RUIN PROBLEMS FOR PHASE-TYPE(2) RISK PROCESSES

David C M Dickson, The University of Melbourne

and

Christian Hipp, The University of Karlsruhe

Abstract

In this paper we consider a risk process in which claim inter-arrival times have a phase-type(2) distribution, a distribution with a density satisfying a second order linear differential equation. We consider some ruin related problems. In particular we consider the compound geometric representation of the infinite time survival probability, as well as the (defective) distributions of the surplus immediately prior to ruin and of the deficit at ruin. We also consider explicit solutions for the infinite time ruin probability in the case where the individual claim amount distribution is phase-type.

Keywords: Sparre Andersen model, ruin probability, surplus prior to ruin, severity of ruin, phase-type distribution.

1. Introduction

In this paper we shall consider a special Sparre Anderson model for the risk process. In particular we shall assume that the times between claims (and the time until the first claim) form a sequence of independent and identically distributed random variables, denoted $\{T_i\}_{i=1}^{\infty}$, with density function k satisfying the following differential equation:

$$k(t) + A_1 k'(t) + A_2 k''(t) = 0 \text{ for } t > 0, \tag{1.1}$$

where

$$A_2 > 0. \tag{1.2}$$

Our conditions (1.1) and (1.2) are satisfied for all mixtures and convolutions of two exponential distributions (with not necessarily equal means). These two types of distribution are special phase-type distributions; general phase-type distributions, introduced by Neuts (1975), are defined via a continuous time homogeneous Markov chain $X(t)$ on a finite state space $\{0, 1, ..., I\}$, $I \geq 1$. When state 0 is absorbing, and the Markov chain is assumed to be irreducible, then the random variable

$$T = \inf\{t \geq 0 : X(t) = 0\}$$

is finite almost everywhere, and its distribution is called a *phase-type distribution* with parameters $\pi^* = (\pi_0, ..., \pi_I)$ $-$ the starting distribution - and $B = (b_{i,j})_{i,j=1,...,I}$ - the infinitesimal operator - defined by

$$\mathbb{P}\{X(t+h) = j|X(t) = i\} = b_{i,j}h + o(h),\ h \to 0,\ i \neq j,\ i,j = 1, ..., I$$
$$\mathbb{P}\{X(t+h) = i|X(t) = i\} = 1 - b_{i,i}h + o(h),\ h \to 0, i = 1, ..., I.$$

Whenever $\pi_0 = 0$, then $P\{T = 0\} = 0$ and the distribution of T has a density. In this case we write $\pi = (\pi_1, ..., \pi_I)$ and call the distribution *proper* phase-type. The tail probability of a proper phase-type distribution is given by

$$\Pr(T > t) = \pi' exp(tB)e$$

where e is the $I-$vector with all entries equal to 1, and the exponential of the matrix tB is defined via the Taylor series expansion for exp. The density is given by

$$k(t) = -\pi' B\, exp(tB)e,$$

and its Laplace transform is

$$k^*(s) = \int_0^\infty exp(-st)k(t)dt = -\pi' B\,(s\,Id - B)^{-1}\,e$$

with Id the $I \times I-$identity matrix. Phase-type(2) distributions are phase-type distributions with $I = 2$ and $\pi_0 = 0$. Their infinitesimal operator can be written

$$B = \begin{pmatrix} -\lambda & \alpha\lambda \\ \beta\mu & -\mu \end{pmatrix},$$

where $\lambda, \mu > 0$ and $0 \leq \alpha, \beta \leq 1$, $\alpha\beta < 1$. For $\alpha = 0$ and $\beta = 1$, k is the density of a mixture of $Exp(\lambda)$ and $Exp(\lambda) * Exp(\mu)$.

In Section 2 we shall show that condition (1.2) will be satisfied for all non-exponential densities for which a differential equation (1.1) holds, and these densities are precisely all possible phase-type(2) densities. The exponential density $k(t) = \theta e^{-\theta t}$ satisfies the two differential equations

$$\begin{aligned} k(t) + \frac{1}{\theta}k'(t) &= 0 \\ k(t) - \frac{1}{\theta}k'(t) - \frac{2}{\theta^2}k''(t) &= 0; \end{aligned} \tag{1.3}$$

the first one with $A_2 = 0$ will be considered the canonical differential equation.

In this paper we extend and generalise the work of Dickson and Hipp (1998) who consider the probability of ultimate ruin when claim inter-arrival times have an Erlang(2) distribution - a member of the class of phase-type(2) distributions. In Section 2, after we discuss phase-type(2) distributions, we set out some basic formulae relating to the infinite time survival probability, denoted δ. In section 3 we use these formulae to find $\delta(0)$, and hence find the compound geometric representation of δ. Our approach in this section differs from that of Willmot (1998) who derives the compound geometric representation starting from a result in queueing theory. Our strating point applies not only to the problem discussed in this section, but also to the problems discussed in Sections 5 and 6. In Section 4, we discuss the case of phase-type individual claim distributions. We conclude the paper by discussing the (defective) distribution of the surplus prior to ruin in Section 5, and of the severity of ruin in Section 6.

We remark that the results obtained in Section 4 can also be obtained by methods presented in papers by Asmussen (1992) and Asmussen and Rolski (1991). Our approach is different in that we simply exploit the differential equation (1.1) in the same way that we would exploit (1.3) when solving ruin and related problems for the classical risk model.

2. Preliminaries

2.1. Phase-type(2) distributions

First, we show that all phase-type(2) distributions have a density satisfying a second order differential equation of type (1.1). Our density is $k(t) = -\pi' B\, exp(tB)e$, which, with an appropriate matrix C can be written as

$$k(t) = -\pi' B\, exp(tC\Lambda C^{-1})e = -\pi' BC\, exp(t\Lambda)C^{-1}e,$$

where Λ is the Jordan normal form:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } \Lambda = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

In the first case, k is a linear combination of two exponential densities; in the second it is a linear combination of an exponential and an Erlang(2) density with the same scale parameter. In both cases, k satisfies a second order differential equation of type (1.1). For the proof of (1.2) for general non-exponential waiting time densities k with $k(t) = 0$ for $t < 0$ we derive the Laplace Transform of k from (1.1):

$$\begin{aligned} 0 &= k^*(s) + A_1 \int_0^\infty e^{-st} k'(t)dt + A_2 \int_0^\infty e^{-st} k''(t)dt \\ &= k^*(s) + A_1\{-k(0) + sk^*(s)\} + A_2\{-k'(0) - sk(0) + s^2 k^*(s)\} \end{aligned}$$

or, with

$$0 = \int_0^\infty \left(k(t) + A_1 k'(t) + A_2 k''(t)\right) dt = 1 - A_1 k(0) - A_2 k'(0) \tag{2.1}$$

$$k^*(s) = \frac{A_1 k(0) + A_2 k'(0) + A_2 s k(0)}{1 + A_1 s + A_2 s^2} = \frac{1 + A_2 s k(0)}{1 + A_1 s + A_2 s^2}. \tag{2.2}$$

Let μ_1 denote the mean inter-arrival time. By differentiating (2.2) with respect to s and setting $s = 0$, we find that $\mu_1 = A_1 - A_2 k(0)$.

If $A_2 < 0$, then the denominator would have a positive zero s_1 and a negative zero s_2. Since $k^*(s)$ does not have a singularity in $s > 0$, $1 + A_2 s k(0)$ must have the same positive zero s_1. But $k^*(s)$ can then be written

$$k^*(s) = \frac{K}{s_2 - s}$$

which is the Laplace Transform of an exponential distribution. ■

The parameters of our inter-arrival densities are A_1, A_2, and $k(0)$. For exponential densities we choose the version with $A_2 = 0$.

We shall now show that the possible parameter set for $k(0)$, A_1 and A_2 is the one corresponding to the set of proper phase-type(2) distributions. If $1 + A_1 s + A_2 s^2$ has a complex zero, then $k(t)$ has the form

$$C_1 \exp(r_1 t) \sin(r_2 t) + C_2 \exp(r_1 t) \cos(r_2 t)$$

which will be negative for some t. Now let $0 < \lambda \leq \mu$ be chosen such that $-\lambda, -\mu$ are the zeroes of $1 + A_1 s + A_2 s^2$. Then $k^*(s)$ is defined and positive for $s > -\lambda$ and hence $1 - \lambda A_2 k(0) \geq 0$. Equation (2.2) implies that

$$k^*(s) = \frac{1 - \lambda A_2 k(0) + A_2 k(0)(\lambda + s)}{A_2(\lambda + s)(\mu + s)}$$

which is the Laplace Transform of a mixture of $Exp(\lambda) * Exp(\mu)$ and $Exp(\mu)$, a phase-type(2) distribution. ■

We remark that this class of distributions is referred to by Willmot (1998) as Coxian-2; see also Tijms (1994).

2.2. Some ruin theory

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables, independent of the sequence $\{T_i\}_{i=1}^{\infty}$ defined in Section 1, where X_i denotes the amount of the i-th claim. Let F be the distribution function, with $F(0) = 0$, and, when it exists, let f denote the density function of X_i. We denote the mean individual claim amount by m_1. Let c denote the insurer's premium income per unit time. We will assume that

$$\rho = cE(T_i) - E(X_i) = c\mu_1 - m_1 > 0.$$

For this risk process, we define the probability of ultimate ruin from initial surplus u as

$$\psi(u) = \Pr\left(u + \sum_{i=1}^{n} (cT_i - X_i) < 0 \quad \text{for some } n,\ n = 1, 2, 3, ... \right).$$

Let $\delta(u) = 1-\psi(u)$ denote the survival probability. If the moment generating function of X_i exists, then the adjustment coefficient for this risk process is the unique positive number R such that

$$E[\exp\{-cRT_i\}]E[\exp\{RX_i\}] = 1. \tag{2.3}$$

Conditioning on the time and the amount of the first claim we have

$$\begin{aligned}\delta(u) &= \int_0^\infty k(t) \int_0^{u+ct} \delta(u+ct-x)dF(x)dt \\ &= \frac{1}{c}\int_u^\infty k(\frac{z-u}{c}) \int_0^z \delta(z-x)dF(x)dz,\end{aligned}$$

giving

$$\frac{d}{du}\delta(u) = -\frac{1}{c}k(0)\int_0^u \delta(u-x)dF(x) - \frac{1}{c}\left[\frac{1}{c}\int_u^\infty k'(\frac{z-u}{c})\int_0^z \delta(z-x)dF(x)dz\right],$$

and

$$\begin{aligned}\frac{d^2}{du^2}\delta(u) &= -\frac{1}{c}k(0)\left(\int_0^u \delta(u-x)dF(x)\right)' + \\ &\frac{1}{c^2}k'(0)\int_0^u \delta(u-x)dF(x) + \frac{1}{c^2}\left[\frac{1}{c}\int_u^\infty k''(\frac{z-u}{c})\int_0^z \delta(z-x)dF(x)dz\right].\end{aligned}$$

Using (1.1) we obtain

$$\begin{aligned}&A_2c^2\frac{d^2}{du^2}\delta(u) - A_1c\frac{d}{du}\delta(u) + \delta(u) \\ =\ &(A_1k(0)+A_2k'(0))\int_0^u \delta(u-x)dF(x) - A_2ck(0)\left(\int_0^u \delta(u-x)dF(x)\right)'\end{aligned} \tag{2.4}$$

Further, defining

$$\delta^*(s) = \int_0^\infty e^{-su}\delta(u)du \qquad \text{and} \qquad f^*(s) = \int_0^\infty e^{-sx}dF(x)$$

we obtain with (2.4) and (2.1)

$$\begin{aligned}&A_2c^2[s^2\delta^*(s) - s\delta(0) - \delta'(0)] - A_1c[s\delta^*(s) - \delta(0)] + \delta^*(s) \\ =\ &\delta^*(s)f^*(s) - A_2ck(0)s\delta^*(s)f^*(s).\end{aligned} \tag{2.5}$$

Notice that for (2.5) we do not need that δ is twice differentiable (which is true if F is smooth). Finally, we define L to be the maximum aggregate loss, so that $\delta(u) = \Pr(L \le u)$. Then the Laplace transform of L is given by

$$\phi^*(s) = E[e^{-sL}] = s\delta^*(s) \tag{2.6}$$

As L has a compound geometric distribution we can also write

$$\phi^*(s) = \frac{\delta(0)}{1 - \psi(0) r^*(s)} \tag{2.7}$$

where

$$r^*(s) = \int_0^\infty e^{-sx} dR(x)$$

and where R denotes the ladder height distribution. For convenience, we introduce the distribution Q defined by

$$Q(x) = \frac{1}{m_1} \int_0^x (1 - F(y)) dy$$

with Laplace transform

$$q^*(s) = \frac{1 - f^*(s)}{m_1 s}.$$

Note that Q is the ladder height distribution associated with the classical risk model.

With this notation, (2.5) reads

$$\phi^*(s)[m_1 q^*(s) - A_1 c + A_2 c^2 s + f^*(s) A_2 c k(0)] = -A_1 c \delta(0) + A_2 c^2 \delta'(0) + A_2 c^2 \delta(0) s.$$

For $s = 0$ we obtain the identity

$$m_1 - A_1 c + A_2 c k(0) = -A_1 c \delta(0) + A_2 c^2 \delta'(0),$$

and this in turn yields

$$\phi^*(s) = \frac{m_1 - A_1 c + A_2 c k(0) + A_2 c^2 \delta(0) s}{m_1 q^*(s) - A_1 c + A_2 c^2 s + f^*(s) A_2 c k(0)}. \tag{2.8}$$

3. Compound Geometric Representation

In this section we derive a formula for $\delta(0)$ and for the ladder height distribution R. We start from formula (2.8) which can be written as

$$\phi^*(s) = \frac{A_2 c^2 \delta(0) s - \rho}{m_1 q^*(s) - A_1 c + A_2 c^2 s + f^*(s) A_2 c k(0)} \tag{3.1}$$

where $\rho = c\,(A_1 - A_2 k(0)) - m_1$. From $\rho = c\mu_1 - m_1 > 0$ we see that the numerator

$$A_2 c^2 \delta(0) s - \rho$$

will have a positive zero at $s_0 = \rho / (A_2 c^2 \delta(0))$. Since $\phi^*(s)$ is positive for all $s > 0$, s_0 must also be a solution of the equation

$$I(s) = m_1 q^*(s) - A_1 c + A_2 c^2 s + f^*(s) A_2 c k(0) = 0. \tag{3.2}$$

Notice that (3.2) is the defining equation for the adjustment coefficient for the problem, given in (2.3). However, in the case without an adjustment coefficient, (3.2) will also have a *positive* solution, and this solution will be unique. To see this, notice that

$$\begin{aligned} I(0) &= -\rho < 0, \\ \lim_{s\to\infty} I(s) &= \infty \ , \text{ and} \\ I''(s) &= \int_0^\infty x^2 e^{-sx}(1-F(x))dx + A_2ck(0)\int_0^\infty x^2 e^{-sx} dF(x) > 0. \end{aligned}$$

So if $I'(0) \geq 0$, the function I will be increasing on $(0,\infty)$, and if $I'(0) < 0$ then the function I will be decreasing up to some point, and it will be increasing from this point on, so in both cases the solution to (3.2) exists, and it is unique. Hence $\delta(0)$ can be identified:

$$\delta(0) = \frac{\rho}{A_2c^2 s_0} \ . \tag{3.3}$$

We can now write (3.1) as

$$\phi^*(s) = \frac{A_2c^2(s-s_0)\delta(0)}{A_2c^2(s-s_0) + m_1[q^*(s)-q^*(s_0)] + A_2ck(0)[f^*(s)-f^*(s_0)]},$$

and hence

$$\begin{aligned} r^*(s) &= \frac{s_0}{s_0-s}\frac{1}{A_2c^2s_0-\rho}\left[m_1\left(q^*(s)-q^*(s_0)\right) + A_2ck(0)(f^*(s)-f^*(s_0))\right] \\ &= \sigma\left(m_1\frac{q^*(s)-q^*(s_0)}{s_0-s} + A_2ck(0)\frac{f^*(s)-f^*(s_0)}{s_0-s}\right) \end{aligned}$$

with

$$\sigma = \frac{s_0}{A_2c^2s_0-\rho} \ .$$

From Dickson and Hipp (1998, Section 3) it follows that if G is an arbitrary distribution with Laplace transform $g^*(s)$ then

$$\frac{g^*(s)-g^*(s_0)}{s_0-s}$$

is the Laplace transform of

$$\hat{g}(y) = \int_y^\infty e^{-s_0(x-y)}dG(x), \ y > 0.$$

Hence the Lebesgue density of the ladder height distribution, denoted r, is given by

$$r(y) = \sigma\left[m_1\hat{q}(y) + A_2ck(0)\hat{f}(y)\right], \ y > 0. \tag{3.4}$$

We note that in two cases, $r(y)$ will be proportional to $\hat{q}(y)$. First, if $k(0) = 0$ we have $r(y) = \sigma m_1\hat{q}(y)$. Second, if $F(x) = 1-\exp(-\alpha x)$, we have $Q = F$ and it is easy to see that $r(y)$ is proportional to $\exp(-\alpha y)$, giving $R = F$.

4. Phase-type Distributions

For phase-type claim size distributions, the infinite time survival probability can be given in explicit form for both exponential and phase-type(2) inter-arrival times. For the case of exponential or Erlang(2) inter-arrival times see Dickson and Hipp (1998). For the case of a phase-type(2) inter-arrival time distribution the formulae are similar, but a bit more lengthy. If the claim size F has an arbitrary proper phase-type distribution with parameters (π, B), then the Laplace transform of Q is given by

$$q^*(s) = \frac{1}{m_1}\pi'(s\,Id - B)^{-1}e,$$

where $m_1 = -\pi' B^{-1}e$ and Id is the $I \times I-$identity matrix. Writing $f^*(s)$ in terms of $q^*(s)$ in (3.2), the defining equation for s_0 is

$$\pi'(s\,Id - B)^{-1}e\,(1 - A_2ck(0)s) - c\mu_1 + A_2c^2s = 0, \tag{4.1}$$

giving

$$\begin{aligned} \psi(0) &= \frac{A_2c^2s_0 - \rho}{A_2c^2s_0} \qquad (4.2)\\ &= \frac{1}{A_2c^2s_0}\left(\pi'(s_0\,Id - B)^{-1}e\,(A_2ck(0)s_0 - 1) + m_1\right). \end{aligned}$$

The ladder height distribution is again phase-type with parameters $(\widetilde{\pi}, B)$, where

$$\widetilde{\pi} = \zeta\left[\frac{1}{m_1}\left(-B'\right)^{-1}\left(s_0\,Id - B'\right)^{-1}\pi + A_2ck(0)\left(s_0\,Id - B'\right)^{-1}\pi\right] \tag{4.3}$$

and where the parameter ζ has to be chosen such that $\widetilde{\pi}$ is a stochastic vector. To derive this, consider an arbitrary phase-type distribution G with parameters π_G, B_G. Let $\hat{G}$ be the distribution with density proportional to

$$\hat{g}(y) = \int_y^\infty e^{-s_0(x-y)}dG(x).$$

The distribution $\hat{G}$ is again phase-type with parameters

$$(s_0\,Id - B_G')^{-1}\pi_G \text{ and } B.$$

For this, notice that

$$\begin{aligned} g(x) &= -\pi_G' B_G \exp(xB_G)e,\\ e^{-s_0x}g(x) &= -\pi_G' B_G \exp(x(B_G - s_0\,Id))e,\\ \int_y^\infty e^{-s_0x}g(x)dx &= \pi_G' B_G(B_G - s_0\,Id)^{-1}\exp(y(B_G - s_0\,Id))e,\\ e^{s_0y}\int_y^\infty e^{-s_0x}g(x)dx &= \pi_G' B_G(B_G - s_0\,Id)^{-1}\exp(yB_G)e,\\ \int_u^\infty e^{s_0y}\int_y^\infty e^{-s_0x}g(x)dxdy &= -\pi_G'(B_G - s_0\,Id)^{-1}\exp(uB_G)e. \end{aligned}$$

The density $r(y)$ of R is proportional to

$$m_1\hat{q}(y) + A_2ck(0)\hat{f}(y),$$

and since Q and F have the same intensity matrix B, R also has intensity matrix B and an initial vector $\widetilde{\pi}$ proportional to

$$\frac{1}{m_1}\left(-B'\right)^{-1}\left(s_0\, Id - B'\right)^{-1}\pi + A_2ck(0)\left(s_0\, Id - B'\right)^{-1}\pi$$

The ruin probability is again the tail probability of an (improper) phase-type distribution in the form of

$$\psi(u) = \psi(0)\widetilde{\pi}'\exp(u\hat{B})e$$

with $\hat{B} = \left(\hat{b}_{ij}\right)$ defined by

$$\hat{b}_{ij} = b_{ij} + \psi(0)\widetilde{\pi}_j b_{i0},$$

b_{i0} being the intensity for a transition from state i to the absorbing state 0.

Example 4.1. *Let F be an Erlang$(2,1)$ distribution,*

$$k(t) = \frac{1}{2}\exp(-t) + \exp(-2t),\ t \geq 0,$$

and $c = 4$. Then we have $k(0) = 3/2, A_1 = 3/2, A_2 = 1/2, I = 2,\ \pi = (1,0), m_1 = 2, \mu_1 = 3/4, \rho = 1$ and

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{4.4}$$

The number s_0 is the positive solution of

$$\frac{(s+2)(1-3s)}{(s+1)^2} - 3 + 8s = 0$$

which gives

$$s_0 = 0.40974 \text{ and } \psi(0) = 0.69493.$$

According to (4.3), the vector $\widetilde{\pi}$ is given by

$$\widetilde{\pi} = (0.51037, 0.48963), \tag{4.5}$$

the matrix $\hat{B}$ is given by

$$\hat{B} = \begin{pmatrix} -1 & 1 \\ 0.35467 & -0.65975 \end{pmatrix}$$

and the ruin probability is

$$\psi(u) = \psi(0)\widetilde{\pi}'\exp(u\widehat{B})e = 0.7292\exp(-0.2105u) - 0.0343\exp(-1.4492u).$$

This is in accordance with the fact that 0.2105 is the adjustment coefficient.

5. The Distribution of the Surplus Prior to Ruin

In this section we consider the distribution of the surplus prior to ruin. Dickson (1992) considers this problem for the classical surplus process and uses probabilistic arguments to derive simple expressions for the (defective) density of the surplus prior to ruin. Since the risk processes we have been considering do not have the same properties as the classical surplus process, we have to adopt a more direct approach.

Let us define $H(u,x)$ to be the probability that ruin occurs from initial surplus u and that the surplus immediately prior to ruin is strictly less than x. We can write down expressions for $H(u,x)$ by conditioning on the time and the amount of the first claim, noting that $H(u,x)$ takes different forms depending on the relationship between u and x. For convenience we will assume in this section and the next that the individual claim amount distribution is continuous with density function $f(y)$, $y > 0$.

For $0 \le u < x$, we have

$$H(u,x) = \int_0^\infty k(t) \int_0^{u+ct} f(y) H(u+ct-y,x)\,dy\,dt + \int_0^{\tau_x} k(t)(1-F(u+ct))\,dt$$

where $\tau_x = (x-u)/c$. Thus, if the first claim occurs before time τ_x and that claim causes ruin, then the surplus immediately prior to ruin must be less than x. Changing the variable of integration we get

$$\begin{aligned} cH(u,x) &= \int_u^\infty k\left(\frac{s-u}{c}\right) \int_0^s f(y)H(s-y,x)\,dy\,ds + \int_u^x k\left(\frac{s-u}{c}\right)(1-F(s))\,ds \\ &= \int_u^\infty k\left(\frac{s-u}{c}\right)\omega(s,x)\,ds + \int_u^x k\left(\frac{s-u}{c}\right)(1-F(s))\,ds \end{aligned}$$

where

$$\omega(u,x) = \int_0^u f(y)H(u-y,x)\,dy.$$

For $u \ge x$, ruin at the first claim implies a surplus greater than x immediately prior to ruin. Hence for $u \ge x$

$$cH(u,x) = \int_u^\infty k\left(\frac{s-u}{c}\right)\omega(s,x)\,ds.$$

Thus, for $u \ge 0$ we can write

$$cH(u,x) = \int_u^\infty k\left(\frac{s-u}{c}\right)\omega(s,x)\,ds + 1_{\{u<x\}} \int_u^x k\left(\frac{s-u}{c}\right)(1-F(s))\,ds$$

where $1_{\{.\}}$ is the usual indicator function. Differentiation yields

$$\begin{aligned} c\frac{d}{du}H(u,x) &= -k(0)\omega(u,x) - \frac{1}{c}\int_u^\infty k'\left(\frac{s-u}{c}\right)\omega(s,x)\,ds \\ &\quad -1_{\{u<x\}}\left(k(0)(1-F(u)) + \frac{1}{c}\int_u^x k'\left(\frac{s-u}{c}\right)(1-F(s))\,ds\right), \end{aligned}$$

and hence

$$\begin{aligned} c\frac{d^2}{du^2}H(u,x) &= -k(0)\omega'(u,x) + \frac{1}{c}k'(0)\omega(u,x) + \frac{1}{c^2}\int_u^\infty k''\left(\frac{s-u}{c}\right)\omega(s,x)ds \\ &\quad +1_{\{u<x\}}\Bigg(k(0)f(u) + \frac{1}{c}k'(0)\left(1-F(u)\right) \\ &\qquad +\frac{1}{c^2}\int_u^x k''\left(\frac{s-u}{c}\right)(1-F(s))ds\Bigg). \end{aligned}$$

Proceeding as in Section 2, we employ (1.1) to get

$$\begin{aligned} & A_2c^2\frac{d^2}{du^2}H(u,x) - A_1c\frac{d}{du}H(u,x) + H(u,x) \\ = & \left(A_1k(0) + A_2k'(0)\right)\omega(u,x) - A_2ck(0)\omega'(u,x) \\ & +1_{\{u<x\}}\left(\left[A_1k(0)+A_2k'(0)\right](1-F(u)) + A_2ck(0)f(u)\right) \\ = & \omega(u,x) - A_2ck(0)\omega'(u,x) + \gamma(u,x) \end{aligned} \tag{5.1}$$

using (2.1) and defining

$$\gamma(u,x) = 1_{\{u<x\}}\left(1-F(u) + A_2ck(0)f(u)\right).$$

If we now define

$$\begin{aligned} H^*(s,x) &= \int_0^\infty e^{-su}H(u,x)du, \\ \omega^*(s,x) &= \int_0^\infty e^{-su}\omega(u,x)du = f^*(s)H^*(s,x), \\ \gamma^*(s,x) &= \int_0^\infty e^{-su}\gamma(u,x)du, \end{aligned}$$

and take the Laplace transform of (5.1) we get

$$\begin{aligned} & A_2c^2\left(s^2H^*(s,x) - sH(0,x) - H'(0,x)\right) - A_1c(sH^*(s,x) - H(0,x)) + H^*(s,x) \\ = & f^*(s)H^*(s,x) - A_2ck(0)sf^*(s)H^*(s,x) + \gamma^*(s,x), \end{aligned}$$

giving

$$H^*(s,x) = \frac{A_2c^2\left(sH(0,x)+H'(0,x)\right) - A_1cH(0,x) + \gamma^*(s,x)}{A_2c^2s^2 - A_1cs + 1 - f^*(s) + A_2ck(0)sf^*(s)}. \tag{5.2}$$

As in Section 2 we can write the denominator of (5.2) in terms of $q^*(s)$ giving

$$H^*(s,x) = \frac{A_2c^2\left(sH(0,x)+H'(0,x)\right) - A_1cH(0,x) + \gamma^*(s,x)}{s(A_2c^2s - A_1c + m_1q^*(s) + A_2ck(0)f^*(s))}.$$

Now use (2.8) and recall that $\phi^*(s) = s\delta^*(s)$ to get

$$
\begin{aligned}
H^*(s,x) &= \delta^*(s)\frac{A_2c^2\left(sH(0,x)+H'(0,x)\right)-A_1cH(0,x)+\gamma^*(s,x)}{m_1-A_1c+A_2ck(0)+A_2c^2\delta(0)s} \\
&= \delta^*(s)\frac{A_2c^2\left(sH(0,x)+H'(0,x)\right)-A_1cH(0,x)+\gamma^*(s,x)}{A_2c^2\delta(0)s-\rho} \\
&= \delta^*(s)\frac{A_2c^2\left(sH(0,x)+H'(0,x)\right)-A_1cH(0,x)+\gamma^*(s,x)}{(\rho/s_0)(s-s_0)}.
\end{aligned}
$$

Since $\delta^*(s_0) > 0$ and $H(s_0, x) > 0$ it follows that

$$
A_2c^2\left(s_0H(0,x)+H'(0,x)\right)-A_1cH(0,x)+\gamma^*(s_0,x)=0,
$$

giving

$$
\begin{aligned}
H^*(s,x) &= \delta^*(s)\frac{A_2c^2(s-s_0)H(0,x)+\gamma^*(s,x)-\gamma^*(s_0,x)}{(\rho/s_0)(s-s_0)} \\
&= \delta^*(s)\left(\frac{A_2c^2H(0,x)}{\rho/s_0}+\frac{s_0}{\rho}\frac{\gamma^*(s,x)-\gamma^*(s_0,x)}{s-s_0}\right) \\
&= \frac{\delta^*(s)}{\delta(0)}\left(H(0,x)-\frac{1}{A_2c^2}\frac{\gamma^*(s,x)-\gamma^*(s_0,x)}{s_0-s}\right). \qquad (5.3)
\end{aligned}
$$

As in Section 3 we find that

$$
\frac{\gamma^*(s,x)-\gamma^*(s_0,x)}{s_0-s}
$$

is the Laplace transform of

$$
\hat{\gamma}(u,x)=\int_u^\infty e^{-s_0(z-u)}\gamma(z,x)dz
$$

and so

$$
H(u,x)=\frac{1}{\delta(0)}\left(\delta(u)H(0,x)-\frac{1}{A_2c^2}\int_0^u\hat{\gamma}(r,x)\delta(u-r)dr\right).
$$

This expression contains the term $H(0,x)$ which we can find from (5.3). Multiply this equation by s and let $s \to 0^+$. Using

$$
\lim_{s\to 0^+} sH^*(s,x)=0 \quad \text{and} \quad \lim_{s\to 0^+} s\delta^*(s)=1
$$

we get

$$
\begin{aligned}
H(0,x) &= \frac{1}{A_2c^2s_0}\left(\gamma^*(0,x)-\gamma^*(s_0,x)\right) \\
&= \frac{\delta(0)}{\rho}\int_0^x\left(1-e^{-s_0u}\right)\left[1-F(u)+A_2ck(0)f(u)\right]du.
\end{aligned}
$$

Example 5.1. *Let us consider the situation when $k(t) = (1/2)e^{-t} + e^{-2t}$, as in Example 4.1, so that $A_2 = 1/2$ and $k(0) = 3/2$, and again let $c = 4$. Let the individual claim amount distribution be exponential with mean 1. Then*

$$\gamma(u, x) = 1_{\{u<x\}} 4e^{-u}$$

and

$$\hat{\gamma}(u, x) = 1_{\{u<x\}} \frac{4}{1+s_0} e^{s_0 u} \left(e^{-(1+s_0)u} - e^{-(1+s_0)x} \right).$$

If we consider the (defective) density defined as $h(u, x) = \frac{d}{dx} H(u, x)$, then

$$\begin{aligned} h(u, x) &= \frac{1}{\delta(0)} \left(\delta(u) h(0, x) - \frac{1}{A_2 c^2} \int_0^u \hat{\gamma}'(r, x) \delta(u - r) dr \right) \\ &= \frac{1}{R} \left(\delta(u) h(0, x) - \frac{1}{8} \int_0^u \hat{\gamma}'(r, x) \delta(u - r) dr \right), \end{aligned} \tag{5.4}$$

with

$$\begin{aligned} h(0, x) &= \frac{\delta(0)}{\rho} (1 - e^{-s_0 x})(1 + A_2 c k(0)) e^{-x} \\ &= 2R(1 - e^{-s_0 x}) e^{-x}, \end{aligned}$$

since $\rho = 2$ and $\delta(u) = 1 - (1 - R)e^{-Ru}$, where R is the adjustment coefficient as defined by (2.3) (see, for example, Grandell (1991)). We find that $R = 0.64039$ and $s_0 = 0.39039$. Now

$$\hat{\gamma}'(u, x) = \frac{d}{dx} \hat{\gamma}(u, x) = 1_{\{u<x\}} 4e^{s_0 u} e^{-(1+s_0)x},$$

and so for $0 < u < x$,

$$\begin{aligned} & \int_0^u \hat{\gamma}'(r, x) \delta(u - r) dr \\ = \;& 4e^{-(1-s_0)x} \int_0^u e^{s_0 r} \left(1 - (1 - R) e^{-R(u-r)} \right) dr \\ = \;& 4e^{-(1-s_0)x} \left[\frac{1}{s_0} \left(e^{s_0 u} - 1 \right) - \frac{1-R}{s_0 + R} \left(e^{s_0 u} - e^{-Ru} \right) \right], \end{aligned}$$

and for $u > x$,

$$\begin{aligned} \int_0^u \hat{\gamma}'(r, x) \delta(u - r) dr &= \int_0^x \hat{\gamma}'(r, x) \delta(u - r) dr \\ &= 4e^{-(1-s_0)x} \left[\frac{1}{s_0} \left(e^{s_0 u} - 1 \right) - \frac{1-R}{s_0 + R} e^{-Ru} \left(e^{(s_0+R)x} - 1 \right) \right]. \end{aligned}$$

Thus we have all the components of the density as specified by (5.4).

6. The Probability and Severity of Ruin

We define $G(u, y)$ to be the probability that ruin occurs from initial surplus u and that the deficit at ruin is strictly less than y. We define $g(u, y)$ to be the (defective) density of the deficit at ruin. We have already evaluated a special case of this density since $g(0, y) = \psi(0)r(y)$. In this section we shall find general expressions for $G(u, y)$ from which explicit solutions can be obtained for certain individual claim amount distributions. Our first approach also leads to the solution for the ladder height distribution given in Section 3, whilst the second can be used to produce explicit solutions for $G(u, y)$ only if we already know this distribution.

For our first approach, we again we start by conditioning on the time and the amount of the first claim, giving

$$G(u, y) = \int_0^\infty k(t) \int_0^{u+ct} f(x)G(u + ct - x, y)dxdt + \int_0^\infty k(t) \int_{u+ct}^{u+ct+y} f(x)dxdt.$$

Thus

$$cG(u, y) = \int_u^\infty k\left(\frac{s-u}{c}\right) \varphi_1(s, y)ds + \int_u^\infty k\left(\frac{s-u}{c}\right) \varphi_2(s, y)ds,$$

where

$$\varphi_1(u, y) = \int_0^u f(x)G(u - x, y)dx, \quad \varphi_2(u, y) = F(u + y) - F(u).$$

Proceeding as in previous sections, we find that

$$\begin{aligned} & A_2c^2\frac{d^2}{du^2}G(u, y) - A_1c\frac{d}{du}G(u, y) + G(u, y) \\ = \; & \varphi_1(u, y) + \varphi_2(u, y) - cA_2k(0)\left[\varphi_1'(u, y) + \varphi_2'(u, y)\right]. \end{aligned}$$

If we now define

$$\begin{aligned} G^*(s, y) &= \int_0^\infty e^{-su}G(u, y)du, \\ \varphi_1^*(s, y) &= \int_0^\infty e^{-su}\varphi_1(u, y)du = f^*(s)G^*(s, y), \\ \varphi_2^*(s, y) &= \int_0^\infty e^{-su}\varphi_2(u, y)du, \end{aligned}$$

we get

$$\begin{aligned} & A_2c^2\left(s^2G^*(s, y) - sG(0, y) - G'(0, y)\right) - A_1c(sG^*(s, y) - G(0, y)) + G^*(s, y) \\ = \; & (1 - cA_2k(0)s)[\varphi_1^*(s, y) + \varphi_2^*(s, y)] + cA_2k(0)\varphi_2(0, y). \end{aligned}$$

Writing $\nu(s, y) = (1 - cA_2k(0)s)\varphi_2^*(s, y) + cA_2k(0)\varphi_2(0, y)$ this gives

$$\begin{aligned} G^*(s, y) &= \frac{A_2c^2\left(sG(0, y) + G'(0, y)\right) - A_1cG(0, y) + \nu(s, y)}{A_2c^2s^2 - A_1cs + 1 - f^*(s) + A_2ck(0)sf^*(s)} \\ &= \frac{\delta^*(s)}{(\rho/s_0)}\left(\frac{A_2c^2\left(sG(0, y) + G'(0, y)\right) - A_1cG(0, y) + \nu(s, y)}{s - s_0}\right). \end{aligned}$$

Noting that

$$A_2c^2s_0G(0,y)+\nu(s_0,y)=-A_2c^2G'(0,y)+A_1cG(0,y)$$

we get

$$\begin{aligned} G^*(s,y) &= \frac{\delta^*(s)}{(\rho/s_0)}\left(A_2c^2G(0,y)+\frac{\varphi_2^*(s,y)-\varphi_2^*(s_0,y)}{s-s_0}-cA_2k(0)\frac{\varphi_3^*(s,y)-\varphi_3^*(s_0,y)}{s-s_0}\right) \\ &= \frac{\delta^*(s)}{\delta(0)}\left(G(0,y)-\frac{1}{A_2c^2}\frac{\varphi_2^*(s,y)-\varphi_2^*(s_0,y)}{s_0-s}+\frac{k(0)}{c}\frac{\varphi_3^*(s,y)-\varphi_3^*(s_0,y)}{s_0-s}\right), \end{aligned} \tag{6.1}$$

where

$$\varphi_3(u,y)=\frac{d}{du}\varphi_2(u,y)=f(u+y)-f(u),\quad \varphi_3^*(s,y)=\int_0^\infty e^{-su}\varphi_3(u,y)du.$$

Define

$$\hat{\varphi}_2(u,y)=\int_u^\infty e^{-s_0(x-u)}\varphi_2(x,y)dx=\int_u^\infty e^{-s_0(x-u)}\left[F(x+y)-F(x)\right]dx,$$

and

$$\hat{\varphi}_3(u,y)=\int_u^\infty e^{-s_0(x-u)}\varphi_3(x,y)dx=\int_u^\infty e^{-s_0(x-u)}\left[f(x+y)-f(x)\right]dx.$$

Then, inverting as in previous sections we get

$$\begin{aligned} G(u,y) &= \frac{1}{\delta(0)}\left(\delta(u)G(0,y)-\frac{1}{A_2c^2}\int_0^u\hat{\varphi}_2(x,y)\delta(u-x)dx\right. \\ &\qquad \left.+\frac{k(0)}{c}\int_0^u\hat{\varphi}_3(x,y)\delta(u-x)dx\right). \end{aligned} \tag{6.2}$$

We can use this expression to find explicit solutions for $G(u,y)$. This requires knowledge of $G(0,y)$ which we have already found in Section 3. However, if we had not done so, we could solve for $G(0,y)$ through (6.1). If we multiply each side of this equation by s, then let $s\to 0^+$, we get

$$G(0,y)=\frac{1}{A_2c^2s_0}\left(\varphi_2^*(0,y)-\varphi_2^*(s_0,y)\right)-\frac{k(0)}{cs_0}\left(\varphi_3^*(0,y)-\varphi_3^*(s_0,y)\right) \tag{6.3}$$

with

$$\varphi_2^*(0,y)-\varphi_2^*(s_0,y)=m_1\left(Q(y)-q^*(s_0)+\hat{q}(y)\right)$$

and

$$\varphi_3^*(0,y)-\varphi_3^*(s_0,y)=-F(y)-\hat{f}(y)+f^*(s_0).$$

Differentiation of (6.3) leads to

$$g(0,y) = \frac{1}{A_2 c^2}\left(m_1 \hat{q}(y) + cA_2 k(0)\hat{f}(y)\right),$$

consistent with (3.4).

Although (6.2) leads to explicit solutions for $G(u,y)$, it is perhaps not the most convenient approach. An alternative approach is write down an equation for $G(u,y)$ by conditioning on the amount of the first ladder height. Thus, we get

$$\begin{aligned} G(u,y) &= \int_0^u \psi(0) r(x) G(u-x,y)dx + \int_u^{u+y} \psi(0) r(x)dx \\ &= \psi(0)\left(\int_0^u r(x)G(u-x,y)dx + V(u,y)\right) \end{aligned}$$

where $V(u+y) = R(u+y) - R(u)$. Taking Laplace transforms we get

$$G^*(s,y) = \frac{\psi(0)V^*(s,y)}{1-\psi(0)r^*(s)}$$

where $V^*(s,y)$ is the Laplace transform of $V(u,y)$. Then, by (2.7) we have

$$G^*(s,y) = \frac{\psi(0)}{\delta(0)}\phi^*(s)V^*(s,y),$$

giving

$$\begin{aligned} G(u,y) &= \frac{\psi(0)}{\delta(0)} \int_0^u V(u-x,y)d\delta(x) \\ &= \frac{\psi(0)}{\delta(0)} \int_0^u \left[R(u-x+y) - R(u-x)\right] d\delta(x). \qquad (6.4) \end{aligned}$$

The derivation of (6.4) essentially follows the proof of Theorem 1 of Willmot and Lin (1997) who consider the same problem for the classical risk model. Hence we have

$$g(u,y) = \frac{\psi(0)}{\delta(0)} \int_0^u r(u-x+y)d\delta(x),$$

an expression from which we can easily find $g(u,y)$ if we know r and δ. Note that this approach can be applied only if the ladder height distribution is known. It does not provide a means of finding this distribution, unlike equation (6.1).

Example 6.1. *Let us continue Example 4.1. Using $r(y) = -\tilde{\pi}B\exp(yB)e$ where B and $\tilde{\pi}$ are given by (4.4) and (4.5) respectively, we get*

$$r(y) = c_1 e^{-y} + c_2 y e^{-y}$$

where $c_1 = 0.4896$ *and* $c_2 = 0.5104$. *Let us write*

$$\delta(u) = 1 - \sum_{i=1}^{2} a_i e^{-R_i u}$$

where $a_1 = 0.7292$, $a_2 = -0.0343$, $R_1 = 0.2105$ *and* $R_2 = 1.4492$. *Then*

$$\begin{aligned} g(u,y) &= \psi(0) r(u+y) + \frac{\psi(0)}{\delta(0)} \int_0^u r(u-x+y)\delta'(x)dx \\ &= g(0,u+y) + \frac{\psi(0)}{\delta(0)} \int_0^u r(z+y)\delta'(u-z)dz. \end{aligned}$$

Now

$$\begin{aligned} &\int_0^u r(z+y)\delta'(u-z)dz \\ &= \sum_{i=1}^{2} a_i R_i \int_0^u \left(c_1 e^{-(z+y)} + c_2(z+y)e^{-(z+y)}\right) e^{-R_i(u-z)}dz \\ &= \sum_{i=1}^{2} a_i R_i e^{-y} e^{-R_i u} \left((c_1 + c_2 y) \int_0^u e^{-(1-R_i)z}dz + c_2 \int_0^u z e^{-(1-R_i)z}dz \right) \\ &= \sum_{i=1}^{2} a_i R_i e^{-y} e^{-R_i u} \left(\frac{c_1 + c_2 y}{1-R_i} \left[1 - e^{-(1-R_i)u}\right] \right. \\ &\qquad \left. + \frac{c_2}{(1-R_i)^2} \left[1 - e^{-(1-R_i)u}\left(1 + (1-R_i)u\right)\right] \right) \\ &= \sum_{i=1}^{2} a_i R_i e^{-y} \left(\frac{c_1 + c_2 y}{1-R_i} \left[e^{-R_i u} - e^{-u}\right] \right. \\ &\qquad \left. + \frac{c_2}{(1-R_i)^2} \left[e^{-R_i u} - e^{-u}\left(1 + (1-R_i)u\right)\right] \right) \end{aligned}$$

giving

$$\begin{aligned} e^y g(u,y) &= \psi(0) c_1 e^{-u} + c_2(y+u)e^{-u} \\ &\quad + \frac{\psi(0)}{\delta(0)} \sum_{i=1}^{2} a_i R_i \left(\frac{c_1 + c_2 y}{1-R_i} + \frac{c_2}{(1-R_i)^2} \right) e^{-R_i u} \\ &\quad - e^{-u} \frac{\psi(0)}{\delta(0)} \sum_{i=1}^{2} a_i R_i \left(\frac{c_1 + c_2 y}{1-R_i} + c_2 \frac{1 + (1-R_i)u}{(1-R_i)^2} \right). \end{aligned}$$

Inserting values for the constants we find that

$$g(u,y) = e^{-R_1 u}\left(0.5032e^{-y} + 0.2260ye^{-y}\right) - e^{-R_2 u}(0.1629e^{-y} - 0.1286ye^{-y}),$$

consistent with

$$g(0,y) = 0.3403e^{-y} + 0.3546ye^{-y}$$

and

$$\psi(u) = \int_0^\infty g(u,y)dy = 0.7292e^{-R_1u} - 0.0343e^{-R_2u}.$$

References

[1] Asmussen, S. (1992) Phase-type representations in random walks and queuing problems. The Annals of Probability 20, 772-789.

[2] Asmussen, S. and Rolski, T. (1991) Computational methods in risk theory: a matrix-algorithmic approach. Insurance: Mathematics & Economics 10, 259-274.

[3] Dickson, D.C.M. (1998) On a class of renewal risk processes. North Americam Actuarial Journal, to appear.

[4] Dickson, D.C.M. and Hipp, C. (1998) Ruin probabilities for Erlang(2) risk processes. Insurance: Mathematics and Economics 22, 251-262.

[5] Grandell, J. (1991) Aspects of risk theory. Springer Verlag.

[6] Neuts, M.F. (1975) Probability distributions of phase-type. Liber Amicorum Professor Emeritus H. Florin, Dept. of Mathematics, University of Louvain, Belgium, 173-206.

[7] Tijms, H.C. (1994) Stochastic models: an algorithmic approach. John Wiley & Sons.

[8] Willmot, G.E. (1998) A Laplace transform representation in a class of renewal queueing and risk processes.

[9] Willmot, G.E. and Lin, X. (1998) Exact and approximate properties of the distribution of the surplus before ruin. Insurance: Mathematics & Economics, to appear.