Matrix-form Recursive Evaluation of the Aggregate Claims Distribution Revisited

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This paper aims to evaluate the aggregate claims distribution under the collective risk model when the number of claims follows a so-called generalised $(a, b, 1)$ family distribution. The definition of the generalised $(a, b, 1)$ family of distributions is given first, followed by some detailed discussions on several members of the family. Of particular interest, it can be shown that all discrete phase-type (DPH) distributions belong to the generalised $(a, b, 1)$ family. A simple matrix-form recursion, which is a counterpart of the Panjer's recursion for the $(a, b, 1)$ family, is then derived to calculate the aggregate claims distribution with non-negative individual claims, either discrete or continuous. Recursive formula for calculating the moments of aggregate claims is also obtained in this paper. At last, several numerical examples are presented to illustrate the recursive calculations using *Mathematica*.

Keywords: Discrete phase-type distributions; Generalised $(a, b, 1)$ family; Recursive formula; Compound distribution

1 Introduction

In risk theory literature, how to evaluate the distribution of aggregate claims arising from a portfolio of risks in a certain time period is one of the long-lasting interesting problems. The two models mostly used in addressing the problem are the collective risk model and the individual risk model. In this paper, we will only look at the former model, under which the aggregate claims amount, denoted by S, is defined as

$$S = \sum_{i=1}^{N} X_i.$$

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In the above expression, N is a random variable (r.v.) denoting the number of claims incurred over a fixed time period and it is valued on non-negative integers. It has probability function (p.f.) $p_n = \Pr(N = n)$, $n \geq 0$, and probability generating function (p.g.f.) $\hat{p}(z) = \sum_{n=0}^{\infty} z^n p_n$, $z \in \mathbb{C}$. $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) non-negative r.v.'s , either discrete or continuous, denoting the individual claim sizes. In addition, we assume that N is independent of $\{X\}_{i=1}^{\infty}$. In the following, we first consider the case of discrete individual claims X_i, with p.f. f_x, $x = 0, 1, 2, \ldots$

To evaluate the probability function of the aggregate claims S, one can use the convolution method. Let $g_s = \Pr(S = s)$. A well-known result is, for $s \geq 0$,

$$g_s = \sum_{n=0}^{\infty} p_n f_s^{n*}, \tag{1.1}$$

where f^{n*} is the n-fold convolution of f. Although the above formula provides us a way of calculating the p.f. of S, it is not computationally efficient in practice as it often involves high-order convolutions of f.

To overcome this problem, Panjer (1981) developed a recursive formula to compute the aggregate claims distribution when $\{p_n\}$, the p.f. of N, belongs to the $(a, b, 0)$ family of distributions. In particular, he states that if there exist constants a and b such that the p.f. p_n can be rewritten as for $n \geq 1$,

$$p_n = p_{n-1}\left(a + \frac{b}{n}\right), \tag{1.2}$$

then

$$g_s = \frac{1}{1 - af_0} \sum_{j=1}^{s} \left(a + \frac{bj}{s}\right) f_j g_{s-j}, \qquad s > 0. \tag{1.3}$$

The starting value for the recursion is given by $g_0 = \hat{p}(f_0)$. As shown by Sundt and Jewell (1981), the only counting distributions belonging to the $(a, b, 0)$ family include the Poisson, negative binomial (with geometric distribution as a special case) and binomial distributions.

Sundt and Jewell (1981) further generalised the Panjer's recursion (1.3) to the family of $(a, b, 1)$ distributions, sometimes known as the Sundt-Jewell class of distributions in literature, of which the recursive structure, (1.2), initiates at p_1 rather than p_0. One of their main results says if N follows a $(a, b, 1)$ family distribution, then

$$g_s = \frac{1}{1 - af_0}\left[\sum_{j=1}^{s}\left(a + \frac{bj}{s}\right) f_j g_{s-j} + \left(p_1 - (a+b)p_0\right) f_s\right], \qquad s > 0.$$

The starting value for the recursion is still $g_0 = \hat{p}(f_0)$. Willmot (1988) had shown that the only non-degenerate members of the $(a, b, 1)$ family are the Poisson, negative binomial, binomial, logarithmic series and the extended truncated negative binomial (ETNB) distributions. This was also mentioned in Klugman, Panjer and Willmot (1998, pp 229).

Since then, a large amount of research has been undertaken to allow for cases where N has other types of counting distributions. For example, Schröter (1991) derived a recursive formula for the distribution of S when p_n belongs to the Schröter's family of distributions. Sundt (1992) introduced a broader class of counting distributions which is known as the $\mathcal{R}_k\left[\vec{\mathbf{a}},\vec{\mathbf{b}}\right]$ class. In particular, it can be shown that the $(a,b,0)$ family and the Schröter's family are $\mathcal{R}_1\left[\vec{\mathbf{a}},\vec{\mathbf{b}}\right]$ and $\mathcal{R}_2\left[\vec{\mathbf{a}},\vec{\mathbf{b}}\right]$ respectively. A detailed review on the development of recursive evaluation of aggregate claims distribution can be found in Sundt (2002). Sundt (2003) studied how to calculate the higher moments of S recursively.

The study on aggregate claims distribution has also been extended to include cases where either N follows a discrete phase-type (DPH) distribution or the claim amounts X_i modelled as phase-type r.v.'s. To address the former problem, Eisele (2006) had derived a recursive method in calculating the aggregate claims distribution. However, the derived recursive formulae involve high-order convolutions of individual claim distributions that is similar to formula (1.1). We would argue that the algorithm provided in Eisele (2006) is not computationally efficient. On the other hand, Hipps (2006) derived a simplified recursion algorithm for the aggregate claims distribution where the individual claim amounts are phase-type random variables.

Wu and Li (2010) proposed an alternative method in an attempt to evaluate g_s when N follows a generalised $(a,b,0)$ family distribution. They provided a matrix-form of Panjer's recursive equation. However, only a small proportion of DPH distributions belong to this family of counting distributions. Nonetheless, a similar recursive formula is obtained separately in Wu and Li (2010) to suit general DPH distributions. For better understanding of discussions within the rest of this paper, in the following, we will give a very brief review for the DPH distributions and the generalised $(a,b,0)$ family proposed by Wu and Li (2010).

The DPH distributions form one of the most general classes of distributions of which the p.f., p.g.f., and moments are expressed in terms of matrices. Some well known members are the geometric and negative binomial distributions. The following definition of a DPH distribution was given in Latouche and Ramaswami (1999, pg. 47).

Definition 1 *A discrete phase-type distribution is the distribution of the time to absorption into state* 0 *in a finite discrete Markov chain with a transition probability matrix* $\mathbf{W}$ *of dimension* $m+1$ *given by*

$$\mathbf{W}=\begin{pmatrix} 1 & \vec{\mathbf{0}} \\ \vec{\mathbf{t}}^{\top} & \mathbf{T} \end{pmatrix},$$

where m *is a positive integer,* $\vec{\mathbf{0}}=(0,0,\ldots,0)_{1\times m}$*,* $\vec{\mathbf{t}}=(t_1,t_2,\ldots,t_m)$ *and* $\mathbf{T}$ *is an*

$m \times m$ substochastic matrix with elements $1 \geq T_{i,j} \geq 0$ for $1 \leq i, j \leq m$. Also, we have $\vec{\mathbf{t}}^\top = \vec{\mathbf{1}}^\top - \mathbf{T}\vec{\mathbf{1}}^\top$, where $\vec{\mathbf{1}} = (1, 1, \ldots, 1)_{1\times m}$. The initial probability vector is given as $(\alpha_0, \vec{\boldsymbol{\alpha}})$ where $\vec{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ with $0 \leq \alpha_i \leq 1$ for $0 \leq i \leq m$ and $\alpha_0 + \vec{\boldsymbol{\alpha}}\vec{\mathbf{1}}^\top = 1$. The pair $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$ is called a representation for a DPH distribution.

Thereafter, we denote a DPH distribution with representation $(\vec{\boldsymbol{\alpha}}, \mathbf{T})$ as $\mathrm{PH}_d(\vec{\boldsymbol{\alpha}}, \mathbf{T})$.

If $N \sim \mathrm{PH}_d(\vec{\boldsymbol{\alpha}}, \mathbf{T})$, then we have the following:

$$p_0 = \alpha_0, \qquad p_n = \vec{\boldsymbol{\alpha}}\mathbf{T}^{n-1}\vec{\mathbf{t}}^\top = \vec{\boldsymbol{\alpha}}\mathbf{T}^{n-1}(\mathbf{I} - \mathbf{T})\vec{\mathbf{1}}^\top, \qquad n \geq 1,$$

and

$$\Pr(N \leq n) = 1 - \vec{\boldsymbol{\alpha}}\mathbf{T}^n\vec{\mathbf{1}}^\top.$$

According to Latouche and Ramaswami (1999, pg.54), every finite-support non-negative integer distribution can be rewritten as a DPH distribution. Given that N follows a distribution $\{p_i\}_{i=0}^K$, we can rewrite it as $N \sim \mathrm{PH}_d(\vec{\boldsymbol{\alpha}}, \mathbf{T})$ with $\vec{\boldsymbol{\alpha}} = (p_1, p_2, \ldots, p_K)$ and a matrix $\mathbf{T}$ such that $\mathrm{T}_{i,i-1} = 1$ for $2 \leq i \leq K$, and $\mathrm{T}_{i,j} = 0$ otherwise.

The p.g.f. of N is given by

$$\hat{p}(z) = \alpha_0 + z\vec{\boldsymbol{\alpha}}\left(\mathbf{I} - z\mathbf{T}\right)^{-1}\left(\mathbf{I} - \mathbf{T}\right)\vec{\mathbf{1}}^\top.$$

And the jth factorial moment of N is

$$\mathbb{E}\left[N(N-1)\ldots(N-j+1)\right] = j!\vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{T})^{-j}\mathbf{T}^{j-1}\vec{\mathbf{1}}^\top, \qquad j > 0.$$

Detailed discussions about the DPH distributions can be found in Neuts (1981) and Latouche and Ramaswami (1999).

Definition 2 *Let $\{p_n\}_{n=0}^\infty$ be the p.f. of r.v. N. If p_n can be rewritten as*

$$p_n = \vec{\boldsymbol{\gamma}}\mathbf{P}_n\vec{\mathbf{1}}^\top, \qquad n \geq 0,$$

where $\vec{\boldsymbol{\gamma}} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is a row vector with $\gamma_i \geq 0$ and $\sum_{i=1}^m \gamma_i = 1$, and $\mathbf{P}_n$, $n = 0, 1, \ldots$, is a sequence of $m \times m$ matrices, satisfying the following recursion

$$\mathbf{P}_n = \mathbf{P}_{n-1}\left(\mathbf{A} + \frac{\mathbf{B}}{n}\right), \qquad n \geq 1, \tag{1.4}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two $m \times m$ matrices, then $\{p_n\}_{n=0}^\infty$ is said to belong to a generalised $(a, b, 0)$ family.

The p.g.f. of a generalised $(a, b, 0)$ family distribution becomes

$$\hat{p}(z) = \sum_{n=0}^\infty z^n p_n = \sum_{n=0}^\infty z^n\left(\vec{\boldsymbol{\gamma}}\mathbf{P}_n\vec{\mathbf{1}}^\top\right) = \vec{\boldsymbol{\gamma}}\left(\sum_{n=0}^\infty z^n\mathbf{P}_n\right)\vec{\mathbf{1}}^\top = \vec{\boldsymbol{\gamma}}\hat{\mathbf{P}}(z)\vec{\mathbf{1}}^\top, \tag{1.5}$$

where $\hat{\mathbf{P}}(z) = \sum_{n=0}^{\infty} z^n \mathbf{P}_n$.

Wu and Li (2010) have implicitly suggested that all DPH distributions can be categorised under a similar but broader family of distributions. Motivated by the idea and based on the results obtained in Wu and Li (2010), we aim in this paper to propose a generalised $(a, b, 1)$ family of distributions with matrix parameters and to develop a recursive method to evaluate the distribution and moments of the aggregate claims S.

Firstly, we define the generalised $(a, b, 1)$ family of distributions in Section 2 and show that all DPH distributions are members in this family. Several other members from this family will be discussed as well. In particular, we will discuss the mixtures and linear combinations of zero-modified and zero truncated versions of Logarithmic distributions. A matrix-form recursion is then developed in Section 3 to evaluate the aggregate claims distribution when N belongs to this family of distributions. Section 4 demonstrates how to compute the moments of S recursively based the results from Section 3. Three numerical examples are presented in Section 5 to illustrate the use of the recursive formulas. And lastly, we provide a short summary on some technical issues regarding the generalised $(a, b, 1)$ family of distributions and possible future investigations.

2 The Generalised $(a, b, 1)$ Family of Distributions

Definition 3 *Let $\{p_n\}_{n=0}^{\infty}$ be the p.f. of the r.v. N. If p_n can be rewritten as*

$$p_n = \vec{\boldsymbol{\gamma}} \mathbf{Q}_n \vec{\mathbf{1}}^\top, \qquad n \geq 0, \tag{2.1}$$

where $\mathbf{Q}_n$, $n = 0, 1, \ldots,$ is a sequence of $m \times m$ matrices, satisfying the following recursion:

$$\mathbf{Q}_n = \mathbf{Q}_{n-1} \left(\mathbf{A} + \frac{\mathbf{B}}{n} \right), \qquad n \geq 2, \tag{2.2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two given $m \times m$ matrices, then $\{p_n\}_{n=0}^{\infty}$ is said to belong to the generalised $(a, b, 1)$ family.

When $m = 1$, the recursion (2.2) reduces to the $(a, b, 1)$ family case as mentioned before.

According to (1.5), the p.g.f. of $\{p_n\}_{n=0}^{\infty}$, denoted by $\hat{p}(z)$, can be expressed as

$$\hat{p}(z) = \sum_{n=0}^{\infty} z^n p_n = \vec{\boldsymbol{\gamma}} \hat{\mathbf{Q}}(z) \vec{\mathbf{1}}^\top,$$

where $\hat{\mathbf{Q}}(z) = \sum_{n=0}^{\infty} z^n \mathbf{Q}_n$.

A particularly important class of distributions which belongs to the generalised $(a, b, 1)$ family is the class of DPH distributions. Let $N \sim \text{PH}_d(\vec{\boldsymbol{\alpha}}, \mathbf{T})$. We can rewrite the p.f. p_n in a form of (2.1), by letting $\vec{\gamma} = (1-\alpha_0)^{-1}\vec{\alpha}$ and $\mathbf{Q}_0 = \text{diag}(\alpha_0, \ldots, \alpha_0)_{m\times m}$. It is easy to verify that

$$\begin{aligned} p_0 &= \alpha_0 = \vec{\gamma}\mathbf{Q}_0\vec{\mathbf{1}}^\top, \\ \vec{\alpha} &= \vec{\gamma}(\mathbf{I}-\mathbf{Q}_0) \end{aligned} \tag{2.3}$$

and thus for $n \geq 1$,

$$p_n = \vec{\alpha}\mathbf{T}^{n-1}(\mathbf{I}-\mathbf{T})\vec{\mathbf{1}}^\top = \vec{\gamma}\left[(\mathbf{I}-\mathbf{Q}_0)(\mathbf{I}-\mathbf{T})\mathbf{T}^{n-1}\right]\vec{\mathbf{1}}^\top = \vec{\gamma}\mathbf{Q}_n\vec{\mathbf{1}}^\top.$$

In the above equations $\mathbf{Q}_n = (\mathbf{I}-\mathbf{Q}_0)(\mathbf{I}-\mathbf{T})\mathbf{T}^{n-1}, n \geq 1$, satisfying (2.2) for $\mathbf{A} = \mathbf{T}$ and $\mathbf{B} = \mathbf{0}$. Therefore, all DPH distributions belong to the generalised $(a, b, 1)$ family.

We will now turn our attention into identifying other members of the generalised $(a, b, 1)$ family. To begin with, notice that based on Definition 3, the matrix $\mathbf{Q}_0$ can be chosen independently from the subsequent matrices of $\mathbf{Q}_n$ for $n \geq 1$. This particular feature allows one to tackle the issue of unusual high or low probability observed at $N = 0$, which usually arises in insurance count data as suggested by Klugman, Panjer and Willmot (1998, pp.225). This led us to consider the problem on how to determine the starting value in recursive formula (2.2), i.e., the matrix $\mathbf{Q}_1$. If there is an approach to determine $\mathbf{Q}_1$ (the obtained $\mathbf{Q}_1$ may not be unique), given $\vec{\gamma}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Q}_0$, then it is equivalent to say that sequence $\{\mathbf{Q}_n\}_{n=0}^\infty$ is fully determined. To address this problem, we propose the following method.

Firstly, to make $\{p_n\}_{n=0}^\infty$ a proper distribution, we require

$$\sum_{n=0}^\infty p_n = \vec{\gamma}\sum_{n=0}^\infty \mathbf{Q}_n\vec{\mathbf{1}}^\top = 1,$$

or more specifically,

$$\vec{\gamma}\sum_{n=1}^\infty \mathbf{Q}_n\vec{\mathbf{1}}^\top = 1 - p_0. \tag{2.4}$$

Applying the recursive property of $\mathbf{Q}$ to the left-hand side of equation (2.4) gives

$$\vec{\gamma}\mathbf{Q}_1\left[\mathbf{I} + \sum_{n=1}^\infty \prod_{j=1}^n \left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)\right]\vec{\mathbf{1}}^\top = 1 - p_0.$$

In general, there are an infinite number of solutions of $\mathbf{Q}_1$ to the above equation. (how to find a general system of roots?)

A special solution of $\mathbf{Q}_1$ is $(1-p_0)\left[\mathbf{I} + \sum_{n=1}^\infty \prod_{j=1}^n \left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)\right]^{-1}$ given that the matrix $\left[\mathbf{I} + \sum_{n=1}^\infty \prod_{j=1}^n \left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)\right]$ is invertible. Close-form expressions can be obtained under some special circumstances.

Case 1: Let $\mathbf{A} = \boldsymbol{\Theta}$ and $\mathbf{B} = -\boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is an $m \times m$ non-singular matrix. Then we have

$$\begin{aligned}
\mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(\mathbf{A} + \frac{\mathbf{B}}{j+1} \right) &= \mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(\boldsymbol{\Theta} - \frac{\boldsymbol{\Theta}}{j+1} \right) \\
&= \mathbf{I} + \sum_{n=1}^{\infty} \frac{\boldsymbol{\Theta}^n}{n+1} = \boldsymbol{\Theta}^{-1} \sum_{n=1}^{\infty} \frac{\boldsymbol{\Theta}^n}{n}.
\end{aligned}$$

Assuming that $\mathbf{I} - \boldsymbol{\Theta}$ is invertible and each Jordan block belonging to a negative eigenvalue occurs an even number of times, according to Culver and Walter (1966) we know that the above expression can be further simplified to $-\boldsymbol{\Theta}^{-1}\ln(\mathbf{I} - \boldsymbol{\Theta})$ and here $\ln(\mathbf{I} - \boldsymbol{\Theta})$ is a real matrix. If in addition, the matrix $\ln(\mathbf{I} - \boldsymbol{\Theta})$ is invertible, then we have found an explicit solution for $\mathbf{Q_1}$:

$$\mathbf{Q}_1 = -(1 - p_0) \left[\ln(\mathbf{I} - \boldsymbol{\Theta})\right]^{-1} \boldsymbol{\Theta},$$

and it gives

$$\mathbf{Q}_n = \left(-\frac{1}{n} \right) (1 - p_0) \left[\ln(\mathbf{I} - \boldsymbol{\Theta})\right]^{-1} \boldsymbol{\Theta}^n, \qquad n \geq 1.$$

Therefore, on the other hand, if a given distribution $\{p_n\}_{n=0}^{\infty}$ can be rewritten in the form of

$$\begin{aligned}
p_0 &= \vec{\gamma} \mathbf{Q}_0 \vec{\mathbf{1}}^\top, \\
p_n &= -\frac{1}{n}(1 - p_0) \vec{\gamma} \left[\ln(\mathbf{I} - \boldsymbol{\Theta})\right]^{-1} \boldsymbol{\Theta}^n \vec{\mathbf{1}}^\top, \qquad n \geq 1,
\end{aligned} \tag{2.5}$$

then $\{p_n\}_{n=0}^{\infty}$ belongs to the generalised $(a, b, 1)$ family with parameters $\mathbf{A} = \boldsymbol{\Theta}$ and $\mathbf{B} = -\boldsymbol{\Theta}$. The p.g.f. in this case is given by

$$\begin{aligned}
\hat{p}(z) &= \vec{\gamma} \hat{\mathbf{Q}}(z) \vec{\mathbf{1}}^\top = \vec{\gamma} \Big[\mathbf{Q}_0 + \sum_{n=1}^{\infty} z^n \mathbf{Q}_n \Big] \vec{\mathbf{1}}^\top \\
&= \vec{\gamma} \Big[\mathbf{Q}_0 + (1 - p_0) \left[\ln(\mathbf{I} - \boldsymbol{\Theta})\right]^{-1} \Big(-\sum_{n=1}^{\infty} \frac{(z\boldsymbol{\Theta})^n}{n} \Big) \Big] \vec{\mathbf{1}}^\top \\
&= p_0 + (1 - p_0) \vec{\gamma} \left[\ln(\mathbf{I} - \boldsymbol{\Theta})\right]^{-1} \ln(\mathbf{I} - z\boldsymbol{\Theta}) \vec{\mathbf{1}}^\top.
\end{aligned}$$

For $j > 0$, the jth factorial moments of N can be obtained by differentiating $\hat{p}(z)$ for j times and letting $z = 1$, shown as follows:

$$\mathbb{E}\left[N(N-1)\dots(N-j+1)\right] = (-1)^j (j-1)!(1 - p_0) \vec{\gamma} \left[\ln(\mathbf{I} - \boldsymbol{\Theta})\right]^{-1} (\mathbf{I} - \boldsymbol{\Theta})^{-j} \boldsymbol{\Theta}^j \vec{\mathbf{1}}^\top.$$

Case 2: Let $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is an $m \times m$ invertible matrix. Then

$\mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)$ reduces to

$$\begin{aligned} \mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{\mathbf{\Lambda}}{j+1} &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{\Lambda}^n}{(n+1)!} \\ &= \mathbf{\Lambda}^{-1} \left(\sum_{n=0}^{\infty} \frac{\mathbf{\Lambda}^n}{n!} - \mathbf{I} \right) = \mathbf{\Lambda}^{-1} e^{\mathbf{\Lambda}} \left(\mathbf{I} - e^{-\mathbf{\Lambda}}\right). \end{aligned}$$

(When the matrix exponential function exist?) If, in addition, $\mathbf{I} - e^{-\mathbf{\Lambda}}$ is invertible, then we have an explicit solution for $\mathbf{Q}_1$:

$$\mathbf{Q}_1 = (1-p_0)(\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} e^{-\mathbf{\Lambda}} \mathbf{\Lambda},$$

and hence

$$\mathbf{Q}_n = (1-p_0)(\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} \frac{\mathbf{\Lambda}^n}{n!} e^{-\mathbf{\Lambda}}, \quad n \geq 1.$$

Thus, a distribution $\{p_n\}_{n=0}^{\infty}$ with the following structure

$$\begin{aligned} p_0 &= \vec{\boldsymbol{\gamma}} \mathbf{Q}_0 \vec{\mathbf{1}}^{\top}, \\ p_n &= (1-p_0) \vec{\boldsymbol{\gamma}} (\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} \frac{\mathbf{\Lambda}^n}{n!} e^{-\mathbf{\Lambda}} \vec{\mathbf{1}}^{\top}, \qquad n \geq 1, \end{aligned} \tag{2.6}$$

is said to belong to the generalised $(a, b, 1)$ family with parameters $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{\Lambda}$. Its p.g.f. equals

$$\hat{p}(z) = p_0 + (1-p_0) \vec{\boldsymbol{\gamma}} (\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} (e^{z\mathbf{\Lambda}} - \mathbf{I}) e^{-\mathbf{\Lambda}} \vec{\mathbf{1}}^{\top}. \tag{2.7}$$

The jth factorial moments of N is

$$\mathbb{E}\left[N(N-1)\ldots(N-j+1)\right] = \vec{\boldsymbol{\gamma}} (\mathbf{I} - \mathbf{Q}_0)(\mathbf{I} - e^{-\mathbf{\Lambda}})^{-1} \mathbf{\Lambda}^j \vec{\mathbf{1}}^{\top}, \quad j > 0.$$

Case 3: Now we consider a distribution $\{p_n\}_{n=0}^{k}$, where k is a positive integer. Given $\mathbf{A} = -\mathbf{R}(\mathbf{I} - \mathbf{R})^{-1}$ and $\mathbf{B} = -(k+1)\mathbf{A}$, with $\mathbf{R}$ being an $m \times m$ matrix such that both $\mathbf{I} - \mathbf{R}$ and $\mathbf{R}$ are invertible, then $\mathbf{I} + \sum_{n=1}^{\infty} \prod_{j=1}^{n} \left(\mathbf{A} + \frac{\mathbf{B}}{j+1}\right)$ becomes

$$\begin{aligned} & \mathbf{I} + \sum_{n=1}^{k} \prod_{j=1}^{n} \left[-\mathbf{R}(\mathbf{I} - \mathbf{R})^{-1} + \frac{(k+1)\mathbf{R}(\mathbf{I} - \mathbf{R})^{-1}}{j+1} \right] \\ = \; & \mathbf{I} + \sum_{n=1}^{k-1} \left[\mathbf{R}(\mathbf{I} - \mathbf{R})^{-1}\right]^n \prod_{j=1}^{n} \frac{k-j}{j+1} = \mathbf{I} + \sum_{n=2}^{k} \left[\mathbf{R}(\mathbf{I} - \mathbf{R})^{-1}\right]^{n-1} \frac{(k-1)!}{n!(k-n)!} \\ = \; & \sum_{n=1}^{k} \mathbf{R}^{n-1} (\mathbf{I} - \mathbf{R})^{-(n-1)} \frac{(k-1)!}{n!(k-n)!} = \frac{1}{k} \mathbf{R}^{-1} (\mathbf{I} - \mathbf{R}) \sum_{n=1}^{k} \binom{k}{n} \mathbf{R}^n (\mathbf{I} - \mathbf{R})^{-n} \\ = \; & \frac{1}{k} \mathbf{R}^{-1} (\mathbf{I} - \mathbf{R}) \left[(\mathbf{I} - \mathbf{R})^{-k} - \mathbf{I} \right] = \frac{1}{k} \mathbf{R}^{-1} (\mathbf{I} - \mathbf{R})^{-(k-1)} \left[\mathbf{I} - (\mathbf{I} - \mathbf{R})^k \right]. \end{aligned}$$

If $\mathbf{I}-(\mathbf{I}-\mathbf{R})^k$ is invertible, then we can find an explicit solution for $\mathbf{Q}_1$:

$$\mathbf{Q}_1 = k(1-p_0)\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}\mathbf{R}(\mathbf{I}-\mathbf{R})^{k-1},$$

and hence for $1 \le n \le k$,

$$p_n = (1-p_0)\vec{\gamma}\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}\binom{k}{n}\mathbf{R}^n(\mathbf{I}-\mathbf{R})^{k-n}\vec{\mathbf{1}}^\top. \tag{2.8}$$

It has p.g.f.

$$\hat{p}(z) = p_0 + (1-p_0)\vec{\gamma}\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}\left[\Big(\mathbf{I}-(1-z)\mathbf{R}\Big)^k - (\mathbf{I}-\mathbf{R})^k\right]\vec{\mathbf{1}}^\top.$$

And the jth factorial moments of N is

$$\mathbb{E}\left[N(N-1)\dots(N-j+1)\right] = j!(1-p_0)\vec{\gamma}\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}\mathbf{R}^j\vec{\mathbf{1}}^\top, \quad 0 < j \le k.$$

Case 4: Let k be a positive integer, $\mathbf{A}=\mathbf{R}$ and $\mathbf{B}=(k-1)\mathbf{R}$, where $\mathbf{R}$ is an $m\times m$ invertible matrix such that $\mathbf{I}-\mathbf{R}$ is invertible as well. The expression $\mathbf{I}+\sum_{n=1}^{\infty}\prod_{j=1}^{n}\left(\mathbf{A}+\frac{\mathbf{B}}{j+1}\right)$ can be simplified as

$$\begin{aligned}
&\mathbf{I}+\sum_{n=1}^{\infty}\prod_{j=1}^{n}\left(\mathbf{R}+\frac{(k-1)\mathbf{R}}{j+1}\right)\\
=\;& \mathbf{I}+\sum_{n=1}^{\infty}\mathbf{R}^n\prod_{j=1}^{n}\frac{j+k}{j+1} = \sum_{n=0}^{\infty}\frac{(n+k)!}{(n+1)!k!}\mathbf{R}^n\\
=\;& \frac{1}{k}\sum_{n=1}^{\infty}\binom{n+k-1}{n}\mathbf{R}^{n-1} = \frac{1}{k}\mathbf{R}^{-1}\left[\sum_{n=0}^{\infty}\binom{n+k-1}{n}\mathbf{R}^n - \mathbf{I}\right].
\end{aligned}$$

To simplify the above expression, we use the matrix version of the Maclaurin series expansion for the function $(\mathbf{I}-\mathbf{R})^{-k}$, which gives

$$(\mathbf{I}-\mathbf{R})^{-k} = \sum_{n=0}^{\infty}\binom{k+n-1}{n}\mathbf{R}^n.$$

As a result, the expression simplifies to

$$\frac{1}{k}\mathbf{R}^{-1}\left[(\mathbf{I}-\mathbf{R})^{-k}-\mathbf{I}\right] = \frac{1}{k}(\mathbf{I}-\mathbf{R})^{-k}\mathbf{R}^{-1}\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right].$$

Assume that $\mathbf{I}-(\mathbf{I}-\mathbf{R})^k$ is invertible, then an explicit expression for $\mathbf{Q}_1$ is

$$\mathbf{Q}_1 = k(1-p_0)\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}\mathbf{R}(\mathbf{I}-\mathbf{R})^k.$$

Moreover, the distribution $\{p_n\}$ can be expressed as

$$
\begin{aligned}
p_0 &= \vec{\boldsymbol{\gamma}}\mathbf{Q}_0\vec{\mathbf{1}}^\top, \\
p_n &= (1-p_0)\vec{\boldsymbol{\gamma}}\Big[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\Big]^{-1}\binom{k+n-1}{n}\mathbf{R}^n(\mathbf{I}-\mathbf{R})^k\vec{\mathbf{1}}^\top, \qquad n \geq 1.
\end{aligned}
$$

The p.g.f. of N is given by

$$\hat{p}(z) = p_0 + (1-p_0)\vec{\boldsymbol{\gamma}}\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}\left[(\mathbf{I}-z\mathbf{R})^{-k}-\mathbf{I}\right](\mathbf{I}-\mathbf{R})^k\vec{\mathbf{1}}^\top.$$

The jth factorial moments of N can be found in the usual way. However, due to the complexity of the results, only the expectation is shown below:

$$\mathbb{E}(N) = k(1-p_0)\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}(\mathbf{I}-\mathbf{R})^{-1}\mathbf{R}.$$

As shown by Wu and Li (2010), mixtures and linear combinations of zero-modified $(a, b, 0)$ family of distributions can be obtained by imposing certain restrictions on the matrix parameters. In what follows, we will provide an example of mixtures and linear combinations of zero-modified (or zero-truncated) Logarithmic distributions. Similar approach can be applied to the case for Poisson, binomial and negative binomial distributions.

1. Let $\mathbf{Q}_0 = \mathbf{0}$ and $\mathbf{\Theta} = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_m)$ where $\theta_i, i = 1, \ldots, m$ are m distinct real numbers such that $0 < \theta_i < 1$ for all $1 \leq i \leq m$. Then, for $n \geq 1$,

$$
\begin{aligned}
p_n &= -\frac{1}{n}\vec{\boldsymbol{\gamma}}\ \mathrm{diag}\left[\frac{\theta_1^n}{\ln(1-\theta_1)}, \ldots, \frac{\theta_m^n}{\ln(1-\theta_m)}\right]\vec{\mathbf{1}}^\top \\
&= \sum_{i=1}^{m}\gamma_i\left(-\frac{1}{n}\right)\frac{\theta_i^n}{\ln(1-\theta_i)}
\end{aligned}
$$

is a mixture of m Logarithmic distributions with weights γ_i, $1 \leq i \leq m$.

If instead $\mathbf{Q}_0 = \mathrm{diag}(\alpha_1, \alpha_2, \ldots \alpha_m)$ with $0 \leq \alpha_i \leq 1$ for $1 \leq i \leq m$, then

$$
\begin{aligned}
p_0 &= \vec{\boldsymbol{\gamma}}\ \mathrm{diag}(\alpha_1, \alpha_2, \ldots \alpha_m)\vec{\mathbf{1}}^\top = \sum_{i=1}^{m}\gamma_i\alpha_i, \\
p_n &= \vec{\boldsymbol{\gamma}}\ \mathrm{diag}\left[-\frac{(1-\alpha_1)}{n}\frac{\theta_1^n}{\ln(1-\theta_1)}, \ldots, -\frac{(1-\alpha_m)}{n}\frac{\theta_m^n}{\ln(1-\theta_m)}\right]\vec{\mathbf{1}}^\top \\
&= \sum_{i=1}^{m}-\gamma_i\frac{(1-\alpha_i)}{n}\left(\frac{\theta_i^n}{\ln(1-\theta_i)}\right), \quad n \geq 1,
\end{aligned}
$$

is a mixture of m zero-modified Logarithmic distributions;

2. Let $\mathbf{Q}_0 = \mathbf{0}$ and $\mathbf{\Theta} = \bar{\mathbf{\Theta}}\ \mathrm{diag}(\theta_1, \ldots \theta_m)\bar{\mathbf{\Theta}}^{-1}$ where $\theta_i,\ i = 1, \ldots, m$, are m distinct real numbers such that $0 < \theta_i < 1$ for $1 \leq i \leq m$. Then, for $n \geq 1$,

$$p_n = \vec{\gamma}\bar{\mathbf{\Theta}}\ \mathrm{diag}\left[\left(-\frac{1}{n}\right)\frac{\theta_1^n}{\ln(1-\theta_1)}, \ldots, \left(-\frac{1}{n}\right)\frac{\theta_m^n}{\ln(1-\theta_m)}\right]\bar{\mathbf{\Theta}}^{-1}\vec{\mathbf{1}}^{\top}$$

is a linear combination of m Logarithmic distributions. When $\mathbf{Q}_0 = \mathrm{diag}(\alpha_1, \alpha_2, \ldots \alpha_m)$ with $0 \leq \alpha_i \leq 1$ for $1 \leq i \leq m$, we have

$$\begin{aligned} p_0 &= \vec{\gamma}\ \mathrm{diag}(\alpha_1, \alpha_2, \ldots \alpha_m)\vec{\mathbf{1}}^{\top} = \sum_{i=1}^{\infty} \gamma_i \alpha_i, \\ p_n &= \vec{\gamma}\bar{\mathbf{\Theta}}\ \mathrm{diag}\left[-\frac{(1-\alpha_1)}{n}\frac{\theta_1^n}{\ln(1-\theta_1)}, \ldots, -\frac{(1-\alpha_m)}{n}\frac{\theta_m^n}{\ln(1-\theta_m)}\right]\bar{\mathbf{\Theta}}^{-1}\vec{\mathbf{1}}^{\top}, \quad n \geq 1, \end{aligned}$$

which is a linear combination of m zero-modified Logarithmic distributions.

To end this section, we suggest an alternative approach to determine $\mathbf{Q}_1$ under some special conditions. It can be observed that both the generalised $(a, b, 0)$ family and the generalised $(a, b, 1)$ family of distributions share the same recursive structure for $n > 1$. Therefore, it is a plausible assumption that there exists an $m \times m$ constant matrix $\mathbf{C}$ such that

$$\mathbf{Q}_n = \mathbf{C}\mathbf{P}_n, \qquad n \geq 1, \tag{2.9}$$

where $\mathbf{P}$ and $\mathbf{Q}$ forms part of a generalised $(a, b, 0)$ and a generalised $(a, b, 1)$ distribution, with the same initial vector, respectively. A special solution for $\mathbf{C}$, assuming that $\sum_{n=0}^{\infty} \mathbf{P}_n = \mathbf{I}$ and $\mathbf{I} - \mathbf{P}_0$ is invertible, is

$$\mathbf{C} = (1 - p_0)(\mathbf{I} - \mathbf{P}_0)^{-1},$$

or equivalently,

$$\mathbf{Q}_n = (1 - p_0)(\mathbf{I} - \mathbf{P}_0)^{-1}\mathbf{P}_n, \qquad n \geq 1. \tag{2.10}$$

This method is useful when one intends to extend a generalised $(a, b, 0)$ distribution into the generalised $(a, b, 1)$ one. As a result, Case 2 - 4 above can all be obtained alternatively using this method. We will illustrate this method using Case 3.

Let k be a positive integer and $\mathbf{R}$ be an $m \times m$ matrix such that $\mathbf{I} - \mathbf{R}$ is non-singular. We consider the following generalised $(a, b, 0)$ distribution $\{p_n\}$

$$p_n = \vec{\gamma}\mathbf{P}_n\vec{\mathbf{1}}^{\top}, \qquad 0 \leq n \leq k,$$

where

$$\mathbf{P}_n = \binom{k}{n}\mathbf{R}^n(\mathbf{I} - \mathbf{R})^{k-n}$$

and p_n is a proper probability function. Here $\{p_n\}_{n=0}^{k}$ is a generalised $(a, b, 0)$ distribution with $\mathbf{A} = \mathbf{R}$ and $\mathbf{B} = (\mathbf{k} - \mathbf{1})\mathbf{R}$ by Definition 1. Using the above method, the distribution $\{p_n\}$ can be extended to a generalised $(a, b, 1)$ family distribution, denoted by $\{p'_n\}$, with

$$\begin{aligned} p'_0 &= \vec{\gamma}\mathbf{Q}_0\vec{\mathbf{1}}^\top, \\ p'_n &= \vec{\gamma}\mathbf{Q}_n\vec{\mathbf{1}}^\top = \vec{\gamma}\left[(1-p_0)(\mathbf{I}-\mathbf{P}_0)^{-1}\mathbf{P}_n\right]\vec{\mathbf{1}}^\top \\ &= (1-p_0)\vec{\gamma}\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})^k\right]^{-1}\binom{k}{n}\mathbf{R}^n(\mathbf{I}-\mathbf{R})^{k-n}\vec{\mathbf{1}}^\top, \qquad 1 \le n \le k. \end{aligned}$$

This is exactly the same as formula (2.8) we obtained in Case 3.

3 Recursive Formula for Compound Generalised $(a, b, 1)$ Distributions

In this section, we will derive a recursive formula to calculate the distribution of aggregate claims S defined at the beginning of this paper, given that the number of claims N follows a generalised $(a, b, 1)$ distribution. We first assume that the individual claim amounts are discrete random variables valued on non-negative integers. Secondly, continuous claim amounts will be considered using similar procedures adopted for the discrete case. We will begin with a lemma that is useful in our subsequent derivations.

Lemma 1 *For a generalised $(a, b, 1)$ distribution, $\{p_n\}_{n=0}^{\infty}$, as defined in Definition 3, $\hat{\mathbf{Q}}(z)$ satisfies the following differential equation*

$$\hat{\mathbf{Q}}'(z) = \mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B}) + z\hat{\mathbf{Q}}'(z)\mathbf{A} + \hat{\mathbf{Q}}(z)(\mathbf{A}+\mathbf{B}). \tag{3.1}$$

Proof. We have

$$\begin{aligned} \hat{\mathbf{Q}}'(z) &= \sum_{n=1}^{\infty} nz^{n-1}\mathbf{Q}_n \\ &= \mathbf{Q}_1 + \sum_{n=2}^{\infty} nz^{n-1}\mathbf{Q}_{n-1}\Big(\mathbf{A}+\frac{\mathbf{B}}{n}\Big) \\ &= \mathbf{Q}_1 + \sum_{n=1}^{\infty}(n+1)z^n\mathbf{Q}_n\mathbf{A} + \sum_{n=1}^{\infty} z^n\mathbf{Q}_n\mathbf{B} \\ &= \mathbf{Q}_1 + \Big[z\hat{\mathbf{Q}}'(z) + \hat{\mathbf{Q}}(z) - \mathbf{Q}_0\Big]\mathbf{A} + \Big[\hat{\mathbf{Q}}(z) - \mathbf{Q}_0\Big]\mathbf{B} \\ &= \mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B}) + z\hat{\mathbf{Q}}'(z)\mathbf{A} + \hat{\mathbf{Q}}(z)(\mathbf{A}+\mathbf{B}). \end{aligned}$$

This completes the proof. □

3.1 Discrete Claim Amounts Distribution

Using the notation defined in Section 1, we assume that the individual claims are i.i.d. r.v.'s with p.f. f. Let $\hat{f}(z) = \sum_{x=0}^{\infty} z^x f_x$ and $\hat{g}(z) = \sum_{x=0}^{\infty} z^x g_x$ to be the p.g.f. of f and g respectively, where g is the p.f. of S. Under the generalised $(a, b, 1)$ framework, substituting equation (2.1) into equation (1.1) yields

$$g_x = \sum_{n=0}^{\infty} f_x^{n*} \left(\vec{\gamma} \mathbf{Q}_n \vec{\mathbf{1}}^\top \right) = \vec{\gamma} \left(\sum_{n=0}^{\infty} f_x^{n*} \mathbf{Q}_n \right) \vec{\mathbf{1}}^\top = \vec{\gamma} \mathbf{G}(x) \vec{\mathbf{1}}^\top, \tag{3.2}$$

where $\mathbf{G}(x) = \sum_{n=0}^{\infty} f_x^{n*} \mathbf{Q}_n$. Therefore, $\hat{g}(z)$ can be expressed as

$$\hat{g}(z) = \sum_{x=0}^{\infty} z^x g_x = \sum_{x=0}^{\infty} z^x \left(\vec{\gamma} \mathbf{G}(x) \vec{\mathbf{1}}^\top \right) = \vec{\gamma} \left(\sum_{x=0}^{\infty} z^x \mathbf{G}(x) \right) \vec{\mathbf{1}}^\top.$$

On the other hand, we can rewrite $\hat{g}(z)$ as follows:

$$\hat{g}(z) = \hat{p}\left(\hat{f}(z)\right) = \sum_{n=0}^{\infty} p_n \left[\hat{f}(z)\right]^n = \vec{\gamma} \left[\sum_{n=0}^{\infty} \mathbf{Q}_n \left(\hat{f}(z)\right)^n \right] \vec{\mathbf{1}}^\top = \vec{\gamma} \boldsymbol{\zeta}(z) \vec{\mathbf{1}}^\top, \tag{3.3}$$

where $\boldsymbol{\zeta}(z) = \sum_{n=0}^{\infty} \mathbf{Q}_n \left(\hat{f}(z)\right)^n = \hat{\mathbf{Q}}\left(\hat{f}(z)\right)$. So alternatively, we have $\boldsymbol{\zeta}(z) = \sum_{x=0}^{\infty} z^x \mathbf{G}(x)$.

The expression (3.2) shows us that in order to calculate the p.f. of g_x, we need to find a method to calculate $\mathbf{G}(x)$. If $\mathbf{G}(x)$ can be obtained for all x, then the calculation for g_x is trivial. In what follows, we will derive an recursive equation for $\mathbf{G}(x)$.

Theorem 1 *If the distribution of the number of claims, N, belongs to the generalised $(a, b, 1)$ family of distributions and the individual claim amounts are non-negative integer valued i.i.d. r.v.'s, then the matrix $\mathbf{G}(x)$ defined in (3.2) above satisfies the following recursive formula*

$$\mathbf{G}(x) = \left[\Big(\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A} + \mathbf{B}) \Big) f_x + \sum_{j=1}^{x} f_j \mathbf{G}(x-j) \left(\mathbf{A} + \frac{j}{x}\mathbf{B} \right) \right] (\mathbf{I} - f_0 \mathbf{A})^{-1} \tag{3.4}$$

for $x \geq 1$, and the starting value is $\mathbf{G}(0) = \hat{\mathbf{Q}}(f_0)$.

Proof. From (3.3) we know $\boldsymbol{\zeta}(z) = \hat{\mathbf{Q}}(\hat{f}(z))$. Differentiating with respect to z on both sides of the equation gives

$$\boldsymbol{\zeta}'(z) = \hat{\mathbf{Q}}'\left(\hat{f}(z)\right) \hat{f}'(z). \tag{3.5}$$

Applying (3.1) into the right hand side of (3.5) yields

$$\begin{aligned}
& \left[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B}) + \hat{f}(z)\hat{\mathbf{Q}}'\left(\hat{f}(z)\right)\mathbf{A} + \hat{\mathbf{Q}}\left(\hat{f}(z)\right)(\mathbf{A}+\mathbf{B})\right]\hat{f}'(z) \\
= \; & \left[\mathbf{Q}_1 - \mathbf{Q}_0\left(\mathbf{A}+\mathbf{B}\right)\right]\hat{f}'(z) + \hat{f}(z)\left[\hat{\mathbf{Q}}'\left(\hat{f}(z)\right)\hat{f}'(z)\right]\mathbf{A} + \hat{\mathbf{Q}}\left(\hat{f}(z)\right)(\mathbf{A}+\mathbf{B})\hat{f}'(z) \\
= \; & \left[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B})\right]\hat{f}'(z) + \hat{f}(z)\boldsymbol{\zeta}'(z)\mathbf{A} + \boldsymbol{\zeta}(z)(\mathbf{A}+\mathbf{B})\hat{f}'(z).
\end{aligned}$$

Therefore,

$$\boldsymbol{\zeta}'(z) = \left[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B})\right]\hat{f}'(z) + \hat{f}(z)\boldsymbol{\zeta}'(z)\mathbf{A} + \boldsymbol{\zeta}(z)(\mathbf{A}+\mathbf{B})\hat{f}'(z). \tag{3.6}$$

Expanding both sides of (3.6) in power series and comparing the coefficients of z^{x-1} on both sides yields, for $x > 0$,

$$\begin{aligned}
x\mathbf{G}(x) &= \Big[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B})\Big]xf_x + \sum_{j=0}^{x-1}(x-j)f_j\mathbf{G}(x-j)\mathbf{A} \\
&\quad + \sum_{j=1}^{x} jf_j\mathbf{G}(x-j)(\mathbf{A}+\mathbf{B}) \\
&= \Big[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B})\Big]xf_x + \sum_{j=0}^{x} xf_j\mathbf{G}(x-j)\mathbf{A} + \sum_{j=1}^{x} jf_j\mathbf{G}(x-j)\mathbf{B} \\
&= \Big[\mathbf{Q}_1 - \mathbf{Q}_0(\mathbf{A}+\mathbf{B})\Big]xf_x + xf_0\mathbf{G}(x)\mathbf{A} + \sum_{j=1}^{x} f_j\mathbf{G}(x-j)(x\mathbf{A}+j\mathbf{B}).
\end{aligned}$$

Rearranging the terms on both sides of the above equation and simplifying them we obtain the recursive formula (3.4). The starting value $\mathbf{G}(0)$ of the recursion can be determined as follows:

$$\mathbf{G}(0) = \sum_{n=0}^{\infty} f^{n*}(0)\mathbf{Q}_n = \sum_{n=0}^{\infty}(f_0)^n\,\mathbf{Q}_n = \hat{\mathbf{Q}}(f_0).$$

This completes the proof. □

As commented by Wu and Li (2010), it is worthwhile to further obtain a vector version of recursive equation for $g(s)$. We define $\vec{\mathbf{Q}}_0 = \vec{\gamma}\mathbf{Q}_0$, $\vec{\mathbf{Q}}_1 = \vec{\gamma}\mathbf{Q}_1$ and $\vec{\mathbf{G}}(x) = \vec{\gamma}\mathbf{G}(x)$ to be $1\times m$ row vectors. The matrix-form recursive formula (3.4) has a version in terms of vectors

$$\vec{\mathbf{G}}(x) = \left[\left(\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\right)f_x + \sum_{j=1}^{x} f_j\vec{\mathbf{G}}(x-j)\left(\mathbf{A}+\frac{j}{x}\mathbf{B}\right)\right](\mathbf{I}-f_0\mathbf{A})^{-1}, \tag{3.7}$$

for $x > 0$. The starting vector is $\vec{\mathbf{G}}(0) = \vec{\gamma}\hat{\mathbf{Q}}\left(f_0\right)$.

Remarks:

1. Equation (3.7) can save computational times when calculating the distribution of S.
2. If we assume that the individual claim amounts, X_i, can only take positive integers, i.e. $f_0 = 0$, then equation (3.7) can be further simplified as

$$\vec{\mathbf{G}}(x) = \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\right] f_x + \sum_{j=1}^{x} f_j \vec{\mathbf{G}}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right),$$

with the initial vector is given by $\vec{\mathbf{G}}(0) = \vec{\boldsymbol{\gamma}}\mathbf{Q}_0$.

3. When $m = 1$, we have $\mathbf{A} = a$, $\mathbf{B} = b$, $\vec{\boldsymbol{\gamma}} = 1$ and $\vec{\mathbf{G}}(x) = g(x)$. Then equation (3.7) reduces to the Panjer's recursive formula for the $(a, b, 1)$ family of claim number distributions

$$g(x) = \frac{1}{1 - af_0}\left(q_1 - p_0(a+b)f_x + \sum_{j=1}^{x} f_j g_{x-j}\left(a + \frac{bj}{x}\right)\right),$$

where the initial value is $g(0) = \hat{p}(f_0)$.

3.2 Continuous Claim Amounts Distribution

In this subsection, we consider the continuous individual claim amounts. We still assume that X_i, $i = 1, 2, \ldots$ are i.i.d. r.v.'s with probability density function (p.d.f.) $f(x)$, where $x \in (0, \infty)$. We denote the moment generating functions (m.g.f.) of X_i and S to be $M_X(z)$ and $M_S(z)$ respectively. Then we have

$$M_S(z) = \hat{p}\left(M_X(z)\right) = \vec{\boldsymbol{\gamma}}\hat{\mathbf{Q}}\left(M_X(z)\right)\vec{\mathbf{1}}^\top = \vec{\boldsymbol{\eta}}(z)\vec{\mathbf{1}}^\top,$$

where $\vec{\boldsymbol{\eta}}(z) = \vec{\boldsymbol{\gamma}}\hat{\mathbf{Q}}\left(M_X(z)\right)$. Differentiate $\vec{\boldsymbol{\eta}}(z)$ with respect to z gives

$$\vec{\boldsymbol{\eta}}'(z) = \vec{\boldsymbol{\gamma}}\hat{\mathbf{Q}}'\left(M_X(z)\right)M_X'(z). \tag{3.8}$$

Applying Lemma 1 into the right hand side of (3.8) yields

$$\vec{\boldsymbol{\eta}}'(z) = \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\right] M_X'(z) + M_X(z)\vec{\boldsymbol{\eta}}'(z)\mathbf{A} + M_X'(z)\vec{\boldsymbol{\eta}}(z)(\mathbf{A}+\mathbf{B}). \tag{3.9}$$

Since $\vec{\boldsymbol{\eta}}(z) = \int_0^\infty e^{zx}\vec{\mathbf{G}}(x)dx$ and $M_X(z) = \int_0^\infty e^{zx} f(x)dx$, the above equation can be rewritten as

$$\begin{aligned}
&\int_0^\infty xe^{zx}\vec{\mathbf{G}}(x)dx \\
= &\left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\right]\int_0^\infty xe^{zx}f(x)dx + \int_0^\infty e^{zy}f(y)dy\int_0^\infty xe^{zx}\vec{\mathbf{G}}(x)dx\mathbf{A} \\
&+\left(\int_0^\infty ye^{zy}f(y)dy\int_0^\infty e^{zx}\vec{\mathbf{G}}(x)dx\right)(\mathbf{A}+\mathbf{B}) \\
= &\left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\right]\int_0^\infty xe^{zx}f(x)dx + \int_0^\infty\int_y^\infty (x-y)f(y)e^{zx}\vec{\mathbf{G}}(x-y)dxdy\mathbf{A} \\
&+\int_0^\infty\int_y^\infty yf(y)e^{zx}\vec{\mathbf{G}}(x-y)dxdy(\mathbf{A}+\mathbf{B}). \qquad (3.10)
\end{aligned}$$

Interchanging the order of integration at the right hand side of (3.10) gives

$$\begin{aligned}
\int_0^\infty xe^{zx}\vec{\mathbf{G}}(x)dx \;=\; &\left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\right]\int_0^\infty xe^{zx}f(x)dx \\
&+\int_0^\infty e^{zx}\int_0^x (x-y)f(y)\vec{\mathbf{G}}(x-y)dydx\mathbf{A} \\
&+\int_0^\infty e^{zx}\int_0^x yf(y)\vec{\mathbf{G}}(x-y)dxdy(\mathbf{A}+\mathbf{B}). \qquad (3.11)
\end{aligned}$$

By comparing the coefficients of e^{zx} on both sides of equation (3.11), we obtain the following result.

Theorem 2 *If the distribution of the number of claims, N, belongs to the generalised $(a, b, 1)$ family and the individual claim amounts are i.i.d. continuous non-negative random variables, then for S, $\vec{\mathbf{G}}(x)$ satisfies the following integral equation, for $x > 0$,*

$$\vec{\mathbf{G}}(x) = \left[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\right]f(x) + \int_0^x \vec{\mathbf{G}}(x-y)\left[\mathbf{A}+\frac{y}{x}\mathbf{B}\right]f(y)dy.$$

4 The Moments of the Aggregate Claims

Having discussed the recursive calculation for the p.f. of the aggregate claims S with a generalised $(a, b, 1)$ distributed number of claims, in this section we will consider how to evaluate the moments of S. As shown in the following, the moments of S can also be calculated recursively. Similar results have been obtained by Wu and Li (2010) in the generalised $(a, b, 0)$ case.

Firstly we define row vector $\vec{\mathbf{H}}(r) = \sum_{x=0}^\infty x^r\vec{\mathbf{G}}(x)$, such that, for $r = 0, 1, 2, \ldots$,

$$\mathbb{E}(S^r) = \sum_{x=0}^\infty x^r g(x) = \left(\sum_{x=0}^\infty x^r\vec{\mathbf{G}}(x)\right)\vec{\mathbf{1}}^\top = \vec{\mathbf{H}}(r)\vec{\mathbf{1}}^\top, \qquad (4.1)$$

where $\vec{\mathbf{H}}(0) = \sum_{x=0}^{\infty} \vec{\mathbf{G}}(x)$. Formula (4.1) implies that if $\vec{\mathbf{H}}(r)$ can be calculated, so can $\mathbb{E}(S^r)$. In what follows a method is developed to calculate $\vec{\mathbf{H}}(r)$ recursively when individual claim amounts are discrete.

Theorem 3 *The moment vectors $\vec{\mathbf{H}}(r)$, $r \geq 1$, defined above satisfy the following recursive equation:*

$$\begin{aligned}\vec{\mathbf{H}}(r) &= \Bigg[\Big(\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\Big)\mathbb{E}\left(X^r\right) \\ &\quad + \sum_{k=0}^{r-1}\mathbb{E}\left(X^{r-k}\right)\vec{\mathbf{H}}(k)\left(\binom{r}{k}\mathbf{A} + \binom{r-1}{k}\mathbf{B}\right)\Bigg]\left[\mathbf{I}-\mathbf{A}\right]^{-1}. \qquad (4.2)\end{aligned}$$

Proof. Based on recursive formula (3.7) and the definition of $\vec{\mathbf{H}}(r)$, we have

$$\vec{\mathbf{H}}(r) = \sum_{x=1}^{\infty} x^r \left[\Big(\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\Big) f_x + \sum_{j=1}^{x} f_j \vec{\mathbf{G}}(x-j)\left(\mathbf{A} + \frac{j}{x}\mathbf{B}\right)\right]\left[\mathbf{I} - f_0\mathbf{A}\right]^{-1}.$$

Using a similar approach as in Wu and Li (2010) gives

$$\begin{aligned}&\vec{\mathbf{H}}(r)\left[\mathbf{I} - f_0\mathbf{A}\right] \\ =\ & \Big[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\Big]\mathbb{E}(X^r) + \sum_{x=1}^{\infty} x^r \sum_{j=1}^{x} f_j \vec{\mathbf{G}}(x-j)\mathbf{A} + \sum_{x=1}^{\infty} x^{r-1}\sum_{j=1}^{x} j f_j \vec{\mathbf{G}}(x-j)\mathbf{B} \\ =\ & \Big[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\Big]\mathbb{E}(X^r) + \sum_{j=1}^{\infty} f_j \sum_{x=0}^{\infty}(x+j)^r \vec{\mathbf{G}}(x)\mathbf{A} + \sum_{j=1}^{\infty} j f_j \sum_{x=0}^{\infty}(x+j)^{r-1}\vec{\mathbf{G}}(x)\mathbf{B}.\end{aligned}$$

Using the binomial expansion, the right hand side of the above equation can be rewritten and further simplified as follows

$$\begin{aligned}&\Big[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\Big]\mathbb{E}(X^r) + \sum_{j=1}^{\infty} f_j \sum_{x=0}^{\infty}\left(\sum_{k=0}^{r}\binom{r}{k}x^k j^{r-k}\right)\vec{\mathbf{G}}(x)\mathbf{A} \\ &+ \sum_{j=1}^{\infty} j f_j \sum_{x=0}^{\infty}\left(\sum_{k=0}^{r-1}\binom{r-1}{k}x^k j^{r-1-k}\right)\vec{\mathbf{G}}(x)\mathbf{B} \\ =\ & \Big[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\Big]\mathbb{E}(X^r) + \sum_{j=1}^{\infty} f_j \sum_{k=0}^{r}\binom{r}{k}j^{r-k}\left(\sum_{x=0}^{\infty}x^k\vec{\mathbf{G}}(x)\right)\mathbf{A} \\ &+ \sum_{j=1}^{\infty} j f_j \sum_{k=0}^{r-1}\binom{r-1}{k}j^{r-1-k}\left(\sum_{x=0}^{\infty}x^k\vec{\mathbf{G}}(x)\right)\mathbf{B} \\ =\ & \Big[\vec{\mathbf{Q}}_1 - \vec{\mathbf{Q}}_0(\mathbf{A}+\mathbf{B})\Big]\mathbb{E}\left(X^r\right) + \sum_{k=0}^{r-1}\binom{r}{k}\mathbb{E}\left(X^{r-k}\right)\vec{\mathbf{H}}(k)\mathbf{A} + (1-f_0)\vec{\mathbf{H}}(r)\mathbf{A} \\ &+ \sum_{k=0}^{r-1}\binom{r-1}{k}\mathbb{E}\left(X^{r-k}\right)\vec{\mathbf{H}}(k)\mathbf{B}.\end{aligned}$$

Rearranging the terms on both sides yields equation (4.1). This completes the proof.

5 Numerical Examples

In this section, we will provide three numerical examples to demonstrate how to calculate the distribution and moments of S using recursive formulae (3.7) and (4.2).

Example 1. We will first assume that N follows a DPH distribution, i.e. $N \sim \mathrm{PH}_d(\vec{\boldsymbol{\alpha}}, \mathbf{T})$. We remark that although a more general DPH distribution could be chosen for $\{p_n\}$, for the convenience of comparisons, we will adopt the one considered in the Example 7 of Wu and Li (2010), where $\alpha_0 = 0$. Note that $\alpha_0 = 0$ is not going to have any negative impact on comparing the recursive formula (3.7) derived for compound generalised $(a, b, 1)$ distributions and the one, formula (19) in Wu and Li (2010), specifically developed for compound phase-type distributions.

Let $\vec{\boldsymbol{\alpha}} = (0.1, 0.2, 0.5, 0.05, 0.15)$ and $\mathbf{T}$ is given as follows:

$$\mathbf{T} = \begin{pmatrix} 0.2 & 0.4 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}.$$

We let $\mathbf{Q}_0$ to be $\mathbf{0}$ such that $\vec{\gamma}\mathbf{Q}_0\vec{\mathbf{1}} = \alpha_0 = 0$. Having known $\mathbf{Q}_0$, $\vec{\gamma}$ can be determined using identity (2.3), which is

$$\vec{\gamma} = \vec{\boldsymbol{\alpha}}(\mathbf{I} - \mathbf{Q}_0)^{-1} = \vec{\boldsymbol{\alpha}}.$$

The individual claims follow a negative binomial distribution with parameters 5 and 0.75. The p.f. has the form $f_x = \binom{x+4}{4}(0.25)^5(0.75)^x$, $x = 0, 1, \ldots$.

A direct application of equation (3.7) allows us to calculate the p.f. of the S, $g(x)$ recursively. In particular, the starting vector is given as

$$\vec{\mathbf{G}}(0) = \vec{\gamma}\hat{\mathbf{Q}}(f_0) = \vec{\gamma}\mathbf{Q}_0 + \vec{\gamma}(\mathbf{I} - \mathbf{Q}_0)(\mathbf{I} - f_0\mathbf{T})^{-1}(\mathbf{I} - \mathbf{T})f_0.$$

We use *Mathematica* as our main computational tool for all calculations involved in this section. The results for this example produces exactly the same results as Table 3 of Wu and Li (2010).

When we calculate the rth moments of S using Equation (4.2), one difficulty exists in determining the initial vector $\vec{\mathbf{H}}(0)$ as there is not a closed-form expression for it. We used a numerical approach that requires the summation of $\vec{\mathbf{G}}(x)$ for x from 0 up to N, where N can be determined making use of the tail probability of g and a certain accuracy requirement. For instance in this example, we want the total tail probability to be less than 10^{-5}, then choosing $N = 300$ is sufficient (tail prob.$= 3 \times 10^{-6}$). The results obtained are again identical to those provided in Table 4 of Wu and Li (2010).

Example 2. This example corresponds to Case 2 that is discussed in Section 3. Let $\vec{\gamma} = (0.1, 0.15, 0.25, 0.45, 0.05)$ and N follow a generalised $(a, b, 1)$ distribution with parameters $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} = \mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is an invertible matrix given as follows

$$\mathbf{\Lambda} = \begin{pmatrix} 0.4 & 0.1 & 0.2 & 0.0 & 0.2 \\ 0.1 & 0.35 & 0.0 & 0.25 & 0.2 \\ 0.2 & 0.0 & 0.3 & 0.1 & 0.2 \\ 0.2 & 0.2 & 0.1 & 0.55 & 0.05 \\ 0.2 & 0.1 & 0.1 & 0.15 & 0.3 \end{pmatrix}.$$

Also, we assume

$$\mathbf{Q}_0 = \begin{pmatrix} 0.3 & 0.0 & 0.25 & 0.0 & 0.0 \\ 0.0 & 0.3 & 0.0 & 0.3 & 0.0 \\ 0.1 & 0.0 & 0.3 & 0.0 & 0.2 \\ 0.0 & 0.6 & 0.0 & 0.3 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.4 \end{pmatrix}.$$

The individual claims follow a Logarithmic distribution with parameter $\theta = 0.95$, i.e., having a p.f. with the form

$$f_x = \left(-\frac{1}{x}\right) \frac{0.95^x}{\ln(0.05)}, \qquad x > 0.$$

Since f_x is only defined for $x > 0$, the initial vector $\vec{\mathbf{G}}(0)$ can be determined using the reduced formula

$$\vec{\mathbf{G}}(0) = \vec{\gamma}\mathbf{Q}_0.$$

In the following, Table 1 includes values of the vector $\vec{\mathbf{G}}(x)$ and $g(x)$ for some x values from 0 to 100. The total tail probability for $S > 100$ is 0.000265. The first four moments of S are presented in Table 2.

Example 3. Our last example considers the case where N follows a generalised $(a, b, 1)$ distribution discussed in Case 3, Section 3. In particular, p_n satisfies

$$\begin{aligned} p_0 &= \vec{\gamma}\mathbf{Q}_0\vec{\mathbf{1}}^\top, \\ p_n &= \vec{\gamma}\left[(\mathbf{I} - \mathbf{Q}_0)\left[\mathbf{I} - (\mathbf{I} - \mathbf{R})^k\right]^{-1} \binom{k}{n} \mathbf{R}^n (\mathbf{I} - \mathbf{R})^{k-n}\right] \vec{\mathbf{1}}^\top, \quad 1 \le n \le k, \end{aligned}$$

where $k = 10$,

$$\mathbf{Q}_0 = \begin{pmatrix} 0.1 & 0.3 & 0.0 & 0.4 & 0.0 \\ 0.0 & 0.1 & 0.0 & 0.1 & 0.0 \\ 0.0 & 0.9 & 0.1 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.6 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0.7 & 0.1 & 0.2 & 0.0 & 0.0 \\ 0.1 & 0.4 & 0.0 & 0.2 & 0.2 \\ 0.2 & 0.0 & 0.3 & 0.1 & 0.2 \\ 0.3 & 0.0 & 0.1 & 0.5 & 0.1 \\ 0.0 & 0.3 & 0.1 & 0.0 & 0.6 \end{pmatrix},$$

Table 1: Vector $\vec{\mathbf{G}}$ values and the p.f. of S in Example 2

x	$\vec{\mathbf{G}}(x)$	$g(x)$
0	(0.055000, 0.340000, 0.100000, 0.180000, 0.070000)	0.745000
1	(0.003588, -0.057945, 0.036393, 0.069502, -0.006710)	0.044826
2	(0.004158, -0.028585, 0.020126, 0.037195, -0.003525)	0.029369
3	(0.003645, -0.018369, 0.013829, 0.025121, -0.002239)	0.021987
4	(0.003149, -0.013188, 0.010421, 0.018713, -0.001563)	0.017532
5	(0.002739, -0.010068, 0.008267, 0.014717, -0.001153)	0.014501
10	(0.001528, -0.003920, 0.003649, 0.006332, -0.000369)	0.007220
20	(0.000627, -0.001166, 0.001247, 0.002112, -0.000068)	0.002752
30	(0.000296, -0.000460, 0.000538, 0.000900, -0.000013)	0.001261
40	(0.000149, -0.000204, 0.000256, 0.000424, 0.000000)	0.000624
50	(0.000078, -0.000097, 0.000128, 0.000211, 0.000002)	0.000322
100	(0.000004, -0.000004 ,0.000006, 0.000009, 0.000000)	0.000016

Table 2: Vector $\vec{\mathbf{H}}$ values and $\mathbb{E}(S^r)$ in Example 2

r	$\vec{\mathbf{H}}(r)$	$\mathbb{E}(\mathbb{S}^\frown)$
1	(0.572807, -1.222330, 1.228880, 2.106730, -0.080291)	2.60578
2	(16.26330, -24.17530, 29.92750, 48.24820, -0.355619)	68.9081
3	(762.4480, -924.9410, 1238.570, 2035.780, 28.66220)	7983.54
4	(50319.90, -53039.40, 77236.10, 125741.0, 3804.560)	204062

and $\vec{\gamma} = (0.1, 0.2, 0.5, 0.05, 0.15)$. Note that the parameters k, $\mathbf{R}$ and $\vec{\gamma}$ specified above are the same as those used in Example 5 of Wu and Li (2010). However, due to the difference between the recursive starting points of the generalised $(a, b, 1)$ and the generalised $(a, b, 1)$ families and the imposing of $\mathbf{Q}_0$ for the latter family, the claim number distribution $\{q\}$ generated under the two examples are totally different. For instance, in Example 5 of Wu and Li (2010), $p_0 = 0.0214405$, but in this example $p_0 = 0.84$.

Again, we employ the negative binomial distribution in Example 1 to model the individual claims. Based on the discussions made in Case 3, Section 3, the starting vector $\vec{\mathbf{G}}(0)$ can be calculated as

$$\vec{\mathbf{G}}(0) = \vec{\gamma}\mathbf{Q}_0 + (1-p_0)\vec{\gamma}\left[\mathbf{I} - (\mathbf{I}-\mathbf{R})^{10}\right]^{-1}\left[\Big(\mathbf{I} - (1-f_0)\mathbf{R}\Big)^{10} - (\mathbf{I}-\mathbf{R})^{10}\right].$$

The computed values for the vector $\vec{\mathbf{G}}(x)$ and $g(x)$ are summarised into Table 3 for some chosen x between 0 and 200. The total tail probability for $S > 200$ is 0.00045. Since the computations for moments of S will just repeat what we did for Example 2 except using different numbers, we will omit here.

Table 3: Vector $\vec{\mathbf{G}}$ values and the p.f. of S in Example 3

x	$\vec{\mathbf{G}}(x)$	$g(x)$
0	(0.009974, 0.519986, 0.050066, 0.170002, 0.089982)	0.840010
1	(-0.000098, -0.000053, 0.000248, 0.000007, -0.000067)	0.000038
2	(-0.000220, -0.000112, 0.000559, 0.000016, -0.000149)	0.000086
3	(-0.000387, -0.000212, 0.000981, 0.000029, -0.000259)	0.000152
4	(-0.000583, -0.000326, 0.001479, 0.000045, -0.000385)	0.000231
5	(-0.000792, -0.000454, 0.002011, 0.000066, -0.000512)	0.000319
10	(-0.001635, -0.001214, 0.004157, 0.000234, -0.000764)	0.000777
20	(-0.001962, -0.003441, 0.005278, 0.000903, 0.000974)	0.001751
30	(-0.001633, -0.005190, 0.005893, 0.001080, 0.002132)	0.002283
40	(-0.000681, -0.005582, 0.006114 ,0.000384, 0.001849)	0.002084
50	(0.000584, -0.005171, 0.005742, -0.000720, 0.001187)	0.001621
100	(0.001718, -0.000628, 0.001382, -0.001853, -0.000100)	0.000519
150	(0.000305, 0.000081, 0.000146, -0.000081, 0.000042)	0.000493
200	(0.000018, 0.000006, 0.000008, 0.000003, 0.000004)	0.000039

6 Conclusions

In this paper, we proposed a generalised $(a, b, 1)$ family of distributions that is a broad class of counting distributions, which employ matrices as parameters and satisfy a matrix version of recursive structure of the $(a, b, 1)$ family. On one hand, one can benefit from the great flexibility embedded in setting up matrix parameters. On the other hand, it makes it very difficult to propose appropriate conditions upon how to select proper matrices to build up claim number distributions under the structure. Also, the issue of identifying all members within the family remains unsolved. As a result, only some very special members of the generalised $(a, b, 1)$ family are examined in this paper. It leaves a number of open problems for interested readers to explore.

References

[1] Culver, Walter J. (1966), "On the existence and uniqueness of the real logarithm of a matrix", Proceedings of the American Mathematical Society 17: 11461151.

[2] Eisele, K.T. (2006) Recursions for compound phase distributions. *Insurance: Mathematics and Economics* 38, 149-156.

[3] Hipp, C. (2006) Speedy convolution algorithms and Panjer recursions for phase-type distributions. *Insurance: Mathematics and Economics* 38, 176-188.

[4] Klugman, S.A., Panjer, H.H., and Willmot, G.E. (1998) *Loss Models: From Data to Decisions*. Wiley Series in Probability and Statistics, John Wiley and Sons, Inc.

[5] Latouche, G. and Ramaswami, V. (1999) *Introduction to matrix analytic methods in stochastic modeling*. ASA SIAM, Philadelphia.

[6] Neuts, M.F. (1981) *Matrix-geometric solutions in stochastic models: An algorithmic approach*. Johns Hopkins University Press, Baltimore.

[7] Panjer, H. (1981) Recursive evaluation of a family of compound distributions. *ASTIN Bulletin* 12, 22-26.

[8] Schröter, K.J. (1991) On a family of counting distributions and recursions for the related compound distributions. *Scandinavian Actuarial Journal*, 161-175.

[9] Sundt, B. (1992) On some extensions of Panjer's class of counting distributions. *ASTIN Bulletin* 22, 61-80.

[10] Sundt, B. (2002) Recursive evaluation of aggregate claims distributions. *Insurance: Mathematics and Economics* 30, 297-332.

[11] Sundt, B. (2003) Some recursions for moments of compound distributions. *Insurance: Mathematics and Economics* 33, 486-496.

[12] Sundt, B. and Jewell, W.S. (1981) Further results on recursive evaluation of compound distributions. *ASTIN Bulletin* 12, 27-39.

[13] Willmot, G. (1988) Sundt and Jewell's family of discrete distributions. *ASTIN Bulletin* 18, 17-29.

[14] Wu, X. and Li, S. (2010) Matrix-form recursions for a family of compound distributions. *ASTIN Bulletin*, in press.