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Interest Model**

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Ruin Probabilities with a Markov Chain Interest Model

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Abstract

Ruin probabilities in two generalized discrete time risk processes with a Markov chain interest model are studied. Recursive and integral equations for the ruin probabilities are given. Generalized Lundberg inequalities for the ruin probabilities are derived both by inductive and martingale approaches. The relationships between these inequalities are discussed. A numerical example is given to illustrate these results.

Keywords: Risk process; Markov chain; ruin probability; rate of interest; Lundberg's inequality; Jensen's inequality; NWUC; supermartingale; optional stopping theorem.

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1 Introduction

Let $\{X_n, n = 1, 2, \dots\}$ and $\{Y_n, n = 1, 2, \dots\}$ be two sequences of independent and identically distributed (i.i.d.) non-negative random variables. They have common distribution functions $H(x) = \Pr\{X_1 \leq x\}$ and $F(y) = \Pr\{Y_1 \leq y\}$, respectively, with $F(0) = 0$. Define

$$\psi(u) = \Pr \left\{ \bigcup_{k=1}^{\infty} (U_k < 0) \right\},$$

where

$$U_k = u + \sum_{t=1}^k (X_t - Y_t), \quad k = 1, 2, \dots,$$

or, equivalently, the stochastic process $\{U_k, k = 1, 2, \dots\}$ satisfies

$$U_k = U_{k-1} + X_k - Y_k, \quad k = 1, 2, \dots, \quad (1.1)$$

with $U_0 = u \geq 0$. Then $\psi(u)$ is the ultimate ruin probability in the classical risk model (1.1).

Let $\{I_n, n = 0, 1, 2, \dots\}$ be another sequence of non-negative random variables. Cai (2002a, 2002b) has considered two generalized discrete time risk processes by defining

$$U_k = (U_{k-1} + X_k)(1 + I_k) - Y_k, \quad k = 1, 2, \dots, \quad (1.2)$$

and

$$U_k = U_{k-1}(1 + I_k) + X_k - Y_k, \quad k = 1, 2, \dots, \quad (1.3)$$

respectively, with initial surplus $U_0 = u$.

Mathematically, (1.2) and (1.3) are generalizations of the classical risk model (1.1). Practically, models (1.2) and (1.3) can be used to include the effect of timing of payments and interest on a surplus process as explained in Cai (2002a, 2002b). We study these effects by considering ruin probabilities.

It has been shown in Cai (2002a) that (1.2) implies

$$U_k = u \prod_{j=1}^k (1 + I_j) + \sum_{j=1}^k \left((X_j(1 + I_j) - Y_j) \prod_{t=j+1}^k (1 + I_t) \right), \quad k = 1, 2, \dots, \quad (1.4)$$

while (1.3) is equivalent to

$$U_k = u \prod_{j=1}^k (1 + I_j) + \sum_{j=1}^k \left((X_j - Y_j) \prod_{t=j+1}^k (1 + I_t) \right), \quad k = 1, 2, \dots, \quad (1.5)$$

where throughout this paper we denote $\prod_{t=a}^b x_t = 1$ and $\sum_{t=a}^b x_t = 0$ if $a > b$.

The effect of interest on ruin probabilities has been discussed by various authors. Sundt and Teugels (1995, 1997) have studied the effect of a constant rate of interest on the ruin probability in the compound Poisson risk model. Yang (1998) has considered a special case of (1.5) when $\{I_n, n = 0, 1, 2, \dots\}$ are identical constants. Cai (2002a) has discussed i.i.d. rates of interest. However, the assumption of constant or i.i.d. rates is not particularly realistic since rates of interest are usually statistically dependent over time. Cai (2002b) has considered a dependent model for rates of interest, in which the rates are assumed to have an AR(1) structure.

In this paper we consider another dependent model for $\{I_n, n = 0, 1, 2, \dots\}$, in which $\{I_n, n = 0, 1, 2, \dots\}$ are assumed to follow a Markov chain. We assume that for all $n = 0, 1, 2, \dots$, I_n takes a finite or countable number of possible values. This set of possible values is denoted by $\{i_0, i_1, i_2, \dots\}$. Suppose that for all $n = 0, 1, 2, \dots$, and all states $i_s, i_t, i_{t_0}, i_{t_1}, \dots, i_{t_{n-1}}$,

$$\begin{aligned} & \Pr\{I_{n+1} = i_t \mid I_n = i_s, I_{n-1} = i_{t_{n-1}}, \dots, I_1 = i_{t_1}, I_0 = i_{t_0}\} \\ &= \Pr\{I_{n+1} = i_t \mid I_n = i_s\} = p_{st} \geq 0, \quad s, t = 0, 1, 2, \dots, \end{aligned} \quad (1.6)$$

where $\sum_{t=0}^{\infty} p_{st} = 1$, for $s = 0, 1, 2, \dots$.

Equation (1.6) means that $\{I_n, n = 0, 1, 2, \dots\}$ constitute a Markov chain and satisfy the Markov property, i.e. the conditional distribution of any future state I_{n+1} given the past states $I_0, I_1, I_2, \dots, I_{n-1}$ and the present state I_n , is independent of the past states and depends only on the present state.

This Markov chain interest model was introduced by Norberg (1997). For a continuous-time Markov chain interest model, see Norberg (1995). Further, $\{Y_n, n = 1, 2, \dots\}$, $\{X_n, n = 1, 2, \dots\}$, and $\{I_n, n = 0, 1, 2, \dots\}$ are assumed to be independent.

We define the finite time and ultimate ruin probabilities in model (1.4) with the interest model (1.6), initial surplus u , and a given $I_0 = i_s$, respectively, by

$$\phi_n(u, i_s) = \Pr \left\{ \bigcup_{k=1}^n (U_k < 0) \mid I_0 = i_s \right\}$$

$$= \Pr \left\{ \bigcup_{k=1}^n \left(u \prod_{j=1}^k (1 + I_j) + \sum_{j=1}^k (X_j(1 + I_j) - Y_j) \prod_{t=j+1}^k (1 + I_t) < 0 \right) \mid I_0 = i_s \right\}$$

and

$$\begin{aligned} \phi(u, i_s) &= \Pr \left\{ \bigcup_{k=1}^{\infty} (U_k < 0) \mid I_0 = i_s \right\} \\ &= \Pr \left\{ \bigcup_{k=1}^{\infty} \left(u \prod_{j=1}^k (1 + I_j) + \sum_{j=1}^k (X_j(1 + I_j) - Y_j) \prod_{t=j+1}^k (1 + I_t) < 0 \right) \mid I_0 = i_s \right\}. \end{aligned}$$

Similarly, we define the finite time and ultimate ruin probabilities in model (1.5) with the interest model (1.6), initial surplus u , and a given $I_0 = i_s$, respectively, by

$$\begin{aligned} \varphi_n(u, i_s) &= \Pr \left\{ \bigcup_{k=1}^n (U_k < 0) \mid I_0 = i_s \right\} \\ &= \Pr \left\{ \bigcup_{k=1}^n \left(u \prod_{j=1}^k (1 + I_j) + \sum_{j=1}^k (X_j - Y_j) \prod_{t=j+1}^k (1 + I_t) < 0 \right) \mid I_0 = i_s \right\}, \end{aligned}$$

and

$$\begin{aligned} \varphi(u, i_s) &= \Pr \left\{ \bigcup_{k=1}^{\infty} (U_k < 0) \mid I_0 = i_s \right\} \\ &= \Pr \left\{ \bigcup_{k=1}^{\infty} \left(u \prod_{j=1}^k (1 + I_j) + \sum_{j=1}^k (X_j - Y_j) \prod_{t=j+1}^k (1 + I_t) < 0 \right) \mid I_0 = i_s \right\}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \phi_n(u, i_s) = \phi(u, i_s) \quad \text{and} \quad \lim_{n \rightarrow \infty} \varphi_n(u, i_s) = \varphi(u, i_s). \quad (1.7)$$

Like in the cases of constant, i.i.d., and AR(1) interest rates, it is intuitive (and straightforward to prove) that

$$\phi(u, i_s) \leq \varphi(u, i_s) \leq \psi(u), \quad u \geq 0, \quad (1.8)$$

which states that the ruin probability $\psi(u)$ in the classical risk model is reduced by adding interest income to the surplus. Also, (1.8) shows the effects of the timing of payments on the ruin probabilities $\phi(u, i_s)$ and $\varphi(u, i_s)$.

It is very difficult to obtain exact expressions for $\phi(u, i_s)$ and $\varphi(u, i_s)$. An analytic analysis commonly used in ruin theory is to derive inequalities for ruin probabilities.

For the ruin probability in the classical risk model, we have the well-known Lundberg inequality which states that if $E(X_1) > E(Y_1)$ and there is a unique constant $R > 0$ satisfying

$$E \left[e^{-R(X_1 - Y_1)} \right] = 1, \quad (1.9)$$

then

$$\psi(u) \leq e^{-Ru}, \quad u \geq 0.$$

Thus, any useful upper bounds for $\phi(u, i_s)$ and $\varphi(u, i_s)$, say

$$\phi(u, i_s) \leq \Delta(u, i_s) \quad \text{and} \quad \varphi(u, i_s) \leq \Lambda(u, i_s), \quad u \geq 0,$$

should satisfy

$$\Delta(u, i_s) \leq \Lambda(u, i_s) \leq e^{-Ru}, \quad u \geq 0. \quad (1.10)$$

In this paper, we derive probability inequalities for $\phi(u, i_s)$ and $\varphi(u, i_s)$, which both generalize Lundberg's upper bound and satisfy (1.10). In Section 2, we first give recursive equations for $\phi_n(u, i_s)$ and $\varphi_n(u, i_s)$ and integral equations for $\phi(u, i_s)$ and $\varphi(u, i_s)$. We then derive probability inequalities for $\phi(u, i_s)$ and $\varphi(u, i_s)$ in Section 3 by an inductive approach. In Section 4, we obtain different probability inequalities for $\phi(u, i_s)$ and $\varphi(u, i_s)$ by a martingale approach. The relationships between Lundberg's inequality and the inequalities derived in this paper are also discussed. Finally, a numerical example is given to illustrate these results in Section 5.

2 Recursive and integral equations for ruin probabilities

Throughout this paper, we denote the tail of a distribution function B by $\bar{B}(x) = 1 - B(x)$. We first give a recursive equation for $\phi_n(u, i_s)$ and an integral equation for $\phi(u, i_s)$.

Lemma 2.1 For $n = 1, 2, \dots$,

$$\begin{aligned} \phi_{n+1}(u, i_s) &= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \bar{F}((u+x)(1+i_t)) dH(x) \\ &+ \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{(u+x)(1+i_t)} \phi_n((u+x)(1+i_t) - y, i_t) dF(y) dH(x) \end{aligned}$$

and

$$\begin{aligned}\phi(u, i_s) &= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \bar{F}((u+x)(1+i_t)) dH(x) \\ &\quad + \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{(u+x)(1+i_t)} \phi((u+x)(1+i_t) - y, i_t) dF(y) dH(x).\end{aligned}$$

Proof. Given $Y_1 = y$, $X_1 = x$, and $I_1 = i_t$, from (1.4), we have

$$U_1 = (u + X_1)(1 + I_1) - Y_1 = (u + x)(1 + i_t) - y = h - y,$$

where $h = (u + x)(1 + i_t)$. Thus, if $y > h$, then

$$\Pr \{U_1 < 0 \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s\} = 1,$$

which implies that for $y > h$,

$$\Pr \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s \right\} = 1,$$

while if $0 \leq y \leq h$, then

$$\Pr \{U_1 < 0 \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s\} = 0. \quad (2.1)$$

Let $\{\tilde{Y}_n, n = 1, 2, \dots\}$, $\{\tilde{X}_n, n = 1, 2, \dots\}$, and $\{\tilde{I}_n, n = 0, 1, 2, \dots\}$ be independent copies of $\{Y_n, n = 1, 2, \dots\}$, $\{X_n, n = 1, 2, \dots\}$, and $\{I_n, n = 0, 1, 2, \dots\}$, respectively.

Thus, (2.1) and (1.4) imply that for $0 \leq y \leq h$,

$$\begin{aligned}& \Pr \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s \right\} \\ &= \Pr \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s \right\} \\ &= \Pr \left\{ \bigcup_{k=2}^{n+1} \left((h - y) \prod_{j=2}^k (1 + I_j) + \sum_{j=2}^k (X_j(1 + I_j) - Y_j) \prod_{t=j+1}^k (1 + I_t) < 0 \right) \mid I_1 = i_t \right\} \\ &= \Pr \left\{ \bigcup_{k=1}^n \left((h - y) \prod_{j=1}^k (1 + \tilde{I}_j) + \sum_{j=1}^k (\tilde{X}_j(1 + \tilde{I}_j) - \tilde{Y}_j) \prod_{t=j+1}^k (1 + \tilde{I}_t) < 0 \mid \tilde{I}_0 = i_t \right) \right\} \\ &= \phi_n(h - y, i_t) = \phi_n((u + x)(1 + i_t) - y, i_t),\end{aligned}$$

where the second equality follows from the Markov property of $\{I_n, n = 0, 1, 2, \dots\}$ and the independence of $\{X_n, n = 1, 2, \dots\}$, $\{Y_n, n = 1, 2, \dots\}$, and $\{I_n, n = 0, 1, 2, \dots\}$.

Therefore, by conditioning on Y_1, X_1 , and I_1 , we get

$$\begin{aligned}
\phi_{n+1}(u, i_s) &= \Pr \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid I_0 = i_s \right\} \\
&= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{\infty} \Pr \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s \right\} dF(y) dH(x) \\
&= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_{(u+x)(1+i_t)}^{\infty} dF(y) dH(x) \\
&+ \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{(u+x)(1+i_t)} \phi_n((u+x)(1+i_t) - y, i_t) dF(y) dH(x) \\
&= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \bar{F}((u+x)(1+i_t)) dH(x) \\
&+ \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{(u+x)(1+i_t)} \phi_n((u+x)(1+i_t) - y, i_t) dF(y) dH(x), \tag{2.2}
\end{aligned}$$

which also yields the integral equation for $\phi(u, i_s)$ in Lemma 2.1 by letting $n \rightarrow \infty$ in (2.2) using the Lebesgue dominated convergence theorem and (1.7). \square

We now give a recursive equation for $\varphi_n(u, i_s)$ and an integral equation for $\varphi(u, i_s)$.

Lemma 2.2 For $n = 1, 2, \dots$,

$$\begin{aligned}
\varphi_{n+1}(u, i_s) &= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \bar{F}(u(1+i_t) + x) dH(x) \\
&+ \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{u(1+i_t)+x} \varphi_n(u(1+i_t) + x - y, i_t) dF(y) dH(x)
\end{aligned}$$

and

$$\begin{aligned}
\varphi(u, i_s) &= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \bar{F}(u(1+i_t) + x) dH(x) \\
&+ \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{u(1+i_t)+x} \varphi(u(1+i_t) + x - y, i_t) dF(y) dH(x).
\end{aligned}$$

Proof. In this case,

$$U_k = u \prod_{t=1}^k (1 + I_t) + \sum_{j=1}^k (X_j - Y_j) \left(\prod_{t=j+1}^k (1 + I_t) \right).$$

By similar arguments to those in the proof of Lemma 2.1 and by conditioning on Y_1, X_1 , and I_1 , we get

$$\begin{aligned}
\varphi_{n+1}(u, i_s) &= \Pr \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid I_0 = i_s \right\} \\
&= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \int_0^{\infty} \Pr \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid Y_1 = y, X_1 = x, I_1 = i_t, I_0 = i_s \right\} dF(y) dH(x) \\
&= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \left(\int_{u(1+i_t)+x}^{\infty} dF(y) + \int_0^{u(1+i_t)+x} \varphi_n(u(1+i_t) + x - y, i_t) dF(y) \right) dH(x) \\
&= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \left(\bar{F}(u(1+i_t) + x) + \int_0^{u(1+i_t)+x} \varphi_n(u(1+i_t) + x - y, i_t) dF(y) \right) dH(x),
\end{aligned}$$

which also gives the integral equation for $\varphi(u, i_s)$ in Lemma 2.2 by letting $n \rightarrow \infty$. \square

3 Probability inequalities by an inductive approach

Using the recursive equations for $\phi_n(u, i_s)$ and $\varphi_n(u, i_s)$, we can derive probability inequalities for $\phi(u, i_s)$ and $\varphi(u, i_s)$ by an inductive approach. We first define a generalized adjustment coefficient with the interest model (1.6) and discuss its relationship with the classical adjustment coefficient R given by (1.9).

Proposition 3.1 Let $E(X_1) > E(Y_1)$. Suppose that there exists $R > 0$ in (1.9) and $\tau_s > 0$ satisfying

$$E(e^{\tau_s Y_1}) E(e^{-\tau_s X_1(1+I_1)} \mid I_0 = i_s) = 1, \quad s = 0, 1, 2, \dots \quad (3.1)$$

Then,

$$R_1 = \min_{0 \leq s < \infty} \{\tau_s\} \geq R \quad (3.2)$$

and for all $s = 0, 1, 2, \dots$,

$$E(e^{R_1 Y_1}) E(e^{-R_1 X_1(1+I_1)} \mid I_0 = i_s) \leq 1. \quad (3.3)$$

Proof. For any $s = 0, 1, 2, \dots$, by considering the following functions

$$f_s(r) = E(e^{r Y_1}) E(e^{-r X_1(1+I_1)} \mid I_0 = i_s) - 1 = E(e^{r(Y_1 - X_1(1+I_1))} \mid I_0 = i_s) - 1$$

and

$$g(r) = E(e^{r(Y_1 - X_1)}) - 1, \quad (3.4)$$

we have,

$$f_s''(r) = E\left(\left(Y_1 - X_1(1 + I_1)\right)^2 e^{r(Y_1 - X_1(1 + I_1))} \mid I_0 = i_s\right) \geq 0,$$

which implies that $f_s(r)$ is a convex function with $f_s(0) = 0$ and

$$f_s'(0) = E(Y_1 - X_1(1 + I_1) \mid I_0 = i_s) \leq E(Y_1 - X_1 \mid I_0 = i_s) = E(Y_1 - X_1) < 0.$$

Similarly, $g(r)$ is a convex function with $g(0) = 0$ and $g'(0) = E(Y_1 - X_1) < 0$. Therefore, τ_s and R are the unique positive roots of the equations $f_s(r) = 0$ and $g(r) = 0$ respectively on $(0, \infty)$. Furthermore, if $r > 0$ and $g(r) \geq 0$, then $r \geq R$; if $0 < \tau \leq \tau_s$, then $f_s(\tau) \leq 0$. However,

$$1 = E(e^{\tau_s(Y_1 - X_1(1 + I_1))} \mid I_0 = i_s) \leq E(e^{\tau_s(Y_1 - X_1)} \mid I_0 = i_s) = E(e^{\tau_s(Y_1 - X_1)}),$$

i.e. $g(\tau_s) \geq 0$, which implies that $\tau_s \geq R$ and $R_1 = \min_{0 \leq s < \infty} \tau_s \geq R$, i.e. (3.2) holds. In addition, for all $s = 0, 1, 2, \dots$, $R_1 = \min_{0 \leq t < \infty} \tau_t \leq \tau_s$, which implies that $f_s(R_1) \leq 0$, i.e. (3.3) holds. \square

We now obtain a probability inequality for $\phi(u, i_s)$ by an inductive approach.

Theorem 3.1 Under the conditions of Proposition 3.1, for all $s = 0, 1, 2, \dots$,

$$\phi(u, i_s) \leq \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1(u + X_1)(1 + I_1)} \mid I_0 = i_s), \quad u \geq 0 \quad (3.5)$$

where

$$\beta_1^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{R_1 y} dF(y)}{e^{R_1 t} \bar{F}(t)}. \quad (3.6)$$

Proof. For any $x \geq 0$, we have

$$\begin{aligned} \bar{F}(x) &= \left(\frac{\int_x^\infty e^{R_1 y} dF(y)}{e^{R_1 x} \bar{F}(x)} \right)^{-1} e^{-R_1 x} \int_x^\infty e^{R_1 y} dF(y) \\ &\leq \beta_1 e^{-R_1 x} \int_x^\infty e^{R_1 y} dF(y) \end{aligned} \quad (3.7)$$

$$\leq \beta_1 e^{-R_1 x} E(e^{R_1 Y_1}). \quad (3.8)$$

Then, for any $u \geq 0$ and any $i_s \geq 0$,

$$\begin{aligned}\phi_1(u, i_s) &= \Pr\{Y_1 > (u + X_1)(1 + I_1) \mid I_0 = i_s\} \\ &= \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} \bar{F}((u+x)(1+i_t)) dH(x),\end{aligned}$$

which, together with x replaced by $(u+x)(1+i_t)$ in (3.8), implies that

$$\begin{aligned}\phi_1(u, i_s) &\leq \sum_{t=0}^{\infty} p_{st} \left(\beta_1 E(e^{R_1 Y_1}) \int_0^{\infty} e^{-R_1(u+x)(1+i_t)} dH(x) \right) \\ &\leq \beta_1 E(e^{R_1 Y_1}) \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} e^{-R_1(u+x)(1+i_t)} dH(x) \\ &= \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1(u+X_1)(1+I_1)} \mid I_0 = i_s).\end{aligned}$$

Under an inductive hypothesis, we assume for any $u \geq 0$ and any $i_s \geq 0$,

$$\phi_n(u, i_s) \leq \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1(u+X_1)(1+I_1)} \mid I_0 = i_s). \quad (3.9)$$

Thus, for $0 \leq y \leq (u+x)(1+i_t)$, with u and i_s replaced by $(u+x)(1+i_t) - y$ and i_t respectively in (3.9), $I_1 \geq 0$, and (3.3), we have

$$\begin{aligned}&\phi_n((u+x)(1+i_t) - y, i_t) \\ &\leq \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1((u+x)(1+i_t)-y+X_1)(1+I_1)} \mid I_0 = i_t) \\ &= \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1((u+x)(1+i_t)-y)(1+I_1)-R_1 X_1(1+I_1)} \mid I_0 = i_t) \\ &\leq \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1 X_1(1+I_1)} \mid I_0 = i_t) e^{-R_1((u+x)(1+i_t)-y)} \\ &\leq \beta_1 e^{-R_1((u+x)(1+i_t)-y)}.\end{aligned} \quad (3.10)$$

Therefore, by Lemma 2.1, (3.7), and (3.10), we get

$$\begin{aligned}\phi_{n+1}(u, i_s) &\leq \sum_{t=0}^{\infty} p_{st} \left(\beta_1 \int_0^{\infty} e^{-R_1(u+x)(1+i_t)} \int_{(u+x)(1+i_t)}^{\infty} e^{R_1 y} dF(y) dH(x) \right) \\ &\quad + \sum_{t=0}^{\infty} p_{st} \left(\beta_1 \int_0^{\infty} e^{-R_1(u+x)(1+i_t)} \int_0^{(u+x)(1+i_t)} e^{R_1 y} dF(y) dH(x) \right) \\ &= \sum_{t=0}^{\infty} p_{st} \left(\beta_1 \int_0^{\infty} e^{-R_1(u+x)(1+i_t)} \int_0^{\infty} e^{R_1 y} dF(y) dH(x) \right) \\ &= \beta_1 E(e^{R_1 Y_1}) \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} e^{-R_1(u+x)(1+i_t)} dH(x) \\ &= \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1(u+X_1)(1+I_1)} \mid I_0 = i_s).\end{aligned}$$

Hence, for any $n = 1, 2, \dots$, (3.9) holds. Therefore, (3.5) follows by letting $n \rightarrow \infty$ in (3.9). \square

Similarly, we can obtain the following probability inequality for $\varphi(u, i_s)$.

Theorem 3.2 Let $R > 0$ be the constant satisfying (1.9). Then,

$$\varphi(u, i_s) \leq \beta E(e^{-Ru(1+I_1)} | I_0 = i_s), \quad u \geq 0 \quad (3.11)$$

where

$$\beta^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{Ry} dF(y)}{e^{Rt} \bar{F}(t)}. \quad (3.12)$$

Proof. Similarly to (3.7) and (3.8), we have for any $x \geq 0$,

$$\bar{F}(x) \leq \beta e^{-Rx} \int_x^\infty e^{Ry} dF(y) \quad (3.13)$$

$$\leq \beta e^{-Rx} E(e^{RY_1}). \quad (3.14)$$

Then, for any $u \geq 0$ and any $i_s \geq 0$,

$$\begin{aligned} \varphi_1(u, i_s) &= \Pr\{Y_1 > u(1 + I_1) + X_1 | I_0 = i_s\} \\ &= \sum_{t=0}^{\infty} p_{st} \int_0^\infty \bar{F}(u(1 + i_t) + x) dH(x), \end{aligned}$$

which implies by (3.14) that

$$\begin{aligned} \varphi_1(u, i_s) &\leq \sum_{t=0}^{\infty} p_{st} \left(\beta E(e^{RY_1}) \int_0^\infty e^{-R(u(1+i_t)+x)} dH(x) \right) \\ &= \beta E(e^{RY_1}) \sum_{t=0}^{\infty} p_{st} \int_0^\infty e^{-R(u(1+i_t)+x)} dH(x) \\ &= \beta E(e^{RY_1}) E(e^{-R(u(1+I_1)+X_1)} | I_0 = i_s) \\ &= \beta E(e^{RY_1}) E(e^{-RX_1}) E(e^{-Ru(1+I_1)} | I_0 = i_s) \\ &= \beta E(e^{-Ru(1+I_1)} | I_0 = i_s). \end{aligned}$$

Under an inductive hypothesis, we assume for any $u \geq 0$ and any $i_s \geq 0$,

$$\varphi_n(u, i_s) \leq \beta E(e^{-Ru(1+I_1)} | I_0 = i_s). \quad (3.15)$$

Thus, for $0 \leq y \leq u(1 + i_t) + x$, with u and i_s replaced by $u(1 + i_t) + x - y$ and i_t respectively in (3.15), and $I_1 \geq 0$, we have

$$\begin{aligned} \varphi_n(u(1 + i_t) + x - y, i_t) &\leq \beta E(e^{-R(u(1+i_t)+x-y)(1+I_1)} | I_0 = i_t) \\ &\leq \beta e^{-R(u(1+i_t)+x-y)}. \end{aligned} \quad (3.16)$$

Therefore, by Lemma 2.2, (3.13), and (3.16), we get

$$\begin{aligned} \varphi_{n+1}(u, i_s) &\leq \sum_{t=0}^{\infty} p_{st} \left(\beta \int_0^{\infty} e^{-R(u(1+i_t)+x)} \int_{u(1+i_t)+x}^{\infty} e^{Ry} dF(y) dH(x) \right) \\ &\quad + \sum_{t=0}^{\infty} p_{st} \left(\beta \int_0^{\infty} \int_0^{u(1+i_t)+x} e^{-R(u(1+i_t)+x-y)} dF(y) dH(x) \right) \\ &= \sum_{t=0}^{\infty} p_{st} \left(\beta \int_0^{\infty} e^{-R(u(1+i_t)+x)} \int_0^{\infty} e^{Ry} dF(y) dH(x) \right) \\ &= \beta E(e^{RY_1}) \sum_{t=0}^{\infty} p_{st} \int_0^{\infty} e^{-R(u(1+i_t)+x)} dH(x) \\ &= \beta E(e^{RY_1}) E(e^{-RX_1}) E(e^{-Ru(1+I_1)} | I_0 = i_s) \\ &= \beta E(e^{-Ru(1+I_1)} | I_0 = i_s). \end{aligned}$$

Hence, for any $n = 1, 2, \dots$, (3.15) holds. Therefore, (3.11) follows by letting $n \rightarrow \infty$ in (3.15). \square

Refinements of the upper bounds in Theorems 3.1 and 3.2 can be obtained when F is new worse than used in convex ordering (NWUC). A lifetime distribution B is said to be NWUC if for all $x \geq 0, y \geq 0$,

$$\int_{x+y}^{\infty} \bar{B}(t) dt \geq \bar{B}(x) \int_y^{\infty} \bar{B}(t) dt.$$

The class of NWUC distributions is larger than the class of decreasing failure rate distributions. See Shaked and Shanthikumar (1994) for properties of NWUC and other classes of lifetime distributions.

Corollary 3.1 Under the conditions of Theorems 3.1 and 3.2, if F is NWUC, then,

$$\phi(u, i_s) \leq E(e^{-R_1(u+X_1)(1+I_1)} | I_0 = i_s), \quad u \geq 0 \quad (3.17)$$

and

$$\varphi(u, i_s) \leq (E(e^{RY_1}))^{-1} E(e^{-Ru(1+I_1)} | I_0 = i_s), \quad u \geq 0. \quad (3.18)$$

Proof. By Proposition 6.1.1 of Willmot and Lin (2001), we know that if F is NWUC, then $\beta_1^{-1} = E(e^{R_1 Y_1})$ and $\beta^{-1} = E(e^{R Y_1})$. Thus (3.17) and (3.18) follow from (3.5) and (3.11), respectively. \square

We denote the upper bound in Theorem 3.1 by $A(u, i_s)$, i.e.

$$A(u, i_s) = \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1(u+X_1)(1+I_1)} | I_0 = i_s),$$

and denote the upper bound in Theorem 3.2 by $B(u, i_s)$, i.e.

$$B(u, i_s) = \beta E(e^{-Ru(1+I_1)} | I_0 = i_s).$$

Proposition 3.2 For any $u \geq 0$,

$$A(u, i_s) \leq e^{-Ru} \text{ and } B(u, i_s) \leq e^{-Ru}.$$

Further, if F is NWUC, then for any $u \geq 0$,

$$A(u, i_s) \leq B(u, i_s) \leq e^{-Ru}. \quad (3.19)$$

Proof. First, by $I_1 \geq 0$, (3.2), (3.3), and $0 \leq \beta_1 \leq 1$, we have for $u \geq 0$,

$$\begin{aligned} A(u, i_s) &= \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1 u(1+I_1) - R_1 X_1(1+I_1)} | I_0 = i_s) \\ &\leq \beta_1 E(e^{R_1 Y_1}) E(e^{-R_1 u - R_1 X_1(1+I_1)} | I_0 = i_s) \\ &= \beta_1 e^{-R_1 u} E(e^{R_1 Y_1}) E(e^{-R_1 X_1(1+I_1)} | I_0 = i_s) \\ &\leq e^{-Ru}. \end{aligned}$$

Then, $0 \leq \beta \leq 1$ implies that for $u \geq 0$,

$$B(u, i_s) = \beta e^{-Ru} E(e^{-R I_1 u} | I_0 = i_s) \leq e^{-Ru}.$$

Further, by (3.2) we have that for any $t \geq 0$,

$$\frac{\int_t^\infty e^{R_1 y} dF(y)}{e^{R_1 t} \bar{F}(t)} = \frac{\int_t^\infty e^{R_1(y-t)} dF(y)}{\bar{F}(t)} \geq \frac{\int_t^\infty e^{R(y-t)} dF(y)}{\bar{F}(t)},$$

which, using (3.6) and (3.12), implies that

$$\beta_1^{-1} \geq \beta^{-1}, \text{ or } \beta_1 \leq \beta.$$

Thus, if F is NWUC, by Corollary 3.1, (3.2), and (1.9), we have for $u \geq 0$,

$$\begin{aligned}
A(u, i_s) &= E(e^{-R_1 X_1(1+I_1)} e^{-R_1 u(1+I_1)} | I_0 = i_s) \\
&\leq E(e^{-RX_1(1+I_1)} e^{-Ru(1+I_1)} | I_0 = i_s) \\
&\leq E(e^{-RX_1}) E(e^{-Ru(1+I_1)} | I_0 = i_s) \\
&= (E(e^{RY_1}))^{-1} E(e^{-Ru(1+I_1)} | I_0 = i_s) \\
&= B(u, i_s).
\end{aligned}$$

Hence, (3.19) holds. □

Proposition 3.2 means that the upper bounds in Theorem 3.1 and 3.2 are less than the Lundberg upper bound while the upper bound in Theorem 3.1 for $\phi(u, i_s)$ is less than the upper bound in Theorem 3.2 for $\varphi(u, i_s)$ if F is NWUC.

We remark that although our results apply when I_n takes a countable number of values, in practice we would apply a model under which I_n takes a finite number of values. This practice allows calculation of the constants $\{\tau_s\}$ and hence R_1 . The same comment applies to the constants R_2 and R_3 which are defined in the next section.

Further, if $I_s = 0$ for all $s = 0, 1, 2, \dots$, then $R_1 = \tau_s = R$ and $\beta_1 = \beta$. Thus, the upper bounds in Theorems 3.1 and 3.2 reduce to βe^{-Ru} , which yields an improvement on the Lundberg upper bound since $0 \leq \beta \leq 1$. For further refinements of the Lundberg inequality in different applied probability models, see Grandell (1991), Willmot *et al* (2001), Willmot and Lin (2001), and references therein.

4 Probability inequalities by the martingale approach

Another tool for deriving probability inequalities for ruin probabilities is the martingale approach. The ruin probability associated with either the risk process given by (1.4) or by (1.5) is equal to the ruin probability associated with its discounted risk process $\{V_n, n = 1, 2, \dots\}$, i.e.

$$\Pr \left\{ \bigcup_{k=1}^n (U_k < 0) \mid I_0 = i_s \right\} = \Pr \left\{ \bigcup_{k=1}^n (V_k < 0) \mid I_0 = i_s \right\},$$

where

$$V_k = U_k \prod_{j=1}^k (1 + I_j)^{-1}, \quad k = 1, 2, \dots.$$

In the classical risk model, $\{e^{-rU_n}, n = 1, 2, \dots\}$ is a martingale. However, for the generalized risk processes (1.4) and (1.5), there is no constant $r > 0$ such that $\{e^{-rU_n}, n = 1, 2, \dots\}$ is a martingale. However, there exists a constant $r > 0$ such that $\{e^{-rV_n}, n = 1, 2, \dots\}$ is a supermartingale, which allows us to derive probability inequalities by the optional stopping theorem. Such a constant is defined in the following proposition.

Proposition 4.1 Let $E(X_1) > E(Y_1)$. Suppose that there exists $R > 0$ satisfying (1.9) and there exists $\kappa_s > 0$ satisfying

$$E(e^{-\kappa_s(X_1 - Y_1(1+I_1)^{-1})} | I_0 = i_s) = 1, \quad s = 0, 1, 2, \dots. \quad (4.1)$$

Then,

$$R_2 = \min_{0 \leq s < \infty} \{\kappa_s\} \geq R \quad (4.2)$$

and for all $s = 0, 1, 2, \dots$,

$$E(e^{-R_2(X_1 - Y_1(1+I_1)^{-1})} | I_0 = i_s) \leq 1. \quad (4.3)$$

Proof. Similarly to Proposition 3.1, for any $s = 0, 1, 2, \dots$, we know that

$$h_s(r) = E(e^{r(Y_1(1+I_1)^{-1} - X_1)} | I_0 = i_s) - 1, \quad (4.4)$$

is a convex function with $h_s(0) = 0$ and

$$h'_s(0) = E(Y_1(1+I_1)^{-1} - X_1 | I_0 = i_s) \leq E(Y_1 - X_1 | I_0 = i_s) = E(Y_1 - X_1) < 0.$$

Therefore, κ_s is the unique positive root of the equation $h_s(r) = 0$ on $(0, \infty)$. Further, if $0 < \kappa \leq \kappa_s$, then $h_s(\kappa) \leq 0$. However,

$$1 = E(e^{\kappa_s(Y_1(1+I_1)^{-1} - X_1)} | I_0 = i_s) \leq E(e^{\kappa_s(Y_1 - X_1)} | I_0 = i_s) = E(e^{\kappa_s(Y_1 - X_1)}),$$

or $g(\kappa_s) \geq 0$, where $g(r)$ is defined in (3.4). Hence, $\kappa_s \geq R$ and $R_2 = \min_{0 \leq s < \infty} \kappa_s \geq R$, i.e. (4.2) holds. In addition, for all $s = 0, 1, 2, \dots$, $R_2 = \min_{0 \leq t < \infty} \kappa_t \leq \kappa_s$, which implies that $h_s(R_2) \leq 0$, i.e. (4.3) holds. \square

Theorem 4.1 Under the conditions of Proposition 4.1, for all $s = 0, 1, 2, \dots$,

$$\phi(u, i_s) \leq e^{-R_2 u}, \quad u \geq 0. \quad (4.5)$$

Proof. For the process $\{U_k\}$ given by (1.4), we denote

$$V_k = U_k \prod_{j=1}^k (1 + I_j)^{-1} = u + \sum_{j=1}^k \left((X_j(1 + I_j) - Y_j) \prod_{t=1}^j (1 + I_t)^{-1} \right) \quad (4.6)$$

and $S_n = e^{-R_2 V_n}$. Then

$$S_{n+1} = S_n e^{-R_2(X_{n+1} - Y_{n+1}(1 + I_{n+1})^{-1}) \prod_{t=1}^n (1 + I_t)^{-1}}.$$

Thus, for any $n \geq 1$,

$$\begin{aligned} & E(S_{n+1} | X_1, \dots, X_n, Y_1, \dots, Y_n, I_1, \dots, I_n) \\ &= S_n E(e^{-R_2(X_{n+1} - Y_{n+1}(1 + I_{n+1})^{-1}) \prod_{t=1}^n (1 + I_t)^{-1}} | X_1, \dots, X_n, Y_1, \dots, Y_n, I_1, \dots, I_n) \\ &= S_n E(e^{-R_2(X_{n+1} - Y_{n+1}(1 + I_{n+1})^{-1}) \prod_{t=1}^n (1 + I_t)^{-1}} | I_1, \dots, I_n) \\ &\leq S_n (E(e^{-R_2(X_{n+1} - Y_{n+1}(1 + I_{n+1})^{-1}) \prod_{t=1}^n (1 + I_t)^{-1}} | I_1, \dots, I_n)) \prod_{t=1}^n (1 + I_t)^{-1} \\ &= S_n, \end{aligned}$$

where the inequality follows from $0 \leq \prod_{t=1}^n (1 + I_t)^{-1} \leq 1$ and Jensen's inequality while the last equality follows from

$$\begin{aligned} & E(e^{-R_2(X_{n+1} - Y_{n+1}(1 + I_{n+1})^{-1}) \prod_{t=1}^n (1 + I_t)^{-1}} | I_1, \dots, I_n) = E(e^{-R_2(X_{n+1} - Y_{n+1}(1 + I_{n+1})^{-1})} | I_n) \\ &= E(e^{-R_2(X_1 - Y_1(1 + I_1)^{-1})} | I_0) \leq 1. \end{aligned}$$

Hence, $\{S_n, n = 1, 2, \dots\}$ is a supermartingale with respect to the filtration $\mathcal{F}_n = \sigma\{X_1, \dots, X_n, Y_1, \dots, Y_n, I_1, \dots, I_n\}$.

Let $T_s = \min\{n : V_n < 0 | I_0 = i_s\}$, where V_n is given by (4.6). Then T_s is a stopping time and $n \wedge T_s = \min(n, T_s)$ is a finite stopping time. Thus, by the optional stopping theorem for supermartingales, we get

$$E(S_{n \wedge T_s}) \leq E(S_0) = e^{-R_2 u}.$$

Hence,

$$\begin{aligned}
e^{-R_2 u} &\geq E(S_{n \wedge T_s}) \geq E(S_{n \wedge T_s} I(T_s \leq n)) \\
&= E(S_{T_s} I(T_s \leq n)) \\
&= E(e^{-R_2 V_{T_s}} I(T_s \leq n)) \\
&\geq E(I(T_s \leq n)) \tag{4.7}
\end{aligned}$$

$$= \Pr\{T_s \leq n\} = \phi_n(u, i_s), \tag{4.8}$$

where (4.7) follows from $V_{T_s} < 0$ and (4.8) follows from

$$\phi_n(u, i_s) = \Pr\left\{\bigcup_{k=1}^n (U_k < 0) \mid I_0 = i_s\right\} = \Pr\left\{\bigcup_{k=1}^n (V_k < 0) \mid I_0 = i_s\right\} = \Pr\{T_s \leq n\}.$$

Thus, (4.5) follows by letting $n \rightarrow \infty$ in (4.8). \square

Proposition 4.2 Let $E(X_1) > E(Y_1)$. Suppose that there exists $R > 0$ satisfying (1.9) and there exists $\rho_s > 0$ satisfying

$$E(e^{-\rho_s (X_1 - Y_1)(1 + I_1)^{-1}} \mid I_0 = i_s) = 1, \quad s = 0, 1, 2, \dots \tag{4.9}$$

Then,

$$R \leq R_3 = \min_{0 \leq s < \infty} \{\rho_s\} \leq R_2 \tag{4.10}$$

and for all $s = 0, 1, 2, \dots$,

$$E(e^{-R_3 (X_1 - Y_1)(1 + I_1)^{-1}} \mid I_0 = i_s) \leq 1. \tag{4.11}$$

Proof. For any $s = 0, 1, 2, \dots$, we know that

$$l_s(r) = E(e^{r(Y_1 - X_1)(1 + I_1)^{-1}} \mid I_0 = i_s) - 1$$

is a convex function with $l_s(0) = 0$ and

$$\begin{aligned}
l'_s(0) &= E((Y_1 - X_1)(1 + I_1)^{-1} \mid I_0 = i_s) \\
&= E(Y_1 - X_1 \mid I_0 = i_s) E((1 + I_1)^{-1} \mid I_0 = i_s) \\
&= E(Y_1 - X_1) E((1 + I_1)^{-1} \mid I_0 = i_s) < 0.
\end{aligned}$$

Therefore, ρ_s is the unique positive root of the equation $l_s(r) = 0$ on $(0, \infty)$. Further, if $0 < \rho \leq \rho_s$, then $l_s(\rho) \leq 0$. However,

$$1 = E(e^{\rho_s(Y_1 - X_1)(1+I_1)^{-1}} | I_0 = i_s) \geq E(e^{\rho_s(Y_1(1+I_1)^{-1} - X_1)} | I_0 = i_s),$$

or $h_s(\rho_s) \leq 0$, which implies that $\rho_s \leq \kappa_s$, where $h_s(r)$ and κ_s are given by (4.4) and (4.1), respectively. Therefore,

$$R_3 = \min_{0 \leq s < \infty} \rho_s \leq \min_{0 \leq s < \infty} \kappa_s = R_2.$$

Further, by Jensen's inequality and (1.9), we have

$$\begin{aligned} E(e^{R(Y_1 - X_1)(1+I_1)^{-1}} | I_0 = i_s) &= \sum_{t=0}^{\infty} p_{st} E(e^{R(Y_1 - X_1)(1+I_t)^{-1}}) \\ &\leq \sum_{t=0}^{\infty} p_{st} (E(e^{R(Y_1 - X_1)}))^{(1+I_t)^{-1}} = \sum_{t=0}^{\infty} p_{st} = 1, \end{aligned}$$

which implies that $l_s(R) \leq 0$. Hence $R \leq \rho_s$ and $R_3 = \min_{0 \leq s < \infty} \rho_s \geq R$. Thus, (4.10) holds. In addition, for all $s = 0, 1, 2, \dots$, $R_3 = \min_{0 \leq t < \infty} \rho_t \leq \rho_s$, which implies that $l_s(R_3) \leq 0$, i.e. (4.11) holds. \square

Theorem 4.2 Under the conditions of Proposition 4.2, for all $s = 0, 1, 2, \dots$,

$$\varphi(u, i_s) \leq e^{-R_3 u}, \quad u \geq 0. \quad (4.12)$$

Proof. For the process $\{U_k\}$ given by (1.5), we denote

$$V_k = U_k \prod_{j=1}^k (1 + I_j)^{-1} = u + \sum_{j=1}^k \left((X_j - Y_j) \prod_{t=1}^j (1 + I_t)^{-1} \right) \quad (4.13)$$

and $S_n = e^{-R_3 V_n}$. Then

$$S_{n+1} = S_n e^{-R_3(X_{n+1} - Y_{n+1}) \prod_{t=1}^{n+1} (1+I_t)^{-1}}.$$

Thus, for any $n \geq 1$,

$$\begin{aligned} &E(S_{n+1} | X_1, \dots, X_n, Y_1, \dots, Y_n, I_1, \dots, I_n) \\ &= S_n E(e^{-R_3(X_{n+1} - Y_{n+1}) \prod_{t=1}^{n+1} (1+I_t)^{-1}} | X_1, \dots, X_n, Y_1, \dots, Y_n, I_1, \dots, I_n) \\ &= S_n E(e^{-R_3(X_{n+1} - Y_{n+1})(1+I_{n+1})^{-1} \prod_{t=1}^n (1+I_t)^{-1}} | I_1, \dots, I_n) \\ &\leq S_n (E(e^{-R_3(X_{n+1} - Y_{n+1})(1+I_{n+1})^{-1}} | I_1, \dots, I_n)) \prod_{t=1}^n (1+I_t)^{-1} \\ &\leq S_n, \end{aligned}$$

which implies that $\{S_n, n = 1, 2, \dots\}$ is a supermartingale. Let $T_s = \min\{n : V_n < 0 \mid I_0 = i_s\}$ where V_n is given by (4.13). Then T_s is a stopping time and $n \wedge T_s = \min(n, T_s)$ is a finite stopping time. Thus, (4.12) follows from the same arguments as those for (4.5). \square

It is clear that the upper bounds derived by the martingale approach satisfy (1.10), i.e.

$$e^{-R_2 u} \leq e^{-R_3 u} \leq e^{-R u}, \quad u \geq 0$$

since (4.10) holds. In addition, if for all $s = 0, 1, 2, \dots, I_s = 0$, then $R_2 = R_3 = R$ and the upper bounds in Theorems 4.1 and 4.2 reduce to Lundberg's upper bound.

It seems that the inequalities in Section 3 cannot be derived by the martingale approach. In addition, like the case of i.i.d. rates of interest discussed by Cai (2002a), the numerical results of Section 5 suggest that upper bounds derived by the inductive approach of Section 3 are tighter than those obtained by the martingale approach of this section.

5 Numerical illustration

In this section we give a numerical example to illustrate the bounds derived in Sections 3 and 4. Without loss of generality we can work in monetary units equal to $E(Y_1)$. Let Y_1 have a gamma distribution with each parameter equal to 1/2, so that $E(Y_1) = 1$ and $V(Y_1) = 2$. Let $\Pr(X_1 = 1.1) = 1$ so that there is deterministic premium income with a loading of 10%. Consider an interest model with three possible interest rates: $i_0 = 6\%$, $i_1 = 8\%$ and $i_2 = 10\%$. Let $\mathbf{P} = \{p_{st}\}$ be given by

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.15 & 0.7 & 0.15 \\ 0 & 0.8 & 0.2 \end{pmatrix}.$$

Thus, our interest rate model incorporates mean reversion to a level of 8%.

We can easily find that the constant R which satisfies (1.9) is $R = 0.08807$. Similarly, we find that $R_1 = 0.14665$ by solving equation (3.1) for τ_0 , τ_1 and τ_2 , where, for example, the equation satisfied by τ_1 is

$$0.15e^{-1.166\tau_1} + 0.7e^{-1.188\tau_1} + 0.15e^{-1.21\tau_1} = (1 - 2\tau_1)^{1/2}.$$

As our gamma distribution is DFR and hence NWUC, we can apply the results of Corollary 3.1 with, for example,

$$\phi(u, i_1) \leq 0.15e^{-1.06R_1(u+1.1)} + 0.7e^{-1.08R_1(u+1.1)} + 0.15e^{-1.1R_1(u+1.1)}.$$

Table 1 shows values of $\phi(u, i_1)$, $\varphi(u, i_1)$ and e^{-Ru} for a range of values of u . We note that the upper bounds are ordered in accordance with Proposition 3.2.

Similarly, we can find from equations (4.1) and (4.9) that $R_2 = 0.15773$ and $R_3 = 0.09475$, where, for example, we find κ_1 as the solution of

$$e^{-1.1\kappa_1} (0.15M_Y(\kappa_1/1.06) + 0.7M_Y(\kappa_1/1.08) + 0.15M_Y(\kappa_1/1.1)) = 1$$

where $M_Y(t) = (1 - 2t)^{-1/2}$. Table 2 shows values of e^{-R_2u} , e^{-R_3u} and e^{-Ru} for a range of values of u . The ordering of these bounds is as expected, and we note that $\phi(u, i_1) < e^{-R_2u}$ and $\varphi(u, i_1) < e^{-R_3u}$ in line with our comments in Section 4.

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Table 1: Upper bounds by the inductive approach

u	$\phi(u, i_1)$	$\varphi(u, i_1)$	e^{-Ru}
0	0.8401	0.9077	1.0000
5	0.3806	0.5642	0.6438
10	0.1724	0.3507	0.4145
15	0.0781	0.2180	0.2669
20	0.0354	0.1355	0.1718
25	0.0160	0.0842	0.1106
30	0.0073	0.0523	0.0712

Table 2: Upper bounds by the martingale approach

u	e^{-R_2u}	e^{-R_3u}	e^{-Ru}
0	1.0000	1.0000	1.0000
5	0.4545	0.6227	0.6438
10	0.2065	0.3877	0.4145
15	0.0939	0.2414	0.2669
20	0.0427	0.1503	0.1718
25	0.0194	0.0936	0.1106
30	0.0088	0.0583	0.0712

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