On multivariate Panjer recursions

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Abstract

In the present paper we generalise Panjer's (1981) recursion for compund distribution to a multivariate situation where each claim event generates a random vector. We discuss situations within insurance where such models could be applicable, and consider some special cases of the general algorithm. Finally we deduce from the algorithm a multivariate extension of De Pril's (1985) recursion for convolutions.

1 Introduction

1A. Let N denote the number of claims occurring in an insurance portfolio within a given period, and U_i the amount of the ith of these claims. We assume that these claim amounts are positive, integer-valued, mutually independent and identically distributed with common probability function f, and independent of N. Let p denote the probability function of N. Then the distribution of the aggregate claims $X = \sum_{i=1}^{N} U_i$ is a compound distribution with probability function

$$g = \sum_{n=0}^{\infty} p(n) f^{n*}. \tag{1.1}$$

Panjer (1981) presented an algorithm for recursive evaluation of g when p satisfies a recursion in the form

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1). \qquad (n = 1, 2, \ldots) \tag{1.2}$$

We obtain that N has a Poisson distribution when $a = 0$, a negative binomial distribution when $a > 0$, and a binomial distribution when $a < 0$. Panjer's paper motivated the development of an extensive theory on recursive methods for compound distributions as well other types of distributions that would be appropriate for aggregate claims of insurance portfolios.

1B. Hesselager (1996) presented some bivariate extensions of Panjer recursion, using bivariate generalisations of the counting distribution. He considered a situation with two portfolios. For $j = 1, 2$, let N_j denote the number of claims in portfolio j and W_{ij} the amount of the ith of these claims. We assume that the W_{ij}'s are positive, integer-valued, mutually independent, independent of the claim numbers, and for fixed j identically distributed with common probability function f_j. The aggregate claim amount from portfolio j is $X_j = \sum_{i=1}^{N_j} W_{ij}$. Let q denote the joint probability function of N_1 and N_2 and g the joint probability function of X_1 and X_2. Then

$$g(x_1, x_2) = \sum_{n_1=0}^{x_1} \sum_{n_2=0}^{x_2} q(n_1, n_2) f_1^{n_1 *}(x_1) f_2^{m_2 *}(x_2) . \quad (x_1, x_2 = 0, 1, \ldots)$$

Hesselager deduced recursions for g in some cases where q can interpreted as a bivariate extension of Panjer's counting distributions.

1C. In the present paper we shall study a multivariate generalisation of Panjer's recursion in another direction than Hesselager. We shall still assume that the claim number is one-dimensional, and that its distribution satisfies (1.2). However, we now assume that each claim is an m-dimensional random vector, and that these vectors are mutually independent and identically distributed and independent of the number of claims. This can be interpreted as if the number of claims is now the number of claim events within a portfolio of m policies, and that the severity vector represents the vector of payments to the m policies caused by one claim event. We shall motivate the model further in Section 2.

After the motivation in Section 2 we deduce the main result in Section 3. In Section 4 we discuss some special cases. Finally, in Section 5, we look at an additional special case, from which we deduce a multivariate extension of De Pril's (1985) recursion for convolutions.

2 Motivation

2A. Let N denote the number of claim events,

$$\mathbf{U}_i = (U_{i1}, \ldots, U_{im})' \qquad (i = 1, 2, \ldots)$$

an m-dimensional vector generated by the ith of these events, and

$$\mathbf{X} = (X_1, \ldots, X_m)' = \sum_{i=1}^{N} \mathbf{U}_i$$

(we interpret $\sum_{i=c}^{d} = 0$ when $d < c$). We assume that $\mathbf{U}_1, \mathbf{U}_2, \ldots$ are mutually independent and identically distributed with probability function f, and independent of N. It is further assumed that all the U_{ij}'s are non-negative. Let p and g denote the probability functions of N and $\mathbf{X}$ respectively. Then (1.1) still holds.

2B. In this subsection we shall indicate some situations where our model could be appropriate.

1. As indicated in Section 1, in a portfolio with m policies we can interpret U_{ij} as the claim amount of policy j caused by claim event i. A natural example is windstorm insurance where one windstorm could affect more than one policy. The variable X_j represents the aggregate claims from policy j.

2. If we let U_{ij} be equal to one if claim event i causes a payment on policy j, and zero otherwise, then X_j will be the number of claims of policy j. Analogously we can develop multivariate counting distributions in the following examples.

3. Another application of the model would be to a situation where each claim event can induce various types of claims. These types could have different reinsurance covers. Let m be the number of types and U_{ij} the payments of type j at claim event i. Then X_j represents the aggregate claims of type j. We assume that X_j is covered by a reinsurance such that the insurance company retains $r_j(X_j)$. Thus the total aggregate claims will be $Z = \sum_{j=1}^{m} r_j(X_j)$. From g we can evaluate the distribution of Z. Perhaps Z is covered by an umbrella cover. Then we can apply the distribution of Z to evaluate the premium for the umbrella. A good example of an insurance class where different types of claims could have different reinsurance covers, is motor insurance, where the reinsurance would often be different for vehicle damage and personal injury. Analogously, in workers' compensation insurance one could have different reinsurance for sickness and accident.

4. We now return to the case with only one type of claim and let $m = 1$. Of claim i the ceding company retains U_{i1}, and the reinsurer covers the rest, U_{i2}. Then X_1 and X_2 represent the total payments of the

insurer and the reinsurer respectively, and by our generalised Panjer recursion we can evaluate their joint distribution. In subsection 4G we shall study this situation in the special case of unlimited excess-of-loss reinsurance.

5. Let us now consider the run-off of the claims of an insurance portfolio incurred during a specified year. We assume that all claims will have been settled after m years. Let U_{ij} be the part of the ith claim paid in development year j. Then X_j will be the total payments of development year j.

2C. By conditioning on N we easily obtain that

$$\mathrm{Cov}\,(X_j, X_k) = \mathrm{E}\,N\ \mathrm{Cov}\,(U_{1j}, U_{1k}) + \mathrm{E}\,U_{1j}\,\mathrm{E}\,U_{1k}\,\mathrm{Var}\,N. \qquad (2.1)$$

If U_{1j} and U_{1k} are independent, then (2.1) gives

$$\mathrm{Cov}\,(X_j, X_k) = \mathrm{E}\,U_{1j}\,\mathrm{E}\,U_{1k}\,\mathrm{Var}\,N > 0, \qquad (k \neq j)$$

that is, because N affects all the policies, the aggregate claims of different policies will be positively correlated when the severities of different policies are independent.

We rewrite (2.1) as

$$\mathrm{Cov}\,(X_j, X_k) = \mathrm{E}\,N\ \mathrm{E}\,U_{1j}U_{1k} + \mathrm{E}\,U_{1j}\,\mathrm{E}\,U_{1k}\,(\mathrm{Var}\,N - \mathrm{E}\,N)\,.$$

If N is Poisson distributed, then $\mathrm{Var}\,N = \mathrm{E}\,N$ (cf. e.g. Sundt (1993)), and we obtain

$$\mathrm{Cov}\,(X_j, X_k) = \mathrm{E}\,N\ \mathrm{E}\,U_{1j}U_{1k}.$$

We see that in this case, for $k \neq j$, X_j and X_k are uncorrelated if and only if $\mathrm{E}\,U_{1j}U_{1k} = 0$. This implies that U_{1j} and U_{1k} cannot both be positive. In the situation where U_{ij} denotes the amount caused to policy j by claim event i, this means that a claim event cannot hit more than one policy. In the situation where U_{ij} denotes the amount of type j caused by one claim, it means that a claim cannot cause payments of more than one type; there cannot at the same time be payments on death and disability. In fact, in this case we do not only have that X_j and X_k are uncorrelated, but they are even independent, cf. e.g. Sundt (1993).

Leaving the Poisson assumption, but keeping the assumption that $\mathrm{E}\,U_{1j}U_{1k} = 0$ for $k \neq j$, we obtain

$$\mathrm{Cov}\,(X_j, X_k) = \mathrm{E}\,U_{1j}\,\mathrm{E}\,U_{1k}\,(\mathrm{Var}\,N - \mathrm{E}\,N)\,. \qquad (k \neq j)$$

When N is negative binomially distributed, we have $\mathrm{Var}\,N > \mathrm{E}\,N$, and thus X_j and X_k are positively correlated. On the other hand, if N is binomially distributed, then $\mathrm{Var}\,N < \mathrm{E}\,N$, and X_j and X_k are negatively correlated.

3 Main result

3A. In the following we shall apply the notation

$$\mathbf{x} = (x_1, \ldots, x_m)'; \qquad \mathbf{u} = (u_1, \ldots, u_m)'.$$

We shall always tacitly assume that the elements of $\mathbf{x}$ and $\mathbf{u}$ are non-negative integers. By $\mathbf{u} \leq \mathbf{x}$ we shall mean that $u_j \leq x_j$ for $j = 1, \ldots, m$, and by $\mathbf{u} < \mathbf{x}$ that $u_j \leq x_j$ for $j = 1, \ldots, m$ with strict inequality for at least one j. For $j = 1, \ldots, m$ we define $\mathbf{e}_j$ to be the $m \times 1$ vector whose jth element is 1 and all the other elements are 0.

Under the additional assumption that p satisfies (1.2), we shall deduce multivariate extensions of Panjer's recursion.

Theorem 1. *The probability function g satisfies the recursion*

$$g(\mathbf{0}) = \sum_{n=0}^{\infty} p(n) f(\mathbf{0})^n \tag{3.1}$$

$$x_k g(\mathbf{x}) = \frac{1}{1 - af(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} (ax_k + bu_k) f(\mathbf{u}) g(\mathbf{x} - \mathbf{u}) \quad (\mathbf{x} > \mathbf{0}) \tag{3.2}$$

for $k = 1, 2, \ldots, m$.

Proof. Formula (3.1) follows from (1.1) and the assumption that $\sum_{j=1}^{m} U_{1j}$ is non-negative.

When $\mathbf{x} > \mathbf{0}$, we have

$$x_k g(\mathbf{x}) = \sum_{n=1}^{\infty} x_k p(n) f^{n*}(\mathbf{x}) = \sum_{n=1}^{\infty} x_k p(n-1) \left(a + \frac{b}{n}\right) f^{n*}(\mathbf{x}) =$$
$$\sum_{n=1}^{\infty} p(n-1) \operatorname{E}\left[ax_k + bU_{1k} \middle| \sum_{i=1}^{n} \mathbf{U}_i = \mathbf{x}\right] f^{n*}(\mathbf{x}) =$$
$$\sum_{n=1}^{\infty} p(n-1) \sum_{\mathbf{0}\leq\mathbf{u}\leq\mathbf{x}} (ax_k + bu_k) f(\mathbf{u}) f^{(n-1)*}(\mathbf{x} - \mathbf{u}) =$$
$$\sum_{\mathbf{0}\leq\mathbf{u}\leq\mathbf{x}} (ax_k + bu_k) f(\mathbf{u}) \sum_{n=1}^{\infty} p(n-1) f^{(n-1)*}(\mathbf{x} - \mathbf{u}) =$$
$$\sum_{\mathbf{0}\leq\mathbf{u}\leq\mathbf{x}} (ax_k + bu_k) f(\mathbf{u}) g(\mathbf{x} - \mathbf{u}) =$$
$$ax_k f(\mathbf{0}) g(\mathbf{x}) + \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} (ax_k + bu_k) f(\mathbf{u}) g(\mathbf{x} - \mathbf{u}),$$

and solving for $x_k g(\mathbf{x})$ gives (3.2).

This completes the proof of Theorem 1. Q.E.D.

When $x_k > 0$, we can divide (3.2) by x_k. We obtain

$$g(\mathbf{x}) = \frac{1}{1 - af(\mathbf{0})} \sum_{\mathbf{0} < \mathbf{u} \leq \mathbf{x}} \left(a + b\frac{u_k}{x_k} \right) f(\mathbf{u})\, g(\mathbf{x} - \mathbf{u}), \qquad (\mathbf{x} \geq \mathbf{e}_k) \qquad (3.3)$$

which together with (3.1) can be applied for recursive evaluation of g.

When $f(\mathbf{0}) = 0$, (3.1) and (3.3) reduce to

$$g(\mathbf{0}) = p(0)$$

$$g(\mathbf{x}) = \sum_{\mathbf{0} < \mathbf{u} \leq \mathbf{x}} \left(a + b\frac{u_k}{x_k} \right) f(\mathbf{u})\, g(\mathbf{x} - \mathbf{u}). \qquad (\mathbf{x} \geq \mathbf{e}_k) \qquad (3.4)$$

In the situation with a portfolio with m policies, this assumption means that any claim event will affect at least one policy. When $m = 1$, (3.4) reduces to

$$g(x) = \sum_{u=1}^{x} \left(a + b\frac{u}{x} \right) f(u)\, g(x-u). \qquad (x=1, 2, \ldots) \qquad (3.5)$$

This is the recursion deduced by Panjer (1981).

3B. It is interesting to notice that when $b \neq 0$, the recursion (3.3) is not symmetric in the policies; we have to give one of them, policy k, a special treatment. In practice it seems to be computationally most efficient to choose the policy where the claim amount can take the least number of values. We shall return to this in Section 4.

Also, we see that we have to apply another policy than policy k as the special policy when $x_k = 0$. Let us apply policy l when $x_k = 0$ and $x_l > 0$. However, when $x_k = x_l = 0$, we have to apply a third policy as the special policy, and so on. In the worst case, we will have to involve each of the m policies as the special policy at some stage. This may make the recursion of Theorem 1 awkward to program. However, it will normally involve less arithmetic operations than brute force evaluation by (1.1).

In some cases the problem of $x_k = 0$ is reduced or vanishes completely. We see that the problem arises when $x_k = 0$ and $x_j > 0$ for at least one $j \neq k$. If U_{ik} is always positive when at least one of the other U_{ij}'s is positive, then X_k is also positive only when at least one of the other X_j's is positive. Therefore $g(\mathbf{x}) = 0$ when $x_k = 0$ and some other x_j is positive, and the case $x_k = 0$ does not create any problem for the recursion. In the situation with excess-of-loss reinsurance described in subsection 2B the ceding company will always make payments for own account when there are positive reinsurance payments, and thus, with $k = 1$, the case $x_1 = 1$ is unproblematic. We shall return to this situation in subsection 4G.

It seems that in practice the multivariate recursions would be applicable only when there is a low number of policies in the portfolio, as otherwise the computational work would be prohibitive.

3C. Let $\mathbf{c} = (c_1, \ldots, c_m)'$ be a constant vector with non-negative elements. Multiplication of (3.2) by c_k and summation over k gives

$$g(\mathbf{x})\,\mathbf{c}'\mathbf{x} = \frac{1}{1 - af(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} (a\mathbf{c}'\mathbf{x} + b\mathbf{c}'\mathbf{x})\, f(\mathbf{u})\, g(\mathbf{x}-\mathbf{u})\,. \quad (\mathbf{x} > \mathbf{0}) \quad (3.6)$$

When $\mathbf{c}'\mathbf{x} > 0$, we can divide (3.6) by $\mathbf{c}'\mathbf{x}$, and we obtain

$$g(\mathbf{x}) = \frac{1}{1 - af(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} \left(a + b\frac{\mathbf{c}'\mathbf{u}}{\mathbf{c}'\mathbf{x}}\right) f(\mathbf{u})\, g(\mathbf{x}-\mathbf{u})\,. \quad (\mathbf{c}'\mathbf{x} > \mathbf{0}) \quad (3.7)$$

Normally we would choose the c_k's equal to zero or one. In particular, when $\mathbf{c} = \mathbf{e}_k$, (3.7) reduces to (3.3). Another interesting case is when $c_k = 1$ for all k. Then $\mathbf{c}'\mathbf{x} = \sum_{k=1}^{m} x_k$, which is always positive when $\mathbf{x} > \mathbf{0}$. Thus

$$g(\mathbf{x}) = \frac{1}{1 - af(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} \left(a + b\frac{\sum_{k=1}^{m} u_k}{\sum_{k=1}^{m} x_k}\right) f(\mathbf{u})\, g(\mathbf{x}-\mathbf{u})\,. \quad (\mathbf{x} > \mathbf{0}) \quad (3.8)$$

Under efficient programming, (3.8) is not necessarily significantly more time-consuming than (3.3), and it has the advantage that it can be applied for all $\mathbf{x} > \mathbf{0}$.

When $m = 1$, the recursion (3.8) reduces to Panjer's recursion (3.5).

For the rest of the paper we shall mainly concentrate on recursions with division by x_k and leave to the readers to deduce corresponding recursions with division by $\sum_{k=1}^{m} x_k$ or $\mathbf{c}'\mathbf{x}$.

3D. The way we extended Panjer's univariate recursion, can easily be applied to other univariate recursions. As an example, let us generalise the recursion (3.4) to the situation when p satisfies a recursion

$$p(n) = \sum_{s=1}^{r} \left(a_s + \frac{b_s}{n}\right) p(n-s) \qquad (n = 1, 2, \ldots) \qquad (3.9)$$

for some positive integer r; we have $p(n) = 0$ for $n < 0$. When $r = 1$, (3.9) reduces to (1.2). By modifying the proof of Theorem 9 in Sundt (1992) analogous to the way we modified the proof of Theorem 10.6 in Sundt (1993) for the proof of Theorem 1, we obtain

$$g(\mathbf{x}) = \frac{1}{1 - \sum_{s=1}^{r} a_s f^{s}(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} g(\mathbf{x}-\mathbf{u}) \sum_{s=1}^{r} \left(a_s + \frac{b_s}{s}\frac{u_k}{x_k}\right) f^{s*}(\mathbf{u})\,, \quad (\mathbf{x} \geq \mathbf{e}_k)$$

and analogous to (3.8) we obtain

$$g(\mathbf{x}) = \frac{1}{1 - \sum_{s=1}^{r} a_s f^s(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} g(\mathbf{x}-\mathbf{u}) \sum_{s=1}^{r} \left(a_s + \frac{b_s}{s} \frac{\sum_{k=1}^{m} u_k}{\sum_{k=1}^{m} x_k} \right) f^{s*}(\mathbf{u}) . \qquad (\mathbf{x} > \mathbf{0})$$

These recursions reduce to respectively (3.3) and (3.8) when $r = 1$, and to the recursion of Sundt (1992) when $m = 1$.

The recursions of the present subsection are further analysed in Sundt (1998).

4 Special cases

4A. For the present section we shall for simplicity restrict to the case $m = 2$, and we assume that $f(0,0) = 0$. The examples that we present, can easily be adapted to larger values of m. Some of the examples are primarily of theoretical interest whereas others have practical relevance.

Let $U_i = U_{i1}$, $V_i = U_{i2}$, $X = X_1 = \sum_{i=1}^{N} U_i$, and $Y = X_2 = \sum_{i=1}^{N} V_i$. In this case (3.4) gives

$$g(x,y) = \sum_{u=0}^{x} \left(a + b\frac{u}{x} \right) \sum_{v=0}^{y} f(u,v)\, g(x-u, y-v) \qquad (x = 1,2,\ldots;\ y = 0,1,\ldots) \tag{4.1}$$

$$g(x,y) = \sum_{v=0}^{y} \left(a + b\frac{v}{y} \right) \sum_{u=0}^{x} f(u,v)\, g(x-u, y-v), \qquad (x = 0,1,\ldots;\ y = 1,2,\ldots) \tag{4.2}$$

and from (3.8) we obtain

$$g(x,y) = \sum_{u=0}^{x} \sum_{v=0}^{y} \left(a + b\frac{u+v}{x+y} \right) f(u,v)\, g(x-u, y-v) \qquad ((x,y) \neq (0,0)) \tag{4.3}$$

Notice that, unlike in (3.4), in these formulae we have included $(u,v) = (0,0)$ in the summations to simplify the display of the formulae. However, as by assumption $f(0,0) = 0$, the extra term is equal to zero.

If we were to base our evaluation of $g(x,y)$ on (4.1) for all (x,y)'s such that $x > 0$, then we could evaluate $g(0,y)$ by (4.2), that is,

$$g(0,y) = \sum_{v=0}^{y} \left(a + b\frac{v}{y} \right) f(0,v)\, g(0, y-v) . \qquad (y = 1,2,\ldots)$$

However, if U_i is always positive, then together with $g(0,0)$ the recursion (4.1) specifies g completely.

4B. To assume that U_i and V_i are independent does not seem to bring any substantial simplification to our recursions. In that case, if h and k denote the marginal probability functions of U_i and V_i, then we can write (4.1) as

$$g(x,y) = \sum_{u=0}^{x} \left(a + b\frac{u}{x}\right) h(u) \sum_{v=0}^{y} k(v)\, g(x-u, y-v).$$
$$(x = 1,2,\ldots;\ y = 0,1,\ldots)$$

4C. Let us now consider the situation with two types of claims. We let $U_i = 1$ if claim event i leads to payments of type 1, and $U_i = 0$ if that is not the case. Analogously we let $V_i = 1$ if claim event i leads to payments of type 2, and $V_i = 0$ if that is not the case. Then X and Y are the total numbers of claims of type 1 and 2 respectively. In this case $f(u,v)$ can be positive only when $u, v \in \{0,1\}$, and (4.1) reduces to

$$g(x,y) = \left(a + \frac{b}{x}\right)\left(f(1,0)\, g(x-1,y) + f(1,1)\, g(x-1,y-1)\right) + af(0,1)\, g(x,y-1). \qquad (x = 1,2,\ldots;\ y = 0,1,\ldots) \tag{4.4}$$

4D. We now leave the restriction that U_i and V_i can only take the values 0 and 1, and consider the case when $\mathrm{E}\,U_iV_i = 0$. As pointed out in subsection 2C, in the situation with different types of payments, this corresponds to the case that a claim cannot have payments of more than one type. Let c denote the probability that a claim is of type 1, h the conditional probability function of the claim amount given that the claim is of type 1, and k the conditional probability function of the claim amount given that the claim is of type 2. As $f(0,0)$ should be equal to zero, we have that $h(0) = k(0) = 0$. Then

$$f(u,v) = \begin{cases} ch(u) & (u = 1,2,\ldots;\ v = 0) \\ (1-c)\, k(v) & (u = 0;\ v = 1,2,\ldots) \\ 0. & \text{otherwise} \end{cases}$$

Insertion in (4.1) gives

$$g(x,y) = c\sum_{u=1}^{x} \left(a + b\frac{u}{x}\right) h(u)\, g(x-u,y) + (1-c)\, a \sum_{v=1}^{y} k(v)\, g(x, y-v). \qquad (x = 1,2,\ldots;\ y = 0,1,\ldots) \tag{4.5}$$

In the special case with $a = 0$, that is, N is Poisson distributed with parameter b, the last summation in (4.5) vanishes, and we obtain

$$g(x,y) = \frac{cb}{x} \sum_{u=1}^{x} h(u)\, g(x-u,y). \qquad (x = 1,2,\ldots;\ y = 0,1,\ldots) \tag{4.6}$$

As pointed out in subsection 2C, in the present case we know that X and Y are independent. Thus $g(x,y) = s(x)\,t(y)$, where s and t denote the marginal probability functions of respectively X and Y. Insertion in (4.6) gives

$$s(x)\,t(y) = t(y)\,\frac{cb}{x}\sum_{u=1}^{x} h(u)\,s(x-u)\,. \qquad (x = 1,2,\ldots;\; y = 0,1,\ldots)$$

As there must exist some y such that $t(y) > 0$, we obtain

$$s(x) = \frac{cb}{x}\sum_{u=1}^{x} h(u)\,s(x-u)\,. \qquad (x = 1,2,\ldots) \tag{4.7}$$

This is the univariate Panjer recursion for a compound Poisson distribution with Poisson parameter cb and severity distribution with probability function h. As

$$s(x) = \sum_{y=0}^{\infty} g(x,y)\,, \qquad (x = 1,2,,\ldots)$$

we could also have obtained (4.7) from (4.6) by summation over y.

If we, leaving the Poisson assumption, define U_i and V_i as in subsection 4C, we obtain

$$f(1,0) = c; \qquad f(0,1) = 1-c; \qquad f(1,1) = 0,$$

and insertion in (4.4) gives

$$g(x,y) = \left(a + \frac{b}{x}\right) cf(1,0)\,g(x-1,y) + a(1-c)\,g(x,y-1)\,. \tag{4.8}$$

$$(x = 1,2,\ldots;\; y = 0,1,\ldots)$$

4E. The model of subsection 4C can also be expressed within the framework of subsection 1C. For $j = 1,2$ let N_j be the number of claims of type j and W_{ij} the amount of the ith of these claims. Then the assumptions of subsection 1C are fulfilled with $f_1 = h$ and $f_2 = k$.

The conditional distribution of N_1 given that $N = n$, is binomial with

$$\Pr(N_1 = n_1\,|N = n) = \binom{n}{n_1} c^{n_1}(1-c)^{n-n_1}\,.$$

$$(n_1 = 0,1,\ldots,n;\; n = 0,1,\ldots)$$

Thus for $n_1, n_2 = 0,1,\ldots$ we have

$$\begin{aligned}
&q(n_1,n_2) = \Pr((N_1 = n_1)\cap(N_2 = n_2)) = \\
&\Pr((N_1 = n_1)\cap(N = n_1+n_2)) = \\
&\Pr(N = n_1+n_2)\Pr(N_1 = n_1\,|N = n_1+n_2) = \\
&p(n_1+n_2)\binom{n_1+n_2}{n_1} c^{n_1}(1-c)^{n_2}\,.
\end{aligned}$$

We are now within Model A of Hesselager (1996), and the recursions (4.5) and (4.8) are given in his Therorems 2.2 and 2.1 respectively.

4F. Let us now consider a univariate situation. Let W_i be the amount of the ith claim. We assume that the W_i's are positive, integer-valued, mutually independent and identically distributed with common probability function h, and independent of N. Let s be a positive integer. We say that a claim is of type 1 if it is less than or equal to s, and of type 2 if it is greater than s. We now have a special case of the situation of subsection 4C with

$$f(u,v) = \begin{cases} h(u) & (u = 1,2,\dots s;\ v = 0) \\ h(v) & (u = 0;\ v = s+1, s+2, \dots) \\ 0, & \text{otherwise} \end{cases} \tag{4.9}$$

and insertion in (4.1) gives

$$g(x,0) = \sum_{u=1}^{s} \left(a + b\frac{u}{x}\right) h(u)\, g(x-u,0) \qquad (x = 1,2,\dots)$$

$$g(x,y) = \sum_{u=1}^{s} \left(a + b\frac{u}{x}\right) h(u)\, g(x-u,y) + a \sum_{v=s+1}^{y} h(v)\, g(x,y-v)\,; \\ (x = 1,2,\dots;\ y = s+1, s+2, \dots)$$

Insertion in (4.2) gives

$$g(x,y) = a \sum_{u=1}^{s} h(u)\, g(x-u,y) + \sum_{v=s+1}^{y} \left(a + b\frac{v}{y}\right) h(v)\, g(x,y-v)\,. \\ (x = 0,1,\dots;\ y = s+1, s+2 \dots)$$

It is clear that $g(x,y) = 0$ when $y = 1,\dots,s$.

4G. In the previous subsection, we distinguished between claims less than or equal to s and claims larger than s. A more interesting situation would be to let

$$U_i = \min(W_i, s)\,; \qquad V_i = \max(W_i - s, 0)\,.$$

Under an unlimited excess-of-loss treaty with retention s, U_i and V_i are respectively the retained and reinsured parts the ith claim. Thus X becomes the total payments of the ceding company and Y the total payments of the reinsurer. Analogous to (4.9) we obtain

$$f(u,v) = \begin{cases} h(u) & (u = 1,2,\dots s;\ v = 0) \\ h(v+s) & (u = s;\ v = 1,2,\dots) \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

Insertion in (4.1) gives

$$g(x,y) = \sum_{u=1}^{s}\left(a+b\frac{u}{x}\right)h(u)\,g(x-u,y)+$$
$$\left(a+b\frac{s}{x}\right)\sum_{v=1}^{y}h(v+s)\,g(x-s,y-v)\,. \tag{4.11}$$
$$(x=1,2,\ldots;\ y=0,1,\ldots)$$

As now U_i is always positive, together with $g(0,0)$ this recursion completely specifies g.

Insertion of (4.10) in (4.2) gives

$$g(x,y) = a\sum_{u=1}^{s}h(u)\,g(x-u,y)+\sum_{v=1}^{y}\left(a+b\frac{v}{y}\right)h(v+s)\,g(x-m,y-v)\,.$$
$$(x=1,2,\ldots;\ y=1,2,\ldots)$$

For small y this recursion may be more convenient than (4.11). However, we will still need (4.11) to evaluate $g(x,0)$.

4H. Let is finally consider the case when $U_i = W_i$ and $V_i = 1$. Then X will be the aggregate claims and Y the number of claims, that is, g will be the joint probability function of the aggregate claims and the number of claims. Thus we have

$$g(x,y) = \Pr(Y=y)\Pr(X=x\,|Y=y) = p(y)\,h^{y*}(x)\,. \quad (x,y=0,1,\ldots) \tag{4.12}$$

Furthermore,

$$f(u,v)=\begin{cases} h(u) & (u=1,2,\ldots s;\ v=1)\\ 0, & \text{otherwise}\end{cases}$$

and insertion in respectively (4.1) and (4.2) gives

$$g(x,y) = \sum_{u=1}^{x}\left(a+b\frac{u}{x}\right)h(u)\,g(x-u,y-1) \qquad (x,y=1,2,\ldots) \tag{4.13}$$

$$g(x,y) = \left(a+\frac{b}{y}\right)\sum_{u=1}^{x}h(u)\,g(x-u,y-1)\,. \qquad (x,y=1,2,\ldots) \tag{4.14}$$

As both U_i and V_i are always positive, together with $g(0,0)$ each of the recursions completely specifies g.

The recursion (4.14) can easily be seen more directly. By successively applying (4.12) and (1.2) in the right-hand side of (4.14) we obtain

$$\left(a+\frac{b}{y}\right)\sum_{u=1}^{x} h(u)\, g(x-u, y-1) =$$
$$\left(a+\frac{b}{y}\right)\sum_{u=1}^{x} h(u)\, p(y-1)\, h^{(y-1)*}(x-u) =$$
$$\left(a+\frac{b}{y}\right) p(y-1)\sum_{u=1}^{x} h(u)\, h^{(y-1)*}(x-u) = p(y)\, h^{y*}(x) = g(x).$$

It does not seem possible to give such a simple interpretation of (4.13).

5 Convolutions

5A. Let us now consider the special case of Theorem 1 when p is the binomial probability function

$$p(n) = \binom{r}{n} c^n (1-c)^{r-n}. \quad (n = 0, 1, \ldots, r;\ r = 1, 2, \ldots;\ 0 < c < 1) \quad (5.1)$$

Then

$$a = -\frac{c}{1-c}; \qquad b = (r+1)\frac{c}{1-c}.$$

We assume that $f(0) = 0$. Application of Theorem 1 gives

$$g(\mathbf{0}) = (1-c)^r \quad (5.2)$$

$$g(\mathbf{x}) = \frac{c}{1-c}\sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}}\left((r+1)\frac{u_k}{x_k} - 1\right) f(\mathbf{u})\, g(\mathbf{x}-\mathbf{u}). \qquad (\mathbf{x} \geq \mathbf{e}_k) \quad (5.3)$$

We shall apply this recursion to deduce a multivariate extension of De Pril's (1985) recursion for convolutions.

5B. Let $\mathbf{V}_1, \ldots, \mathbf{V}_r$ be r m-dimensional independent and identically distributed random vectors with non-negative integer-valued elements and common probability function h. It is assumed that $0 < h(\mathbf{0}) < 1$. We want to deduce a recursion for the probability function g of $\mathbf{X} = \sum_{i=1}^{r}\mathbf{V}_i$. We obviously have $g = h^{r*}$.

Let

$$c = 1 - h(\mathbf{0}) \quad (5.4)$$

$$f(\mathbf{y}) = \frac{h(\mathbf{y})}{c}. \qquad (\mathbf{y} > \mathbf{0}) \quad (5.5)$$

The function f can be interpreted as the conditional probability function of $\mathbf{V}_i$ given that at least one of its elements is greater than zero. We now have that $\mathbf{X}$ has a compound binomial distribution with counting distribution given by (5.1) with c given by (5.4) and severity distribution with probability function f. Insertion of (5.4) and (5.5) in (5.2) and (5.3) gives

$$g(\mathbf{0}) = h(\mathbf{0})^r$$

$$g(\mathbf{x}) = \frac{1}{h(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} \left((r+1)\frac{u_k}{x_k} - 1\right) h(\mathbf{u})\, g(\mathbf{x}-\mathbf{u}). \qquad (\mathbf{x} \geq \mathbf{e}_k) \quad (5.6)$$

Analogously, by application of (3.8) we obtain

$$g(\mathbf{x}) = \frac{1}{h(\mathbf{0})} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} \left((r+1)\frac{\sum_{k+1}^m u_k}{\sum_{k+1}^m x_k} - 1\right) h(\mathbf{u})\, g(\mathbf{x}-\mathbf{u}). \qquad (\mathbf{x} > \mathbf{0})$$

When $m = 1$, both these recursion reduce to

$$g(0) = h(0)^r$$

$$g(x) = \frac{1}{h(0)} \sum_{u=1}^{x} \left((r+1)\frac{u}{x} - 1\right) h(u)\, g(x-u). \qquad (x = 1, 2, \ldots) \quad (5.7)$$

This is the recursion deduced by De Pril (1985).

For $j = 1, \ldots, m$, let h_j and g_j denote the marginal probability functions of V_{1j} and X_j respectively. As $h(\mathbf{0}) > 0$, we must have $h_j(0) > 0$ for $j = 1, \ldots, m$. If $V_{11}, \ldots, V_{1m}$ are independent, then $X_1, \ldots, X_m$ are also independent, and we have

$$h(\mathbf{u}) = \prod_{j=1}^{m} h_j(u_j); \qquad g(\mathbf{x}) = \prod_{j=1}^{m} g_j(x_j).$$

Insertion in (5.6) gives

$$\prod_{j=1}^{m} g_j(x_j) = \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} \left((r+1)\frac{u_k}{x_k} - 1\right) \prod_{j=1}^{m} \frac{h_j(u_j)\, g_j(x_j-u_j)}{h_j(0)} \qquad (\mathbf{x} \geq \mathbf{e}_k). \quad (5.8)$$

However, in this case it seems more convenient to evaluate each of the g_j's separately by the univariate recursion (5.7) and then multiply them.

If $\prod_{j\neq k} g_j(x_j) > 0$, we can rewrite (5.8) as

$$g_k(x_k) = \frac{1}{h_k(0)} \sum_{\mathbf{0}<\mathbf{u}\leq\mathbf{x}} \left((r+1)\frac{u_k}{x_k} - 1\right) h_k(u_k)\, g_k(x_k-u_k) \prod_{j\neq k} \frac{h_j(u_j)\, g_j(x_j-u_j)}{h_j(0)\, g_j(x_j)}. \qquad (\mathbf{x} \geq \mathbf{e}_k)$$

As the univariate recursion (5.7) gives

$$g_k(x_k) = \frac{1}{h_k(0)} \sum_{u_k=1}^{x_k} \left((r+1) \frac{u_k}{x_k} - 1 \right) h_k(u_k)\, g_k(x_k - u_k), \qquad (x_k = 1, 2, \ldots) \tag{5.9}$$

it is tempting to conclude that we obtain 1 by a summation over the u_j's of

$$\prod_{j \neq k} \frac{h_j(u_j)\, g_j(x_j - u_j)}{h_j(0)\, g_j(x_j)}.$$

However, in the next subsection we shall see that it is not that simple.

5C. Let us now consider the special case $m = 2$. In that case (5.6) with $k = 1$ gives

$$g(x, y) = \frac{1}{h(0,0)} \left[\sum_{u=1}^{x} \left((r+1) \frac{u}{x} - 1 \right) \sum_{v=0}^{y} h(u, v)\, g(x - u, y - v) - \sum_{v=1}^{y} h(0, v)\, g(x, y - v) \right]; \qquad (x{=}1, 2, \ldots;\ y = 0, 1, \ldots) \tag{5.10}$$

as $h(0,0) > 0$, we cannot include $(u, v) = (0, 0)$ in the summation as in (4.1) and (4.2).

If V_{11} and V_{12} are independent, (5.10) gives that for $x{=}1, 2, \ldots;\ y = 0, 1, \ldots$

$$g_1(x)\, g_2(y) = \frac{1}{h_1(0)\, h_2(0)} \times \left[\left(\sum_{u=1}^{x} \left((r+1) \frac{u}{x} - 1 \right) h_1(u)\, g_1(x - u) \right) \sum_{v=0}^{y} h_2(v)\, g_2(y - v) - h_1(0)\, g_1(x) \sum_{v=1}^{y} h_2(v)\, g_2(y - v) \right],$$

that is,

$$g_1(x)\, g_2(y) = \frac{1}{h_1(0)\, h_2(0)} \left[A \sum_{v=0}^{y} h_2(v)\, g_2(y - v) + h_1(0)\, g_1(x)\, h_2(0)\, g_2(y) \right] \tag{5.11}$$

with

$$A = \sum_{u=1}^{x} \left((r+1) \frac{u}{x} - 1 \right) h_1(u)\, g_1(x - u) - h_1(0)\, g_1(x).$$

>From (5.9) we see that $A = 0$, and thus the right-hand expression in (5.11) reduces to $g_1(x)\, g_2(x)$.

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