\neq 0,\ L_{i,j-1}^{(k)}=\beta\neq 0$ | 0 | 0 | 1 |

where the entries $\bar{*}$ are complementary to the remainder of the rows to give a unit row sum.

Let this matrix be denoted by $M_j^{(k)}$. Appendices A and B develop the methodology for evaluating $M_j^{(k)}$ for various $k(>0)$ and $j$. The following results are reproduced from there.

The quantities found there to be required are:

$Prob[\{j,k,0\}\ \mid\ \{j,k-1,A\},\ \{j-1,k,\alpha,0\}],$

$Prob[\{j,k-1,\gamma>0\}\ \mid\ \{j-1,k-1,0\}],$

$Prob[\{j-1,k-1,\alpha,0\}].$

Denote the three respective sets of these quantities by $N_j^{(k)}$, $R_j^{(k-1)}$, $S_{j-1}^{(k-1)}$. Note that the original target of the investigation, $Prob[\{j,k,\alpha\}]$, may be expressed in terms of $S_j^{(k)}$, thus:

$$Prob[\{j,k,\alpha\}] = Prob[\{j,k,0,\alpha\}] + Prob[\{j,k,\beta>0,\alpha\}]. \qquad (4.3)$$

The $S_j^{(k)}$ and $M_j^{(k)}$ may be evaluated recursively with the following order of evaluation, and with $J$ denoting $I-i+1$, i.e. latest development year:

$$S_j^{(0)}\ (all\ j),\ S_J^{(k)}\ (all\ k)\quad \text{[given]}$$

then

$$M_j^{(1)},\ j=J+1,\ J+2,\ etc.$$

$$S_j^{(1)},\ j=J+1,\ J+2,\ etc.$$

$$M_j^{(2)},\ S_j^{(2)},\ etc.$$

Here $S_j^{(0)}$ is treated as given since, by convention, $\{j,0,\alpha,0\} \equiv \{j,0,0\}$, so that (3.1) gives

$$Prob[\{j,0,0\}] = 0.$$

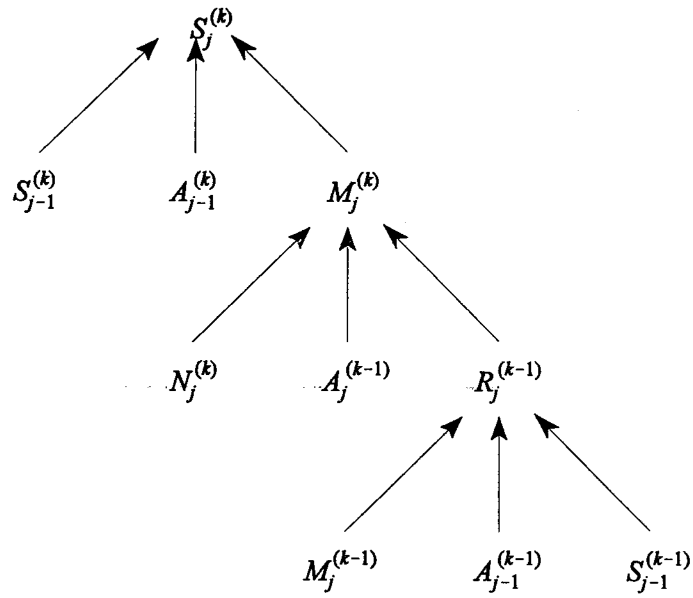
Similarly $M_j^{(1)}$ is a special case of $M_j^{(k)}$. Specifically, (A.6) gives (approximately):

$$Prob[\{j,1,\gamma>0,\delta\} \mid \{j-1,1,\alpha,\beta\}]$$

$$= Prob[\{j,1,\delta\} \mid \{j,0,A\},\ \{j-1,1,\alpha,\beta\}], \tag{4.4}$$

and this last quantity is dealt with below.

The remainder of the recursion follows the schematic appearing hereunder. This is the same as the first schematic of Appendix A, but rearranged to show the $S_j^{(k)}$ as the target of the recursion, in accordance with (4.3).




The equations required for the recursion, in order, are (from Appendix A):

**For $S_j^{(k)}$,**

$$Prob[\{j,k,0,0\}] = Prob[\{j-1,k,0,0\}] \times Prob[\{j,k,0,0\} | \{j-1,k,0,0\}], \tag{4.5}$$

$$Prob[\{j,k,\alpha>0,0\}] = Prob[\{j-1,k,0,0\}] \times Prob[\{j,k,\alpha>0,0\} | \{j-1,k,0,0\}]$$

$$+ Prob[\{j-1,k,\gamma>0,0\}] \times Prob[\{j,k,\alpha>0,0\} | \{j-1,k,A,0\}]. \tag{4.6}$$

**For $R_j^{(k)}$,**

$$Prob[\{j,k,\gamma>0] \; | \; \{j-1,k,0\}]$$

$$= \Big\{Prob[\{j,k,\gamma>0\} \; | \; \{j-1,k,0,0\}] \times Prob[\{j-1,k,0,0\}]$$

$$+ Prob[\{j,k,\gamma>0\} \; | \; \{j-1,k,A,0\}] \times Prob[\{j-1,k,\alpha>0,0\}]\Big\} / Prob[\{j-1,k,0\}]. \tag{4.7}$$

Note the special case for $k=1$,

in which

$$Prob[\{j-1,1,0,0\}] = 0, \tag{4.8}$$

because of (3.1).

**For $M_j^{(k)}$**,

$$Prob[\{j,k,\gamma>0,\delta] \mid \{j-1,k,\alpha(>0),\beta\}]$$

$$= Prob[\{j,k,\delta\} \mid \{j,k-1,A\}, \{j-1,k,\alpha(>0),\beta\}], \tag{4.9}$$

$$Prob[\{j,k,\gamma>0,\delta\} \mid \{j-1,k,0,0\}]$$

$$= Prob[\{j,k,\delta\} \mid \{j,k-1,A\}, \{j-1,k,0,0\}]$$

$$\times\ Prob[\{j,k-1,\gamma>0\} \mid \{j-1,k-1,0\}]. \tag{4.10}$$

The conditional probability which appears on the right sides of (4.4), (4.9) and (4.10) respectively is discussed in Appendix B. There the following approximation is adopted:

$$Prob[\{j,k,0\} \mid \{j,k-1,l_{ij}^{(k-1)}\}, \{j-1,k,l_{i,j-1}^{(k-1)},0\}]$$

$$= \{1 + \exp f\,[q_{ij}^{(k-1)}, \Delta q_{i,j-1}^{(k-1)}]\}^{-1}, \tag{4.11}$$

with

$$\begin{aligned} f(q,p) &= \beta_0[q/(1+q)]\ \log\ (\beta_1 p),\ q\geq 1; \\ &= -\infty,\ q<1, \end{aligned} \tag{4.12}$$

for constants $\beta_0$, $\beta_1 > 0$.

Here $q$ and $\Delta q$ are defined in terms of $l$ in the same way as $Q$ and $\Delta Q$ are defined in terms of $L$ (see (2.3) and (2.5)).

At the end of the recursion illustrated by the earlier schematic, all $S_j^{(k)}$ (and other quantities) will be available. These may be used as inputs to (4.3) to yield $Prob[\{j,k,0\}]$, the probability of nil incurred.

## 4.2 Size of non-zero incurred

Section 4.1 calculates probabilities of zero incurred and of non-zero incurred. In the event that incurred is non-zero, it will be necessary to estimate its expected size.

Amounts of incurred losses are expressed in terms of $R_{ij}^{(k)}$ (see (2.4)), whose evolution with development year is now examined.

If movement occurs between $Q_{ij}^{(k-1)}$ and $Q_{i,j+1}^{(k-1)}$, i.e. $\Delta Q_{ij}^{(k-1)} > 0$, it may generate movement between $Q_{ij}^{(k)}$ and $Q_{i,j+1}^{(k)}$, i.e. $\Delta Q_{i,j-1}^{(k)} > 0$. Even if there is no movement in layer $k-1$, i.e. $\Delta Q_{ij}^{(k-1)} = 0$, movement may still occur in layer $k$.

This may reasonably be modelled as:

$$\Delta R_{ij}^{(k)} = a^{(k)} \Delta R_{ij}^{(k-1)} + R_{ij}^{(k)} g^{(k)}(j), \tag{4.13}$$

for constant $a^{(k)} > 0$ and suitable function $g^{(k)}(.)$.

Note that, in the absence of movements in layer $k-1$, $g^{(k)}(j)$ acts as an age-to-age factor, and must therefore converge to 0 sufficiently rapidly with increasing $j$ if $R_{ij}^{(k)}$ is to be asymptotically constant.

A simple possibility is

$$g^{(k)}(j) = b^{(k)}j^{-s}, \tag{4.14}$$

where $b^{(k)}(>0)$ is a constant, and $s$ would need to exceed 1 for $R_{ij}^{(k)}$ to converge with increasing $j$.

Estimates $\hat{a}^{(k)}$, $\hat{g}^{(k)}(j)$ of $a^{(k)}$, $g^{(k)}(j)$ are made on the basis of observed $R_{ij}^{(k)}$ and the relation (4.13). This is done by means of regression in the present paper.

Relation (4.13) then enables $R_{i,j+1}^{(k)}$ to be predicted recursively on the basis of $\Delta R_{ij}^{(k-1)}$ and $R_{ij}^{(k)}$. This means that **future** values of $R_{ij}^{(k)}$, $j = I - i + 2$, $I - i + 3$, etc. may be predicted on the basis of $R_{i,I-i+1}^{(k)}$ **provided that** future values of $R_{ij}^{(k-1)}$ are available.

# 5 Modelling the bottom layer

By (3.1), it is assumed here that

$$P_{ij}^{(0)} = 0. \tag{5.1}$$

With this assumption, age-to-age factors are meaningful. The bottom layer is modelled here by means of a regression-based variation of the chain ladder.

Consider the variable $Z_{ij}$, defined by (2.6). It is assumed that

$$Z_{ij} + K \sim \log N(\mu_{ij}, \sigma_{ij}^2), \tag{5.2}$$

for a suitable constant $K$, and with $\mu_{ij}$, $\sigma_{ij}^2$ specific parametric forms depending on $i$, $j$.

In fact, it will be assumed here that

$$u_{ij} = h_1(i) + h_2(j), \tag{5.3}$$

$$\sigma_{ij}^2 = \sigma^2 / h_3(j), \tag{5.4}$$

for suitable functional forms $h_1(.)$, $h_2(.)$ and $h_3(.)$, and for constant $\sigma^2 > 0$.

Values of $\sigma^2$ and of the parameters involved in these functional forms are estimated (by regression techniques in the present paper), leading to fitted values $\hat{Y}_{ij}$ corresponding to observations $log(Z_{ij} + K)$. The model values of the age-to-age factors $E[L_{i,j+1}^{(0)} / L_{ij}^{(0)}]$ are calculated as:

$$\exp \{ \exp[\hat{Y}_{ij} + ½ \, \hat{\sigma}^2 / h_3(j)] - K \}, \tag{5.5}$$

where $\hat{\sigma}^2$ is the estimate of $\sigma^2$.

# 6 Assembling the predictions

Sections 4.1, 4.2 and 5 deal with three sets of predictions. Each individual prediction is dependent on data and/or one or more other predictions. They therefore need to be evaluated in a specific order, as set out below.

**Bottom Layer (Section 5)**

For each treaty year, develop the most recent incurred loss figure through future years by means of the chain ladder factors (5.5).

**Upper Layers - size of non-zero incurred (Section 4.2)**

Predict future values of $R_{ij}^{(1)}$ on the basis of the predicted $R_{ij}^{(0)}$. Hence develop future values of $A_{ij}^{(1)}$.

Then develop future $A_{ij}^{(2)}$ from $A_{ij}^{(1)}$, and so on.

**Upper layers - probability of nil incurred (Section 4.1)**

The precedence of the quantities required for prediction of the $P_{ij}^{(k)}$ was examined in detail in Section 4.1. The schematic there indicates that the future $S_j^{(1)}$ can be evaluated recursively over increasing $j$ on the basis of the $S_j^{(0)}$ (which are degenerate because of (3.1)). The $S_j^{(1)}$ then yield predictions of the $P_{ij}^{(1)}$. The $P_{ij}^{(2)}$ are then predicted on the basis of the $S_j^{(1)}$, the $P_{ij}^{(3)}$ from the $S_j^{(2)}$, and so on.

**Upper layers - incurred losses**

These are predicted by means of the simple formula:

$$E[L_{ij}^{(k)}] = [1 - P_{ij}^{(k)}]\, A_{ij}^{(k)}. \tag{6.1}$$

# 7 Numerical example

The following example uses a real data set which has been disguised by adjustment. The adjustment has been made carefully, however, in such a way as not to disturb the main features of the data.

Because the recommended treatment of upper layers differs from that of the bottom layer, the example deals with the bottom layer and one upper layer. Because it is desirable to illustrate the methodology with a sparse data triangle, the upper layer chosen has $k>1$. For the sake of the example, suppose $k=3$.

Although details of layers other than $k=0$, 3 are not given here, they have in fact been used in the estimation of layer-independent parameters, such as appear in (4.12).

The incurred loss data for the example are set out in Appendix C. Those of Layer 3 (Table C.3) are seen to be quite sparse and generally unsuitable in isolation for application of conventional projection techniques. It is here that the activity in the lower layers, e.g. Tables C.1 and C.2, becomes useful.

The schema described in Sections 4 to 6 has been applied and selected numerical details given in Appendix D. Appendix D.1 deals with Layer 0 and Appendix D.2 with Layer 3. The results obtained are as follows.

**Table 7.1**
**Projected claim costs**

| Treaty year | Incurred losses | | | |
|---|---|---|---|---|
| | Layer 0 | | Layer 3 | |
| | to 1995 | projected ultimate | to 1995 | projected ultimate |
| | $M | $M | $M | $M |
| 1983 | 29.4 | 29.9 | | |
| 1984 | 168.1 | 174.3 | 11.6 | 12.2 |
| 1985 | 77.9 | 83.4 | 0 | 1.0 |
| 1986 | 129.6 | 142.4 | 1.8 | 2.7 |
| 1987 | 125.6 | 142.2 | 0 | 2.5 |
| 1988 | 180.5 | 212.7 | 45.0 | 48.8 |
| 1989 | 91.0 | 119.4 | 0 | 3.6 |
| 1990 | 127.2 | 182.9 | 93.7 | 102.8 |
| 1991 | 49.6 | 102.1 | 7.7 | 12.1 |
| 1992 | 55.3 | 157.3 | 10.0 | 17.4 |
| 1993 | 44.6 | 209.7 | 0 | 21.1 |
| 1994 | 7.8 | 122.0 | 10.0 | 41.8 |

In practice, the projected results in respect of the more recent treaty years might require reconsideration due to their relative unreliability, but this has not been done here.

# 8 Further research

All of the above development has been carried out on the basis of aggregate data. In practice, it may be possible to obtain individual loss data, particularly in those layers where simple triangulation provides little information.

Some additional data would enable some exploration of individual claim size distribution, which could then supplement the above models of loss development. The precise form of the additional analysis would depend on the form of additional data available. For example, a complete history of development of each claim might be available; or alternatively, individual estimated loss sizes might be available only at the latest date of experience tabulation.

# APPENDIX A
# Probabilistic relations between layers

## A.1 An approximation

**Lemma.** Let $X$ be a random variable, and $A$, $B$ specific events. Then, as a first order approximation in $X$,

$$Prob[X>0, A|B] = Prob[A|X = E[X|X>0, B],B] \times Prob[X>0|B]. \tag{A.1}$$

**Proof.** Let $p$ denote a generic pdf. Then

$$\begin{aligned} Prob[X>0, A|B] &= \int_{x>0} p(X=x, A|B)dx \\ &= \int_{x>0} p(A|X=x, B)\, p(X=x|B)dx. \end{aligned} \tag{A.2}$$

Now expand $p(A|X=x,B)$ in a Taylor series about $x = E[X|X>0,B] = \mu$ say. Then

$$p(A|X=x,B) = p(A|X=\mu,B) + (x-\mu)\frac{\partial p}{\partial x}(A|X=x,B)|_{x=\mu} + O(x-\mu)^2 \tag{A.3}$$

Substitute (A.3) in (A.2) to obtain as an approximation:

$$\begin{aligned} Prob[X>0, A|B] &= p(A|X=\mu,B)\int_{x>0} p(X=x|B) \\ &+ \frac{\partial p}{\partial x} p(A|X=x,B)|_{x=\mu} \int_{x>0} (x-\mu)p(X=x|B) \end{aligned} \tag{A.4}$$

By definition of $\mu$, the second integral in (A.4) is zero. Then (A.4) is equivalent to result sought.

□

## A.2 Transition probabilities

Consider the matrix $M_j^{(k)}$ set out in Section 4.1. It contains three elements (denoted by *) requiring calculation. Each is of the form:

$$Prob[\{j,k,\gamma>0,\delta\}|\{j-1,k,\alpha,\beta\}]. \tag{A.5}$$

By the lemma, this may be approximated by:

$$Prob[\{j,k,\delta\} \mid \{j,k-1,A\},\{j-1,k,\alpha,\beta\}] \times Prob[\{j,k-1,\gamma>0\} \mid \{j-1,k,\alpha,\beta\}], \qquad (A.6)$$

where the context makes clear that $A$ denotes $A_{ij}^{(k-1)}$.

Consider the specific cases of $\alpha$, $\beta$ that arise in the final member of (A.6). These are enumerated in Section 4.1 just prior to setting out $M_j^{(k)}$. Evidently,

$$Prob[\{j,k-1,\gamma>0\} \mid \{j-1,k,\alpha,\beta\}] = 1 \; for \; \alpha>0. \qquad (A.7)$$

This relates to the second and third rows of $M_j^{(k)}$.

In this case (A.6) may be reduced to give:

$$\begin{aligned} &Prob[\{j,k,\gamma>0,\delta\} \mid \{j-1,k,\alpha,\beta\}] \\ &= Prob[j,k,\delta\} \mid \{j,k-1,A\}, \{j-1,k,\alpha,\beta\} \; for \; \alpha>0. \end{aligned} \qquad (A.8)$$

Consider the alternative cases in which $\alpha = 0$, i.e. the (1, 2) and (1, 3) elements of $M_j^{(k)}$. Here $\beta = 0$ necessarily. Then

$$Prob[\{j,k-1,\gamma>0\} \mid \{j-1,k,0,0\}] = Prob[\{j,k-1,\gamma>0\} \mid \{j-1,k-1,0\}]. \qquad (A.9)$$

From (A.6), (A.8) and (A.9), all transition probabilities may be calculated from probabilities of two forms, viz:

$$Prob[\{j,k,\delta\} \mid \{j,k-1,A\},\{j-1,k,\alpha,\beta\}], \qquad (A.10)$$

and

$$Prob[\{j,k,\gamma>0\} \mid \{j-1,k,0\}]. \qquad (A.11)$$

## A.3 The probability (A.11)

Recall the general result for conditional probabilities:

$$Prob[A \mid B_1 \, or \ldots or \, B_n] = \frac{\sum_{m=1}^{n} Prob[A \mid B_m] \; Prob[B_m]}{\sum_{m=1} Prob[B_m]}, \qquad (A.12)$$

provided that the events $B_1,\ldots,B_n$ are mutually exclusive.

In the present context, this gives:

$$Prob[\{j,k,\gamma>0\} \mid \{j-1,k,0\}]$$

$$= \Big\{Prob[\{j,k,\gamma>0\} \mid \{j-1,k,0,0\}] \times Prob[\{j-1,k,0,0\}]$$

$$+\int Prob[\{j,k,\gamma>0\} \mid \{j-1,k,\alpha(>0),0\}] \times Prob[\{j-1,k,\alpha(>0),0\}]\Big\} / Prob[\{j-1,k,0\}], \quad \text{(A.13)}$$

where the variable of integration is $\alpha$.

The first probability within the integral may be approximated by the replacement of $\alpha$ with its expected value, conditional on positivity (in much the same way as in the lemma), reducing it to:

$$Prob[\{j,k,\gamma,0\} \mid \{j-1,k,A,0\}] \quad \text{(A.14)}$$

Substitution of (A.14) in (A.13) then gives:

$$Prob[\{j,k,\gamma>0\} \mid \{j-1,k,0\}]$$

$$= \Big\{Prob[\{j,k,\gamma>0\} \mid \{j-1,k,0,0\}] \times Prob[\{j-1,k,0,0\}]$$

$$+ Prob[\{j,k,\gamma>0\} \mid \{j-1,k,A,0\}] \times Prob[\{j-1,k,\alpha>0,0\}]\Big\} / Prob[\{j-1,k,0\}]. \quad \text{(A.15)}$$

The conditional probabilities appearing here are in fact the (1, 3) and (2, 3) elements of $M_j^{(k)}$. Their multipliers are unconditional probabilities relating to the end of development year $j-1$.

## A.4 Unconditional probabilities

Consider the unconditional probabilities appearing in (A.15), viz. $Prob\{j-1,k,\alpha,0\}$. These may be calculated by the usual forward recursion involving the transition matrix $M_j^{(k)}$.

$$Prob[\{j-1,k,0,0\}] = Prob[\{j-2,k,0,0\}] \times Prob[\{j-1,k,0,0\} \mid \{j-2,k,0,0\}], \quad \text{(A.16)}$$

$$Prob[\{j-1,k,\alpha>0,0\}] = Prob[\{j-2,k,0,0\} \times Prob[\{j-1,k,\alpha>0,0\} \mid \{j-2,k,0,0\}]$$

$$+ Prob[\{j-2,k,\gamma>0,0\}] \times Prob[\{j-1,k,\alpha>0,0\} \mid \{j-2,k,A,0\}], \quad \text{(A.17)}$$

where the second of the two members on the right is an approximation obtained by application of the lemma to a more precise expression.

Thus, unconditional probabilities at time $j-1$ are expressed in terms of unconditional probabilities at time $j-2$ and the transition matrix $M_{j-1}^{(k)}$.

## A.5 Summary of the recursion

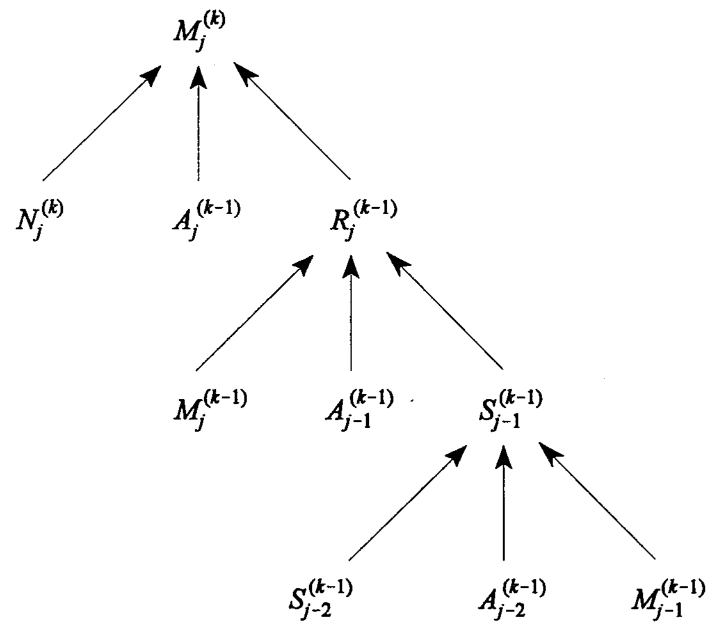
By Appendices A.2 to A.4 (in particular, (A.10), (A.11) and (A.15) - (A.17)), the quantities required for the computation of $M_j^{(k)}$ are:

$$Prob[\{j,k,\delta\} \mid \{j,k-1,A\},\{j-1,k,\alpha,\beta\}], \tag{A.18}$$

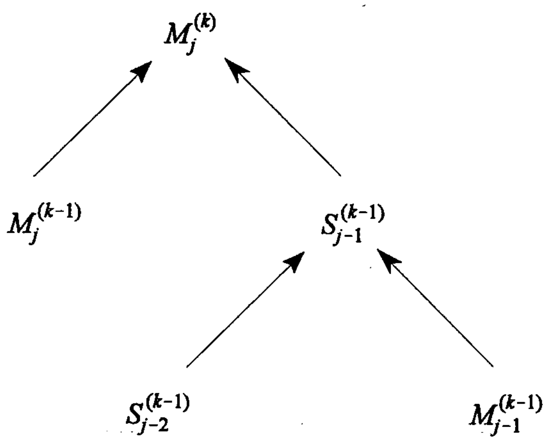
$$Prob[\{j,k-1,\gamma>0\} \mid \{j-1,k-1,0\}], \tag{A.19}$$

$$Prob[\{j-1,k-1,\alpha,0\}]. \tag{A.20}$$

Denote the respective sets of quantities (A.18), (A.19) and (A.20) by $N_j^{(k)}$, $R_j^{(k-1)}$ and $S_{j-1}^{(k-1)}$. Estimation of $N_j^{(k)}$ is the subject of Appendix B. With this notation, the recursion developed in Appendices A.2 to A.4 may be represented diagrammatically as follows.




An abbreviated form of this diagram which recognises just the nodes involving M and S is as follows.




This indicates that the recursion, over $j$ and $k$, must be carried out in the following order:

$S_j^{(0)}$ (all $j$ ), $S_j^{(k)}$ (all $k$) [given]

Then

$M_j^{(1)}$, $j=J+1$, $J+2$, etc.

$S_j^{(1)}$, $j=J+1$, $J+2$, etc.

$M_j^{(2)}$, $S_j^{(2)}$, etc.,

where $J$ denotes $I-i+1$.

Note that the values of $M_j^{(1)}$ require various quantities related to the 0-th layer, e.g.

$$R_j^{(0)} = Prob[L_j^{(0)}>0 \mid L_{j-1}^{(0)} = 0].$$

The modelling of such quantities is treated separately in Section 5.

# APPENDIX B
# Main model for transitions between layers

Appendix A.5 identifies

$$Prob[\{j,k,\delta\} \mid \{j,k-1,A\},\{j-1,k,\alpha,\beta\}], \tag{B.1}$$

as a quantity required for the calculation of $M_j^{(k)}$.

One interpretation of this conditional probability is that the probability of incurred losses in layer $k$ flipping from zero to non-zero in development year $j$ depends on incurred losses in layer $k-1$ in year $j$ and year $j-1$; equivalently, on the **level** of incurred losses in layer in year and their **change** in year $k$.

Thus the probability (B.1) may be expressed in terms of $L_{ij}^{(k-1)}$ and $\Delta L_{i,j-1}^{(k-1)}$; or, by (2.5), equivalently in terms of $Q_{ij}^{(k-1)}$ and $\Delta Q_{i,j-1}^{(k-1)}$.

That is, for example,

$$\begin{aligned} &Prob[\{j,k,0\} \mid \{j,k-1,l_{i,j}^{(k-1)}\},\{j-1,k,l_{i,j-1}^{(k-1)},0\}] \\ &= Prob[L_{ij}^{(k)}=0 \mid L_{ij}^{(k-1)}=l_{ij}^{(k-1)},\ L_{i,j-1}^{(k-1)}=l_{i,j-1}^{(k-1)},\ L_{i,j-1}^{(k)}=0] \\ &= fn[q_{ij}^{(k-1)},\ \Delta q_{i,j-1}^{(k-1)}], \end{aligned} \tag{B.2}$$

where $q$, $\Delta q$ are defined in terms of $l$ in the same way as $Q$, $\Delta Q$ are defined in terms of $L$ (see (2.3) and (2.5)).

Since $Q$ expresses losses $L$ as a multiple of layer width, it is reasonable to adopt the same function for all layers on the right side of (B.2). For convenience this function is written in the logistic form:

$$\begin{aligned} &= Prob[L_{ij}^{(k)}=0 \mid L_{ij}^{(k-1)}=l_{ij}^{(k-1)},\ L_{i,j-1}^{(k-1)}=l_{i,j-1}^{(k-1)},\ L_{i,j-1}^{(k)}=0] \\ &= \{1+exp\ f\ [q_{ij}^{(k-1)},\ \Delta q_{i,j-1}^{(k-1)}]\}^{-1}, \end{aligned} \tag{B.3}$$

where the function $f$ is still to be specified.

Note that $f(q,p)$ will usually be $-\infty$ if $q<1$ (no exhaustions of layer $k-1$). Subject to this,

$$\begin{aligned} f(q,p) &\rightarrow +\infty \text{ as } p \rightarrow \infty \text{ (all } q); \\ &\rightarrow -\infty \text{ as } p \rightarrow 0 \text{ (all } q); \end{aligned} \tag{B.4}$$

and it may further be reasonably assumed that:

$$\partial f / \partial p,\ \partial f / \partial q > 0 \text{ for } q > 0 \text{ and all } p. \tag{B.5}$$

Note that $f(q,p)$ is bounded above for fixed $p$ and $q \rightarrow \infty$; otherwise, the probability in (B.3) would approach 0 by virtue of large $q_{ij}^{(k-1)}$, even if $\Delta q_{i,j-1}^{(k-1)}$ were small.

With this last observation taken into account, together with (B.4) and (B.5), a reasonable form to assume for fitting of $f$ to data is:

$$f(q,p) = \beta_0 [q/(1+q)] \log (\beta_1 p),\ q \geq 1;$$

$$= -\infty,\ q < 1, \tag{B.6}$$

with $\beta_0$, $\beta_1$ positive constants.

# APPENDIX C
# Data for numerical example

## C.1 Incurred Losses

The example is concerned with the modelling of layers $k=0$. The experience in layer 3 is, however, dependent on that in layer 2.

The following are triangulations of incurred losses for layers $k=0,2,3$.

**Table C.1**
**Layer 0**

| Treaty year | Incurred losses to end of | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 |
| | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 |
| 1980 | 3,175 | 2,880 | 10,177 | 10,177 | 13,522 | 13,522 | 13,393 | 13,393 | 13,393 | 13,393 | 12,289 | 12,289 |
| 1981 | 8,255 | 8,380 | 14,380 | 14,380 | 14,630 | 16,491 | 17,426 | 19,426 | 13,191 | 21,016 | 42,871 | 21,103 |
| 1982 | 898 | 878 | 878 | 688 | 688 | 688 | 688 | 688 | 688 | 688 | 688 | 688 |
| 1983 | 2,055 | 20,070 | 23,114 | 26,852 | 31,541 | 29,915 | 26,234 | 27,770 | 29,011 | 28,940 | 29,626 | 29,428 |
| 1984 | 281 | 24,521 | 48,521 | 97,925 | 101,255 | 115,101 | 110,344 | 140,672 | 143,979 | 163,217 | 167,623 | 168,077 |
| 1985 | | 0 | 20,772 | 49,438 | 86,483 | 91,173 | 85,040 | 84,653 | 83,897 | 84,755 | 82,909 | 77,860 |
| 1986 | | | 0 | 10,275 | 71,040 | 99,494 | 113,687 | 121,807 | 112,025 | 121,764 | 129,391 | 129,629 |
| 1987 | | | | 0 | 29,780 | 56,137 | 72,902 | 89,189 | 98,204 | 113,639 | 120,470 | 125,616 |
| 1988 | | | | | 0 | 15,000 | 15,328 | 54,685 | 123,488 | 149,594 | 176,693 | 180,549 |
| 1989 | | | | | | 0 | 0 | 1,250 | 21,231 | 48,202 | 75,254 | 90,957 |
| 1990 | | | | | | | 0 | 16,288 | 78,577 | 68,328 | 121,708 | 127,232 |
| 1991 | | | | | | | | 0 | 0 | 8,175 | 23,873 | 49,615 |
| 1992 | | | | | | | | | 0 | 7,500 | 16,254 | 55,280 |
| 1993 | | | | | | | | | | 0 | 17,295 | 44,586 |
| 1994 | | | | | | | | | | | 0 | 7,750 |

**Table C.2**
**Layer 2**

| Treaty year | Incurred losses to end of | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1990 | 1991 | 1992 | 1993 | 1994 | **1995** |
| | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 |
| 1984 | 0 | 8,750 | 9,688 | 9,892 | 10,072 | 10,075 | 10,075 | 10,075 | 10,075 | 10,076 | 10,076 | 10,076 |
| 1985 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1986 | | | 0 | 0 | 0 | 0 | 0 | 7,500 | 7,500 | 8,657 | 8,792 | 7,500 |
| 1987 | | | | 0 | 0 | 0 | 0 | 11,444 | 11,444 | 11,885 | 8,771 | 8,771 |
| 1988 | | | | | 0 | 0 | 0 | 0 | 1,080 | 9,116 | 15,000 | 22,500 |
| 1989 | | | | | | 0 | 0 | 0 | 0 | 0 | 2,851 | 1,129 |
| 1990 | | | | | | | 0 | 0 | 5,000 | 10,000 | 30,000 | 50,000 |
| 1991 | | | | | | | | 0 | 0 | 0 | 10,000 | 10,000 |
| 1992 | | | | | | | | | 0 | 0 | 0 | 10,847 |
| 1993 | | | | | | | | | | 0 | 0 | 5,000 |
| 1994 | | | | | | | | | | | 0 | 5,000 |

**Table C.3**
**Layer 3**

| Treaty year | Incurred losses to end of | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1984 | 1985 | 1986 | 1987 | 1988 | 1989 | 1990 | 1991 | 1992 | 1993 | 1994 | **1995** |
| | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 |
| 1984 | 0 | 8,750 | 9,688 | 9,892 | 11,620 | 11,624 | 11,624 | 11,624 | 11,624 | 11,625 | 11,625 | 11,625 |
| 1985 | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1986 | | | 0 | 0 | 0 | 0 | 0 | 0 | 2,632 | 1,810 | 1,810 | 1,810 |
| 1987 | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1988 | | | | | 0 | 0 | 0 | 0 | 0 | 15,000 | 27,232 | 45,000 |
| 1989 | | | | | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1990 | | | | | | | 0 | 0 | 10,000 | 20,000 | 52,000 | 93,650 |
| 1991 | | | | | | | | 0 | 0 | 0 | 5,191 | 7,740 |
| 1992 | | | | | | | | | 0 | 0 | 0 | 10,000 |
| 1993 | | | | | | | | | | 0 | 0 | 0 |
| 1994 | | | | | | | | | | | 0 | 10,000 |

The bold heading of the final column of each table draws attention to the fact that this is a **19-month** experience period. All other experience periods are annual.

Note that the data in Table C.1 satisfy assumption (3.1) except in development year 1. This is sufficient for present purposes.

The coverage provided by layers 2 and 3 is as follows.

**Table C.4**
**Layer Limits**

| Year | Layer 2 | | Layer 3 | |
|---|---|---|---|---|
| | Lower limit | Upper limit | Lower limit | Upper limit |
| | $M | $M | $M | $M |
| 1984-88 | 26.25 | 33.75 | 33.75 | 48.75 |
| 1989-94 | 17.5 | 22.5 | 22.5 | 32.5 |

# APPENDIX D
# Numerical Results

## D.1 Bottom Layer

The model is set out in Section 5. The particular forms chosen for $h_1(.)$, $h_2(.)$ and $h_3(.)$ were:

$$h_1(i) = k_1 \max(0, i-1981) + k_2 \max(0, i-1983)$$

$$+ k_3 \max(0, i-1989) + k_4\, I(i=1990)\ , \qquad (D.1)$$

$$h_2(j) = k_5 + k_6 \log j + k_7\, j, \qquad (D.2)$$

$$h_3(j) = 1/5 + (j-6)^2/36 \text{ for } j \geq 2;$$

$$= 25\ [1/5 + (j-6)^2/36] \text{ for } j<2, \qquad (D.4)$$

where $I(.)$ is an indicator function which takes the value 1 when the condition occurring as argument is satisfied, and 0 otherwise.

The function $h_1(.)$ defines a piecewise linear function of treaty year with changes of slope at 1981, 1983 and 1989, and with treaty year 1990 regarded as exceptional.

The function $h_3(j)$ is chosen to produce rough homoscedasticity.

The data set out in Appendix C were analysed by the GLIM package. Outliers, defined as observations producing standardised residuals numerically greater than 3, were given no weight in the regression. This yielded the following parameter estimates.

**Table D.1**

| Parameter | Estimate |
|---|---|
| $k_1$ | -0.3603 |
| $k_2$ | +0.6045 |
| $k_3$ | -0.3796 |
| $k_4$ | -0.4908 |
| $k_5$ | +1.970 |
| $k_6$ | -3.074 |
| $k_7$ | +0.1637 |
| $\sigma^2$ | 1.505 |

The value $K$ in (5.2) was set to zero, and cases of $Z_{ij} \leq 0$ were excluded from the regression.

Note that $h_2(j)$ does **not** converge to zero for increasing $j$. It does however provide reasonable values over the range of $j$ (1 *to* 16) dealt with here. Incurred loss development is ignored beyond development year 16.

The following residual plots indicate reasonable effectiveness of $h_1(.)$ in tracking variations related to treaty year; and of $h_3(.)$ in removing heteroscedasticity.

**Figure D.1**
**Standardised residuals by treaty year**

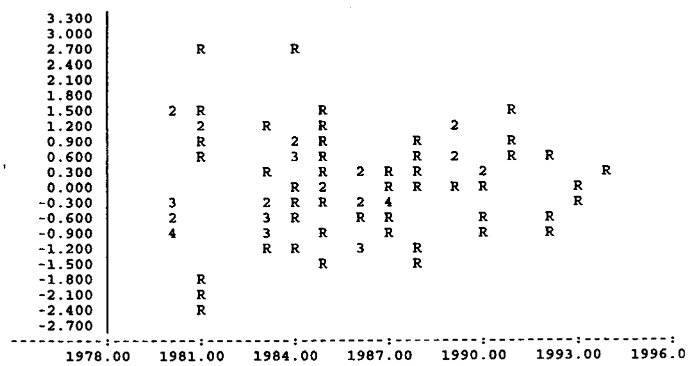
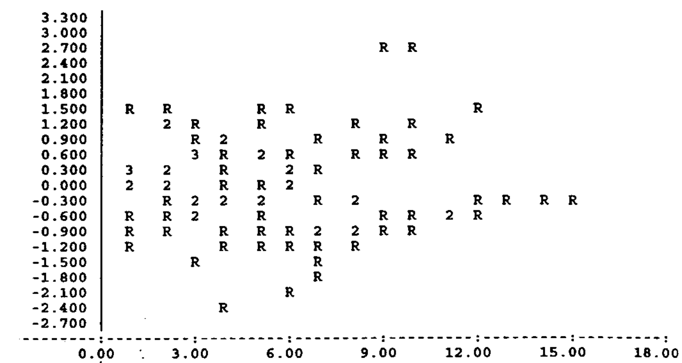



**Figure D.2**
**Standardised residuals by development years**




Model age-to-age factors were produced for this model in accordance with (5.5). These were chained, then applied to the 1995 incurred losses, to give the following results. Further development of treaty years earlier than 1983 has been ignored.

**Table D.2**
**Incurred Losses**

| Treaty year | to 1995 | projected ultimate |
|---|---|---|
| | $000 | $000 |
| 1983 | 29,428 | 29,930 |
| 1984 | 168,077 | 174,306 |
| 1985 | 77,860 | 83,364 |
| 1986 | 129,629 | 142,355 |
| 1987 | 125,616 | 142,212 |
| 1988 | 180,549 | 212,745 |
| 1989 | 90,957 | 119,402 |
| 1990 | 127,232 | 182,878 |
| 1991 | 49,615 | 102,113 |
| 1992 | 55,280 | 157,285 |
| 1993 | 44,586 | 209,722 |
| 1994 | 7,750 | 122,022 |

The reliability of the estimates decreases with increasing treaty years.

## D.2 Upper layers

### *D.2.1 Probability of nil incurred*

The model summarised by (4.11) and (4.12) was fitted to data comprising all transitions from zero incurred at the beginning of a development year to zero or non-zero incurred respectively at the end of that year. Note that $\beta_0$, $\beta_1$ do not depend on layer in (4.12), and so (4.11) was fitted for all $k = 1,2,3,4,5$ simultaneously.

The results were as follows.

**Table D.3**

| Parameter | Estimate |
|---|---|
| $\beta_0$ | 0.8236 |
| $\beta_1$ | 3.6529 |

### *D.2.2 Size of non-zero incurred*

The model summarised by (4.13) and (4.14) was fitted to data comprising all non-zero incurred losses from tables such as C.1 to C.3 for layers $k = 1 \; to \; 5$.

After a small amount of experimentation, $s = 3$ was chosen in (4.14) and $b^{(k)} = b$, independently of $k$. In addition, data became so sparse in the upper layers that $a^{(3)}$, $a^{(4)}$ and $a^{(5)}$ were assumed equal.

The resulting version of (4.13) was fitted to the data using weighted least squares with weight $j^4/(i+j)^2$ associated with observation $\Delta Q_{ij}^{(k)}$ to achieve rough homoscedasticity with respect to development year $j$ and experience year $i+j$ (here $i=1$ corresponds to treaty year 1984, the earliest for which upper layer experience is available).

This weighting factor indicates:

- the strong increase in reliability of the model with increasing development year;
- the fact that variability of experience appears to have increased over past experience years.

The results were as follows.

**Table D.4**

| Parameter | Estimate |
|---|---|
| $a^{(1)}$ | 0.1428 |
| $a^{(2)}$ | 0.3591 |
| $a^{(3)}(=a^{(4)}=a^{(5)})$ | 0.6952 |
| $b$ | 8.265 |

### *D.2.3 Numerical projections*

The models described in Appendices D.2.1 and D.2.2 are applied to produce Tables D.5 and D.6 respectively, applicable to Layer 3.

As indicated by (4.11) and (4.13), these results will depend on year-by-year activity in layers 0 to 2 which is not shown here.

**Table D.5**
**Probability of nil incurred losses in Layer 3**

| Treaty year | Projected probability of nil incurred losses to end of | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
| 1994 | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% |
| 1993 | 100% | 73% | 47% | 31% | 21% | 15% | 12% | 10% | 9% | 8% | 7% | 6% | 5% | |
| 1992 | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | | |
| 1991 | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | | | |
| 1990 | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | | | | |
| 1989 | 100% | 95% | 91% | 88% | 85% | 82% | 80% | 78% | 76% | | | | | |
| 1988 | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | | | | | | |
| 1987 | 100% | 93% | 87% | 81% | 77% | 73% | 70% | | | | | | | |
| 1986 | 0% | 0% | 0% | 0% | 0% | 0% | | | | | | | | |
| 1985 | 100% | 96% | 93% | 89% | 87% | | | | | | | | | |
| 1984 | 0% | 0% | 0% | 0% | | | | | | | | | | |

**Table D.6**
**Expected value of non-zero incurred in Layer 3**

| Treaty year | Model expected value of incurred losses (provided non-zero) to end of | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 |
| | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 |
| 1994 | 10,000 | 24,864 | 28,661 | 31,652 | 34,060 | 35,966 | 37,421 | 38,432 | 39,242 | 39,913 | 40,486 | 40,980 | 41,415 | 41,815 |
| 1993 | **11,339** | 13,102 | 15,130 | 16,888 | 18,326 | 19,411 | 20,062 | 20,582 | 21,017 | 21,393 | 21,722 | 22,016 | 22,295 | |
| 1992 | 10,000 | 11,264 | 12,754 | 13,998 | 14,958 | 15,520 | 15,970 | 16,347 | 16,671 | 16,953 | 17,203 | 17,436 | | |
| 1991 | 7,740 | 8,547 | 9,485 | 10,222 | 10,647 | 10,990 | 11,275 | 11,518 | 11,728 | 11,913 | 12,083 | | | |
| 1990 | 93,650 | 95,674 | 97,310 | 98,587 | 99,622 | 100,476 | 101,190 | 101,795 | 102,312 | 102,766 | | | | |
| 1989 | **12,986** | 13,462 | 13,815 | 14,096 | 14,330 | 14,529 | 14,701 | 14,852 | 14,993 | | | | | |
| 1988 | 45,000 | 45,856 | 46,586 | 47,187 | 47,692 | 48,123 | 48,496 | 48,831 | | | | | | |
| 1987 | **6,717** | 7,021 | 7,300 | 7,539 | 7,747 | 7,930 | 8,100 | | | | | | | |
| 1986 | 1,810 | 2,025 | 2,236 | 2,421 | 2,586 | 2,740 | | | | | | | | |
| 1985 | **6,717** | 6,892 | 7,054 | 7,195 | 7,322 | | | | | | | | | |
| 1984 | 11,625 | 11,837 | 12,041 | 12,227 | | | | | | | | | | |

Comment is required on the bold figures in Table D.6. These are assumed values of $A_{ij}^{(3)}$ for 1995 in those cases where $L_{ij}^{(3)} = 0$. These values are required by (4.13) for the projection of future values of $A_{ij}^{(3)}$.

These artificial values of $A_{ij}^{(3)}$ have been obtained by averaging the values of $R_{ij}^{(3)}$ observed for 1995 in respect of certain treaty years $i$.

With a couple of exceptions, the "certain" treaty years are those preceding the entry under consideration.

Thus, for example, that for $i = 1987$ is taken as the average over $i = 1984, 1986$. The same average is taken for $i = 1985$. The average for $i = 1993$ omits the experience of $i = 1990$ since it appears abnormal. It is calculated as Average (10000, 7740, 45000 × ⅔, 1810 × ⅔, 11625 × ⅔), where the factors of ⅔ reflect the change of layer limits between 1988 and 1989.

The results of Tables D.5 and D.6 are combined by means of (6.1) to produce Table D.7.

**Table D.7**
**Expected value of incurred (zero or non-zero) in Layer 3**

| Treaty year | Model expected value of incurred losses to end of | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 200 |
| | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$000 | US$00 |
| 1994 | 10,000 | 24,864 | 28,661 | 31,652 | 34,060 | 35,966 | 37,421 | 38,432 | 39,242 | 39,913 | 40,486 | 40,980 | 41,415 | 41,81 |
| 1993 | 0 | 3,503 | 7,988 | 11,679 | 14,476 | 16,446 | 17,573 | 18,439 | 19,143 | 19,737 | 20,245 | 20,689 | 21,101 | |
| 1992 | 10,000 | 11,264 | 12,754 | 13,998 | 14,958 | 15,520 | 15,970 | 16,347 | 16,671 | 16,953 | 17,203 | 17,436 | | |
| 1991 | 7,740 | 8,547 | 9,485 | 10,222 | 10,647 | 10,990 | 11,275 | 11,518 | 11,728 | 11,913 | 12,083 | | | |
| 1990 | 93,650 | 95,674 | 97,310 | 98,587 | 99,622 | 100,476 | 101,190 | 101,795 | 102,312 | 102,766 | | | | |
| 1989 | 0 | 648 | 1,197 | 1,687 | 2,135 | 2,550 | 2,934 | 3,294 | 3,646 | | | | | |
| 1988 | 45,000 | 45,856 | 46,586 | 47,187 | 47,692 | 48,123 | 48,496 | 48,831 | | | | | | |
| 1987 | 0 | 491 | 968 | 1,395 | 1,779 | 2,129 | 2,458 | | | | | | | |
| 1986 | 1,810 | 2,025 | 2,236 | 2,421 | 2,586 | 2,740 | | | | | | | | |
| 1985 | 0 | 261 | 524 | 757 | 972 | | | | | | | | | |
| 1984 | 11,625 | 11,837 | 12,041 | 12,227 | | | | | | | | | | |

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| 4 | AUG 93 | RUIN PROBLEMS AND DUAL EVENTS | David CM Dickson<br>Alfredo D Egidio dos Reis |
| 5 | SEP 93 | CONTEMPORARY ISSUES IN AUSTRALIAN SUPERANNUATION - A CONFERENCE SUMMARY | David M Knox<br>John Piggott |
| 6 | SEP 93 | AN ANALYSIS OF THE EQUITY INVESTMENTS OF AUSTRALIAN SUPERANNUATION FUNDS | David M Knox |
| 7 | OCT 93 | A CRITIQUE OF DEFINED CONTRIBUTION USING A SIMULATION APPROACH | David M Knox |
| 8 | JAN 94 | REINSURANCE AND RUIN | David C M Dickson<br>Howard R Waters |
| 9 | MAR 94 | LIFETIME INCOME, TAXATION, EXPENDITURE AND SUPERANNUATION (LITES): A LIFE-CYCLE SIMULATION MODEL | Margaret E Atkinson<br>John Creedy<br>David M Knox |
| 10 | FEB 94 | SUPERANNUATION FUNDS AND THE PROVISION OF DEVELOPMENT/VENTURE CAPITAL: THE PERFECT MATCH? YES OR NO | David M Knox |
| 11 | JUNE 94 | RUIN PROBLEMS: SIMULATION OR CALCULATION? | David C M Dickson<br>Howard R Waters |
| 12 | JUNE 94 | THE RELATIONSHIP BETWEEN THE AGE PENSION AND SUPERANNUATION BENEFITS, PARTICULARLY FOR WOMEN | David M Knox |
| 13 | JUNE 94 | THE COST AND EQUITY IMPLICATIONS OF THE INSTITUTE OF ACTUARIES OF AUSTRALIA PROPOSED RETIREMENT INCOMES STRATEGY | Margaret E Atkinson<br>John Creedy<br>David M Knox<br>Chris Haberecht |
| 14 | SEPT 94 | PROBLEMS AND PROSPECTS FOR THE LIFE INSURANCE AND PENSIONS SECTOR IN INDONESIA | Catherine Prime<br>David M Knox |

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| 15 | OCT 94 | PRESENT PROBLEMS AND PROSPECTIVE PRESSURES IN AUSTRALIA'S SUPERANNUATION SYSTEM | David M Knox |
| 16 | DEC 94 | PLANNING RETIREMENT INCOME IN AUSTRALIA: ROUTES THROUGH THE MAZE | Margaret E Atkinson<br>John Creedy<br>David M Knox |
| 17 | JAN 95 | ON THE DISTRIBUTION OF THE DURATION OF NEGATIVE SURPLUS | David C M Dickson<br>Alfredo D Egidio dos Reis |
| 18 | FEB 95 | OUTSTANDING CLAIM LIABILITIES: ARE THEY PREDICTABLE? | Ben Zehnwirth |
| 19 | MAY 95 | SOME STABLE ALGORITHMS IN RUIN THEORY AND THEIR APPLICATIONS | David C M Dickson<br>Alfredo D Egidio dos Reis<br>Howard R Waters |
| 20 | JUN 95 | SOME FINANCIAL CONSEQUENCES OF THE SIZE OF AUSTRALIA'S SUPERANNUATION INDUSTRY IN THE NEXT THREE DECADES | David M Knox |
| 21 | JUN 95 | MODELLING OPTIMAL RETIREMENT IN DECISIONS IN AUSTRALIA | Margaret E Atkinson<br>John Creedy |
| 22 | JUN 95 | AN EQUITY ANALYSIS OF SOME RADICAL SUGGESTIONS FOR AUSTRALIA'S RETIREMENT INCOME SYSTEM | Margaret E Atkinson<br>John Creedy<br>David M Knox |
| 23 | SEP 95 | EARLY RETIREMENT AND THE OPTIMAL RETIREMENT AGE | Angela Ryan |
| 24 | OCT 95 | APPROXIMATE CALCULATION OF MOMENTS OF RUIN RELATED DISTRIBUTIONS | David C M Dickson |
| 25 | DEC 95 | CONTEMPORARY ISSUES IN THE ONGOING REFORM OF THE AUSTRALIAN RETIREMENT INCOME SYSTEM | David M Knox |
| 26 | FEB 96 | THE CHOICE OF EARLY RETIREMENT AGE AND THE AUSTRALIAN SUPERANNUATION SYSTEM | Margaret E Atkinson<br>John Creedy |
| 27 | FEB 96 | PREDICTIVE AGGREGATE CLAIMS DISTRIBUTIONS | David C M Dickson<br>Ben Zehnwirth |
| 28 | FEB 96 | THE AUSTRALIAN GOVERNMENT SUPERANNUATION CO-CONTRIBUTIONS: ANALYSIS AND COMPARISON | Margaret Atkinson |
| 29 | MAR 96 | A SURVEY OF VALUATION ASSUMPTIONS AND FUNDING METHODS USED BY AUSTRALIAN ACTUARIES IN DEFINED BENEFIT SUPERANNUATION FUND VALUATIONS | Des Welch<br>Shauna Ferris |
| 30 | MAR 96 | THE EFFECT OF INTEREST ON NEGATIVE SURPLUS | David C M Dickson<br>Alfred D Egídio dos Reis |
| 31 | MAR 96 | RESERVING CONSECUTIVE LAYERS OF INWARDS EXCESS-OF-LOSS REINSURANCE | Greg Taylor |

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| 32 | AUG 96 | EFFECTIVE AND ETHICAL INSTITUTIONAL INVESTMENT | Anthony Asher |
| 33 | AUG 96 | STOCHASTIC INVESTMENT MODELS: UNIT ROOTS, COINTEGRATION, STATE SPACE AND GARCH MODELS FOR AUSTRALIA | Michael Sherris<br>Leanna Tedesco<br>Ben Zehnwirth |
| 34 | AUG 96 | THREE POWERFUL DIAGNOSTIC MODELS FOR LOSS RESERVING | Ben Zehnwirth |
| 35 | SEPT 96 | KALMAN FILTERS WITH APPLICATIONS TO LOSS RESERVING | Ben Zehnwirth |