STOCHASTIC VOLATILITY AND OPTION PRICING

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Stochastic volatility models are now routinely used in investments and option pricing. A brief introduction to those models is first given, and then a method for pricing options is described.

The stochastic process that later became known as "Brownian motion" first appeared in Bachelier (1900), as a model for security prices. Bachelier imagined the security price as an arithmetic Brownian motion (defined below); this has the shortcoming of allowing negative security prices. Osborne (1959), apparently unaware of Bachelier's work, proposed geometric Browian motion (GBM) as a model for stock prices, in part because GBM cannot be negative. That model was used in economics from the 1960s, notably to value options. Black and Scholes (1973) also used GBM for their risky asset, and since then Osborne's GBM model for stock prices has often been called the "Black-Scholes model".

In the sequel W will stand for a standard Brownian motion, that is, W has independent increments and continuous trajectories, $W_0 = 0$, $W_t - W_s$ has a $\mathbf{N}(0, t-s)$ distribution for $0 \le s < t$. Arithmetic Brownian motion (or "ABM") is a process of the type $at + bW_t$, where a and b are constants.

The usual financial notation for GBM goes back to Black, Scholes and their predecessors, and is the familiar stochastic differential equation

$$dS_t \;=\; \mu S_t\,dt + \sigma S_t\,dW_t.$$

This means exactly the same thing as

$$S_t \;=\; S_0 \exp[(\mu - \tfrac{\sigma^2}{2})t + \sigma W_t],$$

though the latter shows more clearly that log-prices form an ABM.

A discrete-time equivalent of arithmetic Brownian motion is the random walk

$$X_n \;=\; X_{n-1} + \epsilon_n,$$

where $\{\epsilon_n\}$ is i.i.d. Here X_n stands for the log-price, $\log(S_n)$. In both cases the variance (= squared volatility) of the returns (= difference in log-prices) per time period is a constant; both are *constant volatility models*. Observe that in a random walk ϵ_n may have any distribution, while if S is GBM then $\log(S_t) - \log(S_{t-h})$ necessarily has a normal distribution. (*N.B.* The name "random walk" often has a wider meaning in the finance litterature, I am giving here the classical mathematical definition.)

Whether log-prices actually do follow an arithmetic Brownian motion was challenged very soon after that model was formulated; Mandelbrot (1963) and then Fama (1965) suggested a stable Lévy process as a better model for log-prices. Lévy processes are continuous-time processes with independent increments that are not necessarily normally distributed; ABM is the only Lévy process with normal increments. Stable processes may be a good fit to some return series, but they otherwise have an undesirable property, that the expectation of the exponential of a variable with a stable distribution has infinite expectation; this is fatal for option pricing. Stable processes are not used as models of log-prices in option pricing, but other types of Lévy processes have more recently been proposed (at least in the academic literature) to model returns and also to price options. Note that Lévy processes have independent increments, so the *independence* of returns over time was not contested by Mandelbrot and Fama. Interestingly, Black and Scholes (1972) also questioned whether volatility is constant over time.

"Volatility" is what we denoted σ in the GBM above, or, in the case of the random walk model for log-prices, it is the standard deviation of ϵ_n. Econometric models appeared in the 1970s and 1980s that refined the random walk model by making the variance of ϵ_n a stochastic process, leading to the well-known ARCH processes.

Option pricing

Not only does GBM not fit observed stock prices well, it is also unsatisfactory for option pricing, leading to the well-known "smile effect".

ARCH processes have been suggested for pricing option, but another class of models is found more frequently in the literature, namely those where volatility is a continuous-time process:

$$\begin{aligned} dS_t &= rS_t\,dt + \sqrt{V_t}S_t\,dW_t^{(1)} \\ dV_t &= A_t\,dt + B_t\,dW_t^{(2)}. \end{aligned}$$

Here r is the risk-free rate of interest (since this model is formulated under the risk-neutral measure), $W^{(1)}, W^{(2)}$ are Brownian motions (possibly correlated), and (A_t, B_t) are stochastic processes, that may depend on V_t. In this model the "σ" of the GBM model has been replaced with the stochastic process $\sqrt{V}$; V is the "squared volatility process". Several models of this type have been proposed; they raise two problems: estimation and option computation. If a model is to be used to price options, then logically the model should fit observed option prices; more precisely, what is sought is the distribution of the process S under the risk-neutral measure. This is not easy to estimate, because the number of observed option prices is usually quite small. Only in the case of the GBM model is that not a problem; this is because the quadratic variation of log-prices over t time units is in that case the same *constant*, σt, under both the physical measure and the risk-neutral measure. Therefore, the GBM stock price distribution under the physical measure implies the GBM stock price distribution under the risk-neutral measure, and the σ is the same under both measures; hence, one only needs to estimate σ from past stock prices; but this applies only to the GBM/Black-Scholes model.

Once the stochastic volatility model has been estimated there usually remains another problem, that of computing option prices under the model. The resulting distribution for S often turns out to be either very complicated or just unknown. Monte Carlo simulation is a possibility, but one soon realizes that the simulation of stochastic differential equations is far from trivial; one must deal with errors arising from both randomness of the sample and the discretization of the differential equation. (This is a real problem in applications; to make matters worse, some of the early authors on option pricing under stochastic volatility models have suggested methods, especially series, that are of questionable value; *caveat emptor.*)

Perhaps the best-known stochastic volatility model is the one where the squared volatility V satisfy the so-called "square-root process":

$$dV_t \;=\; (aV_t + b)dt + c\sqrt{V_t}\,dW_t^{(2)},$$

where a, b, c are constants. (This process was studied by William Feller and others in the 1940s; it was chosen as a model for interest rates and squared volatility because it is non-negative and also because many of its properties are known explicitly.) The distribution of V_t is known in this case (see Dufresne, 2001, for details), though this does not help much in finding the distribution of S_t; the reason for this is that the solution of $dS_t = rS_t\,dt + \sqrt{V_t}S_t\,dW_t^{(1)}$ is

$$S_t \;=\; S_0 \exp\Bigl(rt - \tfrac{1}{2}U_t + \int_0^t \sqrt{V_s}\,dW_s^{(1)}\Bigr), \qquad U_t \;=\; \int_0^t V_s\,ds.$$

What matters in finding the distibution of S_t is that of the "integrated squared volatility" U_t, and it is rarely a simple one.

Regime switching stochastic volatility model

Models under which a parameter changes value according to a Markov chain have recently become quite popular, at least in the academic literature. A simple stochastic volatility model consists in letting V be a Markov chain that takes values in a set $\{v_1, \ldots, v_N\}$. Models of this kind almost always have a very small number of possible states (or "regimes"), say $N = 2$ or 3. This model does not appear to match the visual appearance of graphs of observed volatilities, but it is tractable to a certain degree and is a definite improvement over constant volatility models. A notable advantage is that when there are only $N = 2$ possible values for V the distribution of U_t is known explicitly (this result has been in the literature for a while, see Chin and Dufresne (2009) for details).

Pricing options with Fourier integrals

In the case of the square-root volatility model, Heston (1993) showed that a Fourier inversion formula may be used to price European puts and calls. The idea of expressing option prices in that way was not new. A classical result in probability theory is that the characteristic function of X, $\mathsf{E}e^{iuX}$, being the Fourier transform of the probability density function of X, yields the probability density function of X by the inversion formula

$$f(x) \;=\; \frac{1}{2\pi}\int_R e^{-iux}\mathsf{E}e^{iuX}\,du.$$

("R" is the real line and i is the imaginary unit, $i^2 = -1$.) Under some conditions, Parseval's theorem gives a similar expression for the expectation of a function of X:

$$\mathsf{E}g(X) \;=\; \frac{1}{2\pi}\int_R \hat{g}(-u)\mathsf{E}e^{iuX}\,du, \qquad \hat{g}(u) \;=\; \int_R e^{iux}g(x)\,dx.$$

This technique is studied in detail in Dufresne *et al.* (2009). Since the payoffs of some reinsurance contracts are the same as those of call options (*i.e.* $(S-K)_+ = \max(S-K,0)$), the same ideas apply in reinsurance and in option pricing. A formula proved in that paper is

$$\mathsf{E}(S-K)_+ \;=\; \mathsf{E}(S) - \frac{K}{2} + \frac{1}{\pi}\int_R \mathrm{Re}[h(u)]\,du, \qquad h(u) \;=\; \frac{K^{-iu+1}}{iu(iu-1)}\mathsf{E}S^{iu}.$$

("$\mathrm{Re}(z)$" is the real part of z; the only assumptions are: the random variable S is positive and has finite mean, the constant K is positive.) There are similar formulas in terms of $\mathsf{E}\exp(iuS)$. The computational advantage of this type of formula is that it is not necessary to find the distribution of S first before computing the option price (or stop-loss premium); if $\mathsf{E}S^{iu}$ is known, then a single integral gives the price of a European put or call. This is quicker than finding the distribution of S and then computing the expected payoff, and is also faster than Monte Carlo simulation.

One way to apply this idea is described in Chin and Dufresne (2009). Suppose V is independent of the Brownian motion driving S (denoted $W^{(1)}$ above), and that the characteristic function of integrated squared volatility, $\mathsf{E}\exp(iuU_t)$, is known. Then, one may condition on V to get the price of (say) a European call:

$$e^{-rT}\mathsf{E}(S_T-K)_+ \;=\; e^{-rT}\mathsf{E}\{\mathsf{E}[(S_T-K)_+|V]\}.$$

(This means conditioning on the whole process $\{V_s \geq 0\}$.) Now, given V, the values of both V_s and U_T are assumed known, and the stochastic integral

$$\int_0^T \sqrt{V_s}\,dW_s^{(1)}$$

has a normal distribution with mean zero and variance

$$\int_0^T V_s\,ds \;=\; U_T.$$

Hence, the price of the call is

$$e^{-rT}\mathsf{E}\left[\exp\left(rT - \tfrac{1}{2}U_T + \sqrt{U_T}Z\right) - K\right]_+,$$

where Z has a standard normal distribution and is independent of U_T. At this point one recognizes the no-arbitrage price of a call option in the ordinary Black-Scholes model, if σ is replaced with $\sqrt{U_T/T}$. Denote the price of such a call $g(\sqrt{U_T/T})$. We thus see that the price of the call in the stochastic volatility model is the same as $\mathsf{E}g(\sqrt{U_T/T})$. Once the Fourier transform of $g(\cdot)$ is found, Parseval's Theorem may then be used to find the following formula (Chin and Dufresne, 2009, Theorems 2.1, 2.2).

Theorem. *Suppose* $\mathsf{E}e^{\alpha^* U_T} < \infty$ *for some* $\alpha^* > 0$. *Then, for any* $0 < \alpha < \alpha^*$,

$$e^{-rT}\mathsf{E}(S_T - K)_+ \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\alpha, -u)\mathsf{E}e^{(\alpha+iu)U_T}\,du,$$

where

$$\hat{g}(\alpha,-u) = \begin{cases} \frac{S_0(1-\overline{k})}{\alpha+iu} + \frac{S_0\overline{k}^{(1+\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \textit{if } Ke^{-rT} < S_0 \\ \frac{S_0\overline{k}^{(1-\sqrt{1+8\alpha+8iu})/2}}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \textit{if } Ke^{-rT} \geq S_0. \end{cases}$$

A similar formula is obtained for European puts. The regime switching stochastic volatility model was used to test this formula against Monte Carlo simulation and also the explicit formula for the distribution of U_T (in the case where volatility takes $N = 2$ values). The Fourier integral beats both alternatives easily, in computing time as well as coding effort. The only downside is that there is a free parameter α in the theorem, and that some trial and error is required to find a good range for it. This is a common feature of Fourier integrals that involve oscillating functions.

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