On the time and the number of claims when the surplus drops below a certain level

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Abstract: In this paper, same as in Li and Lu (2013), we define $T_z(u)$ to be the first time that the surplus process drops below a certain level z from the initial surplus $u(> z)$ for a risk model with interest. A generalized Gerber-Shiu-type function is then defined based on the first time and the number of claims that the surplus drops below z from u, and other $T_z(u)$-related random variables. Explicit expressions for this function, when $u = z$, and when $u > z$ under exponential claims, are obtained. We then obtain the moments and probability function (with numerical examples) of the number of claims until $T_z(u)$. We also investigate a joint transform function of the two-sided first exit time and the number of claims until then, and obtain the probability of the surplus hitting an upper level from the initial surplus without having dropped below a lower level with Erlang(2) claims.

Keywords: risk model with interest; generalized Gerber-Shiu function; ruin probability; absolute ruin probability; two-sided first exit time, continued fractions, special functions.

1 Introduction

Consider the classical compound Poisson risk model, in which the insurer's surplus process $\{U(t);\, t \geq 0\}$ is described as

$$U(t) = u + ct - S(t)\,, \qquad t \geq 0\,, \tag{1.1}$$

where c denotes the constant premium income rate per unit of time and $u(\geq 0)$ is the initial surplus. The counting process $\{N(t);\, t \geq 0\}$ is assumed Poisson with parameter λ, representing the number of claims that have occurred before time t, and

$$S(t) = \sum_{n=1}^{N(t)} X_n\,, \qquad t \geq 0\,,$$

is the aggregate claim amounts up to time t, where the sequence of random claim amounts $\{X_n;\, n \geq 0\}$ is assumed to have a common probability density function $p(x)$, for $x \geq 0$.

Furthermore, we also assume that $\{X_n;\, n \geq 0\}$ is independent of $\{N(t);\, t \geq 0\}$ and $c > \lambda\mu$ to have a positive loading condition with $\mu = \mathbb{E}[X_1]$.

Assume that the insurer receives interest on its surplus continuously at a constant force of interest δ per unit of time. Then the corresponding surplus process, denoted by $\{U_\delta(t);\, t \geq 0\}$, can be described as

$$U_\delta(t) = ue^{\delta t} + c\bar{s}_{\overline{t}|\delta} - \int_0^t e^{\delta(t-v)} dS(v)\,, \qquad t \geq 0\,, \tag{1.2}$$

where

$$\bar{s}_{\overline{t}|\delta} = \int_0^t e^{\delta v} dv = \begin{cases} \frac{e^{\delta t}-1}{\delta}\,, & \delta > 0 \\ t\,, & \delta = 0 \end{cases}\,.$$

This classical risk model with interest was first treated in 1940s and has attracted a fair amount of attention and interest of many researchers in recent years. Gerber (1971) studied the so-called absolute ruin probability. Sundt and Teugels (1995) investigated the probability of ruin when the initial surplus is zero or when claims are exponentially distributed. Dassios and Embrechts (1989) analyzed some insurance risk processes including (1.2) by using the martingale approach along with the theory of piecewise-deterministic Markov processes (PDMP) where the ruin probability was derived for exponential distributed claim amounts. Embrechts and Schmidli (1994) also used the theory of PDMP to investigate a class of general insurance risk models which includes (1.2) as a special case, and discussed the absolute ruin probability. Cai and Dickson (2002) and Liu and Mao (2006) studied the expected discounted penalty function at ruin for this risk model. For a detailed literature review on this model, see Li and Lu (2013).

As in Li and Lu (2013), we define

$$T_z(u) = \inf\{t \geq 0 : U_\delta(t) < z\}\,, \qquad -\frac{c}{\delta} \leq z < u\,,$$

($T_{u,z}$ in Li and Lu (2013)) to be the first time that the surplus process drops below z from level $u(\geq z)$ with $T_z(u) = \infty$ if $U_\delta(t) \geq z, \forall t \geq 0$. Now define

$$\psi_n(u; z) = \mathbb{P}\left(T_z(u) < \infty, N\big(T_z(u)\big) = n\right)\,, \qquad -\frac{c}{\delta} \leq z < u, n \in \mathbb{N}^+,$$

to be the probability that the surplus will ever drop below z and the number of claims by $T_z(u)$ is n, then $\psi(u; z) = \sum_{n=1}^{\infty} \psi_n(u; z)$ is the probability that the surplus will ever drops below z, and $m_v(u; z) = \sum_{n=1}^{\infty} v^n \psi_n(u; z)$, $0 < v \leq 1$, is the probability generating function of the number of claims by $T_z(u)$ with parameter v.

We remark here that $T_u(u)$ is the first time that the surplus drops below the initial level u, $T_0(u)$ is the time of ruin and $T_{-c/\delta}(u)$ is the time of absolute ruin for $u \geq 0$, which are of interest in practice for insurers to monitor their surplus level and financial solvency for the purpose of risk management. Accordingly, the probabilistic measures that can be used over the course of claim payments for such purpose, for instance, are $\psi_n(u; u)$, the probability that the surplus will ever drop below its initial level u at the nth claim, and $\psi_n(u; 0)$ and

$\psi_n(u;-c/\delta)$, the probabilities of ruin and absolute ruin, respectively, at the nth claim for $u \geq 0$. In addition, we point out that for the classical risk model, i.e., $\delta = 0$, $T_z(u)$ has the same distribution as $T_0(u-z)$ implying $\psi_n(u;z) = \psi_n(u-z;0)$, and hence it is of particular interest and importance to study the random variable $T_z(u)$ and related quantities for the case when $\delta > 0$.

Moreover, we define a Gerber-Shiu-type expected discounted penalty function at $T_z(u)$. Let $w(x,y) \geq 0$ be a bivariate non-negative function and define for $\delta > 0, \alpha \geq 0, 0 < v \leq 1$ and $-(c/\delta) \leq z < u$,

$$\Phi_{\alpha,\delta,v}(u;z) = \mathbb{E}\Big[e^{-\alpha T_z(u)} v^{N(T_z(u))} w\big(U_\delta(T_z(u)-), z - U_\delta(T_z(u))\big) I(T_z(u) < \infty) \big| U_\delta(0) = u\Big], \tag{1.3}$$

where u is the initial surplus. The original such function, first introduced in Gerber and Shiu (1998), is based on the time of ruin, the surplus level just before ruin and the deficit immediately after ruin for the classical risk model (1.1), where the time of ruin is defined as the first time that the surplus drops below zero. Apparently, $\Phi_{\alpha,\delta,v}(u;z)$ can be seen as a generalized expected discounted penalty function at $T_z(u)$ for the number of claims by $T_z(u)$, the surplus prior to $T_z(u)$, and the amount by which the surplus is below level z. When $v = 1$, $\Phi_{\alpha,\delta,1}(u;z)$ is the function studied in Li and Lu (2013) under a constant interest rate, and was used to obtain the distribution of the maximum severity of ruin. When $\alpha = 0$ and $w(x,y) = 1$, $\Phi_{\alpha,\delta,v}(u;z) = m_v(u;z)$ is the probability generating function of the number of claims until the surplus first drops below level z starting from $u(> z)$.

Stanford and Stroiński (1994) presented a recursive method for computing the probability at claim instants that cause ruin for the classical compound Poisson risk process. Explicit formulas were obtained for the probability of ruin at the arrival of the nth claim when the claims are exponentially, mixture of two exponentials and Erlang(2) distributed. Egidio dos Reis (2002) derived the Laplace transform (LT) of the number of claims until ruin for the same model, while Dickson (2012) recently derived an expression for the joint density of the time to ruin and the number of claims until ruin using the probabilistic arguments. Landriault et al. (2011) used the Gerber-Shiu-type function to obtain an expression of the probability function of the number of claims until ruin for a Sparre Andersen risk model with exponential claims.

Further to the results presented in Li and Lu (2013), we want to show in this paper more solvency-related quantities can be obtained based on the newly introduced stopping time $T_z(u)$ for the classical risk model with interest. A generalized Gerber-Shiu-type function (1.3) based on $T_z(u)$ with the number of claims until the surplus dropping below level z involved is introduced and studied for the first time in the literature for the risk model with interest. With the help of this generalized function, we are able to obtain quantities related to the number of claims until the time at which the surplus drops below level z, $N(T_z(u))$, such as the moments and the probability distribution function of $N(T_z(u))$ when claim amounts are exponentially distributed. We further investigate the problem of two-sided first exit time, at which the surplus process either first drops below z from level u without crossing an upper level $b(\geq u)$ or first crosses level b from u without dropping below a lower level z. To do so, we consider a specific expected discounted function with the two-sided exit time and the number of claims by then involved. In the exponential claims case, explicit expressions of the above mentioned specific function and its decomposition are derived. We also derive the probability of the surplus hitting an upper level b from u without having dropped below a lower level z when the claim amounts are Erlang(2) distributed.

In this paper, we first obtain in Section 2 an integro-differential equation for $\Phi_{\alpha,\delta,v}(u,z)$, and then derive an expression for $\Phi_{\alpha,\delta,v}(z;z)$. In Section 3, an explicit expression of $\Phi_{\alpha,\delta,v}(u,z)$ is obtained under exponential claims through which the moments and probability distribution function of $N(T_z(u))$ are derived and numerically illustrated. In Section 4, we investigate a joint transform function of the two-sided first exit time and the number of claim by then and derive some explicit expressions when claims are exponentially and Erlang(2) distributed.

2 General results for $\Phi_{\alpha,\delta,v}(u;z)$

In this section we derive an integro-differential equation satisfied by $\Phi_{\alpha,\delta,v}(u;z)$ and an explicit expression of $\Phi_{\alpha,\delta,v}(z;z)$. Note that the derivation of the former is similar to that in Section 2 of Li and Lu (2013) but simpler as a constant interest rate for both positive and negative surpluses is used, and hence we only show the main steps below, while the derivation of the latter shows slightly different details.

2.1 An integro-differential equation for $\Phi_{\alpha,\delta,v}(u;z)$

For $-c/\delta \leq z < u$, by conditioning on the time of the first claim, t, and the size of the first claim, x, we get

$$\Phi_{\alpha,\delta,v}(u;z) = \int_0^\infty \lambda v e^{-(\lambda+\alpha)t} \left[\int_0^{ue^{\delta t}+c\bar{s}_{\overline{t}|\delta}-z} \Phi_{\alpha,\delta,v}(ue^{\delta t}+c\bar{s}_{\overline{t}|\delta}-x;z)p(x)dx \right. \\ \left. + \int_{ue^{\delta t}+c\bar{s}_{\overline{t}|\delta}-z}^\infty w(ue^{\delta t}+c\bar{s}_{\overline{t}|\delta}, z+x-ue^{\delta t}-c\bar{s}_{\overline{t}|\delta})p(x)dx \right] dt\,. \tag{2.1}$$

Substituting $y = ue^{\delta t}+c\bar{s}_{\overline{t}|\delta} = ue^{\delta t}+c(e^{\delta t}-1)/\delta$ in (2.1), we have

$$\Phi_{\alpha,\delta,v}(u;z) = \lambda v(\delta u+c)^{\frac{\lambda+\alpha}{\delta}} \\ \times \int_u^\infty (\delta y+c)^{-\frac{\lambda+\alpha}{\delta}-1} \left[\int_0^{y-z} \Phi_{\alpha,\delta,v}(y-x;z)p(x)dx + \omega(y;z) \right] dy\,, \tag{2.2}$$

where

$$\omega(u;z) = \int_{u-z}^\infty w(u,z+x-u)p(x)dx\,. \tag{2.3}$$

Differentiating (2.2) with respect to u and rearranging yield, for $-\frac{c}{\delta} \leq z < u$, that

$$(\lambda+\alpha)\Phi_{\alpha,\delta,v}(u;z) = (\delta u+c)\Phi'_{\alpha,\delta,v}(u;z) + \lambda v \int_0^{u-z} \Phi_{\alpha,\delta,v}(u-x;z)p(x)dx + \lambda v\omega(u;z)\,, \tag{2.4}$$

with a boundary condition $\Phi_{\alpha,\delta,v}(\infty; z) = 0$. Note that when $v = 1$, (2.4) is actually (2.8) with $\delta = r$ in Li and Lu (2013).

Furthermore, replacing u by t in (2.4), integrating both sides from z to u with respect to t and doing some simplifications, we obtain

$$(\delta u + c)\Phi_{\alpha,\delta,v}(u; z) = (\delta z + c)\Phi_{\alpha,\delta,v}(z; z) - \lambda v \int_z^u \omega(t; z)dt + \int_z^u [\delta + \alpha + \lambda]\, \Phi_{\alpha,\delta,v}(t; z)dt \\ - \lambda v \int_0^{u-z} p(x) \left(\int_z^{u-x} \Phi_{\alpha,\delta,v}(y; z)dy \right) dx\,. \tag{2.5}$$

Note that the quantity $\Phi_{\alpha,\delta,v}(z; z)$ in (2.5) is essential for the expression of $\Phi_{\alpha,\delta,v}(u; z)$. In the subsection below, we obtain an explicit expression for $\Phi_{\alpha,\delta,v}(z; z)$.

2.2 Explicit expression of $\Phi_{\alpha,\delta,v}(z; z)$

For $z > -c/\delta$, define

$$Y_{\alpha,\delta,v}(u; z) = \begin{cases} \frac{\Phi_{\alpha,\delta,v}(z;z) - \Phi_{\alpha,\delta,v}(u;z)}{\Phi_{\alpha,\delta,v}(z;z)}\,, & u \geq z \\ 0\,, & u < z \end{cases}\,,$$

as an auxiliary function, which implies

$$\Phi_{\alpha,\delta,v}(u; z) = \Phi_{\alpha,\delta,v}(z; z) - \Phi_{\alpha,\delta,v}(z; z)Y_{\alpha,\delta,v}(u; z)\,. \tag{2.6}$$

Let $\tilde{y}_{\alpha,\delta,v}(s; z)$ be the LT of $Y_{\alpha,\delta,v}(u; z)$ with respect to u on the entire real line, that is,

$$\tilde{y}_{\alpha,\delta,v}(s; z) = \int_{-\infty}^{\infty} e^{-su} Y_{\alpha,\delta,v}(u; z)du = \int_z^{\infty} e^{-su} Y_{\alpha,\delta,v}(u; z)du\,.$$

Substituting (2.6) into (2.5) and rearranging, we obtain, for $u \geq z$, that

$$(\delta u + c)Y_{\alpha,\delta,v}(u; z) = (\delta + \alpha + \lambda) \int_z^u Y_{\alpha,\delta,v}(t; z)dt - (\alpha + \lambda)(u - z) \\ + \lambda v \int_0^{u-z} (u - z - x)p(x)dx + \frac{\lambda v}{\Phi_{\alpha,\delta,v}(z; z)} \int_z^u \omega(t; z)dt \\ - \lambda v \int_0^{u-z} p(x) \left(\int_z^{u-x} Y_{\alpha,\delta,v}(y; z)dy \right) dx\,. \tag{2.7}$$

Taking the LTs of both sides of (2.7) with respect to u and rearranging, we get a first-order linear ordinary differential equation for $\tilde{y}_{\alpha,\delta}(s; z)$:

$$\delta \tilde{y}'_{\alpha,\delta,v}(s; z) + \left(\frac{\delta + \alpha + \lambda[1 - v\tilde{p}(s)]}{s} - c \right) \tilde{y}_{\alpha,\delta,v}(s; z) = \frac{\alpha + \lambda[1 - v\tilde{p}(s)]}{s^2} e^{-zs} - \frac{\lambda v \tilde{\omega}(s; z)}{s\Phi_{\alpha,\delta,v}(z; z)}\,,$$

which can be rewritten, for some $s_0 > 0$, as

$$\frac{d}{ds}\left[s^{\frac{\delta+\alpha}{\delta}}\tilde{y}_{\alpha,\delta,v}(s;z)e^{-\frac{1}{\delta}\int_{s_0}^{s}\left(c-\frac{\lambda[1-v\tilde{p}(t)]}{t}\right)dt}\right]$$
$$= \frac{1}{\delta}s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{1}{\delta}\int_{s_0}^{s}\left(c-\frac{\lambda[1-v\tilde{p}(t)]}{t}\right)dt}\left(\frac{\alpha+\lambda[1-v\tilde{p}(s)]}{s^2}e^{-zs}-\frac{\lambda v\tilde{\omega}(s;z)}{s\Phi_{\alpha,\delta,v}(z;z)}\right), \tag{2.8}$$

where $\tilde{p}(s)$ and $\tilde{\omega}(s;z)$ are the LTs of $p(x)$ and $\omega(x;z)$, respectively. Notice, for $s > s_0 > 0$, that

$$\begin{aligned}
0 &< s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{1}{\delta}\int_{s_0}^{s}\left(c-\frac{\lambda[1-v\tilde{p}(t)]}{t}\right)dt} \\
&= s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{c}{\delta}(s-s_0)}e^{\frac{\lambda}{\delta}\int_{s_0}^{s}\frac{1-v+v-v\tilde{p}(t)}{t}dt} \\
&= s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{c}{\delta}(s-s_0)}\left(\frac{s}{s_0}\right)^{\frac{\lambda(1-v)}{\delta}}e^{\frac{\lambda v\mu}{\delta}\int_{s_0}^{s}\tilde{p}_1(s)ds} \\
&\le s^{\frac{\delta+\alpha}{\delta}}e^{-\frac{c}{\delta}(s-s_0)}\left(\frac{s}{s_0}\right)^{\frac{\lambda(1-v)}{\delta}}e^{\frac{\lambda v\mu}{\delta}\int_{s_0}^{s}ds} \\
&= s^{\frac{\delta+\alpha}{\delta}}\left(\frac{s}{s_0}\right)^{\frac{\lambda(1-v)}{\delta}}e^{-\frac{c-\lambda v\mu}{\delta}(s-s_0)},
\end{aligned} \tag{2.9}$$

where $\tilde{p}_1(s) = (1-\tilde{p}(s))/(\mu s)$ is the LT of $p_1(x) = \bar{P}(x)/\mu$, and hence as s is sufficiently large the right-hand side of (2.9) goes to zero because $c - \lambda v\mu > 0$ under $0 < v \le 1$ and the positive loading condition. Since $\lim_{s\to\infty}\tilde{y}_{\alpha,\delta,v}(s;z) = 0$, it follows that

$$\lim_{s\to\infty} s^{\frac{\delta+\alpha}{\delta}}\tilde{y}_{\alpha,\delta,v}(s;z)e^{-\frac{1}{\delta}\int_{s_0}^{s}\left(c-\frac{\lambda[1-v\tilde{p}(t)]}{t}\right)dt} = 0\,.$$

Then, integrating both sides of (2.8) from s_0 to ∞ yields

$$s_0^{\frac{\delta+\alpha}{\delta}}\tilde{y}_{\alpha,\delta,v}(s_0;z)$$
$$= \frac{1}{\delta}\int_{s_0}^{\infty} r^{\frac{\delta+\alpha}{\delta}}e^{-\frac{1}{\delta}\int_{s_0}^{r}\left(c-\frac{\lambda[1-v\tilde{p}(t)]}{t}\right)dt}\left(\frac{\lambda v}{\Phi_{\alpha,\delta,v}(z;z)}\frac{\tilde{\omega}(r;z)}{r}-\frac{\alpha+\lambda[1-v\tilde{p}(r)]}{r^2}e^{-rz}\right)dr\,.$$

By letting $s_0 \to 0+$ we obtain an explicit expression for $\Phi_{\alpha,\delta,v}(z;z)$:

$$\Phi_{\alpha,\delta,v}(z;z) = \lim_{s_0\to 0+}\frac{\lambda v\int_{s_0}^{\infty} r^{\frac{\alpha}{\delta}}e^{-\frac{1}{\delta}\int_{s_0}^{r}\left(c-\frac{\lambda[1-v\tilde{p}(t)]}{t}\right)dt}\tilde{\omega}(r;z)dr}{\int_{s_0}^{\infty} r^{\frac{\alpha}{\delta}-1}e^{-\frac{1}{\delta}\int_{s_0}^{r}\left(c-\frac{\lambda[1-v\tilde{p}(t)]}{t}\right)dt}\left(\alpha+\lambda[1-v\tilde{p}(r)]\right)e^{-rz}dr}, \quad z > -\frac{c}{\delta}, \tag{2.10}$$

which, when $v = 1$, reduces to (3.9) in Li and Lu (2013).

It is in general very difficult to obtain an expression for $\Phi_{\alpha,\delta,v}(u;z)$ by inverting its LT $\tilde{y}_{\alpha,\delta,v}(s;z)$. In Section 4, we discuss the case when claims are exponentially distributed and obtain an explicit expression of $\Phi_{\alpha,\delta,v}(u;z)$ in term of some special functions, of which the expression of $\Phi_{\alpha,\delta,v}(z;z)$ is essential. Below we present some special cases of (2.10).

Remarks:

1. When $z=0$ and $v=1$, (2.10) reduces to

$$\Phi_{\alpha,\delta,1}(0;0)=\frac{\lambda\int_0^\infty r^{\frac{\alpha}{\delta}}e^{-\frac{1}{\delta}\left(cr-\lambda\int_0^r\frac{1-\tilde{p}(t)}{t}dt\right)}\tilde{\omega}(r;0)dr}{\int_0^\infty r^{\frac{\alpha}{\delta}-1}e^{-\frac{c}{\delta}r+\frac{\lambda}{\delta}\int_0^r\frac{1-\tilde{p}(t)}{t}dt}\left(\alpha+\lambda[1-\tilde{p}(r)]\right)dr}\,, \tag{2.11}$$

   which is (3.11) in Li and Lu (2013).

2. When $\alpha=0$ and $w(x,y)=1$, (2.10) simplifies to

$$m_v(z;z)=\lim_{s_0\to0+}\frac{v\int_{s_0}^\infty r^{-1}e^{-\frac{1}{\delta}\left(cr-\lambda\int_{s_0}^r\frac{1-v\tilde{p}(t)}{t}dt\right)}\left[1-e^{-zr}\tilde{p}(r)\right]dr}{\int_{s_0}^\infty r^{-1}e^{-\left(z+\frac{c}{\delta}\right)r+\frac{\lambda}{\delta}\int_{s_0}^r\frac{1-v\tilde{p}(t)}{t}dt}[1-v\tilde{p}(r)]dr}\,,$$

   which is the probability generating function of the number of claims by $T_z(z)$.

3. Consider that claim amounts are exponentially distributed with $p(x)=\beta e^{-\beta x}$, and $w(x,y)=w(y)$ with LT $\tilde{w}(r)$. In this case, $\tilde{p}(r)=\beta/(r+\beta)$, and the LT of function ω in (2.3) is

$$\tilde{\omega}(r;z)=\int_z^\infty e^{-ru}\left[\int_{u-z}^\infty w(z+x-u)\beta e^{-\beta x}dx\right]du=\frac{\beta\tilde{w}(\beta)}{\beta+r}e^{-zr}\,.$$

   Then (2.10) becomes, for $z>-c/\delta$, to

$$\Phi_{\alpha,\delta,v}(z;z)=\frac{\lambda v\tilde{w}(\beta)\int_0^\infty r^{\frac{\alpha}{\delta}+(1-v)\frac{\lambda}{\delta}}\left(\frac{\beta+r}{\beta}\right)^{v\frac{\lambda}{\delta}-1}e^{-\left(z+\frac{c}{\delta}\right)r}dr}{\int_0^\infty r^{\frac{\alpha}{\delta}-1+(1-v)\frac{\lambda}{\delta}}\left(\frac{\beta+r}{\beta}\right)^{v\frac{\lambda}{\delta}}e^{-\left(z+\frac{c}{\delta}\right)r}\left[\alpha+\lambda\frac{r+\beta(1-v)}{\beta+r}\right]dr}\,. \tag{2.12}$$

   By using the confluent hypergeometric function of the second kind $U(a,b,x)$, defined by (Abramowitz and Stegun, 1970, 13.2.5)

$$U(p,q,x)=\frac{1}{\Gamma(p)}\int_0^\infty e^{-xt}t^{p-1}(1+t)^{q-p-1}dt\,, \tag{2.13}$$

   and letting

$$\xi=\frac{\alpha}{\delta}+(1-v)\frac{\lambda}{\delta}\,,\qquad \eta=1+\frac{\lambda+\alpha}{\delta}\,,\qquad z_1=\beta\left(z+\frac{c}{\delta}\right)\,, \tag{2.14}$$

   (2.12) can be rewritten as

$$\Phi_{\alpha,\delta,v}(z;z)=\frac{\left[\frac{\lambda}{\delta}v\beta\tilde{w}(\beta)\right]\xi U\left(\xi+1,\eta,z_1\right)}{\frac{\alpha}{\delta}U\left(\xi,\eta,z_1\right)+\frac{\lambda}{\delta}\xi U\left(\xi+1,\eta,z_1\right)+\frac{\lambda}{\delta}(1-v)U\left(\xi,\eta-1,z_1\right)}\,. \tag{2.15}$$

Applying the following relationship (Abramowitz and Stegun, 1970, 13.4.17)

$$\xi U(\xi+1,\eta,x) = U(\xi,\eta,x) - U(\xi,\eta-1,x)\,,$$

to the middle term in the denominator of (2.15), we further get

$$\Phi_{\alpha,\delta,v}(z;z) = \frac{\left[\frac{\lambda}{\delta}v\beta\tilde{w}(\beta)\right]\xi U\left(\xi+1,\eta,z_1\right)}{(1+\xi-\eta)U\left(\xi,\eta-1,z_1\right)-(1-\eta)U\left(\xi,\eta,z_1\right)}\,. \tag{2.16}$$

Moreover, by the two equations below (Abramowitz and Stegun, 1970, 13.4.24, 13.4.21)

$$(1+\xi-\eta)U(\xi,\eta-1,x)-(1-\eta)U(\xi,\eta,x) = -xU'(\xi,\eta,x) = \xi x U(\xi+1,\eta+1,x)\,,$$

we finally obtain an explicit expression for $\Phi_{\alpha,\delta,v}(z;z)$:

$$\Phi_{\alpha,\delta,v}(z;z) = \frac{\left[\frac{\lambda}{\delta}v\tilde{w}(\beta)\right]U\left(1+\frac{\alpha}{\delta}+(1-v)\frac{\lambda}{\delta},1+\frac{\lambda+\alpha}{\delta},\beta\left(z+\frac{c}{\delta}\right)\right)}{\left(z+\frac{c}{\delta}\right)U\left(1+\frac{\alpha}{\delta}+(1-v)\frac{\lambda}{\delta},2+\frac{\lambda+\alpha}{\delta},\beta\left(z+\frac{c}{\delta}\right)\right)}\,, \tag{2.17}$$

with $z>-c/\delta$. Note that (2.17) reduces to the corresponding expression in Li and Lu (2013, Section 3.1) when $v=1$.

3 Exponentially distributed claims

Now consider the exponential claims case with $p(x)=\beta e^{-\beta x}$ and $w(x,y)=w(y)$, that is, the penalty function depends only on the "deficit" below level z when the surplus drops below z. Because the derivation for the general expression of $\Phi_{\alpha,\delta,v}(u;z)$ is very similar to that showed in Section 4 of Li and Lu (2013), we only show main steps and focus more on deriving the formulas for the moments of $N(T_z(u))$ and the recursive formulas for calculating probability function $\psi_n(u;z)$ and their conditional ones.

First, it is not difficult to get the following homogeneous second order linear differential equation satisfied by $\Phi_{\alpha,\delta,v}(u;z)$ in this case:

$$[\lambda+\alpha-\delta-\beta(\delta u+c)]\Phi'_{\alpha,\delta,v}(u;z) = (\delta u+c)\Phi''_{\alpha,\delta,v}(u;z)-\beta[\alpha+\lambda(1-v)]\Phi_{\alpha,\delta,v}(u;z)\,. \tag{3.1}$$

Then, with $t=\beta(u+c/\delta)$ and $Q_{\alpha,\delta,v}(t;z)=e^t\Phi_{\alpha,\delta,v}(t/\beta-c/\delta;z)$, (3.1) becomes a standard Kummer's equation for $Q_{\alpha,\delta,v}(t)$:

$$tQ''_{\alpha,\delta,v}(t;z)+\left(1-\frac{\lambda+\alpha}{\delta}-t\right)Q'_{\alpha,\delta,v}(t;z)-\left(1-\frac{\lambda}{\delta}v\right)Q_{\alpha,\delta,v}(t;z)=0\,,$$

and its solution can be expressed in term of $M(a,b,t)$ and $U(a,b,t)$, the confluent hypergeometric functions of the first and second kind, namely,

$$Q_{\alpha,\delta,v}(t;z) = t^{\eta-1}\left[C_1(z)M(\xi+1,\eta,t)+C_2(z)U(\xi+1,\eta,t)\right]\,, \tag{3.2}$$

where $C_1(z)$ and $C_2(z)$ are two functions of z to be determined, or

$$\Phi_{\alpha,\delta,v}(u;z) = \left[\beta\left(u+\frac{c}{\delta}\right)\right]^{\eta-1} e^{-\beta\left(u+\frac{c}{\delta}\right)} \left[C_1(z) M\left(\xi+1,\eta,\beta\left(u+\frac{c}{\delta}\right)\right) \right.$$
$$\left. + C_2(z) U\left(\xi+1,\eta,\beta\left(u+\frac{c}{\delta}\right)\right)\right]. \tag{3.3}$$

In (3.2) and (3.3), the function $U(a,b,t)$ is given by (2.13) and $M(a,b,t)$ is defined as

$$M(a,b,t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$ for $n \geq 1$ and $(a)_0 = 1$ (Abramowitz and Stegun, 1970, 13.1.2).

Similar to the discussion in Section 4 of Li and Lu (2013), by the boundary condition $\Phi_{\alpha,\delta,v}(\infty;z) = 0$ and expression of $\Phi_{\alpha,\delta,v}(z;z)$ given by (2.17), we find that $C_1(z) = 0$ and

$$C_2(z) = \frac{\left[\frac{\lambda\beta}{\delta} v\tilde{w}(\beta)\right] \left[\beta\left(z+\frac{c}{\delta}\right)\right]^{-\eta} e^{\beta\left(z+\frac{c}{\delta}\right)}}{U\left(\xi+1,\eta+1,\beta\left(z+\frac{c}{\delta}\right)\right)},$$

yielding, finally, from (3.3) that

$$\Phi_{\alpha,\delta,v}(u;z) = \frac{\frac{\lambda}{\delta} v\tilde{w}(\beta)}{z+\frac{c}{\delta}} \left(\frac{u+\frac{c}{\delta}}{z+\frac{c}{\delta}}\right)^{\eta-1} e^{-\beta(u-z)} \frac{U\left(\xi+1,\eta,\beta\left(u+\frac{c}{\delta}\right)\right)}{U\left(\xi+1,\eta+1,\beta\left(z+\frac{c}{\delta}\right)\right)}, \tag{3.4}$$

which is an explicit expression of $\Phi_{\alpha,\delta,v}(u;z)$ when the claim amounts are exponentially distributed and the penalty function $w(x,y) = w(y)$.

3.1 Moments of $N(T_z(u))$

In this subsection, we consider a special case when $\alpha = 0$ and $w(x,y) = 1$. In this case, $\Phi_{\alpha,\delta,v}(u;z)$ reduces to $m_v(u;z)$, the probability generating function of the number of claims by $T_z(u)$, namely,

$$m_v(u;z) = \mathbb{E}\left[v^{N(T_z(u))} I(T_z(u) < \infty)\right].$$

We derive formulas for the unconditional and conditional, given that the surplus has dropped to level z at the first time from the initial level u, first two moments of $N(T_z(u))$. Note now $\xi = \frac{\lambda}{\delta}(1-v)$ and $\eta = 1 + \frac{\lambda}{\delta}$, and $\tilde{w}(\beta) = 1/\beta$.

We immediately get from (3.4), for $0 < v \leq 1$, that

$$m_v(u;z) = \frac{\frac{\lambda}{\delta} v}{\beta\left(z+\frac{c}{\delta}\right)} \left(\frac{u+\frac{c}{\delta}}{z+\frac{c}{\delta}}\right)^{\frac{\lambda}{\delta}} e^{-\beta(u-z)} \frac{U\left(1+\frac{\lambda}{\delta}(1-v), 1+\frac{\lambda}{\delta}, \beta\left(u+\frac{c}{\delta}\right)\right)}{U\left(1+\frac{\lambda}{\delta}(1-v), 2+\frac{\lambda}{\delta}, \beta\left(z+\frac{c}{\delta}\right)\right)}. \tag{3.5}$$

Note that when $v = 1$, $m_1(u;z) = \psi(u;z)$, that is considered in Li and Lu (2013).

Now by (2.13) and letting

$$\theta(u,z) = \frac{\frac{\lambda}{\delta}}{\beta\left(z+\frac{c}{\delta}\right)}\left(\frac{u+\frac{c}{\delta}}{z+\frac{c}{\delta}}\right)^{\frac{\lambda}{\delta}} e^{-\beta(u-z)}\,, \qquad -c/\delta < z < u\,, \tag{3.6}$$

Equation (3.5) can be rewritten as

$$m_v(u;z) = \theta(u,z)\, v\, \frac{\int_0^\infty e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}\left(\frac{1+t}{t}\right)^{\frac{\lambda}{\delta}v} dt}{\int_0^\infty e^{-\beta\left(z+\frac{c}{\delta}\right)t} t^{\frac{\lambda}{\delta}}\left(\frac{1+t}{t}\right)^{\frac{\lambda}{\delta}v} dt} = \theta(u,z)\, v\, \frac{f(v;u)}{g(v;z)}\,, \tag{3.7}$$

where

$$f(v;u) = \int_0^\infty e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}\left(\frac{1+t}{t}\right)^{\frac{\lambda}{\delta}v} dt\,, \tag{3.8}$$

$$g(v;z) = \int_0^\infty e^{-\beta\left(z+\frac{c}{\delta}\right)t} t^{\frac{\lambda}{\delta}}\left(\frac{1+t}{t}\right)^{\frac{\lambda}{\delta}v} dt\,. \tag{3.9}$$

Denote by $f_i(v;u)$ and $g_i(v;z)$ the ith order derivatives of $f(v;u)$ and $g(v;z)$ with respect to v, respectively. Then for $i = 1, 2, \ldots,$

$$f_i(v;u) = \frac{d^i f(v;u)}{dv^i} = \int_0^\infty e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}\left(\frac{1+t}{t}\right)^{\frac{\lambda}{\delta}v}\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^i dt\,, \tag{3.10}$$

$$g_i(v;z) = \frac{d^i g(v;z)}{dv^i} = \int_0^\infty e^{-\beta\left(z+\frac{c}{\delta}\right)t} t^{\frac{\lambda}{\delta}}\left(\frac{1+t}{t}\right)^{\frac{\lambda}{\delta}v}\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^i dt\,, \tag{3.11}$$

with $f_0(v;u) = f(v;u)$ and $g_0(v;z) = g(v;z)$.

Taking the derivative of both sides of (3.7) with respect to v and letting $v = 1$, we obtain the (unconditional) expected number of claims by $T_z(u)$ as

$$\mathbb{E}\left[N(T_z(u))I(T_z(u)<\infty)\right] = \theta(u,z)\left[\frac{f_0(1;u)+f_1(1;u)}{g_0(1;z)} - \frac{f_0(1;u)g_1(1;z)}{[g_0(1;z)]^2}\right]\,, \tag{3.12}$$

where $\theta(u,z)$ is given in (3.6). Similarly, differentiating both sides of (3.12) with respect to v and letting $v = 1$ give

$$\begin{aligned}&\mathbb{E}\left[N(T_z(u))(N(T_z(u))-1)I(T_z(u)<\infty)\right]\\ &= \theta(u,z)\left[\frac{2f_1(1;u)+f_2(1;u)}{g_0(1;z)} - \frac{2(f_0(1;u)+f_1(1;u))g_1(1;z)}{[g_0(1;z)]^2} + \frac{2f_0(1;u)g_2(1;z)}{[g_0(1;z)]^3}\right]\,.\end{aligned}$$

Then the variance of $N(T_z(u))$ can be obtained from above as

$$\begin{aligned}
&\mathbb{V}\left[N(T_z(u))I(T_z(u)<\infty)\right]\\
&=\theta(u,z)\left[\frac{f_0(1;u)+3f_1(1;u)+f_2(1;u)}{g_0(1;z)}-\frac{(3f_0(1;u)+2f_1(1;u))g_1(1;z)}{[g_0(1;z)]^2}\right.\\
&\quad\left.+\frac{2f_0(1;u)g_2(1;z)}{[g_0(1;z)]^3}\right]-\theta^2(u,z)\left[\frac{f_0(1;u)+f_1(1;u)}{g_0(1;z)}-\frac{f_0(1;u)g_1(1;z)}{[g_0(1;z)]^2}\right]^2, \qquad (3.13)
\end{aligned}$$

in which functions f_i and g_i, $i=0,1,2$, are all integrations defined by (3.8)-(3.11). By taking the nth ($n \geq 3$) derivative of both sides of (3.7) with respect to v and letting $v=1$, we are able to get the nth factorial moment of $N(T_z(u))$. Then the nth raw moment of $N(T_z(u))$ can be calculated using the relationship between the raw and factorial moments (see, Hamming, 1973, page 160).

Accordingly, it follows from (3.12) and (3.13) the conditional mean and variance of the number of claims by $T_z(u)$ given that the surplus has dropped to level z by the first time from the initial level u, using (3.7) with $v=1$, are given by

$$\mathbb{E}\left[N(T_z(u))\,|\,T_z(u)<\infty\right]=1+\frac{f_1(1;u)}{f_0(1;u)}-\frac{g_1(1;z)}{g_0(1;z)}, \qquad (3.14)$$

$$\begin{aligned}
\mathbb{V}\left[N(T_z(u))\,|\,T_z(u)<\infty\right]&=\frac{(f_1(1;u)+f_2(1;u))f_0(1;u)-f_1^2(1;u)}{f_0^2(1;u)}\\
&\quad+\frac{2g_2(1;z)-(g_1(1;z)+g_0(1;z))g_1(1;z)}{g_0^2(1;z)}. \qquad (3.15)
\end{aligned}$$

We remark that by letting $w(x,y)=1$ in (1.3), $\Phi_{\alpha,\delta,v}(u;z)$ reduces to a joint transform of $N(T_z(u))$ and $T_z(u)$, given by

$$\Phi_{\alpha,\delta,v}(u;z)=\mathbb{E}\Big[e^{-\alpha T_z(u)}v^{N(T_z(u))}I(T_z(u)<\infty)\big|U_\delta(0)=u\Big],$$

and a general expression for the conditional covariance of $N(T_z(u))$ and $T_z(u)$, given that the surplus has dropped to level z by the first time from the initial level u, can be obtained by the following formula:

$$\begin{aligned}
&\mathrm{Cov}\left(N(T_z(u)),T_z(u)\,|\,T_z(u)<\infty\right)\\
&=\frac{1}{\psi(u;z)}\left(\frac{\partial^2\Phi_{\alpha,\delta,v}(u;z)}{\partial\alpha\partial v}-\frac{\partial\Phi_{\alpha,\delta,v}(u;z)}{\partial\alpha}\frac{\partial\Phi_{\alpha,\delta,v}(u;z)}{\partial v}\right)\Big|_{\alpha=0,v=1}.
\end{aligned}$$

3.2 Formulas for $\psi_n(u;z)$

In this subsection, we derive a recursive formula for calculating $\psi_n(u;z)$, the probability that the surplus will ever drops below z and the number of claims by $T_z(u)$ is n. We also define $\psi_n^c(u;z)=\mathbb{P}\left(N\big(T_z(u)\big)=n\big|T_z(u)<\infty\right)=\psi_n(u;z)/\psi(u;z)$ to be the corresponding conditional (proper) probability function of $T_z(u)$ given that the surplus drops to level z at the first time from the initial level u.

Now by definition $m_v(u;z)=\sum_{n=1}^{\infty} v^n\psi_n(u;z)$ and using (3.7), we first have

$$g(v;z)\sum_{n=1}^{\infty} v^n\psi_n(u;z)=\theta(u,z)\,v\,f(v;u)\,, \tag{3.16}$$

where functions f and g are defined in (3.8) and (3.9), respectively. Using the Taylor's expansion, f and g can be further expressed as

$$f(v;u)=\int_0^{\infty} e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}e^{v\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]}dt=\int_0^{\infty} e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}\sum_{n=0}^{\infty}\frac{v^n\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^n}{n!}dt,$$
$$g(v;z)=\int_0^{\infty} e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}e^{v\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]}dt=\int_0^{\infty} e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}\sum_{n=0}^{\infty}\frac{v^n\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^n}{n!}dt\,.$$

Then the right-hand-side of (3.16) can be written as a power series in terms of v as

$$\text{RHS}=\theta(u,z)\sum_{n=0}^{\infty} v^{n+1}\int_0^{\infty} e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^n}{n!}dt\,, \tag{3.17}$$

and similarly the left-hand-side of (3.16) can be written as

$$\begin{aligned}\text{LHS}&=\int_0^{\infty} e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}\left(\sum_{n=0}^{\infty} v^n\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^n}{n!}\right)\left(\sum_{n=1}^{\infty} v^n\psi_n(u;z)\right)dt\\&=\int_0^{\infty} e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}\left[\sum_{n=0}^{\infty} v^{n+1}\left(\sum_{m=1}^{n+1}\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^{n+1-m}}{(n+1-m)!}\psi_m(u;z)\right)\right]dt\\&=\sum_{n=0}^{\infty} v^{n+1}\sum_{m=1}^{n+1}\psi_m(u;z)\int_0^{\infty} e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^{n+1-m}}{(n+1-m)!}dt\,.\end{aligned} \tag{3.18}$$

Now equating (3.18) and (3.17), and comparing the coefficients before v^{n+1} of both sides, we get, for $n=0$,

$$\psi_1(u;z)=\frac{\theta(u,z)\int_0^{\infty} e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}dt}{\int_0^{\infty} e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}dt}\,, \tag{3.19}$$

and for $n>1$,

$$\sum_{m=1}^{n}\psi_m(u;z)\int_0^{\infty} e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^{n-m}}{(n-m)!}dt=\theta(u,z)\int_0^{\infty} e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^{n-1}}{(n-1)!}dt\,,$$

which yields

$$\psi_n(u;z) = \frac{\theta(u,z)\int_0^\infty e^{-\beta\left(u+\frac{c}{\delta}\right)t}\frac{t^{\frac{\lambda}{\delta}}}{1+t}\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^{n-1}}{(n-1)!}dt}{\int_0^\infty e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}dt}$$

$$-\sum_{m=1}^{n-1}\psi_m(u;z)\frac{\int_0^\infty e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}\frac{\left[\frac{\lambda}{\delta}\ln\left(\frac{1+t}{t}\right)\right]^{n-m}}{(n-m)!}dt}{\int_0^\infty e^{-\beta\left(z+\frac{c}{\delta}\right)t}t^{\frac{\lambda}{\delta}}dt}, \qquad n = 2,3,\ldots. \tag{3.20}$$

Using the notation in Section 3.1, (3.19) and (3.20) are re-expressed as

$$\psi_1(u;z) = \theta(u,z)\frac{f_0(0;u)}{g_0(0;z)},$$

$$\psi_n(u;z) = \theta(u,z)\frac{f_{n-1}(0;u)}{(n-1)!g_0(0;z)} - \sum_{m=1}^{n-1}\psi_m(u;z)\frac{g_{n-m}(0;z)}{(n-m)!g_0(0;z)}, \qquad n > 1,$$

which is a recursion formula for the probability that the surplus will ever drops below z from initial surplus u and the number of claims by then is n.

Accordingly, by dividing $\psi(u;z)$ given by (3.7) with $v = 1$ we obtain the following recursive formula for calculating the conditional probabilities $\psi_n^c(u;z)$:

$$\psi_1^c(u;z) = \frac{f_0(0;u)g_0(1;z)}{f_0(1;u)g_0(0;z)}, \tag{3.21}$$

$$\psi_n^c(u;z) = \frac{1}{(n-1)!}\frac{f_{n-1}(0;u)g_0(1;z)}{f_0(1;u)g_0(0;z)} - \sum_{m=1}^{n-1}\psi_m^c(u;z)\frac{g_{n-m}(0;z)}{(n-m)!g_0(0;z)}, \quad n > 1. \tag{3.22}$$

3.3 Approximation for $\psi_n(z;z)$ using continued fractions

When $u = z$, we have

$$m_v(z;z) = \frac{\frac{\lambda}{\delta}v}{\beta\left(z+\frac{c}{\delta}\right)}\frac{U\left(1+\frac{\lambda}{\delta}(1-v), 1+\frac{\lambda}{\delta}, \beta\left(z+\frac{c}{\delta}\right)\right)}{U\left(1+\frac{\lambda}{\delta}(1-v), 2+\frac{\lambda}{\delta}, \beta\left(z+\frac{c}{\delta}\right)\right)}. \tag{3.23}$$

Let $a = 1+\frac{\lambda}{\delta}(1-v), b = 1+\frac{\lambda}{\delta}, z_1 = \beta\left(z+\frac{c}{\delta}\right)$. It follows from Cuyt et al. (2008) that

$$\frac{U(a,b,z_1)}{z_1 U(a,b+1,z_1)} = \mathbf{K}_{m=1}^{\infty}\left(\frac{\gamma_m(v)}{1}\right),$$

where $\gamma_1(v) = 1/z_1$, and $\gamma_{2n}(v) = (a+n-1)/z_1, \gamma_{2n+1}(v) = (a-b+n)/z_1$, for $n = 1,2,\ldots$. Here

$$\mathbf{K}_{m=1}^{\infty}\left(\frac{\gamma_m(v)}{1}\right) = \frac{\gamma_1(v)}{1+}\frac{\gamma_2(v)}{1+}\cdots = \frac{\gamma_1(v)}{1+\frac{\gamma_2(v)}{1+\frac{\gamma_3(v)}{1+\ddots}}} := H(v)$$

is the continued fraction. Let $H_m(v) = A_m(v)/B_m(v)$ be the m-th approximant of $H(v)$, where $A_m(v)$ and $B_m(v)$ are the m-th numerator and denominator, respectively. It is shown in Cuyt et al. (2008) that

$$A_m(v) \;=\; A_{m-1}(v) + \gamma_m(v) A_{m-2}(v), \tag{3.24}$$

$$B_m(v) \;=\; B_{m-1}(v) + \gamma_m(v) B_{m-2}(v), \quad m = 1, 2, \ldots, \tag{3.25}$$

with $A_{-1}(v) = 1, B_{-1}(v) = 0, A_0(v) = 0, B_0(v) = 1$. It is easily seen from (3.24) and (3.25) that $A_{2m-1}(v)$ and $A_{2m}(v)$ are polynomials of v of degree $m-1$, $B_{2m}(v)$ and $B_{2m+1}(v)$ are polynomials of v of degree m, with

$$\begin{aligned}
A_{2m-1}(v) &= a_0^{(2m-1)} + \sum_{k=1}^{m-1} a_k^{(2m-1)} v^k, \\
A_{2m}(v) &= a_0^{(2m)} + \sum_{k=1}^{m-1} a_k^{(2m)} v^k, \qquad m = 1, 2, \ldots, \\
B_{2m}(v) &= b_0^{(2m)} + \sum_{k=1}^{m} b_k^{(2m)} v^k, \\
B_{2m+1}(v) &= b_0^{(2m+1)} + \sum_{k=1}^{m} b_k^{(2m+1)} v^k, \quad m = 0, 1, 2, \ldots.
\end{aligned}$$

Then we have an approximation formula for $m_v(z;z)$ by using the Taylor's expansion of rational function as

$$m_v(z;z) \approx \frac{\lambda}{\delta} \frac{v A_{2m}(v)}{B_{2m}(v)} = \frac{\lambda}{\delta} \frac{\sum_{k=1}^{m} a_{k-1}^{(2m-1)} v^k}{b_0^{(2m)} + \sum_{k=1}^{m} b_k^{(2m)} v^k} = \sum_{n=1}^{\infty} c_n^{(m)} v^n \,.$$

Inverting it gives an approximation for $\psi_n(z;z) \approx c_n^{(m)}$. We comment that $c_n^{(m)}$ can be e valued recursively and the larger the m, the better the approimation is.

3.4 Numerical Illustrations

We now illustrate numerically the quantities derived in above two subsections when claim amounts are exponentially distributed. In the first example, we show the values of the conditional mean and standard deviation of the number of claims by $T_z(u)$ given that the surplus has dropped to level z by the first time from u for different values of z and u. In the second example, we show values of the conditional probability of the number of claims by $T_z(u)$ being n for different values of n and z, and especially compare them with the corresponding quantities at ruin ($z = 0$) in the classical risk model case.

Example 1 In this example, we set $\lambda = \beta = 1$ and consider two values of c ($c = 1.1$ and $c = 1.2$) and two values of δ ($\delta = 0.1$ and $\delta = 0.06$), respectively. The conditional mean (μ_N) and standard deviation (σ_N) of the number of claims by $T_z(u)$, for $u = 0, 2, 5, 10, 20, 50$ and $z = 2, 0, -2, -5$, are calculated using formulas (3.14) and (3.15) and displayed in Table

1. Note that all the μ_N values in Table 1 are rounded to be integer numbers. Below are some observations that we observe from Table 1.

(a) In all four cases, for fixed z, the values of μ_N increase as the initial surplus level u increases, while those values of σ_N increase extremely slowly as u increases with a maximum increase of 3 occurring at $c = 1.1$, $\delta = 0.1$, $u = 50$ and $z = -5$.

(b) For fixed u, the values of μ_N increase as level z decreases, which is expected. However, the corresponding values of σ_N decrease significantly as z decreases, implying the variation from the conditional mean so as the potential risk drops when the distance between u and z becomes large.

(c) We now compare pairs of subtables (i) and (ii), and (iii) and (iv), in which each pair has a same δ value. It is found that a higher value of the positive loading factor (in Table 1(i) and (iii), the loading is 20%, while in (ii) and (iv) that is 10%) results a non-decreasing trend in μ_N's and a considerably increasing trend in σ_N's for same sets of (u, z) values.

(d) Comparing pairs of subtables (i) and (iii), and (ii) and (iv), each pair having a same c value, we observe the same trends as described in (c) for μ_N's and σ_N's for same sets of (u, z) values.

Example 2 In this example, we set $c = 1.2$ and $\lambda = \beta = 1$. For $u = 10$ and $\delta = 0.1$, the conditional probabilities $\psi_n^c(u; z)$ are calculated recursively for $z = 2, 0, -2, -5$ using formulas (3.21) and (3.22) and displayed, for $n = 1, 2, 5, 10, 15, 20, 30$ as well as $n > 30$, in Table 2. We observe from Table 2 that for each n in the table, as expected, the conditional probabilities $\psi_n^c(10; z)$ decrease as z decreases. For fixed z, these conditional probabilities increase first then decrease when the number of claims becomes bigger and bigger. In addition, we see that, given the surplus has dropped to level z by the first time from u, with probability 82.42% this dropping occurs at one of the first 30 claim occurrence times when $z = -5$, while for $z = 2$ this probability is 99.54%.

Figure **??** shows the comparison of the conditional probabilities of the number of claims at ruin ($z = 0$) given ruin occurs when $\delta = 0, 0.06, 0.1$ in the exponential claims case with the initial surplus $u = 5$. When $\delta = 0$, the model reduces to the classical compound Poisson risk model (with no interest involved) and the corresponding formulas from Dickson (2012, Section 6) are used for our calculations. It can be seen from the figure that when n is small, the bigger the δ is, the bigger these conditional probabilities are, and when n is relatively large, the situation is reversed. This shows from Shaked and Shanthikumar (1994) that $\mathbb{P}\big(N(T_0(5)) > n \big| T_0(5) < \infty\big)$ is decreasing in δ, for a fixed n, that is to say, $N(T_0(5))|T_0(5) < \infty$ is stochastically smaller when δ becomes larger.

4 The two-sided first exit time

The problems related to one or two-sided exit times in risk models without interest have been studied in Dickson and Gray (1984), Gerber and Shiu (1998), Perry at al. (2005) and Li

Table 1: Conditional mean and standard deviation of the number of claims by $T_z(u)$ for some values of u and z with exponential claims

(i) $c = 1.2$, $\lambda = 1$, $\beta = 1$, and $\delta = 0.1$

	u=0		u=2		u=5		u=10		u=20		u=50	
z	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N
2			2	85.6	5	85.6	9	85.7	14	85.8	22	85.9
0	2	63.0	5	63.1	8	63.1	12	63.2	17	63.3	25	63.4
-2	6	40.9	8	41.0	11	41.2	15	41.3	20	41.5	28	41.6
-5	12	14.4	15	14.8	18	15.2	22	15.6	27	15.9	35	16.4

(ii) $c = 1.1$, $\lambda = 1$, $\beta = 1$, and $\delta = 0.1$

	u=0		u=2		u=5		u=10		u=20		u=50	
z	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N
2			2	74.3	6	74.4	10	74.5	15	74.6	23	74.6
0	2	51.7	5	51.8	9	52.0	13	52.1	18	52.2	26	52.3
-2	6	30.5	9	30.7	12	30.9	17	31.2	22	31.4	30	31.6
-5	13	10.0	16	10.7	19	11.4	23	12.0	29	12.5	37	13.0

(iii) $c = 1.2$, $\lambda = 1$, $\beta = 1$, and $\delta = 0.06$

	u=0		u=2		u=5		u=10		u=20		u=50	
z	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N
2			2	180.1	7	180.1	12	180.2	20	180.3	32	180.4
0	3	145.3	6	145.3	10	145.4	15	145.5	23	145.6	36	145.7
-2	7	110.6	10	110.7	14	110.8	20	110.9	27	111.0	40	111.1
-5	14	61.5	17	61.7	21	61.9	27	62.1	34	62.3	47	62.5

(iv) $c = 1.1$, $\lambda = 1$, $\beta = 1$, and $\delta = 0.06$

	u=0		u=2		u=5		u=10		u=20		u=50	
z	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N	μ_N	σ_N
2			3	151.1	7	151.2	13	151.3	21	151.4	35	151.5
0	3	116.2	6	116.3	11	116.5	17	116.6	25	116.7	39	116.8
-2	7	82.4	11	82.6	16	82.7	22	82.9	30	83.1	43	83.3
-5	15	38.1	19	38.5	24	38.9	30	39.3	38	39.6	51	40.0

(2008) and references therein. In this section, we aim to study the joint Laplace transform and the probability generating function of the two-sided first exit time and the number of claims by this time.

For $-\frac{c}{\delta} \le z < u \le b$, we also define

$$T^b(u) = \inf\{t \ge 0 : U_\delta(t) = b\}$$

Table 2: Conditional probabilities of the number of claims by $T_z(u)$ for $u = 10$ and some values of z when $c = 1.2$, $\lambda = \beta = 1$ and $\delta = 0.1$

n	$z = 2$	$z = 0$	$z = -2$	$z = -5$
1	0.0120	0.0032	0.0007	0.0001
2	0.0361	0.0124	0.0035	0.0004
5	0.0932	0.0573	0.0275	0.0059
10	0.0635	0.0704	0.0607	0.0309
15	0.0258	0.0405	0.0504	0.0454
20	0.0093	0.0188	0.0303	0.0409
30	0.0011	0.0033	0.0080	0.0192
> 30	0.0046	0.0168	0.0511	0.1758

to be the first time that the surplus process reaches b from u. Now denote

$$T_z^b(u) = \min\Big(T_z(u), T^b(u)\Big), \qquad -\frac{c}{\delta} \le z < u \le b\,.$$

Then $T_z^b(u)$ is a two-sided first exit time at which the surplus process either first drops below z from level u without crossing an upper level b or first crosses level b from u without dropping below a lower level z. Let $G_{\delta,\alpha,v}(u; z, b)$ be the joint transform of $T_z^b(u)$ and $N\big(T_z^b(u)\big)$ with initial surplus level u. Then

$$G_{\delta,\alpha,v}(u; z, b) = \mathbb{E}\Big[e^{-\alpha T_z^b(u)} v^{N\left(T_z^b(u)\right)}\Big] = R_{\delta,\alpha,v}(u; z, b) + W_{\delta,\alpha,v}(u; z, b)\,, \tag{4.1}$$

where

$$R_{\delta,\alpha,v}(u; z, b) = \mathbb{E}\Big[e^{-\alpha T^b(u)} v^{N\left(T^b(u)\right)} I\big(T^b(u) < T_z(u)\big)\Big]\,, \qquad -\frac{c}{\delta} \le z < u \le b\,,$$

and

$$W_{\delta,\alpha,v}(u; z, b) = \mathbb{E}\Big[e^{-\alpha T_z(u)} v^{N\left(T_z(u)\right)} I\big(T_z(u) < T^b(u)\big)\Big]\,, \qquad -\frac{c}{\delta} \le z < u \le b\,.$$

Note that when $z = 0$ and $b = \infty$, function (4.1) has been studied in Landriault et al. (2011) and Dickson (2012) for risk models with no interest.

Furthermore, for $\delta, \alpha, v > 0$, and $-(c/\delta) \le z < u$, let

$$\phi_{\delta,\alpha,v}(u; z) = \mathbb{E}\Big[e^{-\alpha T_z(u)} v^{N\left(T_z(u)\right)} I\left(T_z(u) < \infty\right) \big| U_\delta(0) = u\Big]\,.$$

Here $\phi_{\delta,\alpha,v}(u; z)$ is a special case of the function defined by (1.3) when $w(x, y) = 1$. Clearly, we have

$$\phi_{\delta,\alpha,v}(u;z) = W_{\delta,\alpha,v}(u;z,b) + R_{\delta,\alpha,v}(u;z,b)\phi_{\delta,\alpha,v}(b;z)\,, \qquad -\frac{c}{\delta} \le z < u \le b\,. \tag{4.2}$$

In what follows, we evaluate $R_{\delta,\alpha,v}(u;z,b)$. Then $W_{\delta,\alpha,v}(u;z,b)$ can be obtained from (4.2) if knowing $\phi_{\delta,\alpha,v}(u;z)$.

First note that when the initial surplus $u \ge b$, we immediately have $R_{\delta,\alpha,v}(u;z,b) = 1$. For $-\frac{c}{\delta} \le z < u \le b$, by conditioning on the time of the first claim, t, we get

$$\begin{aligned} R_{\delta,\alpha,v}(u;z,b) = &\int_0^{\frac{1}{\delta}\ln\frac{\delta b+c}{\delta u+c}} \lambda v e^{-(\lambda+\alpha)t} \int_0^{ue^{\delta t}+c\bar{s}_{\overline{t}|\delta}-z} R_{\delta,\alpha,v}(ue^{\delta t}+c\bar{s}_{\overline{t}|\delta}-x;z,b)p(x)dxdt \\ &+ \int_{\frac{1}{\delta}\ln\frac{\delta b+c}{\delta u+c}}^{\infty} \lambda v e^{-\lambda t} e^{-\frac{\alpha}{\delta}\ln\frac{\delta b+c}{\delta u+c}} dt\,. \end{aligned} \tag{4.3}$$

Note that if the first claim arrives after $t = \frac{1}{\delta}\ln\frac{\delta b+c}{\delta u+c}$, or, $ue^{\delta t} + c\bar{s}_{\overline{t}|\delta} = b$, the cumulative of the initial surplus and the constant premiums in interest reaches level b at time $\frac{1}{\delta}\ln\frac{\delta b+c}{\delta u+c} = T^b(u)$, implying the second term of (4.3). The first term corresponds to the case when the first claim arrives before $t = \frac{1}{\delta}\ln\frac{\delta b+c}{\delta u+c}$ with the claim size no more than $ue^{\delta t} + c\bar{s}_{\overline{t}|\delta} - z$, and the surplus continues with a new initial surplus level; otherwise, the surplus level drops below $z \ge -c/\delta$.

Similar to the derivation of (2.4), we obtain, for $-\frac{c}{\delta} \le z < u \le b$, that

$$(\delta u + c)R'_{\delta,\alpha,v}(u;z,b) =(\lambda+\alpha)R_{\delta,\alpha,v}(u;,z,b) - \lambda v \int_z^u R_{\delta,\alpha,v}(y;z,b)p(u-x)dx\,. \tag{4.4}$$

Based on the fact that

$$R_{\delta,\alpha,v}(u;z,b) = R_{\delta,\alpha,v}(u;z,d)R_{\delta,\alpha,v}(d;z,b)\,, \qquad -\frac{c}{\delta} \le z < u \le d \le b\,,$$

it is not difficulty to justify that

$$R_{\delta,\alpha,v}(u;z,b) = \frac{A_{\delta,\alpha,v}(u;z)}{A_{\delta,\alpha,v}(b;z)} \tag{4.5}$$

for some differentiable function $A_{\delta,\alpha,v}$, and we can always set $A_{\delta,\alpha,v}(z;z) = 1$. We remark here that when $z = -c/\delta$ and $v = 1$, the above result is Theorem 2.3 in He et al. (2009). Now substituting (4.5) into (4.4) gives, for $-c/\delta \le z < u$, that

$$(\delta u + c)A'_{\delta,\alpha,v}(u;z) =(\lambda+\alpha)A_{\delta,\alpha,v}(u;z) - \lambda v \int_z^u A_{\delta,\alpha,v}(y;z)p(u-x)dx\,. \tag{4.6}$$

Letting $u \to z$ in (4.6) and with $A_{\delta,\alpha,v}(z;z) = 1$, we obtain

$$A'_{\delta,\alpha,v}(z;z) =\frac{\lambda+\alpha}{\delta z + c}\,, \qquad -\frac{c}{\delta} < z < u\,, \tag{4.7}$$

which can be seen as a boundary condition for (4.6). In the following two subsections, we consider claim amounts follow the exponential and Erlang(2) distributions respectively.

4.1 Exponential claims

Consider a special case when $p(x) = \beta e^{-\beta x}$. Then (4.6) becomes

$$(\delta u + c)A'_{\delta,\alpha,v}(u;z) = (\lambda+\alpha)A_{\delta,\alpha,v}(u;z) - \lambda v \int_z^u A_{\delta,\alpha,v}(y;z)\beta e^{-\beta(u-y)}dy\,. \tag{4.8}$$

By differentiating (4.8) with respect to u and using again (4.8) we obtain the following homogeneous second order linear differential equation

$$(\delta u + c)A''_{\delta,\alpha,v}(u;z) = [\lambda+\alpha-\delta-\beta(\delta u + c)]A'_{\delta,\alpha,v}(u;z) + \beta[\alpha + \lambda(1-v)]A_{\delta,\alpha,v}(u;z)\,, \tag{4.9}$$

with the boundary conditions $A_{\delta,\alpha,v}(z;z) = 1$ and (4.7). Note that equation (3.1) is of the same form as (4.9) but with different boundary conditions. Hence, following (3.3) we can write the solution to (4.9) as

$$\begin{aligned} A_{\delta,\alpha,v}(u;z) = {} & \left[\beta\left(u+\frac{c}{\delta}\right)\right]^{\eta-1} e^{-\beta\left(u+\frac{c}{\delta}\right)} \left[C_1(z)M\left(\xi+1,\eta;\beta\left(u+\frac{c}{\delta}\right)\right)\right. \\ & \left. + C_2(z)U\left(\xi+1,\eta;\beta\left(u+\frac{c}{\delta}\right)\right)\right]\,, \end{aligned} \tag{4.10}$$

where ξ and η are defined in (2.14). By $A_{\delta,\alpha,v}(z;z) = 1$, we obtain that

$$1 = z_1^{\eta-1}e^{-z_1}\left[C_1(z)M\left(\xi+1,\eta;z_1\right) + C_2(z)U\left(\xi+1,\eta;z_1\right)\right]\,, \tag{4.11}$$

where $z_1 = \beta(z + c/\delta)$. In order to use (4.7), we now first differentiate (4.10) with respect to u and then let $u \to z$, yielding

$$\begin{aligned} A'_{\delta,\alpha,v}(z;z) = {} & \beta\left(\frac{\eta-1}{z_1} - 1\right)A_{\delta,\alpha,v}(z;z) + \beta z_1^{\eta-1}e^{-z_1} \\ & \times \left[C_1(z)M'\left(\xi+1,\eta;z_1\right) + C_2(z)U'\left(\xi+1,\eta;z_1\right)\right]\,. \end{aligned} \tag{4.12}$$

Furthermore, by the following two properties on M and U (see, Abramowitz and Stegun, 1970, 13.4.8 and 13.4.21)

$$M'(\xi+1,\eta;x) = \frac{\xi+1}{\eta}M(\xi+2,\eta+1;x)\,, \quad U'(\xi+1,\eta;x) = -(\xi+1)U(\xi+2,\eta+1;x)\,,$$

and the boundary condition (4.7), (4.12) can be rewritten as

$$1 = z_1^{\eta-1}e^{-z_1}(\xi+1)\left[\frac{C_1(z)}{\eta}M\left(\xi+2,\eta+1;z_1\right) - C_2(z)U\left(\xi+2,\eta+1;z_1\right)\right]. \tag{4.13}$$

Now, $C_1(z)$ and $C_2(z)$ can be solved from (4.11) and (4.13) as

$$C_1(z) = \eta z_1^{-(\eta-1)} e^{z_1} \frac{U(\xi+1,\eta+1;z_1)}{(\xi+1)D(z)}, \tag{4.14}$$

$$C_2(z) = (\xi-\eta+1) z_1^{-(\eta-1)} e^{z_1} \frac{M(\xi+1,\eta+1;z_1)}{(\xi+1)D(z)}, \tag{4.15}$$

where (13.4.3) and (13.4.17) in Abramowitz and Stegun (1970) are used, and

$$D(z) = M(\xi+2,\eta+1;z_1)U(\xi+1,\eta;z_1) + \eta M(\xi+1,\eta;z_1)U(\xi+2,\eta+1;z_1).$$

Finally, we obtain an explicit expression of $A_{\delta,\alpha,v}(u;z)$ when the claim amounts are exponentially distributed from (4.10) as follows:

$$A_{\delta,\alpha,v}(u;z) = \left(\frac{\beta\left(u+\frac{c}{\delta}\right)}{z_1}\right)^{\eta-1} e^{-\beta(u-z)} \frac{\Pi(u;z)}{(\xi+1)D(z)}, \tag{4.16}$$

where

$$\begin{aligned}\Pi(u;z) =& \eta U(\xi+1,\eta+1;z_1) M\left(\xi+1,\eta;\beta\left(u+\frac{c}{\delta}\right)\right) \\ &+ (\xi-\eta+1) M(\xi+1,\eta+1;z_1) U\left(\xi+1,\eta;\beta\left(u+\frac{c}{\delta}\right)\right).\end{aligned} \tag{4.17}$$

It follows from (4.5) that

$$R_{\delta,\alpha,v}(u;z,b) = \left(\frac{u+\frac{c}{\delta}}{b+\frac{c}{\delta}}\right)^{\frac{\lambda+\alpha}{\delta}} e^{-\beta(u-b)} \frac{\Pi(u;z)}{\Pi(b;z)}, \qquad -\frac{c}{\delta} \le z < u \le b, \tag{4.18}$$

where $\Pi(u;z)$ is given in (4.17). We show some special cases of $R_{\delta,\alpha,v}(u;z,b)$ below.

Remarks:

1. When $z = 0$ and $v = 1$, $R_{\delta,\alpha,v}(u;z,b)$ is the LT of the first time that the surplus process reaches b from u without dropping below zero, given by

$$R_{\delta,\alpha,1}(u;0,b) = \left(\frac{u+\frac{c}{\delta}}{b+\frac{c}{\delta}}\right)^{\frac{\lambda+\alpha}{\delta}} e^{-\beta(u-b)} \frac{\Pi(u;0)}{\Pi(b;0)}, \qquad -\frac{c}{\delta} \le z < u \le b,$$

   in which $\Pi(u;0)$ can be obtained from (4.17) with $\xi = \alpha/\delta$ and $\eta = 1 + (\lambda+\alpha)/\delta$.

2. When $z = 0$ and $\alpha = 0$, $R_{\delta,\alpha,v}(u;z,b)$ is the probability generating function of the number of claims by the first time that the surplus process reaches b from u without dropping below zero, and (4.18) reduces to

$$R_{\delta,0,v}(u;0,b) = \left(\frac{u+\frac{c}{\delta}}{b+\frac{c}{\delta}}\right)^{\frac{\lambda}{\delta}} e^{-\beta(u-b)} \frac{\Pi(u;0)}{\Pi(b;0)}, \qquad -\frac{c}{\delta} \le z < u \le b,$$

where $\Pi(u;0)$ is given by (4.17) with $\xi = (1-v)\lambda/\delta$ and $\eta = 1 + \lambda/\delta$.

3. When $z \to -c/\delta + 0$, i.e., $z_1 \to +0$, by the definition of M and the relationship between functions M and U (see, Abramowitz and Stegun, 1970, 13.1.2 and 13.1.3), we get $M(a,b;0) = 1$ and since $\eta > 0$,

$$\lim_{z_1 \to +0} U(\xi+1, \eta+1; z_1) = -\lim_{z_1 \to +0} \frac{\pi}{\sin[\pi(\eta+1)]} \frac{z_1^{-\eta}}{\Gamma(\xi+1)\Gamma(1-\eta)} = -\infty$$

which leads to

$$\lim_{z \to -\frac{c}{\delta}+0} \frac{\Pi(u;z)}{\Pi(b;z)} = \frac{M\left(\xi+1, \eta; \beta\left(u+\frac{c}{\delta}\right)\right)}{M\left(\xi+1, \eta; \beta\left(b+\frac{c}{\delta}\right)\right)}.$$

Therefore, when $z = -c/\delta$, we have, for $-c/\delta < u \le b$,

$$R_{\delta,\alpha,v}\left(u; -\frac{c}{\delta}, b\right) = \frac{e^{-\beta u}\left(u+\frac{c}{\delta}\right)^{\eta-1} M\left(\xi+1, \eta; \beta\left(u+\frac{c}{\delta}\right)\right)}{e^{-\beta b}\left(b+\frac{c}{\delta}\right)^{\eta-1} M\left(\xi+1, \eta; \beta\left(b+\frac{c}{\delta}\right)\right)}, \tag{4.19}$$

with $\xi = \alpha/\delta + (1-v)\lambda/\delta$ and $\eta = 1 + (\lambda+\alpha)/\delta$. Note that He et al. (2009) also obtained an alternative expression for $R_{\delta,\alpha,v}\left(u; -\frac{c}{\delta}, b\right)$ when $v = 1$.

4.2 Erlang(2) claims

When $\alpha = 0$ and $v = 1$, $R_{\delta,\alpha,v}(u;z,b)$ reduces to the probability of the surplus hitting an upper level b from u without having dropped below a lower level z, denoted by $\chi_\delta(u;z,b)$. In the following, we derive an expression for $\chi_\delta(u;z,b)$ when claim amounts are Erlang(2) distributed with $p(x) = \mu^2 x e^{-\mu x}$. It is easy to verify that $p(x)$ satisfies

$$p''(x) + 2\mu p'(x) + \mu^2 p(x) = 0, \qquad x > 0. \tag{4.20}$$

From (4.5), $\chi_\delta(u;z,b) = A_{\delta,0,1}(u;z)/A_{\delta,0,1}(b;z)$, we then derive $A_{\delta,0,1}(u;z)$ below.

Differentiating (4.6) (with $\alpha = 0$ and $v = 1$) twice with respect to u and using (4.20), we can obtain the following third order differential equation for $A_{\delta,0,1}(u;z)$:

$$\begin{aligned}(\delta u + c)A'''_{\delta,0,1}(u;z) &+ [2\mu(\delta u + c) + 2\delta - \lambda]A''_{\delta,0,1}(u;z) \\ &+ [\mu^2(\delta u + c) + 2\mu(\delta - \lambda)]A'_{\delta,0,1}(u;z) = 0.\end{aligned} \tag{4.21}$$

Similar to the exponential claims case, the boundary conditions to (4.21) are

$$A_{\delta,0,1}(z;z) = 1, \quad A'_{\delta,0,1}(z;z) = \frac{\lambda}{\delta z + c}, \quad A''_{\delta,0,1}(z;z) = \frac{\lambda(\lambda-\delta)}{(\delta z + c)^2}, \tag{4.22}$$

where the last two conditions are obtained from (4.6) and the derivative of equation (4.6) by letting $u = z$, respectively.

Let $B_\delta(u;z) = A'_{\delta,0,1}(u;z)$. Then (4.21) can be rewritten as

$$(\delta u + c)B''_\delta(u;z) + [2\mu(\delta u + c) + 2\delta - \lambda]B'_\delta(u;z) + [\mu^2(\delta u + c) + 2\mu(\delta - \lambda)]B_\delta(u;z) = 0\,,$$

which is a second order linear differential equation whose coefficients a linear function of the independent variable. By letting $s = u + c/\delta$ and $\eta_\delta(s;z) = e^{\mu\left(s-\frac{c}{\delta}\right)}B_\delta\left(s - \frac{c}{\delta};z\right)$, the above equation can be further simplified to

$$s\eta''_\delta(s;z) + \left(2 - \frac{\lambda}{\delta}\right)\eta'_\delta(s;z) - \frac{\lambda\mu}{\delta}\eta_\delta(s;z) = 0\,.$$

which has a general solution in terms of the Bessel functions (see, Abramowitz and Stegun, 1970, 9.1.53) with $\tau = (\lambda - \delta)/\delta$, $q = -\lambda\mu/\delta$ as

$$\eta_\delta(s;z) = s^{\frac{\tau}{2}}\left[C_1(z)J_\tau\left(2\sqrt{q}s^{\frac{1}{2}}\right) + C_2(z)Y_\tau\left(2\sqrt{q}s^{\frac{1}{2}}\right)\right]\,, \tag{4.23}$$

where $C_1(z)$ and $C_2(z)$ are two functions of z to be determined, J_a and Y_a are Bessel functions of the first and second kind, respectively, defined by

$$J_a(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{a+2k}}{k!\Gamma(a+k+1)}\,, \qquad Y_a(x) = \frac{J_a(x)\cos(a\pi) - J_{-a}(x)}{\sin(a\pi)}\,.$$

It follows from (4.23), the solution to (4.21) can be expressed as

$$\begin{aligned} A_{\delta,0,1}(u;z) &= A_{\delta,0,1}(z;z) + C_1(z)\int_z^u e^{\mu y}\left(y + \frac{c}{\delta}\right)^{\frac{\tau}{2}} J_\tau\left(2\sqrt{q}\left(y + \frac{c}{\delta}\right)^{\frac{1}{2}}\right)dy \\ &\quad + C_2(z)\int_z^u e^{\mu y}\left(y + \frac{c}{\delta}\right)^{\frac{\tau}{2}} Y_\tau\left(2\sqrt{q}\left(y + \frac{c}{\delta}\right)^{\frac{1}{2}}\right)dy\,, \qquad -\frac{c}{\delta} < z \le u\,. \end{aligned} \tag{4.24}$$

Differentiating once and twice both sides of (4.24) with respect to u yields

$$\begin{aligned} A'_{\delta,0}(u;z) &= e^{\mu u}\left(u + \frac{c}{\delta}\right)^{\frac{\tau}{2}}\left[C_1(z)J_\tau\left(2\sqrt{q}\left(u + \frac{c}{\delta}\right)^{\frac{1}{2}}\right) + C_2(z)Y_\tau\left(2\sqrt{q}\left(u + \frac{c}{\delta}\right)^{\frac{1}{2}}\right)\right]\,, \\ A''_{\delta,0}(u;z) &= \mu A'_{\delta,0}(u;z) + \sqrt{q}e^{\mu u}\left(u + \frac{c}{\delta}\right)^{\frac{\tau-1}{2}}\left[C_1(z)J_{\tau-1}\left(2\sqrt{q}\left(u + \frac{c}{\delta}\right)^{\frac{1}{2}}\right)\right. \\ &\quad \left. + C_2(z)Y_{\tau-1}\left(2\sqrt{b}\left(u + \frac{c}{\delta}\right)^{\frac{1}{2}}\right)\right]\,, \end{aligned}$$

in which a differentiation property of the Bessel functions is used (see, Abramowitz and Stegun, 1970, 9.1.29). Letting $u = z$ and using (4.22), we immediately get from the above that

$$(\tau + 1)e^{-\mu z}\left(\frac{z_2}{q}\right)^{-\frac{\tau+2}{2}} = C_1(z)J_\tau\left(2z_2^{\frac{1}{2}}\right) + C_2(z)Y_\tau\left(2z_2^{\frac{1}{2}}\right)\,,$$

$$\frac{\tau+1}{\sqrt{q}}\left(\tau-\frac{\mu z_2}{q}\right)e^{-\mu z}\left(\frac{z_2}{q}\right)^{-\frac{\tau+3}{2}} = C_1(z)J_{\tau-1}\left(2z_2^{\frac{1}{2}}\right)+C_2(z)Y_{\tau-1}\left(2z_2^{\frac{1}{2}}\right).$$

where we denote $z_2 = q(z + c/\delta)$ for simplicity. Then $C_1(z)$ and $C_2(z)$ can be solved as

$$C_1(z) = \pi(\tau+1)e^{-\mu z}\left(\frac{z_2}{q}\right)^{-\frac{\tau+1}{2}} E_1(z)\,, \qquad C_2(z) = \pi(\tau+1)e^{-\mu z}\left(\frac{z_2}{q}\right)^{-\frac{\tau+1}{2}} E_2(z)\,,$$

where $E_1(z)$ and $E_2(z)$ are defined as

$$E_1(z) = \sqrt{q}Y_{\tau-1}\left(2z_2^{\frac{1}{2}}\right) - \left(\frac{z_2}{q}\right)^{-\frac{1}{2}}\left(\tau-\frac{\mu z_2}{q}\right)Y_\tau\left(2z_2^{\frac{1}{2}}\right),$$
$$E_2(z) = \left(\frac{z_2}{q}\right)^{-\frac{1}{2}}\left(\tau-\frac{\mu z_2}{q}\right)J_\tau\left(2z_2^{\frac{1}{2}}\right) - \sqrt{q}J_{\tau-1}\left(2z_2^{\frac{1}{2}}\right),$$

and the equality below is used (see, Abramowitz and Stegun, 1970, 9.1.16):

$$J_\tau\left(2z_2^{\frac{1}{2}}\right)Y_{\tau-1}\left(2z_2^{\frac{1}{2}}\right) - J_{\tau-1}\left(2z_2^{\frac{1}{2}}\right)Y_\tau\left(2z_2^{\frac{1}{2}}\right) = \frac{z_2^{-\frac{1}{2}}}{\pi}.$$

Finally, by (4.24) and using (4.22), we obtain the following explicit expression of $\chi_\delta(u; z, b)$ when the claim amounts are Erlang(2) distributed:

$$\chi_\delta(u; z, b) = \frac{\delta + \pi\lambda e^{-\mu\left(z+\frac{c}{\delta}\right)}\left(z+\frac{c}{\delta}\right)^{-\frac{\lambda}{2\delta}}\Theta(u; z)}{\delta + \pi\lambda e^{-\mu\left(z+\frac{c}{\delta}\right)}\left(z+\frac{c}{\delta}\right)^{-\frac{\lambda}{2\delta}}\Theta(b; z)}\,, \qquad -\frac{c}{\delta} < z \le u < b\,, \tag{4.25}$$

where

$$\Theta(u; z) = E_1(z)\int_{z+\frac{c}{\delta}}^{u+\frac{c}{\delta}} e^{\mu t}t^{\frac{\tau}{2}}J_\tau\left(2\sqrt{qt}\right)dt + E_2(z)\int_{z+\frac{c}{\delta}}^{u+\frac{c}{\delta}} e^{\mu t}t^{\frac{\tau}{2}}Y_\tau\left(2\sqrt{qt}\right)dt\,.$$

4.3 A general class of claim amount distributions

Consider in this subsection that the claim amounts has a distribution with rational Laplace transform, namely, the Laplace transform of the distribution function of claim amounts P can be expressed as a ratio of two polynomials of finite degree, according to the definition in Asmussen and Albrecher (2010, page 8). Equivalently, its density function $p(x)$ satisfies the following homogeneous ordinary differential equation,

$$p^{(n)}(x) + \kappa_{n-1}p^{(n-1)}(x) + \cdots + \kappa_1 p'(x) + \kappa_0 p(x) = 0\,, \qquad \kappa_j \in \mathbb{R},\ \kappa_0 \ne 0\,. \tag{4.26}$$

This distribution family is referred to as distributions with rational transforms in Asmussen and Albrecher (2010), which includes the class of phase-type distributions as a special case.

Taking the ith ($i = 0, 1, \ldots, n$) derivative of both sides of (4.6) yields

$$(\delta u + c)A^{(i+1)}_{\delta,\alpha,v}(u; z) + (i\delta - \lambda - \alpha)A^{(i)}_{\delta,\alpha,v}(u; z)$$
$$= -\lambda v \sum_{j=1}^{i} p^{(j-1)}(0)A^{(i-j)}_{\delta,\alpha,v}(u; z) - \lambda v \int_z^u A^{(i)}_{\delta,\alpha,v}(y)p^{(i)}(u - y; z)dy\,, \tag{4.27}$$

where $i = 0$ corresponds to equation (4.6). Now multiplying both sides of (4.27) by κ_i, $i = 0, 1, \ldots, n$, summing over $i = 0, 1, \ldots, n$, and making use of equation (4.26), we obtain, after some manipulations, an ordinary differential equation of degree $(n + 1)$ with non-constant coefficients below:

$$(\delta u + c)A^{(n+1)}_{\delta,\alpha,v}(u; z) + \sum_{i=0}^{n} \nu_i(u)A^{(i)}_{\delta,\alpha,v}(u; z) = 0\,, \qquad -\frac{c}{\delta} \le z < u\,, \tag{4.28}$$

where the coefficient function $\nu_i(u)$ is defined by

$$\nu_i(u) = (i\delta - \lambda - \alpha)\kappa_i + (\delta u + c)\kappa_{i-1} + \lambda v \sum_{l=1}^{n-i} \kappa_{l+i}\, p^{(l-1)}(0)\,, \quad i = 0, 1, \ldots, n\,, \tag{4.29}$$

with the convention that $\sum_{i=s}^{t} = 0$ is $s > t$, $\kappa_n = 1$, $\kappa_{-1} = 0$, and $p^{(0)}(u) = p(u)$.

We now use the series solution approach to find the solution of (4.28). Assume a solution in the form of a power series with unknown coefficients, namely,

$$A_{\delta,\alpha,v}(u; z) = \sum_{k=0}^{\infty} \frac{a_k(z)}{k!}(u - z)^k\,, \qquad -\frac{c}{\delta} \le z < u\,, \tag{4.30}$$

where $a_k(z) = A^{(k)}_{\delta,\alpha,v}(z; z)$, for $k = 0, 1, \ldots$, are constants to be determined. In fact, by two of the boundary conditions for (4.28), $A_{\delta,\alpha,v}(z; z) = 1$ and $A'_{\delta,\alpha,v}(z; z) = (\lambda + \alpha)/(\delta z + c)$ from (4.7), we have $a_0(z) = 1$, $a_1(z) = (\lambda + \alpha)/(\delta z + c)$, and by letting $u = z$ in (4.27), we further obtain for $i = 1, 2, \ldots, n - 1$, that

$$(\delta z + c)a_{i+1}(z) + (i\delta - \lambda - \alpha)a_i(z) = -\lambda v \sum_{j=1}^{i} p^{(j-1)}(0)a_{i-j}(z)\,, \tag{4.31}$$

that is, $a_2(z), a_3(z), \ldots, a_n(z)$ can be calculated recursively from (4.31).

For simplicity, we write $\upsilon_i(u)$ in (4.29) as

$$\nu_i(u) = \zeta_i(z) + \delta\kappa_{i-1}\,(u - z)\,, \qquad i = 0, 1, \ldots, n\,,$$

where

$$\zeta_i(z) = (i\delta - \lambda - \alpha)\kappa_i + \delta\kappa_{i-1}\left(z + \frac{c}{\delta}\right) + \lambda v \sum_{l=1}^{n-i} \kappa_{l+i}\, p^{(l-1)}(0)\,.$$

Substituting (4.30) into equation (4.28), we have

$$\left[\delta(u-z)+\delta\left(z+\frac{c}{\delta}\right)\right]\sum_{k=0}^{\infty}\frac{a_{n+1+k}(z)}{k!}(u-z)^k$$
$$+\sum_{i=0}^{n}\left[\zeta_i(z)+\delta\kappa_{i-1}(u-z)\right]\sum_{k=0}^{\infty}\frac{a_{i+k}(z)}{k!}(u-z)^k=0\,. \tag{4.32}$$

Then the coefficient before $(u-z)^k$ for each k, $k=0,1,\ldots$, must all be zero, implying, for the constant term, that

$$\delta\left(z+\frac{c}{\delta}\right)a_{n+1}(z)+\sum_{i=0}^{n}\zeta_i(z)a_i(z)=0\,, \tag{4.33}$$

and for the coefficient before $(u-z)^k$ $(k\geq 1)$, that

$$\frac{\delta}{(k-1)!}a_{n+k}(z)+\frac{\delta\left(z+\frac{c}{\delta}\right)}{k!}a_{n+k+1}(z)+\sum_{i=0}^{n}\left[\frac{\zeta_i(z)}{k!}a_{i+k}(z)+\frac{\delta\kappa_{i-1}}{(k-1)!}a_{i+k-1}(z)\right]=0\,.$$

which can be rearranged to

$$\delta\left(z+\frac{c}{\delta}\right)a_{n+k}(z)+\sum_{i=0}^{n}[k\delta\kappa_i+\zeta_i(z)]a_{i+k}(z)=0\,. \tag{4.34}$$

Now, by (4.33), we can obtain $a_{n+1}(z)$ which is a function of initial values of $a_0(z),a_1(z),\ldots,a_n(z)$, and then the following $a_{n+2}(z),a_{n+3}(z),\ldots$, can be calculated recursively by (4.34). In this way, we obtain a power series form of expression of $A_{\delta,\alpha,v}(u;z)$, and then

$$R_{\delta,\alpha,v}(u;b,z)=\frac{A_{\delta,\alpha,v}(u;z)}{A_{\delta,\alpha,v}(b;z)}=\frac{\sum_{k=0}^{\infty}\frac{a_k(z)}{k!}(u-z)^k}{\sum_{k=0}^{\infty}\frac{a_k(z)}{k!}(b-z)^k}\,,\qquad -\frac{c}{\delta}\leq z<u\leq b\,,$$

is entirely determined.

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