A DECOMPOSITION OF ACTUARIAL SURPLUS AND APPLICATIONS

by Daniel Dufresne, University of Montreal

Abstract

The actuarial gain is the unexpected increase of the actuarial surplus over a certain period, usually one year (a loss is a negative gain). A Decomposition Theorem for the surplus in terms of past gains is derived. If actuarial assumptions are unbiased, then the arithmetic sum of the gains forms a martingale. The latter has qualitatively the same behaviour as a zero-mean random walk. However, the process obtained by accumulating the gains with interest diverges to plus or minus infinity. Also considered are biased actuarial assumptions, and connections with Risk Theory. Mathematically, all this is closely related to the ruin problem of Probability Theory; the novelties are the dependence between the outcomes of the successive "games" played, and the fact that the player's wealth grows with interest (possibly random).

Gains and Losses; Discounted Sums; Martingales; Risk Theory

1. Introduction

Martingales have been applied to a number of actuarial problems, for example Risk Theory and Hattendorff's Theorem; see Boogaert *et al.* (1988), Gerber (1979), Martin-Löf (1986), Norberg (1992), Papatriandafylou and Waters (1984) for these and other examples. This paper has three goals:

(*i*) show how martingale theory may be used to get a better understanding of actuarial gains and losses ("AGLs" in the sequel);

(*ii*) derive a Decomposition Theorem for the actuarial surplus in terms of actuarial gains and losses;

(*iii*) study the behaviour of the cumulative sum of actuarial gains and losses, with and without interest, and see what the practical consequences are.

Actuaries often refer to AGLs. In words, the actuarial gain is the unexpected increase of the actuarial surplus over a certain period, usually one year (a loss is a negative gain). The concept is a natural one, and relates to all types of insurance. (In Time Series Analysis the same idea appears in the definition of the "innovation process.") Actuarial gains and losses are particularly important in pension funding and accounting in many countries, where they are estimated and separately amortized as part of the actuarial valuation.

There is a basic difference between business or investment gains and AGLs. If one invests \$ 100, and its value grows after one year to \$ 110, then the investment gain is \$ 10. However, if *a priori* the rate of return is *expected* to be 9%, then the "actuarial gain" is just \$ 1, the difference between \$ 110 and the expected amount (as seen at the beginning of the period), \$ 109. Nevertheless, the same mathematical tools may be used to study either types of gains and losses. One noticeable disparity between them is that investment gains can be positive or negative on average, while the very definition of AGLs implies that they should be zero on average. (The latter may not hold if actuarial assumptions are "biased"; see Section 8.) One belief concerning AGLs keeps coming back, that a number of losses should be followed by gains, or that there should be some cancelling out of AGLs over time. A very clear expression of this idea can be found in US pension accounting regulation:

> 184. The [Financial Accounting Standards] Board noted that, if assumptions prove to be accurate estimates of experience over a number of years, gains or losses in one year will be offset by losses or gains in subsequent periods. In that situation, all gains and losses would be offset over time, and amortization of unrecognized gains and losses would be unnecessary.
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> (*Statement of Financial Accounting Standards No. 87*, par. 184)

This brings to mind the famous 18th century mathematician (and philosopher) D'Alembert, who believed that, in a game of heads or tails, after observing two consecutive heads the next result was more likely to be a head as well (see Maistrov, 1974, page 128). D'Alembert's claim is similar to FASB's claim, but the latter is not as simple to analyse mathematically. AGLs are much more complex objects than are Bernoulli trials. There is usually a lot of stochastic dependence between successive AGLs, and there is no simple way to describe their joint distributions. This paper suggests one possible approach to the problem, based on martingale theory.

Section 2 gives an intuitive definition of AGLs; this leads to the more general definition of Section 3. Section 4 shows how to decompose the actuarial surplus in terms of past actuarial gains. Section 5 recalls some very well known properties of random walks, which refute the claim that gains and losses may offset over time, at least in the case of Bernoulli trials. The same analysis is performed in the martingale setting in Sections 6 and 7. In Section 6, it is seen that AGLs are not likely to offset when they are added together (without interest). Section 7 shows that there is no offsetting either when AGLs are accumulated with interest (possibly random); however the gain and loss accumulation process now has a very different qualitative behaviour. Section 8 briefly comments on the case where gains are biased. Section 9 shows how the previous analysis relates to classical Risk Theory, in particular with regard to a paper by Harrison (1977). Section 10 interprets the mathematical results obtained and concludes the paper. Theorems 3 and 4 of Section 7, concerning the asymptotic behaviour of some "weighted martingale differences," appear to be new. The Decomposition Theorem 1 is more or less obvious from general reasoning, but had not been given a precise expression up to now.

Remark. The expression "random walk" has acquired new (and confusing) meanings over the past few years; sometimes it even includes diffusion processes. In this paper a random walk will be the cumulative sum of independent and identically distributed (i.i.d.) random variables, not necessarily with mean zero. □

2. Intuitive definition of actuarial gains and losses

Consider an insurance business (life, non-life or pensions) and define

V_t : total mathematical reserve at time t

F_t : total assets at time t

$S_t \;=\; F_t - V_t.$

S_t will be called the *actuarial surplus* (or simply *surplus*) at time t. (In the context of pensions, V is often called the "actuarial liability," and $-S$ the "unfunded actuarial liability.") The surplus S is a measure of the financial soundness of the business at a particular time. A negative surplus is often called a *deficiency*.

Actuarial gains and losses, by contrast, relate to the financial success of the business over a certain period, usually on year. They may be defined as follows:

$$\text{gain during } (t-1,t) \;=\; (\text{surplus at time } t) - \begin{pmatrix}\text{value of surplus at time } t \text{ IF all actuarial as-}\\ \text{sumptions had been correct during } (t-1,t)\end{pmatrix}.$$

In other words, the actuarial gain is the unexpected increase of the surplus over a certain period. A negative gain is a loss, or the unexpected decrease of the surplus. Actuarial gains or losses arise whenever actuarial assumptions are not exactly realized, that is to say, all the time.

Mortality is the most obvious cause of gains and losses, but there are many others. Rates of return on assets are an important source of gains and losses, often surpassing in magnitude those arising from mortality. In pension funding there is a wealth of other possible causes: salary increases, inflation (with respect to benefit increases for retired members), withdrawals, new entrants (since they appear in a valuation with a fraction of year of service, meaning a positive reserve), contributions different in amount or in timing from those expected, investment and other fees, and so on.

A simple example will be given (more detailed examples may be found in Dufresne (1994)). Consider a pension plan with only one active member age x at time 0, whose benefit payments are due to start at age r, and let us define explicitly the random variable "actuarial gain" during the coming year. For the sake of simplicity, it will be assumed that the return on assets is precisely equal to the valuation rate of interest, and that the plan provides no benefits if the member dies or withdraws from the pension plan before retirement. (The more realistic situation where rates of return are uncertain and where termination or death benefits are provided is essentially the same.)

Let I be the indicator variable of the event "the member leaves the plan during the year" (for any reason). Thus

$$I = \begin{cases} 0 & \text{if member is present at time } t=1 \text{ (probability } p_x) \\ 1 & \text{otherwise (probability } q_x = 1-p_x). \end{cases}$$

Let V_x and π_x respectively denote the reserve (= actuarial liability) and net premium (= normal cost) at age x. We have

$$(V_x + \pi_x)(1+i_v) = p_x V_{x+1}, \quad x \le r-1,$$

where i_v is the valuation rate of interest.

Now turn to observed liabilities at time 1:

$$\text{(reserve at } t=1) = \begin{cases} V_{x+1} & \text{if } I=0 \text{ (probability } p_x) \\ 0 & \text{if } I=1 \text{ (probability } q_x). \end{cases} \tag{1}$$

On the assets side there is only one possibility,

$$F_1 = (1+i_v)(F_0+\pi_x),$$

assuming the normal cost is contributed at the beginning of the year. Bringing assets and liabilities together, we find the surplus: at $t=0$, $S_0 = F_0 - V_x$, and at $t=1$,

$$S_1 = \begin{cases} F_1 - V_{x+1} & \text{if } I=0 \\ F_1 & \text{if } I=1. \end{cases} \tag{2}$$

In order to evaluate the actuarial gain we need the expected value of S_1, given that all actuarial assumptions are exactly realized during the previous year. We get

$$\begin{aligned} \mathsf{E}S_1 &= p_x(F_1 - V_{x+1}) + q_x(F_1) \\ &= F_1 - p_x V_{x+1}. \end{aligned} \tag{3}$$

Define the actuarial gain as $G_1 = S_1 - \mathsf{E}S_1$. From the above

$$G_1 = \begin{cases} -q_x V_{x+1} & \text{if } I = 0 \\ p_x V_{x+1} & \text{if } I = 1. \end{cases} \tag{4}$$

According to actuarial assumptions, on average a fraction p_x of one member survives to age $x+1$. This explains the gain $p_x V_{x+1}$ if the member has left the plan, and the loss $q_x V_{x+1} = V_{x+1} - p_x V_{x+1}$ if the member remains in the plan.

Let us calculate the expected value of the gain:

$$\mathsf{E}G_1 = p_x(-q_x V_{x+1}) + q_x p_x V_{x+1} = 0.$$

This is not surprising, because by definition G_1 is equal to $S_1 - \mathsf{E}S_1$, and plainly $\mathsf{E}(S_1 - \mathsf{E}S_1) = 0$.

Remarks. 1. Currently, most pension plans provide for withdrawal and death benefits, which in effect mean that the "0" in case $I = 1$ (see Eq. (1)) is replaced with some positive amount — possibly the entire reserve V — which in turn modifies equations (2) to (4). Nevertheless, the results that losses are nil on average, and that they are uncorrelated (see below) remain unaffected, IF actuarial assumptions are indeed correct on average, that is to say, if the valuation basis consists of "best-estimate" (that is to say "unbiased") death and withdrawal probabilities (as well as benefits). If the valuation basis excludes some benefits, then clearly we cannot claim that $\mathsf{E}G_1 = 0$, because actuarial assumptions are then "biased."

2. In general one has to be very careful with the use of the expectation operator "$\mathsf{E}(\cdot)$," which may have two meanings: expectation with respect to "true" (unbiased) probabilities, or expectation with respect to actuarial assumptions (possibly biased). In Section 8, where biased assumptions are considered, the latter operation will be denoted "$\mathsf{E}^A(\cdot)$." □

3. A general definition of actuarial gains and losses

An actuarial gain was seen to be the "unexpected increase in the actuarial surplus over a certain period." This leads to the following mathematical formulation. Let

$\mathcal{F}_t$: information known at time t (consists of $F_t, F_{t-1}, \ldots, V_t, V_{t-1}, \ldots$, *etc.*).

Definition. The gain during the period $(t-1, t)$ is defined as

$$G_t \;=\; S_t - \mathsf{E}(S_t \,|\, \mathcal{F}_{t-1}). \tag{5}$$

An immediate consequence is $\mathsf{E}(G_t \,|\, \mathcal{F}_{t-1}) = 0$, which implies $\mathsf{E}G_t = 0$. An example will be given showing the relationship between gains and surpluses. It is formulated in the context of pension funding, but the idea is the same for any type of insurance. Let

${}^{r}B_{t-1,t}$: benefits paid during $(t-1, t)$, accumulated to end of year (using the pension fund's rate of return)

π : pure premium for whole population (= normal cost)

C : overall amount invested at beginning of year

ADJ : adjustment made to pure premium, equal to $C - \pi$.

When actuarial assumptions are unbiased, we have

$$\begin{aligned} \mathsf{E}(V_t \,|\, \mathcal{F}_{t-1}) &= (1+i_v)(V_{t-1} + \pi_{t-1}) - \mathsf{E}({}^{r}B_{t-1,t} \,|\, \mathcal{F}_{t-1}) \\ \mathsf{E}(F_t \,|\, \mathcal{F}_{t-1}) &= (1+i_v)(F_{t-1} + C_{t-1}) - \mathsf{E}({}^{r}B_{t-1,t} \,|\, \mathcal{F}_{t-1}). \end{aligned} \tag{6}$$

(The first equation is the usual recursive formula for reserves, see for example Chapter 7 of Bowers *et al.* (1986).) This implies

$$\begin{aligned} \mathsf{E}(S_t \,|\, \mathcal{F}_{t-1}) &= (1+i_v)(S_{t-1} + C_{t-1} - \pi_{t-1}) \\ &= (1+i_v)(S_{t-1} + ADJ_{t-1}), \end{aligned}$$

and

$$\begin{aligned} G_t &= S_t - \mathsf{E}(S_t \,|\, \mathcal{F}_{t-1}) \\ &= S_t - (1+i_v)(S_{t-1} + ADJ_{t-1}). \end{aligned}$$

4. Decomposition of actuarial surplus

The last equation may be rewritten as

$$S_t = (1+i_v)(S_{t-1}+ADJ_{t-1})+G_t.$$

The valuation interest rate does not have to be constant over time. Suppose $i_v(s)$ is the valuation rate of interest for the period $(s, s+1)$. The interest assumption is unbiased if, given what is known at time s, the conditional expectation of the rate of return on assets is equal to $i_v(s)$, for all s. In symbols this becomes

$$\mathsf{E}(R_{s+1}\,|\,\mathcal{F}_s) = i_v(s) \quad \text{almost surely, for } s=0,1,\ldots,$$

where R_{s+1} is the rate of return on assets for the period $(s, s+1)$. The symbol i_v may now be replaced with $i_v(t-1)$ in the previous equations. Define the "valuation accumulation factor from s to t"

$$u_{s,t} = [1+i_v(s)]\cdots[1+i_v(t-1)].$$

Discounting for valuation purposes from time t back to time s is now achieved by multiplying by $u_{s,t}^{-1}$. (It is natural to suppose that $i_v(t)$ is $\mathcal{F}_t$-measurable. Then $u_{s,t}$ is $\mathcal{F}_{t-1}$-measurable; in other words, the process $\{u_{s,t}; t>s\}$ is *predictable*.)

If $ADJ_t = 0$ for all $t \geq 0$, that is to say if only the pure premium π is paid into the fund, then the latter is the accumulated value, at the valuation interest rates $\{i_v(\cdot)\}$, of all previous gains or losses, together with the initial deficiency. We obtain the following *Decomposition Theorem*:

Theorem 1. *Suppose actuarial assumptions are unbiased. The definition of actuarial gain (5) and Eqs. (6) imply*

$$\begin{aligned} S_t &= [1+i_v(t-1)](S_{t-1}+ADJ_{t-1})+G_t \\ &= G_t + u_{t-1,t}(G_{t-1}+ADJ_{t-1}) + \cdots + u_{1,t}(G_1+ADJ_1) + u_{0,t}(S_0+ADJ_0). \end{aligned} \tag{7}$$

If only the pure premium (or normal cost) is invested into the insurance business, then

$$S_t = G_t + u_{t-1,t}\,G_{t-1} + \cdots + u_{1,t}\,G_1 + u_{0,t}\,S_0. \tag{8}$$

If $i_v(s) \equiv i_v$ the last equation boils down to

$$S_t = S_t + (1+i_v)G_{t-1} + \cdots + (1+i_v)^{t-1}G_1 + (1+i_v)^t S_0.$$

It is interesting to relate decomposition (7) to the usual Doob decomposition of $\{S_t; t \geq 0\}$. The latter says that

$$S_t = M_t + A_t,$$

where (assuming S is integrable) M is the martingale

$$M_t = \sum_{k=1}^{t} [S_k - \mathsf{E}(S_k \,|\, \mathcal{F}_{k-1})] = \sum_{k=1}^{t} G_k$$

and A is the predictable process

$$A_t = S_0 + \sum_{k=1}^{t} \mathsf{E}(S_k - S_{k-1} \,|\, \mathcal{F}_{k-1}).$$

Then Theorem 1 says that $G_t = M_t - M_{t-1}$ and

$$A_t = u_{t-1,t}\, ADJ_{t-1} + (u_{t-1,t} - 1)G_{t-1} + \cdots \\ + u_{1,t}\, ADJ_1 + (u_{1,t} - 1)G_1 + u_{0,t}(S_0 + ADJ_0).$$

Remarks 1. Extend the deferred annuity example to subsequent years, and consider the gains G_1, $G_2, \ldots$. It is clear that those variables are not independent in general, simply because they are functions of the single variable K_x representing the number of years lived by the member. Actuarial gains and losses are dependent random variables, unless the insurance portfolio consists entirely of one-year contracts issued at the beginning of each fiscal year. In the case of life insurance and pensions, the stochastic dependence extends for at least as long as any particular individual remains insured with the same company (or plan), whether or not the initial contract is modified along the way. Thus it is a little surprising that, in the case of unbiased assumptions, actuarial gains and losses are *uncorrelated* random variables. The reader will notice the similarity with Hattendorff's Theorem.

2. Actuarial gains may also be defined in contexts where there is no reserve. For example, in the classical risk model

$$U_t = u + ct - \sum_{k=1}^{t} B_k$$

(where the $\{B_k\}$ are independent) the actuarial gain becomes

$$G_t = U_t - \mathsf{E}(U_t \,|\, \mathcal{F}_{t-1}) = \mathsf{E}B_t - B_t.$$

Note that this is different from the financial gain $c - B_t$. □

5. Do gains and losses offset over time? The i.i.d case

Intuitively, one might reason as follows (italics indicate those claims which are not justified mathematically so far, and which may end up being incorrect). If actuarial assumptions are consistently "conservative," then actuarial gains *should* appear, and a surplus *should* arise eventually. The reverse *should* hold for consistently "pessimistic" actuarial assumptions. Going one step further, the intermediate case where actuarial assumptions are "best estimates" (that is to say, unbiased) *should* lead to neither a surplus or a deficiency; some have believed that in that case gains and losses should cancel off, as is explicitly expressed in the quotation from FAS 87 in the Introduction. This section examines the simplest possible case, that of independent and identically independent gains and losses, with a zero interest rate. The focus will be on the behaviour of the cumulative sum of gains and losses.

Consider the game of heads or tails. A coin is tossed, and one of the players wins \$ 1 if the outcome is head, and looses \$ 1 otherwise. If both outcomes have the same probability, then the expected increase in the player's wealth due to a single toss is 0; therefore the "actuarial gain" here is a random variable G taking values ± 1 with equal probabilities. The gains from distinct tosses are independent. Consider W_t the cumulative sum of the first t actuarial gains:

$$W_t \;=\; G_1 + \cdots + G_t, \quad W_0 \;=\; 0.$$

Let us find out whether gains and losses cancel off over time.

W_t is seen to have a symmetrical binomial distribution over the integers $-t, -t+2, \ldots, t-2, t$. Its variance is equal to t. Thus, as t increases the distribution of W_t is more and more spread out around the common mean value 0. This does not appear to agree with gain and loss cancellation over time.

Of course, the Strong Law of Large Numbers says that with probability one the *average* of the first t gains will converge to their expectation as t tends to infinity:

$$\mathsf{P}\left(\lim_{t\to\infty} \frac{W_t}{t} = 0\right) \;=\; 1.$$

But the wealth of the player is not the average but the *sum* of the gains, so the Law of Large Numbers is irrelevant.

Among other facts known concerning the wealth process W, two will be mentioned:

(1) With probability one

$$\limsup_{t\to\infty} W_t \;=\; +\infty\,; \quad \liminf_{t\to\infty} W_t \;=\; -\infty.$$

In other words, for any number $M > 0$, no matter how large, it is certain that at some time t_0 (depending on the particular path) we will observe $W_{t_0} > M$. The same thing holds for a large negative number. (*N.B.* A short, direct proof of this result — not involving martingales — can be found on p. 375 of Loève (1977).)

(2) Suppose that at some time t_0 we know that $W_{t_0} = k$ for some $k \neq 0$. The previous property implies that with certainty there will be some subsequent moment t_1 when $W_{t_1} = 0$. Let T be the first such t_1. Then it is known that $\mathsf{E}T = +\infty$ (this is a consequence of Doob's Optional Stopping Theorem, see Section 10.10 of Williams, 1991). In words, it is certain that W will return to zero, but on average this takes an infinite length of time.

The conclusion is that, at least in the i.i.d. unbiased case, gains and losses do not offset over time. The cumulative sum of the gains ($\{W_t\}$) becomes a *random walk*, with the totally unpredictable behaviour just described.

6. Do gains and losses offset over time? The general unbiased case

In this section we go back to the general definition of actuarial loss (Section 3) and consider the behaviour of their cumulative sum over time, when actuarial assumptions are unbiased ("correct on average"). The arithmetic sum (that is to say, the sum without interest) will first be studied, and then the case where gains and losses are accumulated with interest. An essential assumption is that the insurance business exists indefinitely, so that we can take limits as $t \to \infty$.

Let

$$W_t \;=\; G_1 + \cdots + G_t\,, \quad W_0 \;=\; 0.$$

It is immediate that the process $\{W_t\,, t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\,, t \geq 0\}$; this is because $\mathsf{E}(G_m \,|\, \mathcal{F}_{m-1}) = 0$ for all m implies

$$\mathsf{E}(G_m \,|\, \mathcal{F}_j) \;=\; \mathsf{EE}(G_m \,|\, \mathcal{F}_j \,|\, \mathcal{F}_{m-1}) \;=\; 0$$

for all $j < m$, whence

$$\begin{aligned} \mathsf{E}(W_k \,|\, \mathcal{F}_j) \;&=\; \mathsf{E}(G_1 + \cdots + \; G_j + G_{j+1} + \cdots + G_k \,|\, \mathcal{F}_j) \\ &=\; G_1 + \cdots + G_j \\ &=\; W_j\,. \end{aligned}$$

The variables $\{G_t\,,\,t\geq 1\}$ are then *uncorrelated*:

$$\begin{aligned}\mathsf{E}G_j G_k &= \mathsf{E}\mathsf{E}(G_j\,G_k\,|\,\mathcal{F}_j)\\ &= \mathsf{E}[G_j\,\mathsf{E}(G_k\,|\,\mathcal{F}_j)]\\ &= 0.\end{aligned}$$

Thus

$$\mathsf{Var}W_t \;=\; \mathsf{Var}G_1+\cdots+\mathsf{Var}G_t\,,$$

just as in the i.i.d. case. This is surprising, since the variables $\{G_t\}$ are usually dependent (see Remark 1 at the end of Section 4).

That the variance of W_t grows with t is a sure sign that its distribution is more and more spread out, but does not preclude the possibility that W_t converge with probability one. For instance, define

$$Z_t \;=\; \prod_{k=1}^{t} Y_k, \quad t\geq 1,$$

where the variables $\{Y_t\}$ are i.i.d. and take values -2, 0 and 2 with probabilities 1/4, 1/2 and 1/4, respectively. Then $\mathsf{E}Z_t=0$ for all t and $\mathsf{Var}Z_t$ goes to $+\infty$ as t increases. Nevertheless, for almost all ω there exists $N(\omega)$ such that $Z_t(\omega)=0$ for all $t>N(\omega)$. (Also observe that, if T is the first time Z vanishes, then $\mathsf{E}|Z_{T-1}|$ is infinite.)

It will now be shown that the behaviour of W_t is essentially the same as in the i.i.d. case. That is to say, the arithmetic sum of losses has "unpredictable oscillations" between positive and negative values, and cannot remain very long in any pre-determined bounded region.

Two assumptions will be made:

A1. $\mathsf{E}\sup_t G_t^2<\infty$.

A2. $\displaystyle\sum_{t=1}^{\infty}\mathsf{Var}(G_t\,|\,\mathcal{F}_{t-1}) \;=\; +\infty.$

The first assumption is a technical one, required to apply Theorem 2 below. It is satisfied if there is a constant K such that $|G_t|\leq K$ for all t. The author does not believe that A1 seriously restricts the validity of the model (K may be chosen as large as we want). A2 is roughly equivalent to assuming that the operations of the insurance business will not eventually stop or dwindle to nothing. This is a natural requirement if our goal is to study the qualitative behaviour of W_t over time. It would be violated if we knew that by some time t_1 the company would stop writing new business (or, in the case of pensions, the plan would not accept any new members).

The following theorem relates the convergence of a martingale to its "angle-brackets process." Suppose $\{Y_t\}$ is a martingale; then a process A such that

$$Y^2 = M + A$$

satisfying the conditions (a) M is a martingale, (b) $A_0 = 0$, (c) A is non-decreasing, and (d) A_t is $\mathcal{F}_{t-1}$ measurable for $t \geq 1$ exists and is unique (this results from Doob's Decomposition Theorem); the process A is often denoted $\langle Y \rangle$, and called the *angle-brackets* process associated with Y.

Observe that in general

$$\langle Y \rangle_t - \langle Y \rangle_{t-1} = \mathsf{E}[(Y_t - Y_{t-1})^2 \,|\, \mathcal{F}_{t-1}].$$

Therefore

$$\begin{aligned} \langle W \rangle_t &= \mathsf{E}(G_1^2 \,|\, \mathcal{F}_0) + \cdots + \mathsf{E}(G_t^2 \,|\, \mathcal{F}_{t-1}) \\ &= \mathsf{Var}(G_1 \,|\, \mathcal{F}_0) + \cdots + \mathsf{Var}(G_t \,|\, \mathcal{F}_{t-1}). \end{aligned}$$

Theorem 2. *Let $\{Y_t\}$ be a martingale such that $\mathsf{E} \sup_t (Y_t - Y_{t-1})^2 < \infty$. Then with probability one,*

(i) $\lim_{t\to\infty} Y_t(\omega)$ *exists if, and only if,* $\lim_{t\to\infty} \langle Y \rangle_t(\omega) < \infty$.

(ii) $\limsup_{t\to\infty} Y_t(\omega) = +\infty$ *and* $\liminf_{t\to\infty} Y_t(\omega) = -\infty$ *if, and only if,* $\lim_{t\to\infty} \langle Y \rangle_t(\omega) = \infty$.

(For more details and a proof see Neveu (1975).) This theorem shows immediately that $\mathsf{P}(W_t \text{ converges}) = 0$, that is to say gains and losses cannot offset over time. More generally for any bounded interval $\mathcal{I}$ we have

$$\mathsf{P}(W_t \in \mathcal{I} \quad \forall t \geq t_0) = 0,$$

so even approximate cancelling is ruled out. The parallel with random walks goes further; for instance it can be shown that, under A1 and A2 (see Neveu, 1975, and Williams, 1991):

$$\lim_{t\to\infty} \frac{W_t}{\langle W \rangle_t} = 0.$$

This is a "Strong Law of Large Numbers for martingale differences."

In conclusion the behaviour of the cumulative sum of actuarial gains and losses, when actuarial assumptions are unbiased and there is no interest, is the same as that of a random walk, at least when assumptions A1 and A2 are satisfied.

7. The general unbiased case, with interest

This section takes up the general model of last section, with the difference that gains and losses are now accumulated with interest. This is done in two stages, first constant interest, then random interest.

Let X_t be the value of the gains $G_1, \ldots, G_t$ accumulated with interest:

$$X_t \;=\; G_t + (1+i)G_{t-1} + \cdots + (1+i)^{t-1}G_1, \tag{9}$$

where $i > 0$ is not necessarily equal to the valuation rate of interest.

Remark. If $i = i_v$ is the (constant) valuation rate of interest, and only the pure premium is paid, then it follows from Theorem 1 that $S_t = X_t + (1+i_v)^t S_0$. $\Box$

$\{X_t\}$ is not a martingale. Nevertheless, let $\overline{X}_t$ be the discounted value of X_t at time 0:

$$\begin{aligned}\overline{X}_t \;&=\; (1+i)^{-t}X_t \\ &=\; (1+i)^{-t}G_t + (1+i)^{-t+1}G_{t-1} + \cdots + (1+i)^{-1}G_1.\end{aligned} \tag{10}$$

$\{\overline{X}_t\}$ is a martingale, since

$$\begin{aligned}\mathsf{E}(\overline{X}_k \,|\, \mathcal{F}_j) \;&=\; (1+i)^{-j}G_j + (1+i)^{-j+1}G_{j-1} + \cdots + (1+i)^{-1}G_1 \\ &=\; \overline{X}_j.\end{aligned}$$

Furthermore this martingale has a limit as $t \to \infty$, if $\mathsf{E}|G_t| \le K < \infty$ for all t. Indeed,

$$\begin{aligned}\mathsf{E}|\overline{X}_t| \;&\le\; [1 + (1+i)^{-1} + \cdots + (1+i)^{-t}] \sup_{1 \le k \le t} \mathsf{E}|G_k| \\ &\le\; \frac{1+i}{i} \sup_{k \ge 1} \mathsf{E}|G_k| \;<\; \infty.\end{aligned}$$

$\overline{X}_t$ being bounded in $\mathcal{L}^1$, Doob's Convergence Theorem implies the existence of

$$\lim_{t\to\infty} \overline{X}_t \;=\; \overline{X}_\infty.$$

Consequently,

$$\lim_{t\to\infty} X_t \;=\; \lim_{t\to\infty} (1+i)^t\, \overline{X}_t \;=\; \begin{cases} +\infty & \text{if } \overline{X}_\infty > 0 \\ -\infty & \text{if } \overline{X}_\infty < 0. \end{cases}$$

There remains the possibility that $\overline{X}_\infty = 0$. The next theorem shows that even when this happens X_t has to tend (a.s.) to plus or minus infinity.

Theorem 3. *Under assumptions A1 and A2,*

$$\mathsf{P}(\lim_{t\to\infty} X_t = +\infty) + \mathsf{P}(\lim_{t\to\infty} X_t = -\infty) \;=\; 1. \tag{11}$$

Proof. First it will be shown (by contradiction) that $\mathsf{P}(\sup_t |X_t| = \infty) = 1$. For any $c > 0$ define $N_c = \{t \;:\; |X_t| > c\}$ and

$$T_c \;=\; \begin{cases} \inf N_c & \text{if } N_c \neq \emptyset \\ +\infty & \text{if } N_c = \emptyset. \end{cases}$$

If $\mathsf{P}(\sup_t |X_t| = \infty) < 1$, then $\mathsf{P}(\langle W\rangle_\infty = \infty,\ \sup_t |X_t| < \infty) > 0$ and there exists $c > 0$ such that

$$\mathsf{P}(\langle W\rangle_\infty = \infty,\ T_c = \infty) \;>\; 0. \tag{12}$$

Consider Doob's decomposition of the process X^2 :

$$X_t^2 \;=\; A_t + M_t \tag{13}$$

where

$$A_t \;=\; \sum_{k=1}^{t} \mathsf{E}(X_k^2 - X_{k-1}^2 \,|\, \mathcal{F}_{k-1}) \tag{14}$$

and M is a martingale. From Eq. (9)

$$X_t \;=\; (1+i)X_{t-1} + G_t. \tag{15}$$

We get

$$\begin{aligned} \mathsf{E}(X_k^2 - X_{k-1}^2 \,|\, \mathcal{F}_{k-1}) \;&=\; \mathsf{E}\Big\{[(1+i)X_{k-1} + G_k]^2 - X_{k-1}^2 \,|\, \mathcal{F}_{k-1}\Big\} \\ &=\; (2i + i^2)X_{k-1}^2 + \mathsf{E}(G_k^2 \,|\, \mathcal{F}_{k-1}) \\ &\geq\; \mathsf{E}(G_k^2 \,|\, \mathcal{F}_{k-1}) \;=\; \langle W\rangle_k - \langle W\rangle_{k-1}. \end{aligned} \tag{16}$$

Hence $\langle W\rangle_t \leq A_t$ (almost surely) for all t, which implies

$$\mathsf{E}\langle W\rangle_{T_c\wedge t} \;\leq\; \mathsf{E}A_{T_c\wedge t} \;=\; \mathsf{E}X^2_{T_c\wedge t} \;\leq\; \mathsf{E}[(1+i)c + \sup_t |G_t|]^2. \tag{17}$$

By monotone convergence, we infer $\mathsf{E}\langle W\rangle_{T_c} \leq K_1 < \infty$, which contradicts (12). Therefore it is not possible to have $\mathsf{P}(\sup_t |X_t| = \infty) < 1$.

By A1, with probability one there exists $K_2 < \infty$ (possibly depending on ω) such that $|\sup_t G_t| \leq K_2$. By the first part of the proof there also exists t_0 (depending on ω) such that $|X_{t_0}| > K_2/i$. Suppose $X_{t_0} > K_2/i$. Then for $t > t_0$

$$\begin{aligned}
X_t &= (1+i)^{t-t_0} X_{t_0} + G_t + (1+i)G_{t-1} + \cdots + (1+i)^{t-t_0-1} G_{t_0+1} \\
&\geq (1+i)^{t-t_0} X_{t_0} - K_2[1 + (1+i) + \cdots + (1+i)^{t-t_0-1}] \\
&\geq (1+i)^{t-t_0} \left(X_{t_0} - \frac{K_2}{i} \right) \\
&\rightarrow +\infty \quad (\text{as } t \rightarrow \infty).
\end{aligned} \tag{18}$$

In the same way, if there is t_1 such that $X_{t_1} < -K_2/i$, then X_t tends to $-\infty$. □

It is straightforward to extend this result to random rates of interest. Let $\{J_t\}$ be a process adapted to $\{\mathcal{F}_t\}$, representing the rates at which the gains are accumulated. This process may or may not be related to the rates of return on the assets, or to the valuation rates of interest. Define

$$\begin{aligned}
U_{s,t} &= (1 + J_{s+1}) \cdots (1 + J_t) \\
X_t &= G_t + U_{t-1,t}\, G_{t-1} + \cdots + U_{1,t}\, G_1.
\end{aligned} \tag{19}$$

Remark. If X is interpreted as wealth, this setting allows J_t to depend on X_{t-1}, for instance the borrowing rate (when X is negative) may be different from the rate of return (when X is positive). □

Theorem 4. *Define X as in (19), keep assumptions A1 and A2 unchanged, and suppose furthermore that*

A3. $\mathsf{E}(J_t \,|\, \mathcal{F}_{t-1}) \geq 0$ *almost surely for all t.*

A4. $\sup_t J_t < K_3$ *(K_3 a constant).*

A5. There is Q, possibly depending on ω, such that $\sum_{t>s} U_{s,t}^{-1} < Q$ for all s.

Then

$$\mathsf{P}(\lim_{t\to\infty} X_t = +\infty) + \mathsf{P}(\lim_{t\to\infty} X_t = -\infty) \;=\; 1.$$

Proof. The process $\{J_t\}$ being adapted implies the same for $\{X_t\}$, and X is still square-integrable, so decomposition (13) is possible. Eq. (14) is unchanged, while (15) becomes

$$X_t \;=\; (1 + J_t)X_{t-1} + G_t.$$

Inequality (16) follows from A3, and (17) becomes

$$\mathsf{E}\langle W\rangle_{T_c\wedge t} \;\le\; \mathsf{E}[c\sup_t(1+J_t)+\sup_t|G_t|]^2.$$

The steps leading to (18) are now as follows. From A1 there is K_2, depending on ω, such that $|\sup_t G_t| \le K_2$. If there is a t_0 such that $X_{t_0} > K_2 Q$, then for $t > t_0$

$$\begin{aligned} X_t &= U_{t_0,t}\,X_{t_0} + G_t + U_{t-1,t}\,G_{t-1} + \cdots + U_{t_0+1,t}\,G_{t_0+1} \\ &\ge U_{t_0,t}\,X_{t_0} - K_2[1 + U_{t-1,t} + \cdots + U_{t_0+1,t}] \\ &\ge U_{t_0,t}\,(X_{t_0} - K_2 Q) \\ &\to +\infty \quad (\text{as } t\to\infty). \end{aligned}$$

□

Remarks. 1. Assumptions A3 to A5 are somewhat restrictive. For instance, they are not satisfied if the geometric rates $\{\log(1+J_t)\}$ form an ARMA process.They are satisfied, for example, if there are constants $0 < j_1 < j_2 < \infty$ such that $j_1 \le J_t \le j_2$ with probability one for all t.

2. To see that A1 (or something approaching it) cannot be avoided, consider the following example. Suppose $\{e_t\}$ is a zero-mean i.i.d. sequence, and let $G_t = (1+i)^t e_t$ for all t. Then

$$X_t \;=\; (1+i)^t(e_1+\cdots+e_t).$$

It is clear that the conclusions of Theorem 3 and 4 do not apply to $\{X_t\}$. Those conclusions may still hold, however, if A2 is not satisfied. This is so, for example, if $G_t = 0$ for all $t > T$, and if $\mathsf{P}(X_T = 0) = 0$. □

Theorems 3 and 4 show that the behaviour of accumulated gains and losses is different when there is interest: the process is not "recurrent," in that given any bounded interval $\mathcal{I}$, there will be a time after which the process will never come back to $\mathcal{I}$. The process W of Section 6 is recurrent.

8. Biased actuarial assumptions

In this section it is not assumed that actuarial assumptions are "correct on average." The definition of actuarial gain becomes

$$G_t^A \;=\; S_t - \mathsf{E}^A(S_t\,|\,\mathcal{F}_{t-1}),$$

where the superscript A has the meaning "according to actuarial assumptions."

It is natural to say that actuarial assumptions are "optimistic" during the period $(t-1,t)$ if $\mathsf{E}(G_t^A \,|\, \mathcal{F}_{t-1}) < 0$. If the inequality is reversed, we say that assumptions are "pessimistic."

As an example, consider a one-year term insurance contract, with death benefit b (paid at the end of the year), issued to (x) at time t. Let the premium paid be P, and the pure premium be π_x. The latter is based on the true probability of death and rate of return distribution, that is to say

$$\mathsf{E}(1+R_{t+1})\pi_x \;=\; b\,\mathsf{E}(I) \;=\; b\,q_x,$$

where I is the indicator variable for the event "death of insured before age $x+1$." Let F_t be the assets at time t. Since there is no reserve we have

$$\begin{aligned} G_{t+1}^A &= F_{t+1} - \mathsf{E}^A(F_{t+1} \,|\, \mathcal{F}_t) \\ &= P(1+R_{t+1}) - bI - [P(1+i_t^A) - b\,q_x^A] \\ &= b(q_x^A - I) - P(i_t^A - R_{t+1}). \end{aligned}$$

This implies

$$\mathsf{E}(G_{t+1}^A \,|\, \mathcal{F}_t) \;=\; b(q_x^A - q_x) - P[i_t^A - \mathsf{E}(R_{t+1} \,|\, \mathcal{F}_t)]. \tag{20}$$

This may be positive or negative; if actuarial assumptions are unbiased the expression is identically zero. The sign of the average actuarial gain may depend on the past of the processes involved (it may be *path-dependent*). To take a specific example, suppose earned rates of return follow an autoregressive process:

$$R_{t+1} \;=\; r + a(R_t - r) + \varepsilon_{t+1}$$

where r and a are constants, and $\{\varepsilon_s\}$ is a zero-mean i.i.d. sequence such that $R_s > -1$ almost surely for all s. Then the second term on the r.h.s. of (20) becomes $-P[i_t^A - r - a(R_t - r)]$. For some values of the previous year's rate of return R_t the conditional expectation of the gain will be positive, and for some others it will be negative.

The optimism or pessimism of actuarial assumptions is in general path dependent, but it is conceivable that in some cases they may be *uniformly* one or the other, say that

$$\mathsf{E}(G_t^A \,|\, \mathcal{F}_{t-1}) \;>\; 0, \quad \text{almost surely, for all } t. \tag{21}$$

(In other words, whatever the history $\mathcal{F}_{t-1}$, we know that actuarial assumptions during the next period are pessimistic.) Indeed traditional actuarial practice calls for "conservativeness" in most assumptions, which precisely leads to (21).

Consider the behaviour of the sum of actuarial gains, when actuarial assumptions are systematically pessimistic (as in (21)):

$$W_t \;=\; G_1^A + \cdots + G_t^A.$$

This process can be decomposed as $W = M + A$, where

$$M_t \;=\; \sum_{k=1}^{t} [G_k^A - \mathsf{E}(G_k^A \,|\, \mathcal{F}_{k-1})]$$

is a martingale, and

$$A_t \;=\; \sum_{k=1}^{t} \mathsf{E}(G_k^A \,|\, \mathcal{F}_{k-1})$$

is a predictable process. The latter acts as a positive drift, while the behaviour of the martingale M may be studied as in Section 6.

As a trite example, suppose the $\{G_s^A\}$ are i.i.d. with mean $\mu > 0$ and variance $\sigma^2 > 0$. Then the Strong Law of Large Numbers says that W_t/t converges to μ, which implies $W_t \to +\infty$. In this case the positive drift $A_t = \mu t$ is stronger than the erratic behaviour of M. Other examples may easily be constructed where the opposite occurs (that is to say, $\limsup W_t = -\liminf W_t = -\infty$, as in Section 6).

Now introduce interest. Let X_t be as in Eq. (9), with G replaced by G^A and $i > 0$. Under assumptions A1 and A2 of Section 6, can we conclude that $X_t \to +\infty$? The answer is negative. Go back to the i.i.d. example of the previous paragraph, with the restriction $\mathsf{P}(G_s^A < 0) > 0$. (All this is possible, suppose for instance that G_s^A has a uniform distribution over the interval $[-1, 2]$.) Then $\mathsf{P}(\overline{X}_\infty < 0) > 0$ (see Eq. (10)), which implies $\mathsf{P}(X_t \to -\infty) > 0$. In other words, *gains may be experienced on average in every period, the rate of return may be positive, but the business still goes broke with positive probability!*

9. Connections with Risk Theory

Consider the classical discrete-time no-interest risk model (Remark 2 of Section 4). It is assumed that the $\{B_k\}$ are i.i.d.. From Section 5 it is clear that if $c > \mathsf{E}B_k$, then $U_t \to +\infty$, and if $c < \mathsf{E}B_k$, then $U_t \to -\infty$. The results of Section 7 and the comments of Section 8 can be applied to the same model with interest:

$$U_{t+1} \;=\; (1 + R_t)(U_t + c) - B_{t+1}.$$

In particular, $\mathsf{P}(\lim_t U_t = -\infty)$ may be positive, even in cases where $c > \mathsf{E}W_k$.

Remark. It is sometimes suggested that this model (or its continuous counterpart, see below) is appropriate for long-term life insurance and annuities. The author admits being unable to understand that claim. The model is correctly applied if contracts both begin and end within the same period (e.g. one-year term insurance, or temporary annuities with a total duration of at most one year). Any contract on a particular life which may exist for more than one period automatically makes the cash flows $\{B_t\}$ dependent. By considering actuarial gains or losses, instead of cash-flows (as in Section 4), one may remove correlation, but not dependence. □

The classical continuous-time risk model is

$$U_t \;=\; u + ct - S_t,$$

where $\{S_t\}$ is a compound Poisson process. Harrison (1977) studied a generalization of this model, where (i) the process U is cadlag, square-integrable and has stationary and independent increments, and (ii) there is a constant, positive instantaneous interest rate β. The accumulated assets at time t have value

$$X_t \;=\; e^{\beta t}u + e^{\beta t}\int_0^t e^{-\beta s}\,dU_s. \tag{22}$$

Among other things, Harrison proved that X satisfies (11). It would therefore be interesting to see if Theorems 3 and 4 extend to continuous-time processes. The case where U is Brownian Motion with drift has been considered by Harrison in the same paper. Keeping the same assumption concerning U, we may also consider stochastic interest; replace $e^{\beta t}$ in (22) by the exponential of a another Brownian motion Z_t, independent of U. If the drift of Z is strictly positive, then

$$\overline{X}_\infty \;=\; \int_0^\infty e^{-Z_s}\,dU_s$$

exists almost surely. (If the drift of U is zero, then it can be shown that $\overline{X}_\infty$ has a t distribution.) Since $\mathsf{P}(\overline{X}_\infty = 0) = 0$, it is clear that (11) holds.

More generally, it is not clear under what conditions Theorems 3 and 4 hold for continuous-time martingales U. In many cases it is obvious that X_t tends to plus or minus infinity as soon as $\overline{X}_\infty$ is not equal to 0. But what about the cases where $\overline{X}_\infty = 0$? Assumption A1 (Section 6) has no obvious equivalent here. Two other papers of related interest are Gerber (1971), and Whitt (1972).

10. Conclusions

Leaving technical conditions aside, three general conclusions can be drawn from the previous analysis:

1. Gains and losses do not cancel out over time, whether we sum them with or without interest.

2. When we sum them without interest, and each gain has mean value zero, the process obtained is recurrent (which means that it will eventually come back to any preset value), although the time required for this to happen may have infinite expectation.

3. When we sum them with interest, the process obtained diverges either to plus or minus infinity (it is not recurrent).

From a practical point of view, these conclusions mean that we cannot expect that past losses will, by some kind of magic, be offset by future gains. This is the case even when actuarial assumptions are unbiased. The parallel with gambling (where successive gains are independent) is clear.

The third conclusion expresses a fundamental difference between the zero-interest and the positive-interest cases, and is perhaps less obvious intuitively. One way to interpret it is that the larger gains (or losses) become impossible to counterbalance, when their value grows with interest. (This comment is especially significant with regard to Risk Theory, highlighting the lack or realism or the classical no-interest model.)

As to pension accounting, this paper is in complete agreement with the simulation results in Dufresne (1993). The rationale for the "corridor" approach to gains and losses amortization, expressed in paragraph 184 of FAS 87, is incorrect. (In a nutshell, that approach consists in not requiring any amortization or actuarial gains or losses as long as the absolute value of their cumulative sum does not exceed a threshold amount equal to 10% of the greater of the fund value and the mathematical reserve.) Nevertheless, the practical consequences are mitigated by the requirement that some amortization take place as soon as accumulated AGLs become larger than some specified amount. As this paper shows, for an ongoing pension plan this is certain to occur at some future time.

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Daniel Dufresne
Center for Actuarial Studies
Department of Economics
University of Melbourne
Parkville, Vicoria
Australia 3052
Tel.: (03) 9344-5611, FAX:(03) 9344-6899
dufresne@clyde.tis.unimelb.edu.au