INTERPOLATION SCHEMES IN THE DISPLACED-DIFFUSION LIBOR MARKET MODEL AND THE EFFICIENT COMPUTATION OF PRICES AND GREEKS FOR CALLABLE RANGE ACCRUALS

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Abstract. We introduce a new arbitrage-free interpolation scheme for the displaced-diffusion LIBOR market model. Using this new extension, and the Piterbarg interpolation scheme, we study the simulation of range accrual coupons when valuing callable range accruals in the displaced-diffusion LIBOR market model. We introduce a number of new improvements that lead to significant efficiency improvements, and explain how to apply the adjoint-improved pathwise method to calculate deltas and vegas under the new improvements, which was not previously possible for callable range accruals. One new improvement is based on using a Brownian-bridge-type approach to simulating the range accrual coupons. We consider a variety of examples, including when the reference rate is a LIBOR rate, when it is a spread between swap rates, and when the multiplier for the range accrual coupon is stochastic.

1. Introduction

An exotic interest rate derivative which has become very popular is the callable range accrual or floating range note. For such a product, the size of each coupon depends on the number of days during the accrual period where a reference rate is within a pre-specified range. The reference rate is commonly a LIBOR rate, but can also be a swap rate or the spread between two interest rates. As such, the coupon payments are made up of a number of individual digital options.

While callable range accruals are very popular, the accurate pricing and efficient Greek calculations of these products provide a number of challenges, which have not been fully overcome. In particular, we need to observe the yield curve on a daily basis to determine the pay-off, we need to cope with Bermudan optionality and we want to be able to calibrate to a large number of hedging instruments. In addition, the discontinuous nature of the coupons makes computing sensitivities tricky.

The displaced diffusion LIBOR market model (DDLMM) is a standard approach to pricing exotic interest rate derivatives with the virtue that it can be calibrated to large numbers of market instruments, and can also achieve skew. See, for example, [1]. However, the Bermudan optionality, the discontinuities and daily observations make it challenging to apply to this problem.

Our objective in this paper is to develop extensions to the DDLMM that allow the prices and Greeks of callable range accrual notes to be computed in a similar fashion and with similar speed to other exotic interest rate derivatives. We introduce a number of innovations and apply our recent results on pricing breakable contracts by Monte Carlo, [5].

In order to avoid evolving thousands of rates, it is necessary to use an interpolation scheme that allows the deduction of non-tenor rates within a simulation. Whilst schemes have previously been suggested by Piterbarg [22] and Schlögl [24], we will see that if one wishes the scheme to have no internal arbitrages, have stochastic rates at all times before reset and have rates that remain positive (when displacements are zero), it is necessary to introduce a new scheme. We introduce such a scheme and show how the values of rates within it can be approximately computed.

In order to avoid evolution to every observation date which would be very slow, we employ two devices. The first is to clump observations together. We will see that provided the clumping is done symmetrically, it is very accurate. The second is to compute expectations of the accrual coupons conditional on the values of rates during the simulation. We will, in fact, do so in two different ways. For the first, we compute conditional expectations at the start of accrual period. For the second, we also condition on the values of rates at the end of the accrual period. This Brownian bridging approach has the additional virtue that floating coupons are easy to value, and that the sizes of approximation errors are reduced. Since both methods replace discontinuous coupons by expectations which are smooth functions of their parameters, they allow the application of the pathwise method for computing Greeks. See [23] or [11].

We will show that our techniques are effective for a variety of callable range accrual contracts; in particular, we will study the cases when the reference rate is a LIBOR rates and when it is a difference of two swap-rates. We will also allow the coupon to be determined by a floating rate in arrears.

The structure of the paper is as follows. In Section 2 we briefly recap the DDLMM. In Section 3 we provide some additional notation and introduce our main examples. Various interpolation methodologies are introduced in Section 4 and justified. In Section 5, we show how the simulation can be effectively implemented, and in Section 6 we provide numerical results. These demonstrate significant efficiency improvements for both prices and Greeks.

2. The Displaced-Diffusion LIBOR Market Model

Since it was given a firm theoretical base in the fundamental papers by Brace, Gatarek and Musiela, [7], Musiela and Rutkowski, [21], and Jamshidian, [15], the LIBOR market model has become a very popular method for pricing exotic interest rate derivatives. It is based on the idea of evolving the yield curve directly through a set of discrete market observable forward rates, rather than indirectly through use of a single non-observable quantity which is assumed to drive the yield curve. A distinct advantage of this approach is the ability to easily calibrate to a large number of simpler financial contracts, often used in the hedging process for exotics; see [1].

Suppose we have a set of tenor dates, $0 = T_0 < T_1 < \ldots < T_{n+1}$. Let $\delta_j = T_{j+1} - T_j$, and let $P(t, T)$ denote the price at time t of a zero-coupon bond paying one at its maturity, T. For ease of exposition, we will assume throughout that $\delta_j = \delta$ is constant for all j. Let $L(t, T)$ denote the value of the forward rate from T to $T + \delta$ as of time t. Using no-arbitrage arguments,

$$L(t,T) = \frac{\frac{P(t,T)}{P(t,T+\delta)} - 1}{\delta},$$

where $L(t, T)$ is said to reset at time T, after which point it is assumed that it does not change in value. To simplify notation, let

$$L_j(t) = L(t, T_j) \text{ and } P_j(t) = P(t, T_j).$$

We work in the *spot LIBOR measure*, which corresponds to using the discretely-compounded money market account as numeraire, because this has certain practical advantages; see [16]. This numeraire is made up of an initial portfolio of one zero-coupon bond expiring at time T_1, with the proceeds received when each bond expires being reinvested in bonds expiring at the next tenor date, up until T_{n+1}. More formally, the value of the numeraire portfolio at time t will be,

$$N(t) = P(t, T_{\eta(t)}) \prod_{i=1}^{\eta(t)-1} (1 + \delta L_i(T_i)),$$

where $\eta(t)$ is the unique integer satisfying

$$T_{\eta(t)-1} \leq t < T_{\eta(t)},$$

and thus gives the index of the next forward rate to reset.

Under the DDLMM, the forward rates that make up the state variables of the model are assumed to be driven by the following process

$$dL_i(t) = \mu_i(L(t), t)(L_i(t) + \alpha_i)dt + \sigma_i(t)(L_i(t) + \alpha_i)dW_i(t), \tag{2.1}$$

where the $\sigma_i(t)$'s are deterministic functions of time, the α_i's are constant displacement coefficients, the W_i's are standard Brownian motions under the spot LIBOR martingale measure, $L(t)$ denotes the vector of state variables at time t, and the $\mu_i's$ are uniquely determined by no-arbitrage requirements. It is assumed that W_i and W_j have correlation $\rho_{i,j}$, and throughout $\{\mathcal{F}_t\}_{t\geq 0}$ will be used to denote the filtration generated by the driving Brownian motions. Primarily to ease notation, but also to maintain time-homogeneity, we assume that the displacement coefficients are the same for all forward rates, that is

$$\alpha_i = \alpha,$$

for all values of i. However, where it is not obvious, we will explain how to apply the techniques discussed when the displacements of different forward rates are not equal. To further ease notation, let

$$\tilde{L}_j(t) = L_j(t) + \alpha.$$

The requirement that the discounted price processes of the fundamental tradeable assets, that is the zero-coupon bonds associated to each tenor date, be martingales in the pricing measure, dictates that the drift term is

$$\mu_i(L(t), t) = \sum_{j=\eta(t)}^{i} \frac{\tilde{L}_j(t)\delta}{1 + L_j(t)\delta} \sigma_{i,j}(t),$$

see [8], where

$$\sigma_{i,j}(t) = \sigma_i(t)\sigma_j(t)\rho_{i,j},$$

and therefore denotes the instantaneous covariance between the logarithms of $(L(t, T_i) + \alpha)$ and $(L(t, T_j) + \alpha)$. In addition, let $C_{i,j}(s, t)$ denote the integral of $\sigma_{i,j}(.)$ over the interval (s, t). If $t < s$ we make the convention that this equals zero. We also let $V_j(s, t) = C_{j,j}(s, t)$.

When developing analytic approximations in the DDLMM, a useful tool is the common drift-freezing approximation (see, for example, [8]). The idea is to freeze all state-dependence in the drifts of (2.1) at the current time, so as to obtain a lognormal approximation to the forward rates. In particular, conditional on $\mathcal{F}_s$, we have

$$dL_i(t) \approx \mu_i(L(s), t)\tilde{L}_i(t)dt + \sigma_i(t)\tilde{L}_i(t)dW_i(t), \tag{2.2}$$

for $t > s$. We will need this approximation at various stages later on.

Displaced-diffusion is used as a simple way to allow for the skews seen in implied caplet volatilities that have long persisted in interest rate markets; see [16]. In particular, the use of displaced-diffusion allows for the wealth of results concerning calibrating and evolving rates in the standard LIBOR market model to be carried over with only minor changes. The model presented collapses to the standard LIBOR market model when $\alpha = 0$.

3. Notation and Products

We consider a swap where the issuer receives the floating LIBOR rate with a natural time-lag and pays a range accrual coupon, S_i, at each tenor date T_i (for $i \geq 2$), until the time of exercise. The range accrual coupon is given by

$$S_i = \frac{Q_i}{n_i} \sum_{j=1}^{n_i} \mathrm{I}_{B_i < R(T_{i-1}+t_j) < U_i},$$

where n_i denotes the number of observation days for the $(i-1)^{\text{th}}$ range accrual coupon, $R(t)$ the value of the underlying reference rate at time t, and $T_{i-1} + t_j$ the time of the j^{th} observation day. Note that Q_i can be stochastic, whereas B_i and U_i will generally be deterministic. However, the methods introduced in this paper can easily handle the case where B_i and U_i are $\mathcal{F}_{T_{i-1}}$-measurable. Also note that this particular set-up has one-sided range accruals as a special case.

We consider three examples. These are chosen to represent the most common coupon structures, as well as the most difficult to handle. However, the techniques introduced can easily be applied or extended to any similar product. In particular, we consider

- Example 1: $R(t) = L(t,t)$, $Q_i = q$. This is the most common coupon structure, where the multiplier, Q_i, is deterministic, and the reference rate is the spot LIBOR rate, denoted by $L(t,t)$.
- Example 2: $R(t) = \mathrm{SR}_{t,x}(t) - \mathrm{SR}_{t,y}(t)$ for $x > y$, $Q_i = q$, where $\mathrm{SR}_{t,x}(s)$ denotes the value of the swap rate starting at time t and running for x accrual periods, as of time s. As such, the reference rate is a spread between constant maturity spot swap rates.
- Example 3: $R(t) = \mathrm{SR}_{t,x}(t) - \mathrm{SR}_{t,y}(t)$ for $x > y$, $Q_i = qL_i(T_i)$. The multiplier is now stochastic, and is proportional to the forward rate resetting at the end of the accrual period. This feature, together with the reference rate, make this particular coupon structure particularly challenging.

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For simplicity, assume that the product can be exercised on each of the tenor dates $T_1, \ldots, T_n$, where the time of exercise is decided by the issuer. We also assume that the rebate received by the issuer upon exercise is zero. Let $\bar{\tau}$ denote a given exercise strategy taking values in the set

$$\{1, 2, \ldots, n, n+1\},$$

representing the set of possible exercise times. Note that we assume an extra exercise time, indexed by $n+1$, where exercise must occur and zero rebate is received upon exercise. This is done to ensure that we have a finite stopping time as an exercise strategy.

For ease of exposition, let $\mathbb{E}_T^t(.) = \mathbb{E}_T(.|\mathcal{F}_t)$, where the subscript is used to denote that the equivalent martingale measure associated with the numeraire asset $P(t,T)$ is being used. If no subscript is present, it is implicitly assumed that the equivalent martingale measure associated with $N(t)$ is being used.

For brevity later on, we introduce the following notation. Let

$$\theta(t) = T_{\eta(t)} - t \text{ and } \vartheta(t) = t - T_{\eta'(t)},$$

with

$$\eta'(t) = \eta(t) - 1.$$

Finally, throughout we will work with the understanding that when $i > k$,

$$\prod_{j=i}^{k} = 1 \text{ and } \sum_{j=i}^{k} = 0.$$

4. Interpolation Methods

Since callable range accruals depend on non-tenor forward rates and swap rates, a method to extend the DDLMM to continuous tenor is needed. In this paper we consider two approaches. The first is a simple method introduced by Piterbarg, [22], that will be briefly described in Section 4.1. For the second method, we use the more complicated, but theoretically appealing framework of Schlögl, [24], extended to the DDLMM. We introduce a new way to apply this framework that results in more realistic interpolations while maintaining desirable properties for the interpolated bond prices. The standard framework is discussed in Section 4.2 and our extension in Section 4.3. For all approaches considered, the extension to continuous tenor is carried out by interpolating zero-coupon bond prices in the maturity dimension using the discrete tenor forward rates at the current time.

In order to compare the different interpolation schemes, we study three properties:

(1) Absence of internal arbitrage: arbitrage opportunities do not exist by trading with the non-tenor bonds.
(2) Positivity: if displacements are zero, that is $\alpha = 0$, then interpolated forward rates are positive. This property precludes arbitrage with cash in the standard LIBOR market model.
(3) Stochasticity: $P(t_1, t_2)$ is stochastic for all $t_1 < t_2$.

4.1. **Piterbarg Interpolation.**

In order to extend the DDLMM to continuous tenor, Piterbarg, [22], assumes that the forward rates applying over periods surrounded by tenor dates are equal to the forward rates associated with the surrounding tenor dates. As a result, the DDLMM can be extended to continuous tenor in three steps,

- $P(t_1, t_2) = (1 + (t_2 - t_1)L_{\eta'(t_1)}(t_1))^{-1}$, for $t_2 \leq T_{\eta(t_1)}$,
- $P_j(t_1) = P_{\eta(t_1)}(t_1) \prod_{k=\eta(t_1)}^{j-1} (1 + \delta L_k(t_1))^{-1}$, for $j \geq \eta(t_1)$,
- $P(t_1, t_2) = P_{\eta'(t_2)}(t_1)(1 + \vartheta(t_2)L_{\eta'(t_2)}(t_1))^{-1}$, for $t_2 > T_{\eta(t_1)}$.

We therefore have

Theorem 4.1. *The Piterbarg interpolation scheme satisfies Positivity.*

Proof. That Positivity holds is obvious from

$$L(t_1, t_2) = \begin{cases} \frac{1}{\delta}\left[\frac{(1+\theta(t_1)L_{\eta'(t_1)}(t_1))(1+\vartheta(t_2+\delta)L_{\eta(t_1)}(t_1))}{1+(t_2-t_1)L_{\eta'(t_1)}(t_1)} - 1\right], & \text{if } t_2 < T_{\eta(t_1)}; \\ \frac{1}{\delta}\left[\frac{(1+\delta L_{\eta'(t_2)}(t_1))(1+\vartheta(t_2+\delta)L_{\eta(t_2)}(t_1))}{1+\vartheta(t_2)L_{\eta'(t_2)}(t_1)} - 1\right], & \text{otherwise,} \end{cases}$$

since the leading term of the numerator in each fraction is at least as big as the denominator and the second term of the numerator in each fraction is greater than 1. □

It is clear that Stochasticity fails under the Piterbarg interpolation scheme from the definition of $P(t_1, t_2)$ for $t_2 \leq T_{\eta(t_1)}$, and we demonstrate that the scheme is internally arbitrageable with the following example.

Suppose we are at time t and $t < t_1 < t_2 < T_{\eta(t)}$, then we can construct two portfolios which always have the same value at t_1, but have different initial values. The first portfolio is made up on one unit of $P(t, t_2)$. In contrast, the second portfolio is made up of

$$P(t_1, t_2) = \frac{1}{1 + (t_2 - t_1) L_{\eta'(t)}(T_{\eta'(t)})}$$

units of $P(t, t_1)$. The second portfolio is then worth less at time t since

$$P(t_1, t_2) P(t, t_1) < P(t, t_2),$$

and both portfolios are worth $P(t_1, t_2)$ at t_1.

4.2. **Schlögl Interpolation.** We can use the interpolation framework introduced in [24] to extend the DDLMM to continuous tenor in a way that precludes arbitrage and is consistent with the assumptions of the DDLMM. While the framework from [24] is used, we introduce a new way to apply it. This is required to maintain a sufficient level of volatility in the interpolated bond prices, while providing a realistic interpolation scheme that precludes arbitrage with cash. In doing so, we also extend Schlögl's framework from the LIBOR market model to the DDLMM.

First, consider the framework in [24]. Define the short bond at a given time to be the zero-coupon bond maturing on the next tenor date. Under the assumptions of the DDLMM, we have the unique processes for the ratios of zero-coupon bond prices under each equivalent martingale measure. Any choice of processes for the absolute zero coupon bond prices that are consistent with these will therefore not introduce arbitrage amongst the zero-coupon bonds by construction, although some choices may imply unrealistic dynamics that lead to arbitrage opportunities outside the model. If an interpolation method is specified for the short bonds at all times, then the processes for the zero-coupon bonds that are consistent with the DDLMM are completely and uniquely determined. As such, we can specify any interpolation method we want for the short bonds at all times, without introducing internal arbitrage and maintaining consistency with the assumptions of the DDLMM. Once this is done, the continuous tenor model is completely specified by no-arbitrage requirements. In particular, the unique arbitrage-free price for an arbitrary zero-coupon bond at any time, $P(t_1, t_2)$ (where $t_1 < t_2 < T_{n+1}$), is given by

$$\frac{P(t_1, t_2)}{P_{\eta(t_2)}(t_1)} = \mathbb{E}^{t_1}_{T_{\eta(t_2)}} \left[\frac{1}{P_{\eta(t_2)}(t_2)} \right], \tag{4.1}$$

where the values of $P_{\eta(t_2)}(t_2)$ and $P_{\eta(t_2)}(t_1)$ are determined by the specified method for interpolating short bonds, and thus so is the $P(t_1, t_2)$. Applying the arbitrage-free interpolation scheme of [24] then reduces to specifying how the short bonds are interpolated at all times.

In [24], an interpolation method for the short-bonds that allows for volatility in their prices is given. Schlögl assumed that

$$P_{\eta(t_1)}(t_1)^{-1} = 1 + \theta(t_1) \left[\xi(t_1) L_{\eta'(t_1)}(T_{\eta'(t_1)}) + (1 - \xi(t_1)) L_{\eta(t_1)}(t_1) \right], \tag{4.2}$$

for some user-specified function $\xi(t)$ that satisfies

$$\lim_{\Delta \searrow 0} \xi(T_i + \Delta) = 1, \tag{4.3}$$

$$\lim_{\Delta \nearrow T_{i+1} - T_i} \xi(T_i + \Delta) = 0, \tag{4.4}$$

for all $i = 0, 1, \ldots, n$. Note that $\xi(t)$ can be used to control the level of volatility in the short bonds; the smaller $\xi(t)$ is, the greater the dependence on $L_{\eta(t_1)}(t_1)$ in (4.2), and the greater the volatility in $P_{\eta(t_1)}(t_1)$. In addition, due to the flexibility in $\xi(t)$, in effect (4.2) specifies a family of assumptions.

Using this assumption in (4.1) gives

$$\frac{P(t_1,t_2)}{P_{\eta(t_2)}(t_1)} = 1+\theta(t_2)\left[\xi(t_2)L_{\eta'(t_2)}(t_1)+(1-\xi(t_2))L_{\eta(t_2)}(t_1)g(t_1,t_2,L_{\eta(t_2)}(t_1))\right],$$

where

$$g(t_1,t_2,L_{\eta(t_2)}(t_1)) = \frac{1+\delta L_{\eta(t_2)}(t_1)e^{V_{\eta(t2)}(t_1,t_2)}}{1+\delta L_{\eta(t_2)}(t_1)}; \tag{4.5}$$

see [24].

We therefore get the following.

Theorem 4.2. *The Schlögl interpolation scheme satifies Stochasticity and has no internal aribtages.*

Proof. Both these properties follow immediately from the definition of non-tenor bonds provided

$$\xi(t) < 1 \text{ for } T_{\eta'(t)} < t < T_{\eta(t)}. \tag{4.6}$$

□

However, this method can lead to negative forward rates when displacements are zero, and is therefore not fully arbitrage free. This follows because it can lead to unrealistic interpolations, especially for strongly increasing forward-rate curves.

To see why this is the case, assume that $t_1 > T_{\eta'(t_2)}$ and (4.6) holds, and consider

$$(1+\delta L(t_1,t_2)) = \frac{(1+\delta L_{\eta(t_2)}(t))\left[1+\theta(t_2)\left\{\xi(t_2)L_{\eta'(t_2)}(T_{\eta'(t_2)})+(1-\xi(t_2))L_{\eta(t_2)}(t_1)\right\}\right]}{\left[1+\theta(t_2+\delta)\left\{\xi(t_2+\delta)L_{\eta(t_2)}(t_1)+(1-\xi(t_2+\delta))L_{\eta(t_2)+1}(t_1)g(t_1,t_2+\delta,L_{\eta(t_2)+1}(t_1))\right\}\right]}.$$

Since the denominator of the fraction above is significantly weighted by $L_{\eta(t_2)+1}(t_1)$, and this is the only place that values of this forward rate appear, large values of $L_{\eta(t_2)+1}(t_1)$ relative to $L_{\eta'(t_2)}(t_1)$ and $L_{\eta(t_2)}(t_1)$ can lead to very small (and possibly negative) values for $L(t_1,t_2)$. Furthermore, Positivity can fail easily. For example, if

$$\delta = 1, \theta(t_2)=\theta(t_2+\delta)=0.6, \xi(t_2)=\xi(t_2+\delta)=0.6,$$
$$L_{\eta'(t_2)}(t_1)=0.02, L_{\eta(t_2)}(t_1)=0.05, L_{\eta(t_2)+1}(t_1)=0.22,$$

then $L(t_1,t_2)<0$ regardless of the volatility structure.

While this may seems like an unrealistic example, when pricing with a Monte Carlo simulation in a martingale measure, such forward rate curves can arise frequently, making the failure of Positivity a significant problem.

4.3. **Extending the Schlögl Interpolation.** Instead of (4.2), we suggest using

$$P_{\eta(t_1)}(t_1)^{-1} = 1+\theta(t_1)\left[\xi(t_1)L_{\eta'(t_1)}(T_{\eta'(t_1)})+(1-\xi(t_1))\left\{\tilde{L}_{\eta'(t_1)}(T_{\eta'(t_1)})\frac{\tilde{L}_{\eta(t_1)}(t_1)}{\tilde{L}_{\eta(t_1)}(T_{\eta'(t_1)})}-\alpha\right\}\right], \tag{4.7}$$

with $\xi(t)$ $(0 \le \xi(t) \le 1)$ being a user-specified function. We view this as something of a multiplicative version of (4.2); by scaling $L_{\eta'(t_1)}(T_{\eta'(t_1)})$ based on the ratio in (4.7), the problem with strongly increasing forward rate curves is mitigated. As such, we will refer to this interpolation scheme as the *multiplicative interpolation scheme*. While it may seem natural to set

$$\xi(t) = 0 \text{ for all } t,$$

in (4.7), we persist with the assumption above to maintain generality. This provides us control of the volatility of interpolated bond prices. In addition, it means that all results presented apply to the basic application of the arbitrage-free interpolation scheme in [24] when

$$\xi(t) = 1 \text{ for all } t.$$

In addition, as shown by the following, our new interpolation scheme generally satisfies the three properties we wanted.

Theorem 4.3. *Our multiplicative interpolation scheme satisfies Stochasticity and the Absence of Internal Arbitrage. If*

$$\begin{aligned} &V_{\eta(t_2)+1}(T_{\eta(t_2)}, t_2+\delta) < \\ &\log\left(\frac{\delta}{\theta(t_2+\delta)} + L_{\eta'(t_2)}(t_1)\left(\xi(t_2) + (1-\xi(t_2))\frac{L_{\eta(t_2)}(t_1)}{L_{\eta(t_2)}(T_{\eta'(t_2)} \wedge t_1)}\right)\left[\frac{1}{L_{\eta(t_2)}(t_1)} + \delta\right]\right), \end{aligned} \tag{4.8}$$

then it also satisfies Positivity, that is $L(t_1, t_2) > 0$ *when* $\alpha = 0$.

Proof. See Appendix A. □

It is important to emphasize that (4.8) will hold in nearly all practical situations. In particular, if we assume that $L_{\eta(t_2)+1}(s)$ has constant volatility denoted by σ, then in the extreme case that

$$L_{\eta'(t_2)}(t_1) \to 0,$$

for (4.8) to hold we require that

$$\sigma^2 < \frac{1}{\theta(t_2+\delta)}.$$

Provided tenors are no longer than one year, this will hold if instantaneous volatilities are less than 1.

In order to be able to use this interpolation scheme, we need to be able to easily compute $P(t_1, t_2)$ for any t_1 and t_2. This is taken care of by the following.

Proposition 4.1. *Assumption (4.7), together with the no-arbitrage condition (4.1) and the drift-freezing approximation yields*

$$\frac{P(t_1,t_2)}{P_{\eta(t_2)}(t_1)} = 1 + \theta(t_2)\left[\xi(t_2)L_{\eta'(t_2)}(t_1) + (1-\xi(t_2))\left\{\tilde{L}_{\eta'(t_2)}(t_1)h(t_1,t_2,L_{\eta(t_2)}(t_1)) - \alpha\right\}\right], \tag{4.9}$$

where

$$\begin{aligned} h(t_1,t_2,x) &= \frac{x+\alpha}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)} \wedge t_1)} e^{\left(-\frac{\delta(x+\alpha)}{1+\delta x} C_{\eta(t_2),\eta'(t_2)}(t_1, T_{\eta'(t_2)})\right)} \\ &\quad \frac{\left[1 - \delta\alpha + \delta(x+\alpha)e^{C_{\eta(t_2),\eta'(t_2)}(t_1,T_{\eta'(t_2)}) + V_{\eta(t_2)}(T_{\eta'(t_2)} \vee t_1, t_2)}\right]}{1+\delta x}. \end{aligned} \tag{4.10}$$

Proof. See Appendix A. □

5. Improvements to Range Accurals

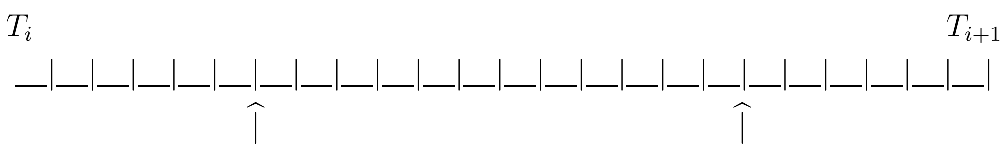
In this section we introduce new improvements that allow for the efficient calculation of prices and Greeks of callable range accruals. Efficiency is improved through a cruder, but accurate, discretisation of the accrual period and by either taking the present value of the range accrual coupons from the preceding tenor date or using our new Brownian-bridge-type approach, removing the need to short step through each day in the accrual period to sample the range accrual coupons. By present valuing the range accrual coupons or using our new Brownian bridge technique, a smoothing effect is applied to the range accrual coupons, removing the discontinuities present if short-stepping is used. As such, the range accrual coupons are sufficiently smooth to use the pathwise method to calculate Greeks, which can result in significant time savings; see [12], [22] and [10].

We also briefly describe the simple method for simulating range accrual coupons in [9]. This method is used for comparison, as it is the only other method for simulating range accrual coupons using the DDLMM present in the literature.

It is worth noting that the techniques used to present value the range accrual coupons can be easily extended to price the non-cancellable swap and can be applied directly to identify approximately sub-optimal points for cancellable range accruals using the method in [4].

5.1. **Improved Discretisation.** When simulating the range accrual coupons, we can improve efficiency by using coarser discretisations of the accrual period. For example, rather than sample the reference rate daily, we can sample it monthly, weekly, or even less frequently. The key to this approach is then deciding how to apply the coarser discretisation.

An effective way is to spread the number of days that the reference rate is sampled evenly over the accrual period. This is equivalent to sampling in the middle of each period under the coarser discretisation. Consider the example shown below. The original observation dates occur at quarterly intervals over the period, marked by the vertical lines with no arrows. Rather than sample on these dates, we can use a coarser discretisation, considering only the digital options at the points marked by the arrowed vertical lines.



We will show in the numerical results that using such a cruder discretisation has little effect on prices and Greeks, but can significantly reduce the number of digital options we need to consider, resulting in significant time savings. This is true for both the short-stepping approach and our new approaches of present valuing the range accrual coupons or bridging, to be described in the next two sections.

5.2. **Calculating the Present Value of Range Accrual Coupons.** By re-writing the value for the cancellable swap, we can simplify its pricing. Consider the value of the net coupon on the i^{th} tenor date, denoted $D_i(0)$. We have,

$$\begin{aligned}\frac{D_i(0)}{N(0)} &= \mathbb{E}\left[\mathrm{I}_{\bar{\tau}>i-1}\frac{(L_{i-1}(T_{i-1})-S_i)\delta}{N(T_i)}\right],\\ &= \delta\mathbb{E}\left[\mathbb{E}^{i-1}\left[\mathrm{I}_{\bar{\tau}>i-1}\frac{(L_{i-1}(T_{i-1})-S_i)}{N(T_i)}\right]\right],\end{aligned}$$

$$= \delta \mathbb{E}\left[\mathrm{I}_{\bar{\tau}>i-1} \left(\frac{L_{i-1}(T_{i-1})P_i(T_{i-1})}{N(T_{i-1})} - \frac{G_i(T_{i-1})}{N(T_{i-1})} \right) \right],$$

where $G_i(t)$ denotes the value of the $(i-1)^{\text{th}}$ range accrual coupon at time t. This says that to value the cancellable range accrual using Monte Carlo, rather than sample the observed range accrual coupon, we can sample the observed value of the coupon at the tenor date preceding the payment date. This clearly removes the need for short-stepping through each day, but it does require an accurate approximation for the value of the range accrual coupons. Since the value of the range accrual coupons in the above expression can be written as

$$\begin{aligned} \frac{G_i(T_{i-1})}{P(T_{i-1}, T_k)} &= \mathbb{E}_{T_k}^{i-1}\left[\frac{S_i}{P(T_i, T_k)} \right], \\ &= \sum_{j=1}^{n_i} \mathbb{E}_{T_k}^{i-1}\left[\frac{Q_i}{n_i} \frac{\mathrm{I}_{B_i<R(T_{i-1}+t_j)<U_i}}{P(T_i, T_k)} \right], \end{aligned} \tag{5.1}$$

and valuing the coupons requires calculating the value of the underlying digital options and adding the results.

We consider a single digital option with expiry t, where $T_{i-1} < t < T_i$. We need to approximate the distribution of the reference rate at the expiry of the digital, $R(t)$. In the case where $R(t)$ depends on more than one rate (for example, the difference between swap rates), we model each rate separately rather than $R(t)$ directly, as this allows for more accurate approximations.

Since the digital pays off at a different time from the reset of the rate, we need a convexity correction. We will use $P(., T_i)$ as numeraire, and this yields a drift for $R(t)$. Since such correction will only be required within accrual periods, it will be effective to use drift freezing.

Let $\hat{f}(s)$ denote the time s value of a variable that agrees with (one of) the rate(s) underlying $R(t)$ at time t. We can then approximate the distribution of $R(t)$ by using an approximation to the underlying process for $\hat{f}(s)$ ($T_{i-1} \leq s \leq t$) to first determine the distribution of the underlying interest rate.

If $f(s)$ denotes the value of a rate underlying $R(t)$ at time s, then $f(s)$ is determined according to each interpolation scheme through a given function g, where

$$f(s) := g(s, L_{i-1}(s), \ldots, L_n(s)).$$

We instead work with

$$\hat{f}(s) = g(t, L_{i-1}(s), \ldots, L_n(s)).$$

In particular, we assume that $\left(\hat{f}(s) + \alpha\right)$ follows a lognormal process, where the drift and volatility are determined by Ito's lemma and the relation dictated by each interpolation scheme. Working with $\hat{f}(s)$ has the advantage that the only time-dependence in the drifts and volatility arises from the time-dependence of the tenor rate volatilities and crucially at expiry f and $\hat{f}$ will agree.

We will derive approximate dynamics for these rates in Section 5.4. Assume we have an approximation for each rate underlying $R(t)$ of the form

$$\frac{d\left(\hat{f}_j(s) + \alpha\right)}{\hat{f}_j(s) + \alpha} \approx \mu_{f_j}(s)ds + \sigma_{f_j}(s)dW_{f_j}(s), \tag{5.2}$$

for $T_{i-1} \leq s \leq T_i$ and $j = 1, 2$, where $W_{f_j}(s)$ is a standard Brownian motion under the spot LIBOR probability measure, and the instantaneous covariance between the logarithms of $(\hat{f}_1(s)+\alpha)$

and $(\hat{f}_2(s)+\alpha)$ is given by

$$\sigma_{f_1,f_2}(s).$$

Then, conditional on $\mathcal{F}_{T_{i-1}}$,

$$\log\left(\hat{f}_j(t)+\alpha\right) \overset{\mathrm{d}}{\approx} N\left(\beta_j,\gamma_j^2\right), \tag{5.3}$$

and

$$\mathrm{Cov}\left(\log\left(\hat{f}_1(t)+\alpha\right),\log\left(\hat{f}_2(t)+\alpha\right)\right)=\int_{T_{i-1}}^{t}\sigma_{f_1,f_2}(s)ds,$$

where

$$\beta_j=\log\left(\hat{f}_j(T_{i-1})+\alpha\right)+\int_{T_{i-1}}^{t}\mu_{f_j}(s)ds-\frac{1}{2}\int_{T_{i-1}}^{t}\sigma_{f_j}(s)^2ds,$$

and

$$\gamma_j=\sqrt{\int_{T_{i-1}}^{t}\sigma_{f_j}(s)^2ds}.$$

For the three examples, we use the following results.

Example 1. We have

$$R(t)=L(t,t)=\hat{f}_1(t).$$

Since we are using $P(.,T_i)$ as numeraire, for each of the digitals in (5.1),

$$\begin{aligned}
\mathbb{E}_{T_i}^{i-1}\left[\frac{Q_i}{n_i}\mathrm{I}_{B_i<R(t)<U_i}\right] &= \frac{Q_i}{n_i}\mathbb{E}_{T_i}^{i-1}\left[\mathrm{I}_{B_i<\hat{f}_1(t)<U_i}\right],\\
&= \frac{Q_i}{n_i}\mathbb{E}_{T_i}^{i-1}\left[\mathrm{I}_{B_i+\alpha<\hat{f}_1(t)+\alpha<U_i+\alpha}\right],\\
&= \frac{Q_i}{n_i}\left(\Phi\left(\frac{\log(U_i+\alpha)-\beta_1}{\gamma_1}\right)-\Phi\left(\frac{\log(B_i+\alpha)-\beta_1}{\gamma_1}\right)\right),
\end{aligned}$$

where $\Phi(.)$ denotes the standard Normal CDF.

Example 2. We have

$$R(t)=\mathrm{SR}_{t,x}(t)-\mathrm{SR}_{t,y}(t)=\hat{f}_1(t)-\hat{f}_2(t).$$

We can write the value of the digitals in (5.1) as

$$\mathbb{E}_{T_i}^{i-1}\left[\frac{Q_i}{n_i}\mathrm{I}_{B_i<R(t)<U_i}\right]=\frac{Q_i}{n_i}\mathbb{E}_{T_i}^{i-1}\left[\mathrm{I}_{B_i<\left(\hat{f}_1(t)+\alpha\right)-\left(\hat{f}_2(t)+\alpha\right)<U_i}\right]. \tag{5.4}$$

Since $\left(\hat{f}_1(t)+\alpha\right)$ and $\left(\hat{f}_2(t)+\alpha\right)$ are lognormally distributed, calculating (5.4) reduces to the case of a spread option in the Black-Scholes model, and we can apply standard results. In particular, we have found the approximate formula developed by Li, Deng and Zhou, [20], to be very useful. It is both efficient and accurate.

Example 3. We now take

$$Q_i=qL_i(T_i).$$

First we re-write the value of the digitals as follows:

$$\mathbb{E}_{T_i}^{i-1}\left[\frac{qL_i(T_i)}{n_i}\mathrm{I}_{B_i<R(t)<U_i}\right]=\frac{q}{n_i}\left(\mathbb{E}_{T_i}^{i-1}\left[\tilde{L}_i(T_i)\mathrm{I}_{B_i<R(t)<U_i}\right]-\alpha\mathbb{E}_{T_i}^{i-1}\left[\mathrm{I}_{B_i<R(t)<U_i}\right]\right). \tag{5.5}$$

The second term on the right hand side of (5.5) can be calculated as in Example 2, so we focus on the first term. To deal with the floating multiplier, we can use the technique introduced by Wu and Chen, [26]. In particular, first use the drift-freezing approximation for $L_i(T_i)$ conditional on $\mathcal{F}_{T_{i-1}}$. Therefore,

$$\mathbb{E}_{T_i}^{i-1}\left[\tilde{L}_i(T_i)\mathrm{I}_{B_i<R(t)<U_i}\right] =$$
$$\tilde{L}_i(T_{i-1})e^{\int_{T_{i-1}}^{T_i}\mu_i(L(T_{i-1}),s)ds}\mathbb{E}_{T_i}^{i-1}\left[e^{-0.5\int_{T_{i-1}}^{T_i}\sigma_i(s)^2ds+\int_{T_{i-1}}^{T_i}\sigma_i(s)dW_i(s)}\mathrm{I}_{B_i<R(t)<U_i}\right].$$

Since the coefficient of the indicator function defines a Radon-Nikodym derivative, we can remove it by performing the corresponding measure change; see, for example, [6]. If bars are used to denote quantities under the resulting measure, then

$$dW_j(s) = d\bar{W}_j(s) + \sigma_i(s)\rho_{i,j}ds, \tag{5.6}$$

and we therefore get an extra drift term in (5.2). As such, we can obtain new approximate dynamics that can be written as

$$\frac{d\left(\hat{f}_j(s)+\alpha\right)}{\hat{f}_j(s)+\alpha} \approx \mu_{f_j}(s)ds + \mu_{f_j}^{(2)}(s)ds + \sigma_{f_j}(s)d\bar{W}_{f_j}(s), \tag{5.7}$$

for $T_{i-1} \leq s \leq T_i$, where $\mu_{f_j}^{(2)}(s)$ gives an extra drift term which we will compute in Section 5.4.

Therefore, the first expectation on the right hand side of (5.5) becomes

$$\mathbb{E}_{T_i}^{i-1}\left[\tilde{L}_i(T_i)\mathrm{I}_{B_i<R(t)<U_i}\right] = \tilde{L}_i(T_{i-1})e^{\int_{T_{i-1}}^{T_i}\mu_i(L(T_{i-1}),s)ds}\bar{\mathbb{E}}^{i-1}\left[\mathrm{I}_{B_i<R(t)<U_i}\right],$$

where the expectation can be valued using the approach from Example 2 with an allowance made for the additional drift term.

5.3. **Bridging Range Accrual Coupons.** We will describe a new Brownian bridging type approach for simulating the range accrual coupons that is very similar to present valuing the coupons. However, we instead calculate the value of the range accrual coupons conditional on the value of the forward rates at the start and end of each accrual period. By taking such an approach, we pin the values of the forward rates at the end of each step; this means we can handle floating rate notes with no additional effort. In addition, since we are bridging over such short intervals (one accrual period), we can also safely ignore the drifts of the interpolated quantities without any loss of accuracy, simplifying the valuation. This follows since the drift has a very minor effect on the distribution of the bridged quantity. In particular, when drifts and volatility are constant, it has no effect. Since we are still calculating the value of the coupons as an expectation, we again obtain a smoothing effect, which means the pathwise method for Greeks can be applied.

Let $\mathcal{H}_i$ denote the sigma-algebra containing the information generated by the state variables at $T_0, T_1, \ldots, T_{i-1}$ and T_i. As such, we can write

$$\begin{aligned}\frac{D_i(0)}{N(0)} &= \mathbb{E}\left[\mathrm{I}_{\bar{\tau}>i-1}\frac{(L_{i-1}(T_{i-1})-S_i)\delta}{N(T_i)}\right],\\ &= \delta\mathbb{E}\left[\mathbb{E}\left[\mathrm{I}_{\bar{\tau}>i-1}\frac{(L_{i-1}(T_{i-1})-S_i)}{N(T_i)}|\mathcal{H}_i\right]\right],\\ &= \delta\mathbb{E}\left[\mathrm{I}_{\bar{\tau}>i-1}\left(\frac{L_{i-1}(T_{i-1})}{N(T_i)}-\frac{\hat{G}_i(T_i)}{N(T_i)}\right)\right],\end{aligned}$$

where $\hat{G}_i(T_i) := \mathbb{E}\left[S_i|\mathcal{H}_i\right]$. Therefore, valuing the range accrual coupons amounts to calculating

$$\mathbb{E}\left[\frac{Q_i}{n_i}\mathrm{I}_{B_i<R(t)<U_i}|\mathcal{H}_i\right] = \frac{Q_i}{n_i}\mathbb{E}\left[\mathrm{I}_{B_i<R(t)<U_i}|\mathcal{H}_i\right], \tag{5.8}$$

for $T_{i-1} < t \leq T_i$ and adding the results for the digitals in the accrual period. The above follows since Q_i is $\mathcal{H}_i$-measurable. This is the case in each Example considered, but would also be expected in practice.

We use a similar approach to that used when taking the present value of the range accrual coupons. We use a displaced-diffusion approximation for the process of $\hat{f}(s)$ $(T_{i-1} \leq s \leq T_i)$. As above, in the case where $R(t)$ depends on more than one rate, we model each rate separately. Since the $\left(\hat{f}_j(s)+\alpha\right)$'s (for $j = 1, 2$) then follow lognormal processes, conditional on $\mathcal{H}_i$, we have the values of the $\hat{f}_j(.)$'s at T_{i-1} and T_i, and we can apply the standard Brownian bridge results to obtain an approximation for the (joint) conditional distribution of the $\left(\hat{f}_j(t)+\alpha\right)$'s. Since the $\hat{f}_j(s)$'s equal the underlying interest rates at the expiry of the digital t, this can be used to calculate (5.8). For Example 1, we only have one underlying interest rate, however, the procedure is the same.

Assuming an approximation of the form in (5.2), for the three examples we calculate the range accrual coupons as follows.

Example 1. We have

$$R(t) = L(t,t) = \hat{f}_1(t).$$

Conditional on $\mathcal{H}_i$ and ignoring drifts,

$$\log\left(\hat{f}_1(t)+\alpha\right) \overset{\mathrm{d}}{\approx} N\left(\beta_1, \gamma_1^2\right),$$

where

$$\beta_1 = \log\left(\hat{f}_1(T_{i-1})+\alpha\right) + \frac{\int_{T_{i-1}}^{t}\sigma_{f_1}(s)^2 ds}{\int_{T_{i-1}}^{T_i}\sigma_{f_1}(s)^2 ds}\left(\log\left(\hat{f}_1(T_i)+\alpha\right) - \log\left(\hat{f}_1(T_{i-1})+\alpha\right)\right),$$

and

$$\gamma_1 = \sqrt{\frac{\int_{T_{i-1}}^{t}\sigma_{f_1}(s)^2 ds \int_{t}^{T_i}\sigma_{f_1}(s)^2 ds}{\int_{T_{i-1}}^{T_i}\sigma_{f_1}(s)^2 ds}},$$

see, for example, [25].

Therefore, the value of the digital in (5.8) is given by

$$\begin{aligned}
\mathbb{E}\left[\frac{Q_i}{n_i}\mathrm{I}_{B_i<R(t)<U_i}|\mathcal{H}_i\right] &= \frac{Q_i}{n_i}\mathbb{E}\left[\mathrm{I}_{B_i+\alpha<\hat{f}_1(t)+\alpha<U_i+\alpha}|\mathcal{H}_i\right], \\
&= \frac{Q_i}{n_i}\left(\Phi\left(\frac{\log(U_i+\alpha)-\beta_1}{\gamma_1}\right) - \Phi\left(\frac{\log(B_i+\alpha)-\beta_1}{\gamma_1}\right)\right).
\end{aligned}$$

Example 2. Now,

$$R(t) = \mathrm{SR}_{t,x}(t) - \mathrm{SR}_{t,y}(t) = \hat{f}_1(t) - \hat{f}_2(t).$$

Using the results for conditional multivariate Normal random variables (see, for example, [25]) and ignoring drifts, we have that conditional on $\mathcal{H}_i$,

$$\begin{pmatrix} \log\left(\hat{f}_1(t)+\alpha\right) \\ \log\left(\hat{f}_2(t)+\alpha\right) \end{pmatrix} \stackrel{\text{d}}{=} N(\beta,\Sigma),$$

where if

$$\beta_1=\beta_2=\begin{pmatrix} \log\left(\hat{f}_1(T_{i-1})+\alpha\right) \\ \log\left(\hat{f}_2(T_{i-1})+\alpha\right) \end{pmatrix},$$

and

$$\Sigma_z=\begin{pmatrix} \int_{T_{i-1}}^{z}\sigma_{f_1}(s)^2ds & \int_{T_{i-1}}^{z}\sigma_{f_1,f_2}(s)ds \\ \int_{T_{i-1}}^{z}\sigma_{f_1,f_2}(s)ds & \int_{T_{i-1}}^{z}\sigma_{f_2}(s)^2ds \end{pmatrix},$$

then

$$\beta=\beta_1+\Sigma_t\Sigma_{T_i}^{-1}\left(\begin{pmatrix} \log\left(\hat{f}_1(T_i)+\alpha\right) \\ \log\left(\hat{f}_2(T_i)+\alpha\right) \end{pmatrix}-\beta_2\right),$$

and

$$\Sigma=\Sigma_t-\Sigma_t\Sigma_{T_i}^{-1}\Sigma_t.$$

Since conditional on $\mathcal{H}_i$, $\log\left(\hat{f}_1(t)+\alpha\right)$ and $\log\left(\hat{f}_2(t)+\alpha\right)$ are jointly Normal, we can then apply the approximate formula from [20] to calculate

$$\mathbb{E}\left[\frac{Q_i}{n_i}\mathrm{I}_{B_i<R(t)<U_i}|\mathcal{H}_i\right]=\frac{Q_i}{n_i}\mathbb{E}\left[\mathrm{I}_{B_i<\left(\hat{f}_1(t)+\alpha\right)-\left(\hat{f}_2(t)+\alpha\right)<U_i}|\mathcal{H}_i\right],$$

as in the previous section.

Example 3. Since conditional on $\mathcal{H}_i$, $L_i(T_i)$ is deterministic, we can apply the procedure that was used for Example 2.

5.4. **Approximate Dynamics for $\hat{f}(s)$.** Here we give the approximate lognormal dynamics for $\left(\hat{f}_j(s)+\alpha\right)$ under each interpolation scheme. Without loss of generality, we consider the period from T_{i-1} to T_i, that is $T_{i-1}\le s\le T_i$. Throughout, hats will have the same meaning as above; they are used to indicate that all time dependence in the interpolation scheme is frozen at the point where the particular digital option expires, denoted by t for concreteness. In particular, all zero coupon bonds in quantities having a hat are obtained as follows. For some fixed $t_2\ge t$, under the Piterbarg interpolation scheme,

- $\hat{P}(s,t_2)=(1+(t_2-t)L_{i-1}(T_{i-1}))^{-1}$, for $t\le t_2\le T_i$,
- $\hat{P}_j(s)=\hat{P}_i(s)\prod_{k=i}^{j-1}(1+\delta L_k(s))^{-1}$, for $j\ge i$,
- $\hat{P}(s,t_2)=\hat{P}_{\eta'(t_2)}(s)(1+\vartheta(t_2)L_{\eta'(t_2)}(s))^{-1}$, for $t_2>T_i$,

and for our multiplicative interpolation scheme,

$$\frac{\hat{P}(s,t_2)}{\hat{P}_{\eta(t_2)}(s)}=1+\theta(t_2)\left[\xi(t_2)L_{\eta'(t_2)}(s)+(1-\xi(t_2))\left\{\tilde{L}_{\eta'(t_2)}(s)h(t,t_2,L_{\eta(t_2)}(s))-\alpha\right\}\right],$$

with

$$\hat{P}_i(s)^{-1} = 1 + \theta(t)\left[\xi(t)L_{i-1}(T_{i-1}) + (1-\xi(t))\left\{\tilde{L}_{i-1}(T_{i-1})\frac{\tilde{L}_i(s)}{\tilde{L}_i(T_{i-1})} - \alpha\right\}\right], \tag{5.9}$$

and the function $h(t_1, t_2, x)$ given by (4.10). Note that as freezing the time-dependence in the interpolation scheme has no effect on the tenor forward rates, we do not alter their notation.

We present the dynamics for swap rates, since forward rates are clearly just a special case. As such, we give the approximate dynamics for the pseudo-swap rate $\widehat{\mathrm{SR}}_{t,x}(s)$, which denotes the time-s value of the swap rate starting at time t and running for x accrual periods calculated using the modified interpolated bond prices obtained as per above. Therefore,

$$\widehat{\mathrm{SR}}_{t,x}(s) = \sum_{j=0}^{x-1} w_j \hat{L}(s, t+j\delta),$$

where

$$w_j = \frac{\delta \hat{P}(s, t+(j+1)\delta)}{\hat{A}_0(s)},$$

with

$$\hat{A}_j(s) = \sum_{k=j}^{x-1} \delta \hat{P}(s, t+(k+1)\delta),$$

and

$$\frac{\partial \widehat{\mathrm{SR}}_{t,x}(s)}{\partial \hat{L}(s, t+j\delta)} = w_j + \frac{\delta}{1+\delta\hat{L}(s,t+j\delta)}\left(\frac{\hat{A}_j(s)}{\hat{A}_0(s)}\widehat{\mathrm{SR}}_{t,x}(s) - \sum_{m=j}^{x-1} w_m \hat{L}(s, t+m\delta)\right), \tag{5.10}$$

see [13] and [8]. In addition, let

$$\widetilde{\widehat{\mathrm{SR}}}_{t,x}(s) = \widehat{\mathrm{SR}}_{t,x}(s) + \alpha.$$

We obtain two sets of approximate dynamics. The first is obtained by freezing all state dependence at T_{i-1}, and will be referred to as Approximation 1. Since we are only applying the approximations over very short periods (at most, one accrual period), this should be adequate; see [3] and [8], for example. For the second set, we derive the dynamics conditional on $\mathcal{F}_{T_{i-1}}$, but set all state dependent quantities to their initial values. This approximation will be referred to as Approximation 2. This has the advantage that the approximate drifts and volatility only need to be computed at the start of each simulation.

Consider an approximation of the form

$$\frac{d\widetilde{\widehat{\mathrm{SR}}}_{t,x}(s)}{\widetilde{\widehat{\mathrm{SR}}}_{t,x}(s)} \approx \mu_{t,x}(L(z), s)ds + \sigma_{t,x}(L(z), s)dW_{t,x}(s), \tag{5.11}$$

with $W_{t,x}(s)$ being a standard Brownian motion under the spot LIBOR probability measure and $z = T_{i-1}, T_0$, for Approximations 1 and 2 respectively. Throughout this section, we work with the understanding that $T_{i-1} \le s \le T_i$. Define

$$X_j^k(L(z)) = \begin{cases} \left.\frac{\partial \hat{L}(s,t+j\delta)}{\partial L(s,T_{i-1+k})}\right|_{L(s)=L(z)}, & \text{otherwise}, \\ 0, & \text{if } j=0 \text{ and } k=0, \end{cases} \tag{5.12}$$

where the X_j^k's are determined by the particular interpolation scheme being used and give the partial derivatives of the interpolated pseudo-forward rates underlying $\widehat{\mathrm{SR}}_{t,x}(s)$ with respect to the non-reset tenor forward rates, with all state dependent quantities set at their time z values. The volatility function is then given by the following.

Proposition 5.1. *The volatility in (5.11) is given by*

$$\sigma_{t,x}(L(z),s) = \frac{1}{\widetilde{\widehat{\mathrm{SR}}}_{t,x}(z)} \left(\sum_{j,k=0}^{x-1} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(z)}{\partial \hat{L}(z,t+j\delta)} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(z)}{\partial \hat{L}(z,t+k\delta)} \right.$$
$$\left. \left(\sum_{m,n=0}^{2} \tilde{L}_{i-1+j+m}(z) \tilde{L}_{i-1+k+n}(z) X_j^m(L(z)) X_k^n(L(z)) \sigma_{i-1+j+m,i-1+k+n}(s) \right) \right)^{0.5}.$$

Proof. See Appendix A. □

To obtain an expression for the drifts, additional approximations are used. We first calculate the drifts assuming that the time-dependence in the interpolation scheme is not frozen at t, and then freeze all time-dependence due to the interpolation scheme in the resulting expression for the drifts. In addition, for the Piterbarg interpolation, we approximate by taking the price processes for the non-tenor bonds to be martingales under the spot LIBOR probability measure. By doing this we obtain a much simpler expression for the drifts using the result from [19], which can lead to significant reductions in computation time. Since we are only considering digital options over short intervals, the importance of the drifts, and thus this additional approximation is not high in any case.

In what follows, let

$$\begin{aligned} \Psi_k(L(z)) &= \left\langle \widehat{\mathrm{SR}}_{t,x}(s), L_k(s) \right\rangle \Big|_{L(s)=L(z)} \\ &= \sum_{j=0}^{x-1} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(z)}{\partial \hat{L}(z,t+j\delta)} \tilde{L}_k(z) \left(\sum_{m=0}^{2} X_j^m(L(z)) \tilde{L}_{i-1+j+m}(z) \sigma_{i-1+j+m,k}(s) \right). \end{aligned}$$

Proposition 5.2. *Under the Piterbarg interpolation scheme, the drift term in (5.11) is given by*

$$\mu_{t,x}(L(z),s) = \frac{1}{\widetilde{\widehat{\mathrm{SR}}}_{t,x}(z)} \frac{1}{\hat{A}_0(z)}$$
$$\left[\sum_{k=1}^{x-1} \frac{\delta \hat{A}_k(z)}{1+\delta L_{i-1+k}(z)} \Psi_{i-1+k}(L(z)) + \sum_{j=1}^{x} \frac{\vartheta(t+j\delta)\delta \hat{P}(z,t+j\delta)}{1+\vartheta(t+j\delta) L_{i-1+j}(z)} \Psi_{i-1+j}(L(z)) \right], \tag{5.13}$$

Proof. See Appendix A. □

Proposition 5.3. *Under our multiplicative interpolation scheme, the drift term in (5.11) is given by*

$$\mu_{t,x}(L(z),s) = \frac{1}{\widetilde{\widehat{\mathrm{SR}}}_{t,x}(z)} \frac{\delta}{\hat{A}_0(z)}$$
$$\sum_{j=1}^{x} \left(\Psi_{i-1+j}(L(z)) \left(\frac{\hat{A}_{j-1}(z)}{1+\delta L_{i-1+j}(z)} - \frac{P_{i+j}(z)}{P_{i-1}(z)} \theta(t+j\delta) \left[\xi(t+j\delta) + \right. \right. \right.$$

$$\left.(1-\xi(t+j\delta))h(t,t+j\delta,L_{i+j}(z))]\right)$$

$$\left.-\frac{P_{i+j}(z)}{P_{i-1}(z)}\theta(t+j\delta)(1-\xi(t+j\delta))\tilde{L}_{i-1+j}(z)\frac{\partial h(t,t+j\delta,L_{i+j}(z))}{\partial L_{i+j}(z)}\Psi_{i+j}(L(z))\right), \tag{5.14}$$

with $h(t_1,t_2,x)$ given by (4.10) and its derivative given by (5.17).

Proof. See Appendix A. □

In order to apply the above results, we still need the X_j^k's under each interpolation scheme. Note that to apply the bridging approach for a different interpolation method to those considered here, one just has to calculate the X_j^k's for that interpolation scheme and substitute into the results above. However, for the present value approach, the drifts will also be different under different interpolation schemes. The X_j^k's for the two interpolation schemes considered in this paper are given by the following.

Proposition 5.4. *Using the Piterbarg interpolation scheme, with $X_j^k(L(s))$ defined in (5.12),*

$$\begin{aligned}
X_j^0(L(s)) &= \begin{cases} 0, & \text{if } j=0,\\ \left(\frac{1}{\delta}\frac{(1+\vartheta(t+(j+1)\delta)L_{i+j}(s))}{(1+\vartheta(t+j\delta)L_{i-1+j}(s))^2}\right) \\ \quad [(1+\vartheta(t+j\delta)L_{i-1+j}(s))\delta-(t+j\delta)(1+\delta L_{i-1+j}(s))], & \text{otherwise,}\end{cases}\\
X_j^1(L(s)) &= \begin{cases} \frac{1}{\delta}\vartheta(t+(j+1)\delta)(1+\delta L_{i-1+j}(s)), & \text{if } j=0,\\ \frac{1}{\delta}\frac{\vartheta(t+(j+1)\delta)(1+\delta L_{i-1+j}(s))}{(1+\vartheta(t+j\delta)L_{i-1+j}(s))}, & \text{otherwise,}\end{cases}\\
X_j^2(L(s)) &= 0.
\end{aligned}$$

Proof. The result follows by taking partial derivatives of

$$\hat{L}(s,t)=\frac{1}{\delta}\left[(1+(T_i-t)L_{i-1}(T_{i-1}))(1+\vartheta(t+\delta)L_i(s))-1\right], \tag{5.15}$$

and

$$\hat{L}(s,t+j\delta) = \frac{1}{\delta}\left[\frac{(1+\delta L_{i-1+j}(s))(1+\vartheta(t+(j+1)\delta)L_{i+j}(s))}{1+\vartheta(t+j\delta)L_{i-1+j}(s)}-1\right], \tag{5.16}$$

for $j=1,2,\ldots,x-1$. □

Proposition 5.5. *Using the multiplicative interpolation scheme, with $X_j^k(L(s))$ defined in (5.12),*

$$X_j^0(L(s)) = \begin{cases} 0, & \text{if } j=0,\\ \frac{(1+\delta L_{i+j}(s))\theta_j[\xi_j+(1-\xi_j)h_j]}{\delta D_j(L(s))}, & \text{otherwise,}\end{cases}$$

$$X_j^1(L(s)) = \begin{cases} \dfrac{\begin{array}{l} D_j(L(s))\left[\delta N_j(L(s))+(1+\delta L_{i+j}(s))\theta_j(1-\xi_j)\frac{\tilde{L}_{i-1+j}(s)}{\tilde{L}_{i+j}(s)}\right]\\ -(1+\delta L_{i+j}(s))N_j(L(s))\theta_{j+1}\left[\xi_{j+1}+(1-\xi_{j+1})h_{j+1}\right]\end{array}}{\delta D_j(L(s))^2}, & \text{if } j=0,\\ \dfrac{\begin{array}{l} D_j(L(s))\left[\delta N_j(L(s))+(1+\delta L_{i+j}(s))\theta_j(1-\xi_j)\frac{\partial h_j}{\partial L_{i+j}(s)}\right]\\ -(1+\delta L_{i+j}(s))N_j(L(s))\theta_{j+1}\left[\xi_{j+1}+(1-\xi_{j+1})h_{j+1}\right]\end{array}}{\delta D_j(L(s))^2}, & \text{otherwise,}\end{cases}$$

$$X_j^2(L(s)) \quad = \quad -\frac{1}{\delta}\frac{(1+\delta L_{i+j}(s))N_j(L(s))}{D_j(L(s))^2}\theta_{j+1}(1-\xi_{j+1})\frac{\partial h_{j+1}}{\partial L_{i+j+1}(s)},$$

where

$$D_j(L(s)) = 1 + \theta_{j+1}\left[\xi_{j+1}L_{i+j}(s) + (1-\xi_{j+1})\left\{\tilde{L}_{i+j}(s)h_{j+1} - \alpha\right\}\right],$$

with,

$$N_j(L(s)) = \begin{cases} \hat{P}_i(s)^{-1}, & \textit{if } j=0, \\ D_{j-1}(L(s)), & \textit{otherwise,} \end{cases}$$

and,

$$\frac{\partial h(t_1,t_2,x)}{\partial x} = \frac{\delta(1-\delta\alpha)}{(1+\delta x)^2} \left(e^{-\frac{\delta(x+\alpha)}{1+\delta x}C_{\eta(t_2),\eta'(t_2)}(t_1,T_{\eta'(t_2)})}\left(e^{C_{\eta(t_2),\eta'(t_2)}(t_1,T_{\eta'(t_2)})+V_{\eta(t_2)}(T_{\eta'(t_2)},t_2)}-1\right)\right.$$
$$\left. -h(t_1,t_2,x)C_{\eta(t_2),\eta'(t_2)}(t_1,T_{\eta'(t_2)})\right), \tag{5.17}$$

using the shorthand notation

$$\theta_j = \theta(t+j\delta),\ \xi_j = \xi(t+j\delta) \textit{ and } h_j = h(t, t+j\delta, L_{i+j}(s)).$$

Proof. The result follows by taking partial derivatives of

$$\hat{L}(s, t+j\delta) = \frac{1}{\delta}\left[\frac{(1+\delta L_{i+j}(s))N_j(L(s))}{D_j(L(s))} - 1\right]. \tag{5.18}$$

□

Note that for ease of exposition, we drop $L_i(T_{i-1})$ from the arguments for X_j^1, X_j^2, and N_0. However, when evaluated at $L(T_0)$, we also assume that this forward rate is set to its time zero value in the listed quantities.

When the reference rate is a spread between two interest rates, we will need the covariance between the interest rates to apply the techniques of Sections 5.2 and 5.3. By following the proof for Proposition 5.1 (see also [3]), it is clear that the instantaneous covariance between the logarithms of $\left(\widehat{\mathrm{SR}}_{t,x}(s)+\alpha\right)$ and $\left(\widehat{\mathrm{SR}}_{t,y}(s)+\alpha\right)$ is given by

$$\frac{1}{\widetilde{\widetilde{\mathrm{SR}}}_{t,x}(z)}\frac{1}{\widetilde{\widetilde{\mathrm{SR}}}_{t,y}(z)}\left(\sum_{j=0}^{x-1}\sum_{k=0}^{y-1}\frac{\partial\widehat{\mathrm{SR}}_{t,x}(z)}{\partial\hat{L}(z,t+j\delta)}\frac{\partial\widehat{\mathrm{SR}}_{t,y}(z)}{\partial\hat{L}(z,t+k\delta)}\right.$$
$$\left.\left(\sum_{m,n=0}^{2}\tilde{L}_{i-1+j+m}(z)\tilde{L}_{i-1+k+n}(z)X_j^m(L(z))X_k^n(L(z))\sigma_{i-1+j+m,i-1+k+n}(s)\right)\right).$$

The terminal covariance that will be needed to value the digitals under each of the new approaches above can then be obtained by integrating over the time-dependent volatilities.

Finally, we derive the adjustment to the drifts that arises due to the measure change when pricing floating rate notes using the present value approach.

Proposition 5.6. *The $\mu_f^{(2)}(L(z), s)$ term from (5.7), under the notation of this section, is given by*

$$\mu_{t,x}^{(2)}(L(z), s) = \frac{1}{\widetilde{\widehat{\mathrm{SR}}}_{t,x}(z)} \sum_{j=0}^{x-1} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(z)}{\partial \hat{L}(z, t+j\delta)} \left(\sum_{m=0}^{2} X_j^m(L(z)) \tilde{L}_{i-1+j+m}(z) \sigma_{i-1+j+m,i}(s) \right).$$

Proof. The result follows from substituting (5.6) and (2.1) into (A.1), and freezing state-dependence. □

5.5. **Non-constant Displacements.** Thus far we have assumed that the displacements for all forward rates are equal. This was done primarily to ease notation, as allowing for non-constant displacements is rather straightforward. However, at the price of time-homogeneity, non-constant forward rate displacements allow greater flexibility when calibrating, and therefore can be desirable. The only difficulty that arises is calculating the displacements to apply to

$$\widehat{\mathrm{SR}}_{t,x}(s).$$

To do this, we set

$$\alpha_{t,x} = -\widehat{\mathrm{SR}}_{t,x}(t)|_{L(t)=-\underline{\alpha}}, \tag{5.19}$$

where $\alpha_{t,x}$ denotes the required displacement and $\underline{\alpha} = (\alpha_0, \ldots, \alpha_n)$. As such, we can pre-compute all of the required displacements at the start of each simulation, and allowing for non-constant forward rate displacements does not lead to any noticeable increases in computation time.

Note that when forward rate displacements are constant,

$$\alpha_{t,x} \approx \alpha.$$

As such, to simplify we have used α to approximate (5.19).

To allow for non-constant forward rate displacements elsewhere, we just have to use the displacement corresponding to the neighboring interest rate, which should be obvious.

5.6. **Using the Pathwise Method.** Although present valuing the range accrual coupons and the Brownian bridge technique generally permit the use of the pathwise method and its extensions (see [12], [22] and [10]), there are still a number of issues that need addressing. We will focus on *elementary vegas*, that is sensitivities to the entries of pseudo-root matrices. This will be sufficient since general vegas can be written as a linear combination of such terms.

First, when present valuing the coupons or using the Brownian bridge technique, the payoffs will depend on the volatilities of the forward rates. As such, we have to take account of this dependence when computing vegas. Extending the adjoint-improved algorithm from [10] to allow payoffs that depend on volatilities is trivial. For each coupon payment we just have to add the derivative of the payoff with respect to the parameter of interest to the overall sensitivity for the current path. However, there is a more subtle issue. The payoffs depend on covariance matrices for subsets of each accrual period. In contrast, when calculating vegas we are usually interested in linear combinations of the sensitivities with respect to covariance or pseudo-square root elements across each accrual period. Clearly these are related, and we need to take this into account to calculate accurate vegas. The following approximation provides a simple way to do this. In particular, if C denotes the covariance matrix across a step of length Δt and C_1 the covariance matrix for a subset of length Δt_1, with A and A_1 denoting the corresponding pseudo-square roots, then

$$C_1 \approx \frac{\Delta t_1}{\Delta t} C,$$

and therefore

$$A_1 \approx \sqrt{\frac{\Delta t_1}{\Delta t}} A.$$

Second, when differentiating the payoff with respect to the parameters of interest for vegas, we suggest calculating derivatives analytically. To do this, we approximate by ignoring all volatility dependence introduced by our multiplicative interpolation scheme.

Third, to apply the pathwise method to calculate deltas (and vegas) we need to be able to compute the partial derivatives of the payoffs with respect to the relevant forward rates for each path. Although this can be done easily using finite differences as suggested in [10], for each finite difference we need to use an additional payoff evaluation. As each payoff evaluation takes considerable time this can be very inefficient, particularly when the reference rate is a CMS spread and we therefore need to compute a large number of finite differences. Instead, we suggest using the following efficient approximation. If we ignore the dependence of the volatility and drifts (where used) on the forward rates, then we can easily calculate the sensitivity with respect to each forward rate analytically largely using quantities that were required for the payoff evaluation under Approximation 1. Applying this approach involves differentiating

$$\widehat{\mathrm{SR}}_{t,x}(z)$$

with respect to each element in $L(z)$ for $z = T_{i-1}$ when present valuing the coupons and $z = T_{i-1}, T_i$ when bridging, and then applying the chain rule to the coupon payoffs. Note that under our multiplicative interpolation scheme, $\widehat{\mathrm{SR}}_{t,x}(T_i)$ also depends on $L_i(s)$ at the previous tenor date, T_{i-1}. The derivatives of $\widehat{\mathrm{SR}}_{t,x}(z)$ are taken care of by the following.

Proposition 5.7. *For $z = T_{i-1}, T_i$, the derivatives of $\widehat{\mathrm{SR}}_{t,x}(z)$ are such that*

$$\frac{\partial \widehat{\mathrm{SR}}_{t,x}(z)}{\partial L_{i-1+k}(z)} = \sum_{j=w}^{\min(k,x-1)} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(z)}{\partial \hat{L}(z, t+j\delta)} \frac{\partial \hat{L}(z, t+j\delta)}{\partial L_{i-1+k}(z)}, \tag{5.20}$$

where $\frac{\partial \widehat{\mathrm{SR}}_{t,x}(z)}{\partial \hat{L}(z,t+j\delta)}$ is given by (5.10).

Under the Piterbarg interpolation scheme, $w = \max(0, k-1)$ and

$$\frac{\partial \hat{L}(z, t+j\delta)}{\partial L_{i-1+k}(z)} = \begin{cases} (T_i - t)(1 + \vartheta(t+\delta)L_i(z)), & \text{if } j = 0, k = 0, \\ X_j^k(L(z)), & \text{otherwise.} \end{cases}$$

Under our multiplicative interpolation scheme, $w = \max(0, k-2)$ and

$$\frac{\partial \hat{L}(z, t+j\delta)}{\partial L_{i-1+k}(z)} =$$

$$\begin{cases} \frac{(1+L_i(z))\theta_0(\xi_0 + (1-\xi_0)\frac{\tilde{L}_i(z)}{\tilde{L}_i(T_{i-1})})}{\delta D_0(L(z))}, & \text{if } j = 0, k = 0, \\ \frac{D_0(L(T_{i-1}))\delta N_0(L(T_{i-1})) - (1+\delta L_i(T_{i-1}))N_0(L(T_{i-1}))\theta_1[\xi_1 + (1-\xi_1)h_1]}{\delta D_0(L(T_{i-1}))^2}, & \text{if } z = T_{i-1}, j = 0, k = 1, \\ X_j^k(L(z)), & \text{otherwise,} \end{cases}$$

where we have used the notation from Proposition 5.5. In addition,

$$\frac{\partial \widehat{\mathrm{SR}}_{t,x}(T_i)}{\partial L_i(T_{i-1})} = -\frac{\partial \widehat{\mathrm{SR}}_{t,x}(T_i)}{\partial \hat{L}(T_i, t)} \frac{(1+\delta L_i(T_i))\theta_0(1-\xi_0)\tilde{L}_{i-1}(T_i)\frac{\tilde{L}_i(T_i)}{(\tilde{L}_i(T_{i-1}))^2}}{\delta D_0(L(T_i))}.$$

Proof. Equation (5.20) follows by the chain rule and that by (5.15), (5.16) and (5.18) $\hat{L}(z, t + j\delta)$ only depends on $L_{i-1+j}(z)$ and $L_{i+j}(z)$ under the Piterbarg interpolation scheme, but depends on $L_{i-1+j}(z)$, $L_{i+j}(z)$ and $L_{i+j+1}(z)$ under our multiplicative interpolation scheme. The remaining results follow by differentiating (5.15), (5.16) and (5.18). □

Fourth, when using the bridging approach, the range accrual coupons depend on the forward rate curve at both of the surrounding tenor dates. As such, applying the pathwise method in this case is the same as applying it to a path-dependent product. Extending the adjoint-improved pathwise algorithm from [10] to handle such a product is simple; at each coupon payment date, we differentiate the payoff with respect to the relevant forward rates at the current and (any relevant) preceding tenor dates and add the results to the stored derivatives at the tenor date being considered. The backwards algorithm can then be applied without adjustment.

Finally, when differentiating the CMS spread option price values, we make use of the closed form approximations from [20].

5.7. **Range Accrual Coupons Using the Chalamandaris Method.** The only alternative method in the literature for simulating range accrual coupons in the DDLMM is due to Chalamandaris, [9]. The idea is simple. The reference rate for the range accrual coupons is determined by spot interest rates. Given the observed spot interest rates on the surrounding tenor dates, we can use linear interpolation through time to get an approximation for the spot rates at each day where the reference rate is needed, allowing us to value the range accrual coupons without short-stepping through each day.

While in [9], the proportion of days the reference rate is in the range is estimated directly, we instead estimate the value of each underlying rate at the relevant times using linear interpolation and then count the proportion of days for which interest is accrued. This is done so that the Chalamandaris method can be used with the improved discretisation technique.

We shall compare this approach to our new improvements in the next section.

6. Results

6.1. **Set-up.** We primarily consider the pricing and Greeks of cancellable range accrual swaps in the DDLMM. An initially increasing forward rate curve is assumed, with

$$L_i(0) = 0.023 + 0.002i.$$

The common "abcd" time-dependent volatility structure is used, with

$$\sigma_i(t) = \begin{cases} 0, & t > T_i; \\ (0.04 + 0.09(T_i - t)) \exp\left(-0.44(T_i - t)\right) + 0.15, & \text{otherwise,} \end{cases}$$

and instantaneous correlation between the driving Brownian motions is assumed to be of the form

$$\rho_{i,j} = \exp\left(-0.06|i - j|\right).$$

In addition, we assume that

$$\alpha = 1.5\%.$$

For all examples, a 5-factor DDLMM is used, where the factor reduction is performed on the covariance matrices across each step.

In evolving the forward rates for pricing, the predictor-corrector drift approximation from [14] is used. However, as is common when calculating Greeks, we instead use the log-Euler drift approximation; see, for example, [12] and [10]. Due to the accuracy of the predictor-corrector method in particular, we evolved the forward rates between tenor dates in a single step. Since we are working in a reduced factor model, the method for computing the drifts from [17] was used.

As mentioned in Section 3, we consider three different types of range accrual. In terms of the notation in that section, we use the following parameter values for each example,

- Example 1: $q = 0.075$, $B_i = 0.02$, and $U_i = 0.07$,
- Example 2: $q = 0.05$, $B_i = 0.005$, $U_i = 0.05$, $x = 10$, and $y = 2$,
- Example 3: $q = 1.5$, $B_i = 0.005$, $U_i = 0.05$, $x = 10$, and $y = 2$.

In each example, a \$1 notional is assumed. In addition, we consider pricing products where coupons are paid yearly with the first coupon due in two years. As such,

$$\delta = 1.$$

We choose yearly tenors because this is as long as would commonly be expected in practice, and it is this length that will place the most pressure on the assumptions used to derive our new approximations.

We consider products with the same number of underlying rates. In particular, we use

$$n = 21 + u.$$

To use the different interpolation techniques, we have to extend the number of tenor forward rates slightly, as indicated by the u in the above expression for n. When using the Piterbarg interpolation, $u = 1$, while $u = 2$ when using our multiplicative method. Note that the additional forward rates needed in the tenor structure do not affect the number of coupon payments or exercise times. As such, when the reference rate is a LIBOR rate, we have 21 coupon payments, and when it is a CMS spread we have 12.

When using our multiplicative interpolation method, in practice we would often use

$$\xi(t) = 0 \text{ for all t}, \tag{6.1}$$

since this choice provides the most natural and intuitively appealing interpolation for short bonds. However, to test our methods for a more general $\xi(t)$, we use a function that satisfies (4.3) and (4.4), but which also goes to zero quickly between tenor dates so as to maintain a sufficient level of short bond volatility and provide an interpolation scheme that is close to the natural choice given by (6.1). A function that achieves this is

$$\xi(t) = \left(\frac{T_{\eta(t)} - t}{T_{\eta(t)} - T_{\eta(t)-1}} \right)^{20},$$

which we use.

6.2. **Numerical Results.** We start by looking at the non-cancellable case to assess the different approaches to simulating the range accrual coupons, with the overall goal being to apply these to the cancellable case. The relevant results concerning prices are contained in Tables B.1-B.4. For Tables B.1-B.3, the column labeled vega contains the change in price (basis points) when the volatility for all forward rates is increased by 1%. It is used to determine whether the different methods are sufficiently accurate from a practical point of view. In particular, provided the error is small compared to a third of the vega, then the price is considered acceptable. All computations for non-cancellable tests were done using single-threaded code on a laptop with a 2 GHz Intel Core2Duo processor.

The results demonstrate that using the improved discretisation described in Section 5.1 leads to significant efficiency improvements with almost no loss of accuracy. The error introduced by using only 2 sampling days as opposed to 365 is much less than a third of the vega, yet reductions in time of a factor of more than 100 are realised in Examples 2 and 3, with a factor of more than 20 improvement for Example 1, even when used in conjunction with the bridging or present value approaches.

We also see that for each example, and each interpolation scheme, the present value and Brownian bridge approaches produce very similar results to one another, and to the results obtained using short-stepping, when using Approximation 1. However, when using Approximation 2, the accuracy of the present value approach can fall away for Examples 2 and 3, to the point where it is not sufficiently accurate depending on the interpolation scheme used. In contrast, the Brownian bridge approach is still extremely accurate, even under Approximation 2. This is because by pinning the forward rates at the beginning and end of the accrual period, the importance of the volatility for the interpolated quantities is reduced.

The present value and Brownian bridge approaches also lead to efficiency improvements over short-stepping, particularly when the reference rate is a LIBOR rate. Even when only two sampling days are used, the new techniques need less than half the time, with reductions above a factor of 10 available if more sampling days are used. This is the case for both Approximation 1 and Approximation 2. For Examples 2 and 3, similar efficiency improvements obtainable if Approximation 2 is used. However, under Approximation 1, for Examples 2 and 3, the efficiency improvements of the new techniques over short-stepping are significantly reduced, to the point where in Example 3 there is little improvement.

We see that in each example, the present value approach is quicker than bridging. However, due to the superior accuracy of the bridging approach, it can consistently be used with Approximation 2, and can be significantly faster than the present value approach used with Approximation 1 for Examples 2 and 3.

Tables B.1-B.3 also demonstrate that the accuracy of the technique due to Chalamandaris, [9], can break down easily. For each example it differs by at least one third of vega from one of the interpolation methods, and sometimes both. As such, for each interpolation scheme, the technique from [9] is not sufficiently accurate.

Finally, we see that the different interpolation methods produce significantly different prices. This is not surprising since under the Piterbarg method, the short bonds have no volatility. As such, the interpolated interest rates also have significantly less volatility, and we would not expect the same values for the underlying digital options.

Table B.4 demonstrates that both of the present value and Brownian bridge approaches, as well as the cruder discretisation, value each of the individual coupons accurately; apart from Example 1, where the vega is very large, the sum of the absolute errors in the individual coupons is close to the total error. Even for Example 1, the sum of the absolute errors is still around one twentieth of the vega. This demonstrates that in all of the examples considered, the accuracy of the new approaches for the overall range accrual does not arise from errors in the individual coupons canceling out. The same cannot be said for the Chalamandaris, [9], technique, where in the few examples where it appears accurate, the accuracy is significantly improved by the errors of individual coupons canceling out.

Consider deltas and elementary vegas for the non-cancellable case, with the results in Tables B.5-B.10. All Greeks are calculated using the adjoint-enhanced pathwise method; see [12] and [10]. As such, we do not provide results when short-stepping or using the Chalamandaris, [9], method, since under these approaches the range accrual coupons have discontinuities and therefore the pathwise method does not apply. For products with discontinuous coupons, we have to use much slower alternative methods; see, for example, [11] and [22]. The results labeled FD are obtained using finite differences to differentiate the payoffs in the pathwise method and Approximation 1, those denoted Aprx are obtained using the approximations from Section 5.6 with Approximation 1, and those denoted PC are obtained using Approximation 2 and the approximations from Section 5.6. All finite differences are calculated using a bump size of 10^{-7}. The numbers in the method column are used to denote the number of sampling days used. The column labeled ”E. Vega" contains the elementary vega giving the sensitivity with respect to parallel shifting the first non-zero element of the pseudo-square root of the covariance matrix across each step in the simulation. Note that over each step, we consider one psuedo-square root element.

Note that as we are using the pathwise method, we calculate deltas and elementary vegas in the same simulation. As such we have reported the time taken to calculate both. However, for each Example and interpolation scheme, the differences in time arise largely due to the different approximations used for differentiating the payoff with respect to the forward rates.

We see some interesting results regarding deltas. First, generally the number of sampling days has little impact on the majority of deltas, but can have a significant impact on the first few deltas. However, by reducing the number of sampling days, we get significant efficiency improvements. In particular, even using 2 sampling days over 12 we can get factor reductions in time of up to 4 in some cases, with improvements of at least approximately 50% in all cases. Importantly, this was the case without using finite differences to differentiate the payoffs. It is also worth noting that even using 6 sampling days will provide a factor reduction in time of at least 10 (and often much more) over using 365 days in all cases.

Second, the deltas from the present value and bridging approaches are generally very close to one another, as we would expect. However, the present value approach is noticeably faster.

Third, the deltas obtained under the different interpolation schemes can be significantly different, especially for the first few deltas under Example 1. However, overall the different interpolation schemes produce similar deltas.

Fourth, the impact on accuracy of ignoring the dependence on the tenor forward rates in the volatility and drifts when computing deltas, as suggested in Section 5.6, varies. It has almost no effect when used with the bridging technique, and this is what we strongly recommend. This is not surprising as when using the bridging technique, the value of the digital options is largely determined by the pinned rates. However, the performance when used in conjunction with the present value approach is inconsistent. In some cases it performs very well (see Example 1 when using our multiplicative interpolation scheme), but in other cases it can break down (see Example 3). Using this approximation also provides significant efficiency improvements, particularly when the reference rate is a CMS spread. In particular, in Examples 2 and 3 reductions in time up to approximately a factor of 20 are possible for the bridging approach and 10 for the present value approach.

Finally, we see that the results for Approximation 2 vary. In Example 1, the deltas calculated under Approximation 2 are very close to those found using Approximation 1. However, the efficiency improvements are almost negligible here. For Examples 2 and 3, the efficiency improvements are significant, with improvements in time of more than 50% possible. In addition, the accuracy is very good when using the bridging approach. However, when using the present value approach the accuracy can break down.

For elementary vegas, there are no significant issues, particularly when using the bridging approach. Over all cases considered, the maximum difference between the elementary vega obtained using finite differencing and 12 sampling days is within 24 basis points of that obtained using Approximation 2 with 4 sampling days when bridging. This suggests that ignoring the volatility dependence introduced through use of our multiplicative interpolation scheme does not introduce significant bias. However, under the present value approach, the accuracy is not as impressive. Here, the maximum difference is 40 basis points. However, in some cases the accuracy under Approximation 2 can break down when only 2 sampling days are used, even when bridging.

We also observe that, before using any additional approximations, the differences in elementary vegas obtained using the bridging and present value approaches are extremely small (often just a few basis points), but can differ significantly under the different interpolation schemes.

Now consider the results obtained for cancellable range accruals, with the relevant results concerning prices contained in Table B.11, and those concerning Greeks in Table B.12. The pricing results were calculated using the Practical Policy Iteration algorithm from [5] with a few minor alterations:

- when using the adaptive basis functions,
  - we included two additional explanatory variables instead of one,
  - we chose the gap between zero-coupon bonds that could be used as explanatory variables to be 3 up until the 14^{th} exercise time, and 2 thereafter,
- when using the delta hedge control variate, to start the first pass from a random point, we took a from [5] to be 0.16 for Examples 1 and 2, and 0.1 for Example 3,
- for Examples 1 and 3, we excluded sub-optimal points using the method from [4] together with the present value approach and Approximation 2. We did not exclude sub-optimal points for Example 2 because it did not improve accuracy, but did increase computation times.

The standard error reductions in Table B.11 were obtained using the control variate from [5]. The control variate is a delta hedge portfolio re-adjusted at each evolution time in our simulation, where the deltas are estimated by differentiating the least-squares continuation value estimates required for the exercise strategy.

All upper bounds were calculated using the extension in [18] to the Andersen–Broadie method, [2], where the input used was the least-squares exercise strategy. As in Tables B.1-B.3, the column labeled vega in Table B.11 contains the change in price (basis points) when the volatility for all forward rates is increased by 1%. It is used to determine whether the lower bounds calculated are sufficiently accurate from a practical point of view, where a difference between lower and upper bounds of less than one vega indicates that the lower bound is accurate from a practical point of view.

For the Greeks, we used the adjoint-improved pathwise method with the extension due to Piterbarg, [22], which takes the sensitivity with respect to the exercise strategy to be zero. In practical terms, for each path we differentiate all coupons and/or rebates as per the usual pathwise method up until the time of exercise. As is commonly the case in practice, we used the least-squares exercise strategy to compute deltas and elementary vegas. As above, in Table B.12, the row labeled ”E. Vega" contains the elementary vega giving the sensitivity with respect to the leading element of each pseudo-square root of the covariance matrix across each step in the simulation.

The times recorded do not include the time taken to develop the least-squares approximate exercise strategy. However, in each case we used 20,000 first pass paths and this always took less than 10 seconds. In addition, when using the pathwise method, it is possible to compute prices and Greeks in the same simulation. However, we have considered prices and Greeks separately to both provide individual results, and so that we could use different sampling days and drift approximations. All computations for cancellable range accruals were done using single-threaded code on a desktop computer with a 3.16 GHz Intel Core2Duo processor.

Table B.11 demonstrates a number of key results regarding prices. First, all methods (except those not sufficiently accurate for the non-cancellable case) produce very similar bounds (up to Monte Carlo error) for a given Example and interpolation scheme. Second, significantly different prices are obtained for Examples 2 and 3 under the different interpolation schemes. Third, except for Example 3, the lower bounds (obtained by adding the policy iteration improvement to the least-squares lower bound) are much less than one vega, and therefore more than adequate from a practical point of view. Even for Example 3, the lower bounds are approximately one vega from the upper bound under the Piterbarg interpolation scheme, and just over for our multiplicative scheme, and are therefore reasonable, especially since the final gap is well under 10 basis points. Finally, all the methods are very quick. To see this, consider the least-squares lower bounds calculated using Sobol quasi-random numbers. The number of paths used for these calculations were chosen so that prices were converged to within approximately 3 basis points, and should represent what would actually be used in practice. Then, using these, the Practical Policy Iteration algorithm could be used in under one minute in each case considered. In addition, if policy iteration is not used, then lower bounds within 30 basis points (and usually much less) of the corresponding upper bound can be obtained in under 20 seconds using single-threaded code.

From Table B.12, we see that the elementary vegas for a given interpolation method and Example are very similar for the different approaches, except for the present value approach and Approximation 2 under Examples 2 and 3. This suggests that the methods which were sufficiently accurate for the non-cancellable case are again producing accurate elementary vegas.

For deltas, the accuracy obtained was very similar to the non-cancellable case. We have not included the results as they do not add much to those given in Tables B.5-B.10.

Table B.12 also demonstrates that deltas and elementary vegas can be computed quickly in each case. Even when using an excessive 524288 paths, it always took under 3 minutes when using the bridging approach with Approximation 2.

Finally, we note that the deltas and elementary vegas obtained for the cancellable case are noticeably different under the different interpolation schemes.

7. Conclusion

We have introduced a new interpolation scheme. Using both this new scheme and the Piterbarg interpolation method, we have introduced and studied a number of new ways to simulate range accrual coupons in the displaced-diffusion LIBOR market model. These lead to significant efficiency improvements when pricing, and allow the use of the pathwise method to calculate the Greeks of callable range accruals, which was not possible previously. From the numerical results, a clear conclusion emerges: use the new bridging approach with Approximation 2 and our multiplicative interpolation scheme. For pricing, 2 sampling days is adequate. However, when calculating Greeks, 4 sampling days should be used. In addition, the present value approach can be used to identify approximately sub-optimal points when handling cancellable range accruals.

References

[1] F. M. Ametrano and M. S. Joshi. Smooth simultaneous calibration of the LIBOR market model to caplets and co-terminal swaptions, 2008. http://ssrn.com/abstract=1092665.

[2] L. Andersen and M. Broadie. A primal-dual simulation algorithm for pricing multi-dimensional American options. *Management Science*, 50:1222–1234, 2004.

[3] D. Belomestny, A. Kolodko, and J. Schoenmakers. Pricing CMS spreads in the LIBOR market model, 2008. http://www.matheon.de/research/show_preprint.asp?action=details&serial=537.

[4] C. J. Beveridge and M. S. Joshi. Juggling snowballs. *Risk*, December:100–104, 2008.

[5] C. J. Beveridge and M. S. Joshi. Practical policy iteration: generic methods for obtaining rapid and tight bounds for Bermudan exotic derivatives using Monte Carlo simulation, 2009. http://ssrn.com/abstract=1331904.
[6] A. Brace. *Engineering BGM*. Chapman and Hall, Sydney, 2007.
[7] A. Brace, D. Gatarek, and M. Musiela. The market model of interest rate dynamics. *Mathematical Finance*, 7:127–155, 1997.
[8] D. Brigo and F. Mercurio. *Interest Rate Models: Theory and Practice*. Springer Verlag, Heidelberg, 2001.
[9] G. Chalamandaris. Pricing multicallable range accruals with the LIBOR market model. *Managerial Finance*, 33:292–308, 2007.
[10] M. Giles and P. Glasserman. Smoking adjoints: fast Monte Carlo Greeks. *Risk*, January:92–96, 2006.
[11] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer, New York, 2004.
[12] P. Glasserman and X. Zhao. Fast Greeks by simulation in forward LIBOR models. *Journal of Computational Finance*, 3:5–39, 1999.
[13] J. Hull and A. White. Forward rate volatilities, swap rate volatilities, and the implementation of the LIBOR market model, 1999. http://www.rotman.utoronto.ca/ãmackay/fin/libormktmodel2.pdf.
[14] C. Hunter, P. Jäckel, and M. S. Joshi. Getting the drift. *Risk*, July:81–84, 2001.
[15] F. Jamshidian. LIBOR and swap market models and measures. *Finance and Stochastics*, 1:293–330, 1997.
[16] M. S. Joshi. *The Concepts and Practice of Mathematical Finance*. Cambridge University Press, London, 2003.
[17] M. S. Joshi. Rapid drift computations in the LIBOR market model. *Wilmott Magazine*, May:84–85, 2003.
[18] M. S. Joshi. Monte Carlo bounds for callable products with non-analytic break costs, 2006. http://ssrn.com/abstract=907407.
[19] M. S. Joshi and L. Liesch. Effective implementation of generic market models. *ASTIN Bulletin*, 37:453–473, 2007.
[20] M. Li, S. Deng, and J. Zhou. Closed-form approximations for spread option prices and greeks. *Journal of Derivatives*, 15:58–81, 2008.
[21] M. Musiela and M. Rutkowski. Continuous-time term structure models: forward-measure approach. *Finance and Stochastics*, 1:261–292, 1997.
[22] V. Piterbarg. Computing Deltas of callable LIBOR exotics in forward LIBOR models. *Journal of Computational Finance*, 7:107–144, 2004.
[23] V. Piterbarg. A practitioner's guide to pricing and hedging callable LIBOR exotics in forward LIBOR models. *Journal of Computational Finance*, 8:65–119, 2004.
[24] E. Schlögl. Arbitrage-free interpolation in models of market observable interest rates. In K. Sandmann and P. Schönbucher, editors, *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann*, pages 197–218. Springer Verlag, Heidelberg, 2002.
[25] L. Tong. *The Multivariate Normal Distribution*. Springer Verlag, New York, 1990.
[26] T.-P. Wu and S.-N. Chen. Valuation of floating range notes in a LIBOR market model. *The Journal of Futures Markets*, 28:697–710, 2008.

Appendix A

Proof of Theorem 4.3. The Absence of Internal Arbitrage is guaranteed to hold by construction, as is Stochasticity provided (4.6) holds.

For Positivity, first consider

$$L(t_1,t_2) =$$
$$\frac{1}{\delta}\left[\frac{(1+\delta L_{\eta(t_2)}(t_1))\mathbb{E}^{t_1}_{T_{\eta(t_2)}}\left(1+\theta(t_2)L_{\eta'(t_2)}(t_2)\left[\xi(t_2)+(1-\xi(t_2))\frac{L_{\eta(t_2)}(t_2)}{L_{\eta(t_2)}(T_{\eta'(t_2)})}\right]\right)}{\mathbb{E}^{t_1}_{T_{\eta(t_2)+1}}\left(1+\theta(t_2+\delta)L_{\eta(t_2)}(t_2+\delta)\left[\xi(t_2+\delta)+(1-\xi(t_2+\delta))\frac{L_{\eta(t_2)+1}(t_2+\delta)}{L_{\eta(t_2)+1}(T_{\eta(t_2)})}\right]\right)}-1\right].$$

Consider the case where $t_1 < T_{\eta'(t_2)}$. Using the Tower property and the standard result for $\mathbb{E}_j\left(L_j(t)\right)$ (see, for example, [24]),

$$1+\delta L(t_1,t_2)=$$
$$\frac{(1+\delta L_{\eta(t_2)}(t_1))\mathbb{E}^{t_1}_{T_{\eta(t_2)}}\left(1+\theta L_{\eta'(t_2)}(t_2)\left[\xi(t_2)+(1-\xi(t_2))g(T_{\eta'(t_2)},t_2,L_{\eta(t_2)}(T_{\eta'(t_2)}))\right]\right)}{\mathbb{E}^{t_1}_{T_{\eta(t_2)+1}}\left(1+\theta L_{\eta(t_2)}(t_2+\delta)\left[\xi(t_2+\delta)+(1-\xi(t_2+\delta))g(T_{\eta(t_2)},t_2+\delta,L_{\eta(t_2)+1}(T_{\eta(t_2)}))\right]\right)},$$

where $\theta:=\theta(t_2)=\theta(t_2+\delta)$ and g is given by (4.5).

Now, as
$$1<g(t_1,t_2,x)<e^{V_{\eta(t_2)}(t_1,t_2)},$$
then
$$L(t_1,t_2)>\frac{1}{\delta}\left[\frac{(1+\delta L_{\eta(t_2)}(t_1))\left(1+\theta L_{\eta'(t_2)}(t_1)\right)}{\left(1+\theta L_{\eta(t_2)}(t_1)\left[\xi(t_2+\delta)+(1-\xi(t_2+\delta))e^{V_{\eta(t_2)+1}(T_{\eta(t_2)},t_2+\delta)}\right]\right)}-1\right].$$

Thus, a sufficient condition for $L(t_1,t_2)>0$ is
$$(1+\delta L_{\eta(t_2)}(t_1))\left(1+\theta L_{\eta'(t_2)}(t_1)\right)>\left(1+\theta L_{\eta(t_2)}(t_1)e^{V_{\eta(t_2)+1}(T_{\eta(t_2)},t_2+\delta)}\right),$$

which gives (4.8).

The case where $t_1\ge T_{\eta'(t_2)}$ can be done in the same way, and can easily be shown to complete the Positivity result.

□

Proof of Proposition 4.1. Substituting (4.7) into (4.1) gives
$$\frac{P(t_1,t_2)}{P_{\eta(t_2)}(t_1)}=1+\theta(t_2)\left[\xi(t_2)L_{\eta'(t_2)}(t_1)+(1-\xi(t_2))\left\{\tilde{L}_{\eta'(t_2)}(t_1)h(t_1,t_2,L_{\eta(t_2)}(t_1))-\alpha\right\}\right],$$
where
$$h(t_1,t_2,L_{\eta(t_2)}(t_1)):=\frac{1}{\tilde{L}_{\eta'(t_2)}(t_1)}\mathbb{E}^{t_1}_{T_{\eta(t_2)}}\left[\tilde{L}_{\eta'(t_2)}(T_{\eta'(t_2)})\frac{\tilde{L}_{\eta(t_2)}(t_2)}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)})}\right].$$

For $t_1\ge T_{\eta'(t_2)}$,
$$\begin{aligned}
h &= \frac{1}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)})}\mathbb{E}^{t_1}_{T_{\eta(t_2)}}\left[\tilde{L}_{\eta(t_2)}(t_2)\right],\\
&= \frac{1}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)})}\frac{1}{1+\delta L_{\eta(t_2)}(t_1)}\mathbb{E}^{t_1}_{T_{\eta(t_2)+1}}\left[\tilde{L}_{\eta(t_2)}(t_2)(1+\delta L_{\eta(t_2)}(t_2))\right],\\
&= \frac{1}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)})}\frac{1}{1+\delta L_{\eta(t_2)}(t_1)}\mathbb{E}^{t_1}_{T_{\eta(t_2)+1}}\left[\tilde{L}_{\eta(t_2)}(t_2)(1-\delta\alpha+\delta\tilde{L}_{\eta(t_2)}(t_2))\right],\\
&= \frac{1}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)})}\frac{1}{1+\delta L_{\eta(t_2)}(t_1)}\left[(1-\delta\alpha)\tilde{L}_{\eta(t_1)}(t_2)+\delta(\tilde{L}_{\eta(t_2)}(t_1))^2e^{V_{\eta(t_2)}(t_1,t_2)}\right],\\
&= \frac{\tilde{L}_{\eta(t_1)}(t_2)}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)})}\frac{1-\delta\alpha+\delta\tilde{L}_{\eta(t_1)}(t_2)e^{V_{\eta(t_2)}(t_1,t_2)}}{1+\delta L_{\eta(t_2)}(t_1)},
\end{aligned}$$
where we have dropped the arguments for h to ease notation.

For $t_1 < T_{\eta'(t_2)}$,

$$
\begin{aligned}
h &= \frac{1}{\tilde{L}_{\eta'(t_2)}(t_1)} \mathbb{E}^{t_1}_{T_{\eta(t_2)}} \left[\frac{\tilde{L}_{\eta'(t_2)}(T_{\eta'(t_2)})}{\tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)})} \mathbb{E}^{T_{\eta(t_2)-1}}_{T_{\eta(t_2)}} \left[\tilde{L}_{\eta(t_2)}(t_2) \right] \right], \\
&= \frac{1}{\tilde{L}_{\eta'(t_2)}(t_1)} \mathbb{E}^{t_1}_{T_{\eta(t_2)}} \left[\tilde{L}_{\eta'(t_2)}(T_{\eta'(t_2)}) \frac{1 - \delta\alpha + \delta \tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)}) e^{V_{\eta(t_2)}(T_{\eta(t_2)-1}, t_2)}}{1 + \delta L_{\eta(t_2)}(T_{\eta'(t_2)})} \right], \\
&= \frac{1}{\tilde{L}_{\eta'(t_2)}(t_1)} \frac{1}{(1 + \delta L_{\eta(t_2)}(t_1))} \\
&\qquad \mathbb{E}^{t_1}_{T_{\eta(t_2)+1}} \left[\tilde{L}_{\eta'(t_2)}(T_{\eta'(t_2)}) \left(1 - \delta\alpha + \delta \tilde{L}_{\eta(t_2)}(T_{\eta'(t_2)}) e^{V_{\eta(t_2)}(T_{\eta(t_2)-1}, t_2)} \right) \right].
\end{aligned}
$$

Now by applying the drift-freezing approximation to $L_{\eta'(t_2)}(.)$ at time t_1, we get

$$
\begin{aligned}
h &= \frac{1}{\tilde{L}_{\eta'(t_2)}(t_1)} \frac{1}{(1 + \delta L_{\eta(t_2)}(t_1))} \Bigg[(1 - \delta\alpha) \tilde{L}_{\eta'(t_2)}(t_1) e^{-\frac{\tilde{L}_{\eta(t_2)}(t_1)\delta}{1 + \delta L_{\eta(t_2)}(t_1)} C_{\eta(t_2), \eta'(t_2)}(t_1, T_{\eta'(t_2)})} + \\
&\qquad \delta \tilde{L}_{\eta'(t_2)}(t_1) \tilde{L}_{\eta(t_2)}(t_1) e^{C_{\eta(t_2), \eta'(t_2)}(t_1, T_{\eta(t_2)-1}) + V_{\eta(t_2)}(T_{\eta(t_2)-1}, t_2)} e^{-\frac{\tilde{L}_{\eta(t_2)}(t_1)\delta}{1 + \delta L_{\eta(t_2)}(t_1)} C_{\eta(t_2), \eta'(t_2)}(t_1, T_{\eta'(t_2)})} \Bigg], \\
&= e^{-\frac{\tilde{L}_{\eta(t_2)}(t_1)\delta}{1 + \delta L_{\eta(t_2)}(t_1)} C_{\eta(t_2), \eta'(t_2)}(t_1, T_{\eta'(t_2)})} \frac{\left[1 - \delta\alpha + \delta \tilde{L}_{\eta(t_2)}(t_1) e^{C_{\eta(t_2), \eta'(t_2)}(t_1, T_{\eta(t_2)-1}) + V_{\eta(t_2)}(T_{\eta(t_2)-1}, t_2)} \right]}{1 + \delta L_{\eta(t_2)}(t_1)}.
\end{aligned}
$$

The result then follows by combining the above. $\square$

Proof of Proposition 5.1. Ignoring drifts and applying Ito's lemma twice,

$$
\begin{aligned}
d\widehat{\mathrm{SR}}_{t,x}(s) &= \sum_{j=0}^{x-1} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(s)}{\partial \hat{L}(s, t + j\delta)} d\hat{L}(s, t + j\delta), \\
&= \sum_{j=0}^{x-1} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(s)}{\partial \hat{L}(s, t + j\delta)} \left(X^0_j(L(s)) dL_{i-1+j}(s) + \right. \\
&\qquad \left. X^1_j(L(s)) dL_{i+j}(s) + X^2_j(L(s)) dL_{i+j+1}(s) \right).
\end{aligned}
$$

Still ignoring drifts,

$$
\begin{aligned}
d \log \left(\widetilde{\widehat{\mathrm{SR}}}_{t,x}(s) \right) &= \frac{1}{\widetilde{\widehat{\mathrm{SR}}}_{t,x}(s)} \sum_{j=0}^{x-1} \frac{\partial \widehat{\mathrm{SR}}_{t,x}(s)}{\partial \hat{L}(s, t + j\delta)} \left(X^0_j(L(s)) dL_{i-1+j}(s) \right. \\
&\qquad \left. + X^1_j(L(s)) dL_{i+j}(s) + X^2_j(L(s)) dL_{i+j+1}(s) \right). \qquad \text{(A.1)}
\end{aligned}
$$

Writing the SDE in terms of a single standard Brownian motion, and freezing all state-dependence at z,

$$d\log\left(\widetilde{\widehat{\mathrm{SR}}}_{t,x}(s)\right) \approx \frac{1}{\widetilde{\widehat{\mathrm{SR}}}_{t,x}(z)}\left(\sum_{j,k=0}^{x-1}\frac{\partial\widehat{\mathrm{SR}}_{t,x}(z)}{\partial\hat{L}(z,t+j\delta)}\frac{\partial\widehat{\mathrm{SR}}_{t,x}(z)}{\partial\hat{L}(z,t+k\delta)}\right.$$
$$\left.\left(\sum_{m,n=0}^{2}\tilde{L}_{i-1+j+m}(z)\tilde{L}_{i-1+k+n}(z)X_j^m(L(z))X_k^n(L(z))\sigma_{i-1+j+m,i-1+k+n}(s)\right)\right)^{0.5} dW_{t,x}(s).$$

□

Proof of Proposition 5.2. We start by deriving the drifts for

$$\mathrm{SR}_{t,x}(s) = \frac{P(s,t)-P(s,t+x\delta)}{\sum_{j=1}^{x}\delta P(s,t+j\delta)},$$

and then freeze all time-dependence due to the interpolation scheme at time t. Using the result from [19],

$$\mu_{t,x}(L(s),s) = -\frac{1}{(\mathrm{SR}_{t,x}(s)+\alpha)}\frac{P(s,T_i)}{\sum_{j=1}^{x}\delta P(s,t+j\delta)}\left\langle \mathrm{SR}_{t,x}(s),\sum_{j=1}^{x}\delta\frac{P(s,t+j\delta)}{P(s,T_i)}\right\rangle, \quad \text{(A.2)}$$

under the spot LIBOR probability measure, where $\langle A(t),B(t)\rangle$ denotes the cross-variation derivative of two stochastic processes $A(t)$ and $B(t)$, and corresponds to the coefficient of dt in $dA(t)dB(t)$. Note that this equation implicitly assumes that the discounted prices processes for the non-tenor bonds are martingales in the spot LIBOR probability measure. This is therefore an additional approximation under the Piterbarg interpolation method.

Under the Piterbarg interpolation scheme,

$$\begin{aligned}
&\mu_{t,x}(L(s),s)\\
&= -\frac{1}{\widetilde{\mathrm{SR}}_{t,x}(s)}\frac{P(s,T_i)}{A_0(s)}\sum_{j=1}^{x}\delta\left\langle \mathrm{SR}_{t,x}(s),\frac{\prod_{k=i}^{i-1+j-1}(1+\delta L_k(s))^{-1}}{(1+\vartheta(t+j\delta)L_{i-1+j}(s))}\right\rangle,\\
&= -\frac{1}{\widetilde{\mathrm{SR}}_{t,x}(s)}\frac{P(s,T_i)}{A_0(s)}\sum_{j=1}^{x}\delta\left(\sum_{k=i}^{i-1+j-1}\delta\left\langle \mathrm{SR}_{t,x}(s),L_k(s)\right\rangle\frac{-1}{(1+\delta L_k(s))}\frac{P(s,t+j\delta)}{P_i(s)}\right.\\
&\qquad\left.+\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+j}(s)\right\rangle\frac{-\vartheta(t+j\delta)}{(1+\vartheta(t+j\delta)L_{i-1+j}(s))}\frac{P(s,t+j\delta)}{P_i(s)}\right),\\
&= \frac{1}{\widetilde{\mathrm{SR}}_{t,x}(s)}\frac{1}{A_0(s)}\left(\sum_{k=1}^{x-1}\frac{\delta A_k(s)}{1+\delta L_{i-1+k}(s)}\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+k}(s)\right\rangle+\right.\\
&\qquad\left.\sum_{j=1}^{x}\frac{\vartheta(t+j\delta)\delta P(s,t+j\delta)}{(1+\vartheta(t+j\delta)L_{i-1+j}(s))}\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+j}(s)\right\rangle\right).
\end{aligned}$$

By freezing all state-dependence at time z, and all time-dependence in the interpolation scheme at time t, the above expression equals (5.13).

□

Proof of Proposition 5.3. We use the same approach as that used in proving Proposition 5.2. Note that since the discounted non-tenor bond prices are martingales by construction in the Schlögl interpolation framework, (A.2) holds without additional approximation. We then have,

$$\begin{aligned}
\mu_{t,x}(L(s),s) &= -\frac{1}{\widetilde{\mathrm{SR}}_{t,x}(s)}\frac{P_i(s)}{A_0(s)}\sum_{j=1}^{x}\delta\left\langle \mathrm{SR}_{t,x}(s), \frac{P(s,T_{i+j})}{P(s,T_i)}\frac{P(s,t+j\delta)}{P(s,T_{i+j})}\right\rangle, \\
&= -\frac{1}{\widetilde{\mathrm{SR}}_{t,x}(s)}\frac{P_i(s)}{A_0(s)}\sum_{j=1}^{x}\left(\frac{P(s,t+j\delta)}{P_{i+j}(s)}\left\langle \mathrm{SR}_{t,x}(s), \prod_{k=i}^{i-1+j}(1+\delta L_k(s))^{-1}\right\rangle + \right. \\
&\quad \frac{P_{i+j}(s)}{P_i(s)}\theta(t+j\delta)\left\langle \mathrm{SR}_{t,x}(s), \xi(t+j\delta)L_{i-1+j}(s)+\right. \\
&\quad\quad \left.\left.(1-\xi(t+j\delta))\tilde{L}_{i-1+j}(s)h(s,t+j\delta,L_{i+j}(s))\right\rangle\right), \\
&= -\frac{1}{\widetilde{\mathrm{SR}}_{t,x}(s)}\frac{P_i(s)}{A_0(s)}\sum_{j=1}^{x}\left(\frac{P(s,t+j\delta)}{P_{i+j}(s)}\sum_{k=i}^{i-1+j}\frac{-\delta\left\langle \mathrm{SR}_{t,x}(s),L_k(s)\right\rangle}{1+\delta L_k(s)}\frac{P_{i+j}(s)}{P_i(s)}+\right. \\
&\quad \frac{P_{i+j}(s)}{P_i(s)}\theta(t+j\delta)\left(\xi(t+j\delta)\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+j}(s)\right\rangle + \right. \\
&\quad (1-\xi(t+j\delta))h(s,t+j\delta,L_{i+j}(s))\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+j}(s)\right\rangle + \\
&\quad \left.\left.(1-\xi(t+j\delta))\tilde{L}_{i-1+j}(s)\frac{\partial h(s,t+j\delta,L_{i+j}(s))}{\partial L_{i+j}(s)}\left\langle \mathrm{SR}_{t,x}(s),L_{i+j}(s)\right\rangle\right)\right), \\
&= \frac{1}{\widetilde{\mathrm{SR}}_{t,x}(s)}\frac{\delta}{A_0(s)}\left[\sum_{k=1}^{x}\frac{A_{k-1}(s)}{1+\delta L_{i-1+k}(s)}\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+k}(s)\right\rangle - \right. \\
&\quad \sum_{j=1}^{x}P_{i+j}(s)\theta(t+j\delta)\left(\xi(t+j\delta)\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+j}(s)\right\rangle + \right. \\
&\quad (1-\xi(t+j\delta))h(s,t+j\delta,L_{i+j}(s))\left\langle \mathrm{SR}_{t,x}(s),L_{i-1+j}(s)\right\rangle + \\
&\quad \left.\left.(1-\xi(t+j\delta))\tilde{L}_{i-1+j}(s)\frac{\partial h(s,t+j\delta,L_{i+j}(s))}{\partial L_{i+j}(s)}\left\langle \mathrm{SR}_{t,x}(s),L_{i+j}(s)\right\rangle\right)\right].
\end{aligned}$$

By rearranging after freezing all state-dependence at time z, and all time-dependence in the interpolation scheme at time t, the above expression equals (5.14). □

Appendix B

Method	Sampling Days							Vega
	365	52	12	6	4	2	1	
mult.								
SS	443.1	444.6	442.2	442.3	444.0	442.7	444.9	
	(100)	(13.41)	(3.17)	(1.71)	(1.25)	(0.75)	(0.54)	
PV, 1	443.2	443.0	442.8	442.6	442.8	444.0	442.8	207.6
	(6.70)	(1.17)	(0.46)	(0.36)	(0.32)	(0.29)	(0.28)	
PV, 2	440.3	440.1	439.9	439.7	440.0	441.2	440.6	
	(5.17)	(0.96)	(0.43)	(0.34)	(0.31)	(0.28)	(0.27)	
BB, 1	443.2	442.9	442.7	442.4	442.7	443.9	442.7	
	(10.05)	(1.65)	(0.57)	(0.41)	(0.36)	(0.30)	(0.28)	
BB, 2	443.2	442.9	442.8	442.5	442.8	443.9	442.8	
	(9.68)	(1.60)	(0.56)	(0.40)	(0.36)	(0.30)	(0.28)	
Chal.	377.6	377.3	377.0	376.3	375.1	368.5	342.1	
	(0.28)	(0.26)	(0.24)	(0.24)	(0.23)	(0.23)	(0.24)	
Pit.								
SS	368.5	366.8	368.4	366.2	366.3	357.9	327.36	
	(100)	(13.19)	(3.08)	(1.68)	(1.20)	(0.73)	(0.53)	
PV, 1	369.0	368.8	368.5	367.8	366.5	359.4	326.3	207.9
	(7.48)	(1.28)	(0.50)	(0.39)	(0.35)	(0.31)	(0.28)	
PV, 2	366.0	365.8	365.5	364.8	363.6	356.5	324.0	
	(6.06)	(1.10)	(0.47)	(0.36)	(0.33)	(0.30)	(0.28)	
BB, 1	368.2	368.0	367.7	367.0	365.7	358.5	325.2	
	(7.95)	(1.35)	(0.52)	(0.39)	(0.35)	(0.30)	(0.29)	
BB, 2	367.9	367.7	367.4	366.7	365.4	358.2	324.7	
	(7.61)	(1.31)	(0.51)	(0.39)	(0.34)	(0.30)	(0.28)	
Chal.	377.6	377.3	377.0	376.3	375.1	368.5	342.1	
	(0.32)	(0.30)	(0.27)	(0.27)	(0.26)	(0.26)	(0.27)	

Table B.1. *Prices and relative times for a non-cancellable range accrual where the reference rate is the spot 1 year LIBOR rate (Example 1). Prices are the non-bracketed numbers and are given in basis points. The bracketed numbers give the proportion of time (%) relative to short-stepping with 365 sampling days. Calculations were done using* 2^{19} *Sobol paths together with Brownian bridging to ensure a given level of convergence. The numbers after PV and BB denote which Approximation was used.*

Method	Sampling Days							Vega
	365	52	12	6	4	2	1	
mult.								
SS	78.8	79.1	79.3	79.5	78.4	78.6	74.4	
	(100)	(13.20)	(3.19)	(1.71)	(1.21)	(0.72)	(0.51)	
PV, 1	77.0	76.9	76.9	76.8	76.6	75.6	71.7	21.4
	(84.57)	(11.18)	(2.76)	(1.48)	(1.08)	(0.64)	(0.43)	
PV, 2	66.3	66.2	66.2	66.2	66.0	64.8	60.4	
	(22.76)	(3.42)	(0.96)	(0.59)	(0.47)	(0.35)	(0.28)	
BB, 1	80.4	80.3	80.3	80.3	80.1	79.3	76.3	
	(97.02)	(13.74)	(3.33)	(1.77)	(1.25)	(0.75)	(0.49)	
BB, 2	77.2	77.1	77.1	77.0	76.8	75.7	71.5	
	(37.27)	(5.54)	(1.45)	(0.84)	(0.64)	(0.44)	(0.33)	
Chal.	65.9	65.8	65.7	65.5	65.0	63.3	55.2	
	(0.31)	(0.30)	(0.27)	(0.26)	(0.26)	(0.26)	(0.26)	
Pit.								
SS	110.6	109.9	110.0	111.4	110.1	113.2	121.9	
	(100)	(12.39)	(3.07)	(1.67)	(1.19)	(0.72)	(0.51)	
PV, 1	108.7	108.6	108.7	109.0	109.4	111.5	119.4	21.9
	(62.56)	(8.71)	(2.23)	(1.22)	(0.91)	(0.57)	(0.40)	
PV, 2	102.4	102.4	102.5	102.7	103.2	105.6	115.0	
	(17.70)	(2.75)	(0.82)	(0.53)	(0.44)	(0.34)	(0.29)	
BB, 1	112.0	111.9	112.0	112.3	112.7	115.0	124.0	
	(65.56)	(9.52)	(2.35)	(1.31)	(0.96)	(0.60)	(0.42)	
BB, 2	110.9	110.8	110.9	111.1	111.5	113.7	122.2	
	(26.09)	(3.88)	(1.09)	(0.67)	(0.53)	(0.39)	(0.32)	
Chal.	65.9	65.8	65.7	65.5	65.0	63.3	55.2	
	(0.35)	(0.34)	(0.31)	(0.30)	(0.30)	(0.29)	(0.30)	

Table B.2. *Prices and relative times for a non-cancellable range accrual where the reference rate is a CMS spread (Example 2). Prices are the non-bracketed numbers and are given in basis points. The bracketed numbers give the proportion of time (%) relative to short-stepping with 365 sampling days. Calculations were done using* 2^{19} *Sobol paths together with Brownian bridging to ensure a given level of convergence. The numbers after PV and BB denote which Approximation was used.*

Method	Sampling Days							Vega
	365	52	12	6	4	2	1	
mult.								
SS	332.2	330.3	330.3	331.3	330.4	331.4	327.04	
	(100)	(13.39)	(3.15)	(1.71)	(1.20)	(0.72)	(0.50)	
PV, 1	328.9	328.5	328.5	328.4	328.4	327.9	324.5	27.6
	(92.97)	(13.42)	(3.29)	(1.77)	(1.25)	(0.75)	(0.45)	
PV, 2	332.5	332.1	332.1	332.1	332.1	331.5	328.6	
	(30.86)	(4.60)	(1.22)	(0.72)	(0.55)	(0.39)	(0.30)	
BB, 1	331.0	330.6	330.6	330.6	330.6	330.4	328.0	
	(96.92)	(13.28)	(3.30)	(1.73)	(1.23)	(0.73)	(0.47)	
BB, 2	327.7	327.3	327.3	327.2	327.2	326.5	322.9	
	(96.92)	(5.43)	(1.42)	(0.82)	(0.62)	(0.42)	(0.32)	
Chal.	336.6	336.2	336.2	336.2	336.0	335.8	335.8	
	(0.30)	(0.29)	(0.26)	(0.26)	(0.26)	(0.25)	(0.25)	
Pit.								
SS	351.2	348.7	348.4	351.5	349.3	351.2	354.7	
	(100)	(12.87)	(3.12)	(1.68)	(1.21)	(0.73)	(0.49)	
PV, 1	349.5	349.2	349.2	349.3	349.5	350.5	353.1	28.3
	(74.48)	(10.63)	(2.77)	(1.47)	(1.05)	(0.65)	(0.44)	
PV, 2	358.6	358.2	358.3	358.4	358.7	360.3	365.9	
	(28.50)	(4.22)	(1.16)	(0.70)	(0.55)	(0.40)	(0.32)	
BB, 1	351.1	350.7	350.8	350.9	351.1	352.2	356.5	
	(65.41)	(9.61)	(2.43)	(1.33)	(0.98)	(0.62)	(0.43)	
BB, 2	351.0	350.6	350.7	350.8	351.0	352.2	356.3	
	(26.46)	(3.93)	(1.11)	(0.68)	(0.54)	(0.40)	(0.32)	
Chal.	336.6	336.2	336.2	336.2	336.0	335.8	335.8	
	(0.35)	(0.35)	(0.31)	(0.31)	(0.30)	(0.30)	(0.29)	

Table B.3. *Prices and relative times for a non-cancellable range accrual where the reference rate is a CMS spread and there is a stochastic multiplier (Example 3). Prices are the non-bracketed numbers and are given in basis points. The bracketed numbers give the proportion of time (%) relative to short-stepping with 365 sampling days. Calculations were done using* 2^{19} *Sobol paths together with Brownian bridging to ensure a given level of convergence. The numbers after PV and BB denote which Approximation was used.*

Method	Example		
	1	2	3
mult.			
SS	5.71 (0.45)	1.72 (0.13)	3.33 (0.81)
PV, 1	6.88 (-0.84)	3.21 (3.16)	4.34 (4.34)
PV, 2	7.60 (1.89)	14.01 (14.01)	5.08 (0.68)
BB, 1	5.75 (-0.76)	1.17 (-0.55)	2.49 (1.85)
BB, 2	5.79 (-0.80)	3.12 (3.12)	5.72 (5.72)
Chal.	74.64 (74.64)	15.52 (15.52)	9.03 (-3.62)
Pit.			
SS	11.62 (10.64)	2.60 (-2.60)	2.39 (-0.01)
PV, 1	9.81 (9.10)	1.21 (-0.91)	2.68 (0.70)
PV, 2	12.18 (11.98)	5.00 (5.00)	10.82 (-9.10)
BB, 1	10.35 (9.98)	4.38 (-4.38)	2.30 (-1.07)
BB, 2	10.61 (10.27)	3.08 (-3.08)	3.26 (-1.07)
Chal.	26.26 (0.04)	47.38 (47.38)	16.70 (15.34)

Table B.4. *Sum of absolute errors (non-bracketed) and errors (bracketed) in the individual coupon values. All numbers are in basis points. Calculations were done using* 2^{19} *Sobol paths together with Brownian bridging to ensure a given level of convergence. The numbers after PV and BB denote which Approximation was used. Exact answers were obtained using short-stepping with 365 sampling days, while 2 sampling days were used to obtain the results for the methods listed in the table.*

Method	Delta Rate							E. Vega	Time
	1	4	7	10	13	16	19		
PV, 12, FD	-.326	-.033	.3539	.4539	.4662	.4474	.4146	.2865	555.92
PV, 6, FD	-.3236	-.0333	.3538	.4538	.4663	.4474	.4147	.2864	394.41
PV, 4, FD	-.3186	-.0334	.3538	.4537	.4664	.4474	.4148	.2865	340.19
PV, 2, FD	-.2951	-.0339	.3536	.4535	.4664	.4472	.4149	.2875	286.47
PV, 12, Aprx	-.3272	-.0324	.3539	.4537	.466	.4473	.4145	.2869	290.94
PV, 6, Aprx	-.3248	-.0327	.3538	.4537	.4661	.4472	.4146	.2868	238.55
PV, 4, Aprx	-.3198	-.0328	.3538	.4536	.4663	.4472	.4146	.2869	219.88
PV, 2, Aprx	-.2965	-.0333	.3537	.4534	.4662	.4471	.4147	.2879	205.31
PV, 12, PC	-.3272	-.033	.3539	.4539	.4663	.4475	.4147	.285	269.5
PV, 6, PC	-.3248	-.0332	.3538	.4539	.4664	.4475	.4148	.2849	228.98
PV, 4, PC	-.3199	-.0333	.3538	.4538	.4665	.4475	.4148	.2851	214.34
PV, 2, PC	-.2966	-.0338	.3536	.4536	.4665	.4473	.4149	.2861	203.08
BB, 12, FD	-.3225	-.032	.3551	.4542	.4695	.4464	.4158	.2867	952.47
BB, 6, FD	-.321	-.0325	.3551	.4541	.4695	.4464	.4157	.2867	610.89
BB, 4, FD	-.3164	-.0328	.3552	.4541	.4695	.4464	.4156	.2869	497.33
BB, 2, FD	-.2933	-.0335	.3554	.4541	.4694	.4466	.4154	.2879	384.72
BB, 12, Aprx	-.3229	-.032	.3551	.4542	.4695	.4464	.4158	.2871	372.97
BB, 6, Aprx	-.3213	-.0325	.3552	.4541	.4695	.4464	.4157	.2871	283.78
BB, 4, Aprx	-.3167	-.0328	.3553	.4541	.4695	.4464	.4156	.2873	258.95
BB, 2, Aprx	-.2937	-.0335	.3555	.4542	.4695	.4466	.4154	.2883	227.06
BB, 12, PC	-.3229	-.032	.3551	.4542	.4695	.4464	.4158	.2871	359.66
BB, 6, PC	-.3214	-.0325	.3551	.4541	.4695	.4464	.4157	.2871	275.8
BB, 4, PC	-.3167	-.0328	.3552	.4541	.4695	.4464	.4156	.2873	252.33
BB, 2, PC	-.2937	-.0335	.3555	.4541	.4694	.4466	.4154	.2883	225.05

Table B.5. *Deltas, elementary vegas and times (in seconds) for a non-cancellable range accrual where the reference rate is the spot 1 year LIBOR rate (Example 1) and our multiplicative interpolation scheme. Calculations were done using* 2^{19} *Sobol paths.*

Method	Delta Rate							E. Vega	Time
	1	4	7	10	13	16	19		
PV, 12, FD	-.4076	-.0536	.3606	.4631	.4741	.4535	.42	.2329	435.94
PV, 6, FD	-.4047	-.0504	.3628	.4636	.4727	.4537	.4199	.2327	309.64
PV, 4, FD	-.4034	-.049	.3621	.464	.4736	.4528	.4194	.2317	265.48
PV, 2, FD	-.3893	-.0539	.3634	.4644	.4747	.4535	.42	.2271	225.02
PV, 12, Aprx	-.3892	-.0552	.3598	.4626	.4736	.4532	.4198	.233	275.78
PV, 6, Aprx	-.3861	-.0519	.362	.463	.4721	.4534	.4196	.2329	226.06
PV, 4, Aprx	-.3845	-.0506	.3612	.4634	.4731	.4525	.4191	.2319	209.73
PV, 2, Aprx	-.3687	-.0557	.3625	.4638	.4741	.4532	.4198	.2273	192.58
PV, 12, PC	-.3876	-.0553	.36	.4627	.4739	.4535	.4201	.2309	264.06
PV, 6, PC	-.3847	-.0517	.3625	.4634	.4727	.4537	.4197	.2308	221.44
PV, 4, PC	-.383	-.0504	.3614	.4637	.4732	.4528	.4194	.2297	207.55
PV, 2, PC	-.3673	-.0558	.3627	.464	.4747	.4534	.42	.2252	191.3
BB, 12, FD	-.4003	-.0454	.362	.4627	.4771	.4543	.4205	.2327	538.26
BB, 6, FD	-.3974	-.048	.363	.4629	.4759	.4535	.4207	.2321	375.92
BB, 4, FD	-.3953	-.0483	.3635	.4626	.4762	.4538	.4203	.2314	316.61
BB, 2, FD	-.3854	-.0487	.3635	.4637	.4771	.4548	.4195	.2267	261.23
BB, 12, Aprx	-.3961	-.0457	.3619	.4625	.477	.4542	.4205	.233	311.23
BB, 6, Aprx	-.3933	-.0484	.3629	.4628	.4758	.4534	.4207	.2324	249.12
BB, 4, Aprx	-.3911	-.0487	.3634	.4625	.4761	.4537	.4203	.2317	227.52
BB, 2, Aprx	-.3814	-.0489	.3634	.4636	.477	.4547	.4195	.227	207.14
BB, 12, PC	-.3956	-.0457	.3619	.4624	.4769	.454	.4206	.2328	306.92
BB, 6, PC	-.3927	-.0485	.3628	.4626	.4759	.4534	.4206	.2323	248.41
BB, 4, PC	-.3906	-.0489	.3634	.4623	.4761	.4538	.4204	.2315	226.73
BB, 2, PC	-.3809	-.049	.3634	.4632	.477	.4547	.4195	.2268	208.22

Table B.6. *Deltas, elementary vegas and times (in seconds) for a non-cancellable range accrual where the reference rate is the spot 1 year LIBOR rate (Example 1) and the Piterbarg interpolation scheme is used. Calculations were done using* 2^{19} *Sobol paths.*

Method	Delta Rate							E. Vega	Time
	1	4	7	10	13	16	19		
PV, 12, FD	1.6276	1.8919	.7067	.0712	-.5126	-.4451	-.1728	-.0753	11068.64
PV, 6, FD	1.627	1.8922	.7067	.0712	-.5126	-.4451	-.1728	-.0754	5576.83
PV, 4, FD	1.6258	1.8929	.7069	.0713	-.5127	-.4453	-.1729	-.0756	3765.31
PV, 2, FD	1.6188	1.8961	.708	.0718	-.5133	-.4459	-.1732	-.0765	1940.41
PV, 12, Aprx	1.6264	1.8975	.7113	.0765	-.5067	-.4397	-.17	-.0736	1005.64
PV, 6, Aprx	1.6258	1.8979	.7113	.0764	-.5068	-.4398	-.1701	-.0738	556.98
PV, 4, Aprx	1.6245	1.8986	.7115	.0765	-.5069	-.4399	-.1701	-.0739	404.98
PV, 2, Aprx	1.6175	1.9021	.7128	.0771	-.5073	-.4406	-.1704	-.0747	256.22
PV, 12, PC	1.6203	1.8875	.7143	.084	-.5024	-.4376	-.1694	-.079	459.31
PV, 6, PC	1.6197	1.888	.7143	.0839	-.5024	-.4377	-.1694	-.0791	281.66
PV, 4, PC	1.6182	1.8887	.7146	.0841	-.5025	-.4378	-.1695	-.0793	227.16
PV, 2, PC	1.6103	1.8922	.7161	.0849	-.5028	-.4384	-.1698	-.0803	165.28
BB, 12, FD	1.6278	1.894	.7083	.0684	-.5147	-.4449	-.1722	-.0753	26221.8
BB, 6, FD	1.6273	1.8945	.7086	.0683	-.5148	-.4451	-.1723	-.0755	13834.81
BB, 4, FD	1.6261	1.8951	.7089	.0682	-.5149	-.4453	-.1724	-.0757	8879
BB, 2, FD	1.6197	1.898	.7104	.0683	-.5153	-.4461	-.1728	-.0769	4481.84
BB, 12, Aprx	1.6271	1.8958	.709	.0688	-.5138	-.4439	-.1716	-.0738	1341
BB, 6, Aprx	1.6266	1.8963	.7093	.0686	-.5139	-.4441	-.1717	-.0739	723.38
BB, 4, Aprx	1.6255	1.897	.7096	.0686	-.5139	-.4442	-.1718	-.0741	526.69
BB, 2, Aprx	1.6189	1.9001	.7112	.0687	-.5142	-.4449	-.1721	-.0751	323.78
BB, 12, PC	1.6234	1.8927	.7102	.0713	-.5119	-.4426	-.1712	-.0754	829.2
BB, 6, PC	1.6229	1.8932	.7104	.0712	-.512	-.4428	-.1713	-.0756	471.28
BB, 4, PC	1.6216	1.8938	.7108	.0712	-.512	-.4429	-.1714	-.0758	352.22
BB, 2, PC	1.6144	1.8967	.7126	.0716	-.5121	-.4435	-.1717	-.077	234.88

Table B.7. *Deltas, elementary vegas and times (in seconds) for a non-cancellable range accrual where the reference rate is a CMS spread (Example 2) and our multiplicative interpolation scheme. Calculations were done using* 2^{19} *Sobol paths.*

Method	Delta Rate							E. Vega	Time
	1	4	7	10	13	16	19		
PV, 12, FD	1.6427	1.9093	.7055	.0626	-.5143	-.4497	-.1755	-.0684	6388.14
PV, 6, FD	1.6421	1.9099	.7056	.0625	-.5144	-.4498	-.1756	-.0685	3207.34
PV, 4, FD	1.6411	1.9109	.7058	.0624	-.5145	-.4501	-.1757	-.0685	2168.92
PV, 2, FD	1.6361	1.9162	.7067	.0617	-.5155	-.4513	-.1763	-.0689	1134.77
PV, 12, Aprx	1.6358	1.9142	.7097	.0692	-.5076	-.4448	-.1727	-.068	757.59
PV, 6, Aprx	1.6351	1.9148	.7098	.0691	-.5077	-.4449	-.1728	-.068	426.69
PV, 4, Aprx	1.634	1.9159	.71	.069	-.5078	-.4451	-.1729	-.0681	317.89
PV, 2, Aprx	1.6283	1.9213	.7109	.0685	-.5086	-.4465	-.1735	-.0684	209.56
PV, 12, PC	1.633	1.9076	.7113	.0743	-.5046	-.4442	-.1727	-.0712	374.55
PV, 6, PC	1.6322	1.9082	.7115	.0742	-.5047	-.4444	-.1728	-.0712	238.94
PV, 4, PC	1.6311	1.9093	.7117	.0741	-.5048	-.4446	-.1729	-.0713	193.55
PV, 2, PC	1.6251	1.915	.7127	.0737	-.5057	-.4461	-.1736	-.0716	146.06
BB, 12, FD	1.6417	1.9112	.7097	.0593	-.5157	-.4499	-.175	-.0678	12216.14
BB, 6, FD	1.6411	1.9119	.7099	.0593	-.5158	-.4501	-.1751	-.0679	5968.95
BB, 4, FD	1.6402	1.9129	.7102	.0591	-.5159	-.4503	-.1752	-.068	4096.88
BB, 2, FD	1.6355	1.9183	.7114	.0581	-.5169	-.4516	-.1758	-.0685	2104.17
BB, 12, Aprx	1.6404	1.9121	.7098	.0601	-.5147	-.449	-.1744	-.0677	888.22
BB, 6, Aprx	1.6397	1.9127	.7099	.0601	-.5148	-.4491	-.1744	-.0678	501.2
BB, 4, Aprx	1.6388	1.9138	.7102	.0599	-.515	-.4493	-.1745	-.0679	369.33
BB, 2, Aprx	1.6337	1.9192	.7115	.059	-.5158	-.4505	-.1751	-.0684	239.66
BB, 12, PC	1.6388	1.9104	.71	.0615	-.5137	-.4484	-.1742	-.0685	634.02
BB, 6, PC	1.6382	1.9111	.7102	.0615	-.5138	-.4485	-.1742	-.0685	377.23
BB, 4, PC	1.6372	1.9121	.7105	.0614	-.5139	-.4487	-.1743	-.0687	285.67
BB, 2, PC	1.6321	1.9173	.7117	.0605	-.5147	-.4498	-.1749	-.0692	200.38

Table B.8. *Deltas, elementary vegas and times (in seconds) for a non-cancellable range accrual where the reference rate is a CMS spread (Example 2) and the Piterbarg interpolation scheme is used. Calculations were done using* 2^{19} *Sobol paths.*

Method	Delta Rate							E. Vega	Time
	1	4	7	10	13	16	19		
PV, 12, FD	1.5379	1.1877	.2076	-.4118	-1.1341	-.6189	-.2352	.0183	14229.84
PV, 6, FD	1.5375	1.1881	.2076	-.4117	-1.1344	-.619	-.2352	.0183	7183.92
PV, 4, FD	1.5367	1.1886	.2078	-.4117	-1.1347	-.6193	-.2353	.0184	4805.5
PV, 2, FD	1.5318	1.1918	.2089	-.4116	-1.1364	-.6205	-.2358	.0189	2512.34
PV, 12, Aprx	1.5388	1.2036	.22	-.406	-1.1281	-.6151	-.234	.021	1212.42
PV, 6, Aprx	1.5384	1.204	.2201	-.406	-1.1283	-.6153	-.234	.021	660.31
PV, 4, Aprx	1.5376	1.2047	.2203	-.4059	-1.1286	-.6155	-.2341	.0212	479.48
PV, 2, Aprx	1.5327	1.2082	.2214	-.4059	-1.1303	-.6168	-.2346	.0219	292.41
PV, 12, PC	1.5307	1.1829	.2157	-.4004	-1.1168	-.6067	-.2309	.0142	647.41
PV, 6, PC	1.5302	1.1833	.2158	-.4004	-1.117	-.6069	-.231	.0142	364.05
PV, 4, PC	1.5292	1.1839	.216	-.4003	-1.1172	-.6071	-.231	.0143	276.67
PV, 2, PC	1.5234	1.1873	.2174	-.4001	-1.1183	-.6083	-.2315	.0147	190.39
BB, 12, FD	1.5384	1.1912	.2114	-.4166	-1.1383	-.6179	-.234	.0178	26205.61
BB, 6, FD	1.538	1.1917	.2118	-.4167	-1.1385	-.6181	-.2341	.0178	13250
BB, 4, FD	1.5373	1.1922	.2121	-.4168	-1.1387	-.6184	-.2342	.0178	8854.48
BB, 2, FD	1.5328	1.1952	.2138	-.4169	-1.1399	-.6197	-.2348	.0179	4499.39
BB, 12, Aprx	1.538	1.1926	.2114	-.4166	-1.1372	-.6162	-.233	.02	1337.23
BB, 6, Aprx	1.5377	1.193	.2117	-.4168	-1.1373	-.6164	-.2331	.0201	728.64
BB, 4, Aprx	1.5369	1.1936	.2121	-.4169	-1.1375	-.6167	-.2332	.0201	524.22
BB, 2, Aprx	1.5324	1.1967	.2138	-.417	-1.1386	-.6179	-.2337	.0205	321.06
BB, 12, PC	1.5345	1.1876	.2105	-.4159	-1.1343	-.6136	-.2321	.0162	817.38
BB, 6, PC	1.5341	1.188	.2109	-.416	-1.1345	-.6138	-.2322	.0162	467.24
BB, 4, PC	1.5333	1.1885	.2112	-.4161	-1.1345	-.614	-.2323	.0162	353.06
BB, 2, PC	1.5282	1.1912	.213	-.4163	-1.1352	-.6149	-.2327	.0161	238.16

Table B.9. *Deltas, elementary vegas and times (in seconds) for a non-cancellable range accrual with stochastic multiplier and the reference rate is a CMS spread (Example 3), and our multiplicative interpolation scheme. Calculations were done using* 2^{19} *Sobol paths.*

Method	Delta Rate							E. Vega	Time
	1	4	7	10	13	16	19		
PV, 12, FD	1.5281	1.2014	.2203	-.3966	-1.1108	-.6208	-.2379	.0032	7320.42
PV, 6, FD	1.5273	1.202	.2206	-.3965	-1.1107	-.6209	-.238	.0031	3714.92
PV, 4, FD	1.5261	1.2029	.2211	-.3963	-1.1106	-.6212	-.2381	.0029	2521.83
PV, 2, FD	1.5201	1.2079	.2234	-.3952	-1.1103	-.6228	-.2389	.0017	1316
PV, 12, Aprx	1.5343	1.2146	.232	-.3901	-1.1062	-.6172	-.2365	.0035	929.59
PV, 6, Aprx	1.5336	1.2152	.2323	-.39	-1.1062	-.6174	-.2366	.0033	526.48
PV, 4, Aprx	1.5324	1.2161	.2328	-.3898	-1.1061	-.6177	-.2367	.0031	384.31
PV, 2, Aprx	1.5269	1.2212	.2352	-.3886	-1.1058	-.6194	-.2375	.0019	245.28
PV, 12, PC	1.529	1.1985	.2282	-.3842	-1.0957	-.6116	-.2345	.0001	526.44
PV, 6, PC	1.5283	1.1991	.2286	-.384	-1.0956	-.6118	-.2346	-.0001	313.12
PV, 4, PC	1.527	1.2	.2291	-.3838	-1.0955	-.6122	-.2348	-.0003	249.36
PV, 2, PC	1.5211	1.2053	.2317	-.3823	-1.095	-.614	-.2356	-.0015	172.86
BB, 12, FD	1.5277	1.2037	.2276	-.4022	-1.1107	-.6212	-.2376	.0036	12119.14
BB, 6, FD	1.5269	1.2043	.2279	-.4019	-1.1107	-.6213	-.2377	.0034	6111.06
BB, 4, FD	1.5258	1.2053	.2285	-.4016	-1.1107	-.6216	-.2378	.0031	4090.44
BB, 2, FD	1.5201	1.2104	.2316	-.4008	-1.1102	-.6232	-.2386	.0016	2114.05
BB, 12, Aprx	1.5276	1.2047	.2276	-.402	-1.1098	-.6198	-.2366	.0035	894.5
BB, 6, Aprx	1.5269	1.2053	.2279	-.4017	-1.1098	-.6199	-.2366	.0033	501.03
BB, 4, Aprx	1.5257	1.2063	.2285	-.4013	-1.1098	-.6202	-.2367	.003	378.59
BB, 2, Aprx	1.5199	1.2114	.2316	-.4006	-1.1093	-.6216	-.2374	.0015	241.58
BB, 12, PC	1.5258	1.2016	.2265	-.4006	-1.1075	-.6183	-.236	.0018	622.81
BB, 6, PC	1.5251	1.2022	.2268	-.4003	-1.1075	-.6184	-.2361	.0015	370.03
BB, 4, PC	1.5239	1.2031	.2274	-.4	-1.1074	-.6186	-.2362	.0012	282.34
BB, 2, PC	1.5179	1.208	.2304	-.399	-1.1067	-.6198	-.2368	-.0004	195.83

Table B.10. *Deltas, elementary vegas and times (in seconds) for a non-cancellable range accrual with stochastic multiplier and the reference rate is a CMS spread (Example 3), and the Piterbarg interpolation scheme is used. Calculations were done using* 2^{19} *Sobol paths.*

	Example											
	1				2				3			
	PV, 1	PV, 2	BB, 1	BB, 2	PV, 1	PV, 2	BB, 1	BB, 2	PV, 1	PV, 2	BB, 1	BB, 2
mult.												
LS	794.3	792.6	789.2	789.5	879.6	875.4	882.2	880.2	709.6	705.2	712.8	707.3
SE	1.6	1.6	1.8	1.8	.7	.7	.8	.8	.9	.9	1	1
SER	2.3	2.3	2.1	2.1	3.3	3.3	2.9	2.9	2	2.1	1.9	2
Time	73.4	72.5	73.1	73.5	73	47.5	81.8	54	118.1	87.8	121.1	91.2
Paths	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288
Sobol	791.5	789.9	792.2	791.9	882.6	878	883.5	881.3	710.5	705.6	713.1	707.9
Time	9.3	9.1	8.9	8.9	4.6	2.9	5.1	3.3	7.2	5.3	7.6	5.5
Paths	65536	65536	65536	65536	32768	32768	32768	32768	32768	32768	32768	32768
PI	13.3	13.3	11.5	12.8	3.1	3.4	3.3	3.1	8.9	9.1	8.7	8.3
SE	2.1	2.3	2.3	2.3	1.3	1.3	1.2	1.2	1.8	1.3	1.9	1.8
Time	22.7	22.4	27.6	26.6	4	2	4.8	2.7	38.5	27.5	46	32.3
Paths	512	512	512	512	128	128	128	128	512	512	512	512
DG	26.7	27.9	26.9	27.3	7.2	7.6	9.9	9.8	15.9	16.6	17.2	17.5
SE	1.5	1.6	1.6	1.6	.7	.7	.9	.9	1.4	1.4	1.7	1.8
Vega	161.1	162.6	158	157.7	35.7	35.9	36.1	36	4.6	5.2	5.5	5.2
Pit.												
LS	762.9	760.9	759.9	759.8	907.2	906.1	908.9	908.4	723.1	724.3	724.3	722
SE	1.7	1.7	1.8	1.8	.7	.7	.7	.7	.9	.9	.9	.9
SER	2.2	2.2	2.1	2.1	3.1	3.2	3.1	3.1	2	2.1	2	2
Time	67.5	67.6	66.3	65.1	59	42	62.3	46.1	95.7	75.4	90.9	74.9
Paths	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288
Sobol	761.3	759.1	759.4	760.1	906.5	905.5	910.2	909.3	722.4	724	729.8	727.2
Time	8.3	8.1	8.1	8	3.7	2.6	3.8	2.9	5.9	4.6	5.6	4.6
Paths	65536	65536	65536	65536	32768	32768	32768	32768	32768	32768	32768	32768
PI	17.4	20.3	19.6	18.1	4.8	6	4.2	3.8	9.5	10.1	10.2	10.8
SE	2	2	2	2	1.8	1.8	1.7	1.6	1.6	1.6	2.2	2.4
Time	42.4	40.9	47.8	47.7	3.3	1.8	3.3	2.1	31.4	23.1	30.8	23.2
Paths	1024	1024	1024	1024	128	128	128	128	512	512	512	512
DG	26.6	25.5	31.1	30.9	5.6	5.4	9.7	9.9	13.7	14	16.2	15
SE	1.7	1.7	2	1.9	.5	.5	.8	.8	1.4	1.5	1.8	1.5
Vega	151.6	151	149.7	148.5	35.4	35.4	36.2	36.1	4.1	6.1	4.2	4.3

Table B.11. *Callable prices for each case considered. All prices and standard errors are in basis points, and times are measured in seconds. All calculations were done using Mersenne Twister random numbers, except for the row labeled Sobol, which contains the least-squares lower bound using Sobol quasi-random numbers. The numbers after PV and BB denote which Approximation was used. LS is used to denote the least-squares lower bound, PI the improvement in the lower bound due to policy iteration, DG the distance between the least-squares lower bound and the upper bound, SE the standard error and SER the factor reduction in standard error obtained using the delta hedge control variate from* [5]. *In each case, 1500 outer and inner paths were used to calculate the duality gaps and* 2^{19} *paths were used to calculate vegas. In each case, 2 sampling days were used when simulating range accrual coupons.*

	Example											
	1				2				3			
	PV, 1	PV, 2	BB, 1	BB, 2	PV, 1	PV, 2	BB, 1	BB, 2	PV, 1	PV, 2	BB, 1	BB, 2
mult.												
E. Vega	.1832	.1819	.1801	.1793	-.0233	-.0267	-.0228	-.0240	.0234	.0187	.0237	.0213
Time	131.5	125.9	139.7	130.9	152.3	90.5	195.1	131.9	214.6	143.1	232.9	168.8
Paths	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288
Pit.												
E. Vega	.1432	.1421	.1404	.1405	-.0174	-.0197	-.0162	-.0168	.0144	.0126	.0154	.0145
Time	119.3	119.2	126.8	121.4	118.1	75.6	135.8	109.2	174.1	121.7	169	138.4
Paths	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288	524288

Table B.12. *Callable Greek results for each case considered. Times are measured in seconds. Calculations were done using Sobol random numbers. The numbers after PV and BB denote which Approximation was used. In each case, 4 sampling days were used.*

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