

The Moments of the Present Value of Total Dividends Under Stochastic Interest Rates

Abstract

We study the moments of the present value of total dividend payments in the compound binomial risk model in the presence of a constant dividend barrier. In the evaluation of the present value of dividends, the interest rates are assumed to follow a Markov chain with finite state space. A system of difference equations with certain boundary conditions for the moments of the present value of total dividend payments prior to ruin is derived and solved. Explicit results can be obtained when the claim sizes are rationally distributed or the claim size distributions have finite support. Numerical results are given for two special claim size distributions: geometric distribution and the degenerate distribution at 2.

Keywords: Compound binomial risk model; Discounted dividend payments; Stochastic interest rate; Markov chain; Time of ruin

1 Introduction

Consider the classical compound binomial risk model

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i, \quad n = 1, 2, \dots,$$

where $u \in \mathbb{N}$ is the initial surplus. The X_i 's are i.i.d. positive integer-valued random variables with common probability function (p.f.) $p(x) = P(X = x)$, for $x = 1, 2, \dots$, denoting claim sizes. Denote by $\mu_k = E[X^k]$ the k -th moment of X and by $\hat{p}(s) = \sum_{i=1}^{\infty} s^i p(i)$, $s \in \mathbb{C}$, its p.g.f.. The counting process $\{N(n); n \in \mathbb{N}\}$ denotes the number of claims up to time n and is defined as $N(n) = \max\{k : W_1 + W_2 + \dots + W_k \leq n\}$, where the claim inter arrival times W_i 's are assumed to be i.i.d. positive integer-valued random variables with common probability function $k(x) = q(1 - q)^{x-1}$, for $x = 1, 2, \dots$ and $0 < q < 1$. Further assume that $\{W_i; i \in \mathbb{N}^+\}$ and $\{X_i; i \in \mathbb{N}^+\}$ are independent, and $E(W) = (1 + \eta)E(X) = (1 + \eta)\mu$, in order to have a positive loading factor.

The compound binomial risk model, as a discrete analogue of the classical compound Poisson risk model, was first proposed by Gerber (1988) and further studied by a number of papers, see Pavlova and Willmot (2004) and references therein for details. Extensions to the compound binomial model can be found in Yuen and Guo (2001), Cossette et al. (2003) and Li (2005a, b).

In this paper, a barrier strategy is considered by assuming that there is a horizontal barrier of level $b \geq u$ and $b \in \mathbb{N}^+$: at the end of each period (immediately after possible claim settlement), when the surplus is greater than level b , dividends

are paid such that the surplus stays at level b until it becomes less than b by a claim. It implies that a dividend of 1 is payable at time n only if the surplus was at b at time $n - 1$ and there are no claims at time n , for $n = 1, 2, \dots$. Let $U_b(n)$ be the modified surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy. Then

$$U_b(n) = u + n - \sum_{i=1}^{N(n)} X_i - D(n), \quad n = 1, 2, \dots, \quad (1)$$

where $D(n)$ is the sum of the total dividend payments in the first n periods, with the definition

$$D(n) = \sum_{i=1}^n D_i,$$

with

$$D_i = \max\{U_b(i) - b, 0\}$$

being the amount of dividend paid out at the end of period i . Define $T_{u,b} = \inf\{n \in \mathbb{N} : U_b(n) < 0\}$ to be the time of ruin.

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies for a compound Poisson risk process have been studied in a number of papers and books. The main focus is on optimal dividend payouts and problems associated with time of ruin, under various barrier strategies and other economic conditions. Claramunt et al. (2003) calculate the expected present value of dividends in a discrete time risk model with a barrier dividend strategy. Dickson and Waters (2004) show how to use the compound binomial risk model to approximate the classical compound Poisson risk model in

calculating the moments of discounted dividend payments. Tan and Yang (2006) study the ruin probability and the distribution of the surplus before ruin and the deficit at ruin in the compound binomial risk model with randomized decisions on paying dividends in which the dividend of 1 unit is paid with certain probability when the surplus process is greater than a constant barrier. Landriault (2007) studies the expected discounted penalty function for the compound binomial risk model under the Markov chain interest rates.

In actuarial literature, it is well known that the optimal dividend strategy which maximizes the expectation of the discounted dividends until ruin is a barrier strategy in the classical compound Poisson risk model and its discrete counterpart, the compound binomial risk model, if the force of interest or the discount factor per period is assumed to be a constant.

In this note, we assume that the interest rates $\{R_n, n \in \mathbb{N}\}$ with R_n being the interest rate in the interval $(n, n+1]$ follow a time homogenous Markov chain with finite state space $\{r_1, r_2, \dots, r_m\}$. The one period transition probability matrix is given by $\mathbf{P} = (p_{i,j})_{i,j=1}^m$, where $p_{i,j} = \mathbb{P}(R_{n+1} = j | R_n = i)$, for $n \in \mathbb{N}$. The one period discount factors are denoted by v_1, v_2, \dots , respectively, where $v_i = 1/(1 + r_i)$.

Under the interest rate model described above, the present value of total dividends until time of ruin $T_{u,b}$ given that the initial surplus is u is denoted by

$$D_{u,b} = \sum_{k=1}^{T_{u,b}} D_k \left(\prod_{i=1}^k \frac{1}{1 + R_i} \right), \quad u = 0, 1, 2, \dots, b.$$

Define

$$V_i(u; b) = \mathbb{E} [D_{u,b} | R_0 = i], \quad i = 1, 2, \dots, m, u = 0, 1, 2, \dots, b$$

to be the expected present value of total dividend payments up to the time of ruin given that the initial interest rate $R_0 = r_i$.

The moments and distributions of present values of stochastic or deterministic payment streams under stochastic interest rates have been studied in a lot of actuarial papers, including Boyle (1976), Waters (1978), Panjer and Bellhouse (1980), Dufresne (1989, 1990, 1992, 2007), Parker (1994a, b), Norberg (1995a, b), and references therein.

The aim of this note is to calculate $V_i(u; b)$ for some special claim amount distributions so to determine whether the barrier strategy is still a barrier strategy.

2 A system of difference equations with boundary conditions

In this section, we will show that $V_i(u; b), i = 1, 2, \dots, m$, satisfy a system of difference equations with certain boundary conditions as follows.

Conditioning on the occurrence (or not) of a claim in the first period, the amount of the claim, and the interest rate at time 1, we have for $u = 0, 1, 2, \dots, b-1$ and $i = 1, 2, \dots, m$ that

$$V_i(u; b) = v_i(1 - q) \sum_{j=1}^m p_{i,j} V_j(u + 1; b) + v_i q \sum_{j=1}^m p_{i,j} \sum_{k=1}^{u+1} V_j(u + 1 - k; b) p(k). \quad (2)$$

When $u = b$, we have for $i = 1, 2, \dots, m$ that

$$V_i(b; b) = v_i(1 - q) \left[1 + \sum_{j=1}^m p_{i,j} V_j(b; b) \right] + v_i q \sum_{j=1}^m p_{i,j} \sum_{k=1}^{b+1} V_j(b+1-k; b) p(k). \quad (3)$$

Now let $W_1(u), W_2(u), \dots, W_m(u)$ satisfy the following difference equations:

$$W_i(u) = v_i(1-q) \sum_{j=1}^m p_{i,j} W_j(u+1) + v_i q \sum_{j=1}^m p_{i,j} \sum_{k=1}^{u+1} W_j(u+1-k) p(k), \quad u \in \mathbb{N}. \quad (4)$$

The solutions of (4) are uniquely determined by the initial conditions $W_i(0)$ for $i = 1, 2, \dots, m$. For an integer value $1 \leq j \leq m$, Let $W_{1,j}(u), W_{2,j}, \dots, W_{m,j}(u)$ be a particular solution of (4) with the initial conditions $W_{i,j}(0) = I(i = j)$, where $I(\cdot)$ is the indicator function. Then the general solution of (4) is of the form:

$$W_i(u) = \sum_{j=1}^m W_i(0) W_{i,j}(u), \quad u \in \mathbb{N}, i = 1, 2, \dots, m.$$

It follows that the solutions to (2) with boundary conditions (3) can be expressed as

$$V_i(u; b) = \sum_{j=1}^m V_j(0; b) W_{i,j}(u), \quad i = 1, 2, \dots, m, u = 0, 1, \dots, b,$$

or in matrix notation

$$\vec{\mathbf{V}}(u; b) = \mathbf{W}(u) \vec{\mathbf{V}}(0; b), \quad u = 0, 1, 2, \dots, b, \quad (5)$$

where $\vec{\mathbf{V}}(u; b) = (V_1(u; b), V_2(u; b), \dots, V_m(u; b))^T$ is an $m \times 1$ column vector and $\mathbf{W}(u) = (W_{i,j}(u))_{i,j=1}^m$ is an $m \times m$ matrix. The value of vector $\vec{\mathbf{V}}(0; b)$ can be obtained from the m boundary conditions in (3) which can be rewritten in the following matrix form

$$[\mathbf{I} - (1 - q - qp(1))\mathbf{vP}] \vec{\mathbf{V}}(b; b) = (1 - q) \vec{\mathbf{v}} + q\mathbf{vP} \sum_{k=1}^b \vec{\mathbf{V}}(b - k; b) p(k + 1), \quad (6)$$

where $\mathbf{v} = \text{diag}(v_1, v_2, \dots, v_m)$, $\vec{\mathbf{v}} = (v_1, v_2, \dots, v_m)^T$ is an $m \times 1$ column vector, and \mathbf{I} is the $m \times m$ identity matrix. Plunging (5) into (6) yields

$$\begin{aligned} \vec{\mathbf{V}}(0; b) &= (1 - q) \left\{ [\mathbf{I} - (1 - q - qp(1))\mathbf{vP}] \mathbf{W}(b) \right. \\ &\quad \left. - q\mathbf{vP} \sum_{k=1}^b \mathbf{W}(b - k)p(k + 1) \right\}^{-1} \vec{\mathbf{v}}. \end{aligned} \quad (7)$$

Then we have the following matrix factorization formula for $\vec{\mathbf{V}}(u; b)$:

$$\begin{aligned} \vec{\mathbf{V}}(u; b) &= (1 - q)\mathbf{W}(u) \left\{ [\mathbf{I} - (1 - q - qp(1))\mathbf{vP}] \mathbf{W}(b) \right. \\ &\quad \left. - q\mathbf{vP} \sum_{k=1}^b \mathbf{W}(b - k)p(k + 1) \right\}^{-1} \vec{\mathbf{v}}, \quad u = 0, 1, 2, \dots, b. \end{aligned} \quad (8)$$

Remarks:

1. When $m = 1$, that is to say, $r_1 = r_2 = \dots = r_m = r$, that is to say, the interest rate is constant r in each period, then the expected present value of total dividend payments up to ruin, $V(u; b)$, with the definition

$$V(u; b) = \mathbb{E} \left[\sum_{k=1}^{T_b} \frac{D_k}{(1 + r)^k} \mid U_b(0) = u \right], \quad u = 0, 1, 2, \dots, b,$$

can be expressed as

$$V(u; b) = \frac{v(1 - q)W(u)}{[1 - v(1 - q - qp(1))]W(b) - vq \sum_{k=1}^b W(b - k)p(k + 1)},$$

where $W(u)$ satisfies the following difference equation:

$$W(u) = v(1 - q)W(u + 1) + vq \sum_{k=1}^{u+1} W(u + 1 - k)p(k), \quad u = 0, 1, 2, \dots,$$

with $W(0) = 1$ and $v = 1/(1 + r)$.

From the expression of $V(u; b)$, we can see that the optimal dividend barrier b^* which maximizes $V(u; b)$ is independent of u .

2. If $m > 1$, it follows from (8) that, for a fixed i and u , the optimal dividend barrier level b^* which maximizes $V_i(u; b)$ depends on both u and i .

3 Higher Moments of $D_{u,b}$

In this section, we study the higher moment of $D_{u,b}$. Define $V_{i,n}(u; b) = \mathbb{E}[D_{u,b}^n | R_0 = r_i]$, for $n \in \mathbb{N}^+$, $i = 1, 2, \dots, m$, and $u = 0, 1, 2, \dots, b$, to be the n -th moment of the present value of dividend payments up to time of ruin, given that the initial interest rate is r_i . Clearly, $V_{i,1}(u; b) = V_i(u; b)$. Conditioning on the events which can occur in the first time period, we have for $u = 0, 1, 2, \dots, b-1$ that

$$\begin{aligned} V_{i,n}(u; b) &= (1-q)v_i^n \sum_{j=1}^m p_{i,j} V_{j,n}(u+1; b) \\ &\quad + qv_i^n \sum_{j=1}^m p_{i,j} \sum_{k=1}^{u+1} V_{j,n}(u+1-k; b)p(k), \quad n \in \mathbb{N}^+. \end{aligned} \quad (9)$$

For $u = b$,

$$\begin{aligned} V_{i,n}(b; b) &= (1-q)v_i^n \sum_{k=0}^n \binom{n}{k} \sum_{j=1}^m p_{i,j} V_{j,k}(b; b) \\ &\quad + qv_i^n \sum_{j=1}^m p_{i,j} \sum_{k=1}^{b+1} V_{j,n}(b+1-k; b)p(k), \quad n \in \mathbb{N}^+. \end{aligned} \quad (10)$$

For $k = 1, 2, \dots, m$, let $W_{1,k}(u; n), W_{2,k}(u; n), \dots, W_{m,k}(u; n)$, with initial conditions $W_{i,k}(0; n) = I(i = k)$, be the solutions of the following equations:

$$\begin{aligned} W_{i,k}(u; n) &= v_i^n (1-q) \sum_{j=1}^m p_{i,j} W_{j,k}(u+1; n), \quad i, k = 1, 2, \dots, m \\ &\quad + v_i^n q \sum_{j=1}^m p_{i,j} \sum_{x=1}^{u+1} W_{j,k}(u+1-x; n)p(x). \end{aligned} \quad (11)$$

Then

$$V_{i,n}(u; b) = \sum_{j=1}^m V_{j,n}(0; b) W_{i,j}(u; n), \quad i = 1, 2, \dots, m, n \in \mathbb{N}^+.$$

In matrix notation,

$$\mathbf{V}_n(u; b) = \mathbf{W}_n(u) \vec{\mathbf{V}}_n(0; b), \quad (12)$$

where $\vec{\mathbf{V}}_n(u; b) = (V_{1,n}(u; b), V_{2,n}(u; b), \dots, V_{m,n}(u; b))^T$ and $\mathbf{W}_n(u) = (W_{i,k}(u; n))_{i,k=1}^m$.

Noting that $\vec{\mathbf{V}}_1(u; b) = \vec{\mathbf{V}}(u; b)$ and $\mathbf{W}_1(u) = \mathbf{W}(u)$. The unknown vector $\vec{\mathbf{V}}_n(0; b)$ can be determined by (10) with a matrix form:

$$\begin{aligned} [\mathbf{I} - (1 - q - qp(1))\mathbf{v}_n\mathbf{P}] \vec{\mathbf{V}}_n(b; b) &= (1 - q)\mathbf{v}_n\mathbf{P} \sum_{k=0}^{n-1} \binom{n}{k} \vec{\mathbf{V}}_k(b; b) \\ &+ q\mathbf{v}_n\mathbf{P} \sum_{k=1}^b \vec{\mathbf{V}}_n(b - k; b)p(k + 1), \end{aligned} \quad (13)$$

where $\mathbf{v}_n = \text{diag}(v_1^n, v_2^n, \dots, v_m^n)$. Setting $u = b$ in (12) and plunging it into (13) give

$$\begin{aligned} \vec{\mathbf{V}}_n(0; b) &= (1 - q) \left\{ [\mathbf{I} - (1 - q - qp(1))\mathbf{v}_n\mathbf{P}] \mathbf{W}_n(b) \right. \\ &\quad \left. - q\mathbf{v}_n\mathbf{P} \sum_{k=1}^b \mathbf{W}_n(b - k)p(k + 1) \right\}^{-1} \mathbf{v}_n\mathbf{P} \sum_{k=0}^{n-1} \binom{n}{k} \vec{\mathbf{V}}_k(b; b), \end{aligned} \quad (14)$$

where $\vec{\mathbf{V}}_0(b; b) = (1, 1, \dots, 1)^T$ is an $m \times 1$ column vector and $\vec{\mathbf{V}}_k(b; b)$ for $k = 1, 2, \dots, n - 1$ can be calculated repeatedly using (12) by setting $u = b$ and $n = k$.

4 Numerical Illustrations

We now apply the tool of generating functions to find the particular solutions $W_{i,j}(u; n)$ to the particular equations (11) for some particular claim amount dis-

tributions.

Let $\hat{W}_{i,j}(s; n) = \sum_{u=0}^{\infty} s^u W_{i,j}(u; n)$ be the generating function of $W_{i,j}(u; n)$.

Taking generating functions on both sides of equations (11) yields

$$\begin{aligned} s\hat{W}_{i,j}(s; n) &= v_i^n(1-q) \sum_{k=1}^m p_{i,k} [\hat{W}_{k,j}(s; n) - W_{k,j}(0; n)] \\ &\quad + v_i^n q \sum_{k=1}^m p_{i,k} \hat{W}_{k,j}(s; n) \hat{p}(s), \quad i, j = 1, 2, \dots, m, \end{aligned} \quad (15)$$

where $W_{k,j}(0; n) = I(k = j)$. In matrix form,

$$\hat{\mathbf{W}}_n(s) = -(1-q)[\mathbf{A}_n(s)]^{-1} \mathbf{v}_n \mathbf{P} = -\frac{(1-q)\mathbf{A}_n^*(s)\mathbf{v}_n \mathbf{P}}{\det[\mathbf{A}_n(s)]}, \quad (16)$$

where $\hat{\mathbf{W}}_n(s) = (\hat{W}_{i,j}(s; n))_{i,j=1}^m$, $\mathbf{v}_n = \text{diag}(v_1^n, v_2^n, \dots, v_m^n)$, $\mathbf{A}_n(s) = s\mathbf{I} - (1-q + q\hat{p}(s))\mathbf{v}_n \mathbf{P}$, and $\mathbf{A}_n^*(s)$ is the adjoint matrix of $\mathbf{A}_n(s)$.

$\hat{\mathbf{W}}_n(s)$ can be inverted if each element is a rational function, while each element is a rational function if and only if claim size distribution has a rational probability generating function or the claim size distribution has a finite support so the probability generating function is a polynomial, as will be seen in the following two examples.

4.1 Constant Claim Amounts

In this subsection, we consider the De Finetti's original model: the claim amounts are of constant size at 2, i.e., $p(x) = I(x = 2)$ and $\hat{p}(s) = s^2$. Clearly, $\det[\mathbf{A}_n(s)]$ is a polynomial of degree $2m$ and each element of $\mathbf{A}_n^*(s)$ is a polynomial of degree

$2m - 2$. Let $\rho_1, \rho_2, \dots, \rho_{2m}$ be the roots of equation $\det[\mathbf{A}_n(s)] = 0$. Then

$$\det[\mathbf{A}_n(s)] = a_{2m-1}^{(n)} \prod_{i=1}^{2m} (s - \rho_i),$$

where $a_{2m-1}^{(n)}$ is the leading coefficient of the polynomial $\det[\mathbf{A}_n(s)]$. For simplicity, we assume that $\rho_1, \rho_2, \dots, \rho_{2m}$ are distinct. It follows from partial fractions that (16) can be rewritten as

$$\hat{\mathbf{W}}_n(s) = (1 - q) \left[\sum_{i=1}^{2m} \frac{\mathbf{M}_i^{(n)}}{\rho_i - s} \right] \mathbf{v}_n \mathbf{P}, \quad (17)$$

where

$$\mathbf{M}_i^{(n)} = \frac{\mathbf{A}_n^*(\rho_i)}{a_{2m-1}^{(n)} \prod_{j=1, j \neq i}^{2m} (\rho_i - \rho_j)}, \quad i = 1, 2, \dots, 2m,$$

is an $m \times m$ matrix. Inverting (17) yields

$$\mathbf{W}_n(u) = (1 - q) \left[\sum_{i=1}^{2m} \mathbf{M}_i^{(n)} \rho_i^{-(u+1)} \right] \mathbf{v}_n \mathbf{P}, \quad n \in \mathbb{N}^+.$$

Once $\mathbf{W}_n(u)$ is obtained, $\mathbf{V}_n(u; b)$ can be calculated using (12).

Example 1 In this example, we assume that the interest rates have three possible values: $r_1 = 2\%$, $r_2 = 5\%$, $r_3 = 10\%$. The transition probability matrix \mathbf{P} is

$$\mathbf{P} = \begin{pmatrix} 0.9 & 0.08 & 0.02 \\ 0.13 & 0.8 & 0.07 \\ 0.05 & 0.3 & 0.65 \end{pmatrix}.$$

We set the premium loading factor to be $\eta = 0.2$ so $q = 5/12$. Table 1 gives the values for $V_i(u; 10)$ for $u = 0, 1, 2, 3, 4, 6, 8, 10$, $b = 1, 2, 3, 4, 6, \dots, 12$ and $i = 1, 2, 3$.

Table 1: $V_i(u; b)$ for $i = 1, 2, 3$

$u \setminus b$	1	2	3	4	6	8	10	12	R_0
0	1.576	1.684	1.653	1.536	1.206	0.815	0.519	0.159	$r_1 = 2\%$
	1.423	1.483	1.439	1.328	1.037	0.751	0.444	0.135	$r_2 = 5\%$
	1.276	1.310	1.263	1.162	0.906	0.655	0.387	0.117	$r_3 = 10\%$
1	2.784	2.984	2.936	2.727	2.144	1.556	0.923	0.283	$r_1 = 2\%$
	2.547	2.647	2.565	2.364	1.845	1.335	0.789	0.240	$r_2 = 5\%$
	2.312	2.349	2.257	2.072	1.612	1.164	0.687	0.208	$r_3 = 10\%$
2		4.080	4.026	3.746	2.948	2.141	1.271	0.391	$r_1 = 2\%$
		3.668	3.545	3.262	2.541	1.837	1.085	0.329	$r_2 = 5\%$
		3.293	3.133	2.865	2.220	1.062	0.944	0.286	$r_3 = 10\%$
3			5.038	4.699	3.706	2.695	1.600	0.493	$r_1 = 2\%$
			4.492	4.122	3.202	2.312	1.363	0.413	$r_2 = 5\%$
			4.012	3.637	2.802	2.016	1.185	0.357	$r_3 = 10\%$
4				5.656	4.478	3.262	1.941	0.600	$r_1 = 2\%$
				5.022	3.865	2.798	1.645	0.496	$r_2 = 5\%$
				4.474	3.407	2.441	1.428	0.426	$r_3 = 10\%$
6					6.209	4.557	2.733	0.858	$r_1 = 2\%$
					5.494	3.921	2.280	0.671	$r_2 = 5\%$
					4.885	3.427	1.968	0.564	$r_3 = 10\%$
8						6.233	3.815	1.246	$r_1 = 2\%$
						5.489	3.104	0.855	$r_2 = 5\%$
						4.873	2.657	0.676	$r_3 = 10\%$
10							5.460	1.955	$r_1 = 2\%$
							4.693	1.095	$r_2 = 5\%$
							4.135	0.768	$r_3 = 10\%$

From Table 1, we can see that for $u = 0, 1, 2$, the optimal dividend barrier b^* which maximizes $V_i(0; b)$ is 2 for $i = 1, 2, 3$. For $u \geq 3$, the optimal dividend barrier b^* is equal to the initial surplus u for $i = 1, 2, 3$. Furthermore, as expected, $V_i(u; b)$ is increasing in u for fixed b and i .

4.2 Geometric Claim Amounts

In this subsection, we assume that the claim amounts are zero-truncated geometrically distributed with $p(x) = (1 - \theta)\theta^{x-1}$, $x = 1, 2, \dots$, $0 < \theta < 1$ and $\hat{p}(s) = (1 - \theta)s/(1 - \theta s)$. Now that

$$\begin{aligned} (s - 1/\theta)^m \det[\mathbf{A}_n(s)] &= \det[(s - 1/\theta)\mathbf{A}_n(s)] \\ &= \det \left[\left(s^2 - \frac{s}{\theta} \right) \mathbf{I} + \left(\frac{q - \theta}{\theta} s + \frac{1 - q}{\theta} \right) \mathbf{v}_n \mathbf{P} \right] \end{aligned}$$

is a polynomial of degree $2m$ with leading coefficient 1 and it must have $2m$ zeros, say, $\rho_1, \rho_2, \dots, \rho_{2m}$, i.e., $(s - 1/\theta)^n \det[\mathbf{A}_n(s)] = \sum_{i=1}^{2m} (s - \rho_i)$. Multiplying both the denominator and numerator of (16) by $(s - 1/\theta)^m$, noting that each element of $(s - 1/\theta)^m \mathbf{A}_n^*(s)$ is a polynomial of degree $2m - 1$, and using partial fractions, we can rewritten (16) as

$$\hat{\mathbf{W}}_n(s) = -(1 - q) \frac{(s - 1/\theta)^m \mathbf{A}_n^*(s)}{\prod_{i=1}^{2m} (s - \rho_i)} \mathbf{v}_n \mathbf{P} = (1 - q) \sum_{i=1}^{2m} \frac{\mathbf{B}_i^{(n)}}{\rho_i - s} \mathbf{v}_n \mathbf{P}, \quad (18)$$

where

$$\mathbf{B}_i^{(n)} = \frac{(\rho_i - 1/\theta)^m \mathbf{A}_n^*(\rho_i)}{\prod_{j=1, j \neq i}^{2m} (\rho_i - \rho_j)}, \quad i = 1, 2, \dots, 2m,$$

are $m \times m$ matrices. Inverting it gives

$$\mathbf{W}_n(u) = (1 - q) \left[\sum_{i=1}^{2m} \mathbf{B}_i^{(n)} \rho_i^{-(u+1)} \right] \mathbf{v}_n \mathbf{P}, \quad n \in \mathbb{N}^+, u \in \mathbb{N}.$$

Example 2 In this example, we assume that interest rates take two possible values: $r_1 = 4\%$ and $r_2 = 8\%$. The transition probability \mathbf{P} is

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}.$$

Table 2: Means and variances of $D_{u,10}$ for $m = 2$ and $m = 1$

u	$m = 2$	$R_0 = r_1$	$m = 2$	$R_0 = r_2$	$m = 1$	$R_0 = \frac{1-v}{v}$
	mean	variance	mean	variance	mean	variance
0	0.599	1.313	0.556	1.127	0.568	1.154
1	0.766	1.688	0.710	1.451	0.726	1.487
2	0.949	2.113	0.879	1.816	0.899	1.865
3	1.152	2.599	1.068	2.233	1.093	2.298
4	1.379	3.162	1.279	2.715	1.309	2.800
5	1.636	3.809	1.517	3.273	1.554	3.384
6	1.927	4.558	1.787	3.919	1.833	4.059
7	2.258	5.416	2.096	4.658	2.150	4.840
8	2.635	6.389	2.449	5.507	2.513	5.732
9	3.064	7.472	2.857	6.461	2.930	6.737
10	3.547	8.652	3.331	7.525	3.409	7.848

The stationary probability $\boldsymbol{\pi} = (\pi_1, \pi_2)$ associated with \mathbf{P} is $\boldsymbol{\pi} = (0.4, 0.6)$. The claim amounts are zero-truncated geometric distributed with mean 4, e.g., $p(x) = 0.25 \times 0.75^{x-1}$, $x = 1, 2, \dots$. The loading factor $\eta = 0.2$ so that $q = 1/4.8$. Table 2 gives the means and variances of $D_{u,10}$ given the initial interest rates r_i for $i = 1, 2$, together with means and variances of $D_{u,10}$ calculated based on the constant discount factor which is equal to $v = \pi_1 v_1 + \pi_2 v_2 = 0.94$, the stationary discount factor in the Markovian environment.

From Table 2, we can observe that both mean and variance of $D_{u,10}$ are increasing in u for $m = 2$ and $m = 1$. $V(u; 10)$ evaluated at the constant discount factor $v = 0.94$ is between $V_1(u; 10)$ and $V_2(u; 10)$ for each u , so is true for the variance of $D_{u,10}$.

Concluding Remarks

This paper studies how to compute the expected discounted dividend payments prior to ruin under stochastic interest rates in the compound binomial risk model. The results show that, unlike the constant interest case, the optimal dividend barrier level depends on both the initial surplus and the initial interest rate. The formulas are readily programmable in practice and they can be used to approximate the corresponding results in the compound Poisson risk model under stochastic interest rates.

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