

The Diffusion Perturbed Compound Poisson Risk Model with a Dividend Barrier

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Abstract

We consider a diffusion perturbed classical compound Poisson risk model in the presence of a constant dividend barrier. Integro-differential equations with certain boundary conditions for the expected discounted penalty (Gerber-Shiu) functions (caused by oscillations or by a claim) are derived and solved. Their solutions can be expressed in terms of the Gerber-Shiu functions in the corresponding perturbed risk model without a barrier. Finally, explicit results are given when the claim sizes are rationally distributed.

Keywords: Compound Poisson process; Diffusion process; Gerber-Shiu function; Integro-differential equation; Time of ruin; Surplus before ruin; Deficit at ruin

1 Introduction

Consider the following classical continuous time surplus process perturbed by a diffusion

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma B(t), \quad t \geq 0, \quad (1)$$

where $\{N(t); t \geq 0\}$ is a Poisson process with parameter λ , denoting the total number of claims from an insurance portfolio. X_1, X_2, \dots , independent of

$\{N(t); t \geq 0\}$, are positive i.i.d. random variables with common distribution function (df) $P(x) = 1 - \bar{P}(x) = P(X \leq x)$, density function $p(x)$, moments $\mu_j = \int_0^\infty x^j p(x) dx$, for $j = 0, 1, 2, \dots$, and the Laplace transform $\hat{p}(s) = \int_0^\infty e^{-sx} p(x) dx$. $\{B(t); t \geq 0\}$ is a standard Wiener process that is independent of the aggregate claims process $S(t) := \sum_{i=1}^{N(t)} X_i$ and $\sigma > 0$ is the dispersion parameter. In the above model, $u = U(0) \geq 0$ is the initial surplus, $c = \lambda \mu_1 (1 + \theta)$ is the premium rate per unit time, and $\theta > 0$ is the relative security loading factor.

The classical risk model perturbed by a diffusion was first introduced by Gerber (1970) and has been further studied by many authors during the last few years. In this paper, a barrier strategy is considered by assuming that there is a horizontal barrier of level $b \geq u$ such that when the surplus reaches level b , the ‘‘overflow’’ will be paid as dividend. Same as in Gerber and Shiu (2004), a formal definition can be given in terms of the running maximum

$$M(t) = \max_{0 \leq s \leq t} U(s), \quad t \geq 0.$$

Then the aggregate dividends paid by time t are

$$D(t) = (M(t) - b)_+ = \begin{cases} 0, & \text{if } M(t) \leq b, \\ M(t) - b, & \text{if } M(t) > b. \end{cases}$$

Let $U_b(t)$ be the modified surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy. Then

$$U_b(t) = U(t) - D(t) \quad t \geq 0. \quad (2)$$

Define now $T_b = \inf\{t : U_b(t) \leq 0\}$ to be the time of ruin and

$$\Psi_b(u) = P(T_b < \infty | U_b(0) = u), \quad 0 \leq u \leq b,$$

to be the ultimate ruin probability. Further, define

$$\Psi_{b,d}(u) = P(T_b < \infty, U_b(T_b) = 0 | U_b(0) = u), \quad 0 \leq u \leq b,$$

to be the probability of ruin caused by the oscillations in $U_b(t)$ due to the Wiener process $B(t)$ and

$$\Psi_{b,s}(u) = P(T_b < \infty, U_b(T_b) < 0 | U_b(0) = u), \quad 0 \leq u \leq b,$$

to be the probability of ruin caused by a claim. We have that $\Psi_b(u) = \Psi_{b,d}(u) + \Psi_{b,s}(u)$, with $\Psi_{b,d}(0) = 1$ and $\Psi_{b,s}(0) = 0$.

Next, for $\delta > 0$, define

$$\phi_{b,d}(u) = E[e^{-\delta T_b} I(T_b < \infty, U_b(T_b) = 0) | U_b(0) = u], \quad 0 \leq u \leq b,$$

with $\phi_{b,d}(0) = 1$, to be the Laplace transform of the ruin time T_b with respect to δ if the ruin is due to the oscillations. Let $w(x, y)$, for $x, y \geq 0$, be the non-negative values of a penalty function and define

$$\phi_{b,s}(u) = E[e^{-\delta T_b} w(U_b(T_b^-), |U_b(T_b)|) I(T_b < \infty, U_b(T_b) < 0) | U_b(0) = u],$$

with $\phi_{b,s}(0) = 0$, to be the expected discounted penalty (Gerber-Shiu) function if the ruin is caused by a claim. Then

$$\phi_b(u) = \phi_{b,d}(u) + \phi_{b,s}(u)$$

is the expected discounted penalty function.

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies for a compound Poisson risk process have been studied in a number of papers and books. These references include Bühlmann (1970), Segerdahl (1970), Gerber (1972, 1979, 1981), Paulsen and Gjessing (1997), Albrecher and Kainhofer (2002), Højgaard (2002), Dickson and Waters (2004), Gerber and Shiu (2004), and Albrecher et al. (2005). The main focus is on optimal dividend payouts and the time of ruin, under various barrier strategies and other economic conditions. More recently, there are some research papers studying the ruin related quantities such as the surplus before ruin and the deficit at ruin by using the Gerber-Shiu function under a barrier strategy, in the classical risk model or Sparre Andersen risk models, e.g., Lin et al. (2003), Li and Garrido (2004).

For the risk model defined in (2), Li (2006) studies the moments and distribution of dividend payments until ruin. The main goal of this paper is to evaluate the Gerber-Shiu function $\phi_b(u)$ and its decompositions of $\phi_{b,s}(u)$ and $\phi_{b,d}(u)$ in the above defined diffusion perturbed classical risk model in the presence of a constant dividend barrier b and analyze several of its special cases.

2 Integro-differential Equations and Their Solutions

In this section, we will show that $\phi_{b,d}(u)$ and $\phi_{b,s}(u)$ both satisfy an integro-differential equation with certain boundary conditions. First, the conditions satisfied by $\phi_{b,d}(b)$ and $\phi_{b,s}(b)$ are given in the following two theorems.

Theorem 1 If the initial surplus is b , then we have the following equation

$$\frac{1}{2}\sigma^2\phi''_{b,d}(b-) - c\phi'_{b,d}(b-) = (\lambda + \delta)\phi_{b,d}(b) - \lambda \int_0^b \phi_{b,d}(b-x)p(x)dx. \quad (3)$$

Proof:

First, we define for $u = b$ and $t \geq 0$

$$\begin{aligned} \tilde{U}(t) &= b + ct + \sigma B(t), \\ \tilde{M}(t) &= \max_{0 \leq s \leq t} \tilde{U}(s) = \max_{0 \leq s \leq t} \{b + cs + \sigma B(s)\}, \\ \tilde{D}(t) &= |\tilde{M}(t) - b|_+ = \max_{0 \leq s \leq t} \{cs + \sigma B(s)\}, \\ \tilde{U}_b(t) &= \tilde{U}(t) - \tilde{D}(t) = b + ct + \sigma B(t) - \max_{0 \leq s \leq t} \{cs + \sigma B(s)\}. \end{aligned}$$

It follows from Borodin and Salminen (2002, page 73) that

$$\max_{0 \leq s \leq t} \{(c/\sigma)s + B(s)\} - (c/\sigma)t - B(t)$$

is identical in law with a reflecting Brownian motion with drift $-c/\sigma$, that is to say,

$$\max_{0 \leq s \leq t} \{(c/\sigma)s + B(s)\} - (c/\sigma)t - B(t) = |\tilde{B}(t) - (c/\sigma)t|,$$

where $\tilde{B}(t)$ is a standard Brownian motion independent of $B(t)$ and the aggregate claim process $\sum_{i=1}^{N(t)} X_i$. Therefore,

$$\tilde{U}_b(t) = b - |\sigma\tilde{B}(t) - ct|, \quad t \geq 0. \quad (4)$$

Now we give an heuristic proof of (3) which is essentially similar to that of Gerber and Landry (1998). Consider an infinitesimal interval from 0 to dt . The discount factor for this interval is $1 - \delta dt$, the probability of having no claims is $1 - \lambda dt$, and the probability of exactly one claim is λdt . By conditioning on these and the amount of the first claim (if it occurs), we have for $u = b$ that

$$\begin{aligned} \phi_{b,d}(b) &= (1 - \lambda dt)(1 - \delta dt)E \left[\phi_{b,d}(\tilde{U}_b(dt)) \right] \\ &\quad + \lambda dt(1 - \delta dt)E \left[\phi_{b,d}(\tilde{U}_b(dt) - X_1) \right]. \end{aligned} \quad (5)$$

Substituting (4) into (5) yields

$$\begin{aligned} \phi_{b,d}(b) &= (1 - \lambda dt)(1 - \delta dt)E \left[\phi_{b,d}(b - |\sigma\tilde{B}(dt) - c dt|) \right] \\ &\quad + \lambda dt(1 - \delta dt)E \left[\int_0^{b - |\sigma\tilde{B}(dt) - c dt|} \phi_{b,d}(b - |\sigma\tilde{B}(dt) - c dt| - x)p(x)dx \right]. \end{aligned} \quad (6)$$

Since

$$E[\phi_{b,d}(b - |\sigma\tilde{B}(dt) - c dt|)] = \phi_{b,d}(b) + \mathcal{G}\phi_{b,d}(b)dt + o(dt),$$

where $\lim_{dt \rightarrow 0} o(dt)/dt = 0$ and \mathcal{G} is the infinitesimal generator of the process $\{|\sigma\tilde{B}(t) - ct|; t \geq 0\}$. It is straightforward to show from Borodin and Salminen (2002, page 129) that

$$\mathcal{G}\phi_{b,d}(b) = -c\phi'_{b,d}(b-) + \frac{1}{2}\sigma^2\phi''_{b,d}(b-).$$

Then

$$E[\phi_{b,d}(b - |\sigma\tilde{B}(dt) - c dt|)] = \phi_{b,d}(b) - c\phi'_{b,d}(b-)dt + \frac{1}{2}\sigma^2\phi''_{b,d}(b-)dt + o(dt). \quad (7)$$

Substituting (7) into (6), subtracting $\phi_{b,d}(b)$ from both sides, interpreting dt and $o(dt)$ terms, dividing both sides by dt , and letting dt go to 0, we have

$$\frac{1}{2}\sigma^2\phi''_{b,d}(b-) - c\phi'_{b,d}(b-) = (\lambda + \delta)\phi_{b,d}(b) - \lambda \int_0^b \phi_{b,d}(b-x)p(x)dx.$$

This completes the proof. \square

Theorem 2 If the initial surplus is b , then we have the following equations:

$$\frac{1}{2}\sigma^2\phi''_{b,s}(b-) - c\phi'_{b,s}(b-) = (\lambda + \delta)\phi_{b,s}(b) - \lambda \int_0^b \phi_{b,s}(b-x)p(x)dx - \lambda\omega(b), \quad (8)$$

where $\omega(b) = \int_b^\infty w(b, x-b)p(x)dx$.

Proof: We use the similar arguments as in the proof of Theorem 1. For an infinitesimal interval $(0, dt)$, we have

$$\begin{aligned} \phi_{b,s}(b) &= (1 - \lambda dt)(1 - \delta dt)E[\phi_{b,s}(b - |\sigma\tilde{B}(dt) - c dt|)] \\ &\quad + \lambda dt(1 - \delta dt)E\left[\int_0^{b - \sigma|\tilde{B}(dt) - c dt|} \phi_{b,s}(b - |\sigma\tilde{B}(dt) - c dt| - x)p(x)dx \right. \\ &\quad \left. + \omega(b - \sigma|\tilde{B}(dt) - c dt|)\right]. \end{aligned} \quad (9)$$

Since

$$E[\phi_{b,s}(b - |\sigma\tilde{B}(dt) - c dt|)] = \phi_{b,s}(b) - c\phi'_{b,s}(b-)dt + \frac{1}{2}\sigma^2\phi''_{b,s}(b-)dt + o(dt),$$

then similar arguments show that (8) holds. \square

Theorem 3 Suppose $p(x)$ is continuously differentiable on $(0, \infty)$, then $\phi_{b,d}(u)$ satisfies the following homogenous integro-differential equation for $0 < u < b$:

$$\frac{\sigma^2}{2}\phi''_{b,d}(u) + c\phi'_{b,d}(u) = (\lambda + \delta)\phi_{b,d}(u) - \lambda \int_0^u \phi_{b,d}(u-x)p(x)dx, \quad (10)$$

with the boundary conditions

$$\phi_{b,d}(0) = 1, \quad (11)$$

$$\phi'_{b,d}(b-) = 0. \quad (12)$$

Proof: The proof of the integro-differential equation (10) is exactly the same as that of the integro-differential equation satisfied by $\phi_{\infty,d}(u)$, see Gerber and Landry (1998, pp. 265-266). The boundary condition (11) is from the definition of $\phi_{b,d}$. To prove the boundary condition (12), letting u goes to b from below in (10) gives

$$\frac{1}{2}\sigma^2\phi''_{b,d}(b-) + c\phi'_{b,d}(b-) = (\lambda + \delta)\phi_{b,d}(b) - \lambda \int_0^b \phi_{b,d}(b-x)p(x)dx. \quad (13)$$

Comparing (3) with (13) gives

$$\phi'_{b,d}(b-) = 0.$$

This completes the proof. □

Theorem 4 Suppose $p(x)$ is continuously differentiable on $(0, \infty)$ and $\omega(u)$ is twice continuously differentiable on $(0, \infty)$, then $\phi_{b,s}(u)$ satisfies the following non-homogenous integro-differential equation for $0 < u < b$:

$$\frac{\sigma^2}{2}\phi''_{b,s}(u) + c\phi'_{b,s}(u) = (\lambda + \delta)\phi_{b,s}(u) - \lambda \int_0^u \phi_{b,s}(u-x)p(x)dx - \lambda\omega(u), \quad (14)$$

with boundary conditions

$$\phi_{b,s}(0) = 0, \quad (15)$$

$$\phi'_{b,s}(b-) = 0. \quad (16)$$

Proof: The proof of the integro-differential equation (14) is exactly the same as that of the integro-differential equation satisfied by $\phi_{\infty,s}(u)$, see Tsai and Willmot (2002a, pp. 53-54) or Chiu and Yin (2003, pp. 63-64). The boundary condition

(15) is from the definition of $\phi_{b,s}$. To prove the boundary condition (16), let u go to b from below in (14), we have

$$\frac{1}{2}\sigma^2\phi''_{b,s}(b-) + c\phi'_{b,s}(b-) = (\lambda + \delta)\phi_{b,s}(b) - \lambda \int_0^b \phi_{b,s}(b-x)p(x)dx - \lambda\omega(b). \quad (17)$$

Comparing (8) with (17) gives

$$\phi'_{b,s}(b-) = 0.$$

This completes the proof. \square

Remark: When $u \geq b$, dividends $u - b$ are paid immediately so

$$\phi_{b,d}(u) = \phi_{b,s}(b), \quad \phi'_{b,d}(u) = \phi'_{b,d}(b), \quad u \geq b.$$

Thus we have

$$\phi'_{b,d}(b+) = \phi'_{b,s}(b+) = 0.$$

Theorems 3 and 4 show that both $\phi'_{b,d}(u)$ and $\phi'_{b,s}(u)$ are continuous at $u = b$ and

$$\phi'_{b,d}(b) = \phi'_{b,s}(b) = 0.$$

The solutions of the above integro-differential equations with boundary conditions heavily depend on the solutions of the following homogenous integro-differential equation:

$$\frac{\sigma^2}{2}v''(u) + cv'(u) = (\lambda + \delta)v(u) - \lambda \int_0^u v(u-x)p(x)dx, \quad u \geq 0. \quad (18)$$

It follows from the general theory of differential equations (e.g., see Petrovski, 1996, p. 119) that the general solution of equation (18) is of the form

$$v(u) = \eta_1 v_1(u) + \eta_2 v_2(u), \quad u \geq 0, \quad (19)$$

where $v_1(u)$ and $v_2(u)$ are two linearly independent solutions of (18), which will be discussed in the next section, and η_1, η_2 are any real numbers. Then the solution of the integro-differential equation (10) with boundary conditions (11) and (12) is

$$\phi_{b,d}(u) = \eta_1(b) v_1(u) + \eta_2(b) v_2(u), \quad 0 \leq u \leq b, \quad (20)$$

where $\eta_1(b)$ and $\eta_2(b)$ can be determined by solving the following linear equation system

$$\begin{cases} \eta_1(b) v_1(0) + \eta_2(b) v_2(0) = 1, \\ \eta_1(b) v'_1(b) + \eta_2(b) v'_2(b) = 0. \end{cases} \quad (21)$$

Let $\phi_{\infty, s}(u)$ be the expected discounted penalty function if the ruin is caused by a claim in the perturbed compound Poisson risk model (1) without a barrier. Tsai and Willmot (2002a) show that it satisfies the following integro-differential equation for $u \geq 0$:

$$\frac{\sigma^2}{2}\phi''_{\infty, s}(u) + c\phi'_{\infty, s}(u) = (\lambda + \delta)\phi_{\infty, s}(u) - \lambda \int_0^u \phi_{\infty, s}(u-x)p(x)dx - \lambda\omega(u).$$

We note that this equation is the same as equation (14) except $b = \infty$. Then $\phi_{\infty, s}(u)$ can be viewed as a particular solution of (14). It follows from the general theory of differential equations that the solution of non-homogeneous integro-differential equation (14) with boundary conditions (15) and (16) can be expressed as

$$\phi_{b, s}(u) = \phi_{\infty, s}(u) + \vartheta_1(b)v_1(u) + \vartheta_2(b)v_2(u), \quad 0 \leq u \leq b, \quad (22)$$

where $\vartheta_1(b)$ and $\vartheta_2(b)$ can be determined by solving the following linear equation system

$$\begin{cases} \vartheta_1(b)v_1(0) + \vartheta_2(b)v_2(0) = 0, \\ \vartheta_1(b)v'_1(b) + \vartheta_2(b)v'_2(b) = -\phi'_{\infty, s}(b). \end{cases} \quad (23)$$

Tsai and Willmot (2002a) have shown that $\phi_{\infty, s}(u)$ satisfies a defective renewal equation, described as follows. Let $\rho = \rho(\delta)$ be the unique non-negative root of the following generalized Lundberg equation:

$$\lambda\hat{p}(s) = \lambda + \delta - cs - \sigma^2 s^2/2, \quad (24)$$

with $\rho(0) = 0$. Let

$$h(y) = \frac{2c}{\sigma^2}e^{-(\rho + \frac{2c}{\sigma^2})y},$$

$$\gamma(y) = \frac{\lambda}{c} \int_y^\infty e^{-\rho(x-y)}p(x)dx,$$

and

$$\gamma_\omega(y) = \frac{\lambda}{c} \int_y^\infty e^{-\rho(x-y)}\omega(x)dx.$$

Then $\phi_{\infty, s}(u)$ satisfies the following defective renewal equation:

$$\phi_{\infty, s}(u) = \int_0^u \phi_{\infty, s}(u-y)g(y)dy + g_\omega(u), \quad u \geq 0, \quad (25)$$

where $g(y) = h * \gamma(y)$ and $g_\omega(u) = h * \gamma_\omega(u)$, with $*$ denoting the convolution operation.

Properties of $\phi_{\infty, s}(u)$ and its applications have been studied extensively by Tsai (2001, 2003), Tsai and Willmot (2002a, 2002b), Chiu and Yin (2003), and Li and Garrido (2005) for $n = 1$. Therefore, we may use the properties of $\phi_{\infty, s}(u)$ to analyze $\phi_{b, s}(u)$.

3 Analysis of the Function $v(u)$

In this section, we use the Laplace transform to solve the homogenous equation (18) and to find the two particular solutions $v_1(u)$ and $v_2(u)$ by specifying their initial conditions at $u = 0$. Let $\hat{v}(s) = \int_0^{\infty} e^{-sx} v(x) dx$ be the Laplace transform of $v(u)$. Taking Laplace transforms on both sides of (18) gives

$$\left[\frac{1}{2} \sigma^2 s^2 + cs - (\lambda + \delta) + \lambda \hat{p}(s) \right] \hat{v}(s) = \frac{\sigma^2}{2} v(0) s + c v(0) + \frac{\sigma^2}{2} v'(0). \quad (26)$$

Since $\sigma^2 \rho^2/2 + c\rho - (\lambda + \delta) + \lambda \hat{p}(\rho) = 0$, then (26) can be rewritten as

$$\begin{aligned} & \left\{ 1 - \left(\frac{2\lambda/\sigma^2}{s + \rho + 2c/\sigma^2} \right) \left[\frac{\hat{p}(\rho) - \hat{p}(s)}{s - \rho} \right] \right\} \hat{v}(s) \\ &= \frac{v(0)}{s + \rho + 2c/\sigma^2} + \frac{v(0)(\rho + 2c/\sigma^2) + v'(0)}{(s - \rho)(s + \rho + 2c/\sigma^2)}. \end{aligned} \quad (27)$$

Inverting it yields

$$\begin{aligned} v(u) &= \int_0^u v(u-y) g(y) dy + \frac{\sigma^2 v(0)}{2c} h(u) \\ &+ \frac{\sigma^2 [v(0)(\rho + 2c/\sigma^2) + v'(0)]}{2c} e^{\rho u} * h(u), \quad u \geq 0. \end{aligned} \quad (28)$$

We remark that equation (28) is defective renewal equation, since $g(y)$ is a defective density function with $\int_0^{\infty} g(y) dy = (c\rho + \sigma^2 \rho^2/2 - \delta)/(c\rho + \sigma^2 \rho^2/2) < 1$, see Gerber and Landry (1998, eq. (16)). It can be seen from Eq. (28) that $v(u)$ is uniquely determined by initial conditions $v(0)$ and $v'(0)$. One can find two linearly independent solutions $v_1(u)$ and $v_2(u)$ by specifying the initial conditions $v_i(0)$ and $v'_i(0)$ for $i = 1, 2$. For example, setting $v_1(0) = 1$ and $v'_1(0) = -(\rho + 2c/\sigma^2)$ yields

$$v_1(u) = \int_0^u v_1(u-y) g(y) dy + \frac{\sigma^2}{2c} h(u), \quad u \geq 0, \quad (29)$$

and setting $v_2(0) = 0$ and $v'_2(0) = 1$ yields

$$v_2(u) = \int_0^u v_2(u-y) g(y) dy + \frac{\sigma^2}{2c} e^{\rho u} * h(u), \quad u \geq 0. \quad (30)$$

To prove that $v_1(u)$ and $v_2(u)$ are linearly independent, we assume that there are constants c_1 and c_2 such that $c_1 v_1(u) + c_2 v_2(u) \equiv 0$, for any $u \geq 0$. Then we have $c_1 v_1(0) + c_2 v_2(0) = 0$ and $c_1 v_1'(0) + c_2 v_2'(0) = 0$. Solving these two equations gives $c_1 = c_2 = 0$. This proves that $v_1(u)$ and $v_2(u)$ are linearly independent.

Gerber and Landry (1998, eq. (17)) have shown that $\phi_{\infty,d}(u)$ with $\phi_{\infty,d}(0) = 1$ also satisfies the defective renewal equation (29). By the uniqueness of the solution of the defective renewal equation (29), we have $v_1(u) = \phi_{\infty,d}(u)$. By comparing (29) and (30), we can easily prove that

$$v_2(u) = e^{\rho u} * v_1(u) = e^{\rho u} * \phi_{\infty,d}(u) = \int_0^u \phi_{\infty,d}(u-x)e^{\rho x} dx, \quad u \geq 0.$$

It follows from (28), (29) and (30) that $v(u)$ can be expressed as a linear combination of $v_1(u)$ and $v_2(u)$ as follows:

$$v(u) = v(0)v_1(u) + [v'(0) + v(0)(\rho + 2c/\sigma^2)]v_2(u), \quad u \geq 0. \quad (31)$$

Formula (31) further justifies that the general solution of integro-differential equation (18) is of the form given in (19).

4 Main Results

For $v_1(u) = \phi_{\infty,d}(u)$ and $v_2(u) = \int_0^u \phi_{\infty,d}(u-x)e^{\rho x} dx$ obtained in the previous section, eq. (20) and (21) give for $0 \leq u \leq b$:

$$\begin{aligned} \phi_{b,d}(u) &= v_1(u) - \frac{v_1'(b)}{v_2'(b)}v_2(u) \\ &= \phi_{\infty,d}(u) - \frac{\phi'_{\infty,d}(b)}{\rho \int_0^b \phi_{\infty,d}(b-x)e^{\rho x} dx + \phi_{\infty,d}(b)} e^{\rho u} * \phi_{\infty,d}(u), \end{aligned} \quad (32)$$

and (22) and (23) give for $0 \leq u \leq b$:

$$\begin{aligned} \phi_{b,s}(u) &= \phi_{\infty,s}(u) - \frac{\phi'_{\infty,s}(b)}{v_2'(b)}v_2(u) \\ &= \phi_{\infty,s}(u) - \frac{\phi'_{\infty,s}(b)}{\rho \int_0^b \phi_{\infty,d}(b-x)e^{\rho x} dx + \phi_{\infty,d}(b)} e^{\rho u} * \phi_{\infty,d}(u). \end{aligned} \quad (33)$$

In particular, if $\delta = 0$ and $w(x, y) = 1$, then $\rho = 0$, and $\phi_{b,d}(u)$ and $\phi_{b,s}(u)$ simplify to the ruin probabilities $\Psi_{b,d}(u)$ and $\Psi_{b,s}(u)$, respectively. We have the following

results for $0 \leq u \leq b$:

$$\begin{aligned}\Psi_{b,d}(u) &= \Psi_{\infty,d}(u) - \frac{\Psi'_{\infty,d}(b)}{\Psi_{\infty,d}(b)} \int_0^u \Psi_{\infty,d}(x) dx, \\ \Psi_{b,s}(u) &= \Psi_{\infty,s}(u) - \frac{\Psi'_{\infty,s}(b)}{\Psi_{\infty,d}(b)} \int_0^u \Psi_{\infty,d}(x) dx.\end{aligned}$$

Dufresne and Gerber (1991, Eq. (4.7)) show that

$$\Phi'_{\infty}(u) = \frac{2(c - \lambda\mu_1)}{\sigma^2} \Psi_{\infty,d}(u),$$

where $\Phi_{\infty}(u)$ is the non-ruin probability of the risk model (1). Then $\Psi_{b,d}(u)$ and $\Psi_{b,s}(u)$ can be expressed for $0 \leq u \leq b$ as

$$\Psi_{b,d}(u) = \Psi_{\infty,d}(u) - \frac{\Psi'_{\infty,d}(b)}{\Phi'_{\infty}(b)} \Phi_{\infty}(u), \quad (34)$$

$$\Psi_{b,s}(u) = \Psi_{\infty,s}(u) - \frac{\Psi'_{\infty,s}(b)}{\Phi'_{\infty}(b)} \Phi_{\infty}(u). \quad (35)$$

Remark: Since $\Psi_{\infty,s}(u) + \Psi_{\infty,d}(u) = \Psi_{\infty}(u) = 1 - \Phi_{\infty}(u)$, it follows from (34) and (35) that $\Psi_{b,d}(u) + \Psi_{b,s}(u) = 1$ for $0 \leq u \leq b$. This shows that ruin is certain under the constant dividend barrier strategy.

Next, we will show that if p is rationally distributed then both v_1 and v_2 have a rational Laplace transform, which can be inverted explicitly by partial fractions as follows. Let us assume that claim size X is rationally distributed, i.e.,

$$\hat{p}(s) = \frac{Q_{m-1}(s)}{Q_m(s)}, \quad \Re(s) \in (h_X, \infty), \quad (36)$$

where $m \in \mathbb{N}^+$, $h_X := \inf\{s \in \mathbb{R} : E[e^{-sX}] < \infty\}$, Q_m is a polynomial of degree m with leading coefficient 1, Q_{m-1} is a polynomial of degree $m-1$ or less, and Q_m and Q_{m-1} do not have any common zeros. Further, since $\hat{p}(s)$ is finite for all s , with $\Re(s) > 0$, equation $Q_m(s) = 0$ has no roots with positive real parts. This class of distributions is widely used in applied probability applications, which includes, as special cases, Erlangs and phase-type, Coxian distributions, as well as mixture of them. Further discussions on rational distributions can be found in Cox (1955) and Neuts (1981, Chapter 2).

Substituting (36) into (26) with $v_1(0) = 1$ and $v'_1(0) = -(\rho + 2c/\sigma^2)$ and multiplying $Q_m(s)$ to both the denominator and numerator yields

$$\hat{v}_1(s) = \frac{(\sigma^2/2)(s - \rho)Q_m(s)}{[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s)}. \quad (37)$$

Since $[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s)$ is a polynomial of degree $m + 2$ with leading coefficient $\sigma^2/2$, it can be factored as

$$[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s) = (\sigma^2/2)(s - \rho) \prod_{i=1}^{m+1} (s + R_i),$$

where $\rho > 0$ and $-R_1, -R_2, \dots, -R_{m+1}$, with $\Re(R_i) > 0, i = 1, 2, \dots, m + 1$, are all the roots of the equation $[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s) = 0$ on the whole complex plane. We remark ρ is also the unique positive root of the generalized Lundberg equation (24), since $[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s) = [\sigma^2 s^2/2 + cs - (\lambda + \delta) + \lambda \hat{p}(s)]Q_m(s)$. If R_1, R_2, \dots, R_{m+1} are distinct, then by partial fractions, (37) can be rewritten as

$$\hat{v}_1(s) = \frac{Q_m(s)}{\prod_{i=1}^{m+1} (s + R_i)} = \sum_{i=1}^{m+1} \frac{\alpha_i}{s + R_i}, \quad (38)$$

where $\alpha_i = Q_m(-R_i)/\prod_{j=1, j \neq i}^{m+1} (R_j - R_i), i = 1, 2, \dots, m + 1$. Inverting it gives

$$v_1(u) = \sum_{i=1}^{m+1} \alpha_i e^{-R_i u}, \quad u \geq 0. \quad (39)$$

Then we have for $u \geq 0$

$$v_2(u) = \int_0^u v_1(u-x) e^{\rho x} dx = \frac{Q_m(\rho)}{\prod_{i=1}^{m+1} (\rho + R_i)} e^{\rho u} - \sum_{i=1}^{m+1} \frac{\alpha_i}{(\rho + R_i)} e^{-R_i u}. \quad (40)$$

5 An Example

In this section, we will illustrate some explicit results when the claim sizes are exponentially distributed, i.e.,

$$p(x) = \kappa e^{-\kappa x}, \quad x \geq 0,$$

and the Laplace transform $\hat{p}(s) = \kappa/(s + \kappa)$. The equation

$$[\sigma^2 s^2/2 + cs - (\lambda + \delta)](s + \kappa) + \lambda \kappa = 0 \quad (41)$$

has one positive root, say ρ , and two negative roots, say $-R_1, -R_2$. Then (39) gives

$$v_1(u) = \phi_{\infty, d}(u) = \frac{\kappa - R_1}{R_2 - R_1} e^{-R_1 u} + \frac{\kappa - R_2}{R_1 - R_2} e^{-R_2 u}, \quad u \geq 0,$$

and

$$\begin{aligned}
v_2(u) &= \int_0^u v_1(u-x)e^{\rho x} dx \\
&= \frac{\rho + \kappa}{(\rho + R_1)(\rho + R_2)} e^{\rho u} + \frac{R_1 - \kappa}{(\rho + R_1)(R_2 - R_1)} e^{-R_1 u} \\
&\quad + \frac{R_2 - \kappa}{(\rho + R_2)(R_1 - R_2)} e^{-R_2 u}, \quad u \geq 0.
\end{aligned}$$

Then (32) gives

$$\begin{aligned}
\phi_{b,d}(u) &= \frac{\kappa - R_1}{R_2 - R_1} \left(1 + \frac{\xi}{\rho + R_1}\right) e^{-R_1 u} + \frac{\kappa - R_2}{R_1 - R_2} \left(1 + \frac{\xi}{\rho + R_2}\right) e^{-R_2 u} \\
&\quad - \frac{\xi(\kappa + \rho)}{(\rho + R_1)(\rho + R_2)} e^{\rho u}, \quad 0 \leq u \leq b,
\end{aligned}$$

where $\xi = v_1'(b)/v_2'(b)$.

The evaluation of $\phi_{b,s}(u)$ depends on the choice of the penalty function $w(x, y)$, e.g., if $w(x, y) = 1$, then $\omega(u) = \bar{P}(u) = e^{-\kappa u}$, and Tsai and Willmot (2002a) show that the Laplace transform of $\phi_{\infty,s}(u)$ can be expressed as

$$\hat{\phi}_{\infty,s}(s) = \frac{\lambda(s + \kappa)[\hat{\omega}(\rho) - \hat{\omega}(s)]}{[\sigma^2 s^2/2 + cs - (\lambda + \delta)](s + \kappa) + \lambda\kappa} = \frac{2\lambda/[\sigma^2(\rho + \kappa)]}{(s + R_1)(s + R_2)}, \quad (42)$$

inverting it yields

$$\phi_{\infty,s}(u) = \frac{2\lambda}{\sigma^2(\rho + \kappa)(R_2 - R_1)} [e^{-R_1 u} - e^{-R_2 u}], \quad u \geq 0. \quad (43)$$

Then (33) gives

$$\begin{aligned}
\phi_{b,s}(u) &= \frac{1}{(R_2 - R_1)} \left[\frac{2\lambda}{\sigma^2(\rho + \kappa)} - \frac{\varsigma(R_1 - \kappa)}{\rho + R_1} \right] e^{-R_1 u} \\
&\quad + \frac{1}{(R_1 - R_2)} \left[\frac{2\lambda}{\sigma^2(\rho + \kappa)} - \frac{\varsigma(R_2 - \kappa)}{\rho + R_2} \right] e^{-R_2 u} - \frac{\varsigma(\rho + \kappa)}{(\rho + R_1)(\rho + R_2)} e^{\rho u},
\end{aligned}$$

where $\varsigma = \phi'_{\infty,s}(b)/v_2'(b)$.

Now, let $c = 1.1$, $\lambda = 1$, $\kappa = 1$, $\sigma = 0.5$, $\delta = 0.05$, $b = 10$. The roots of equation (41) are: $\rho = 0.1812$, $-R_1 = -0.2264$, $-R_2 = -9.7548$. Then

$$\begin{aligned}
\phi_{10,d}(u) &= 0.08 e^{-0.2264 u} + 0.9183 e^{-9.7548 u} + 0.0017 e^{0.1812 u}, \\
\phi_{10,s}(u) &= 0.7006 e^{-0.2264 u} - 0.7155 e^{-9.7548 u} + 0.0148 e^{0.1812 u}, \quad 0 \leq u \leq 10.
\end{aligned}$$

If the discount factor δ is set to be $\delta = 0$, then $\rho = 0$, $R_1 = 0.0823$, $R_2 = 9.7177$. In this case, $\phi_{10,d}(u)$ simplifies to the ruin probability due to oscillations $\Psi_{10,d}(u)$ and $\phi_{10,s}(u)$ simplifies to the ruin probability caused by a claim $\Psi_{10,s}(u)$, which are given as follows.

$$\begin{aligned}\Psi_{10,d}(u) &= 0.1029 + 0.8971 e^{-9.7177 u}, \\ \Psi_{10,s}(u) &= 0.8971 - 0.8971 e^{-9.7177 u}, \quad 0 \leq u \leq 10.\end{aligned}$$

The total probability of ruin $\Psi_{10}(u) = \Psi_{10,d}(u) + \Psi_{10,s}(u) = 1$, for all $0 \leq u \leq 10$, this is because ruin is certain under the constant barrier strategy.

6 Concluding Remarks

We have shown that the two types of expected discounted penalty functions (due to oscillations and caused by a claim), for the diffusion perturbed compound Poisson risk model in presence of a constant dividend barrier, can be expressed in terms of that in the corresponding risk model without a barrier. These results can be used to analyzed some ruin related quantities, e.g., the surplus before ruin, the deficit at ruin, and the time of ruin, as did in Tsai (2001, 2003) and Tsai and Willmot (2002b), for the classical risk model perturbed by diffusion.

The results can be extended to the preturbed Sparre Andersen risk models with Erlang interclaim times in presence of of a constant dividend barrier.

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