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ASYMPTOTIC STATISTICAL PROPERTIES OF THE NEOCLASSICAL OPTIMAL GROWTH MODEL

by

John Stachurski

Department of Economics
The University of Melbourne
Melbourne   Victoria   3010
Australia.

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ABSTRACT. The standard one-sector stochastic optimal growth model is shown to be not just ergodic but geometrically ergodic. In addition, it is proved that the time series generated by the optimal path satisfy the Law of Large Numbers and the Central Limit Theorem.

1. INTRODUCTION

Brock and Mirman (1972) is widely recognized to be one of the most important studies in modern macroeconomics. The stochastic neoclassical infinite horizon growth model they consider has become the foundation and common language for a vast and growing literature, spanning such fields as economic development, public finance, fiscal policy, environmental and resource economics, monetary policy and asset pricing.

A central result of Brock and Mirman's study is that, given "Inada" type conditions, the optimizing behavior of agents implies convergence for the sequence of distributions describing per capita income (equivalently, capital) to a unique limiting distribution, or stochastic steady state, which is independent of initial income. In other words, the Markov process for the state variable is ergodic.

This paper strengthens Brock and Mirman's main conclusion in several directions. First, we prove that the optimal process is not only ergodic but geometrically ergodic. That is, for any given starting point, the distance between the current distribution and the limiting distribution decreases at a geometric rate.

In addition, we prove under standard econometric assumptions on the noise process that the series for the state variable also satisfies both the Law of Large Numbers (LLN) and Central Limit Theorem (CLT). The former states that sample means converge asymptotically to their long-run expected value. The latter associates asymptotic distributions to estimators, from which confidence intervals and hypothesis tests are constructed.

It is shown in this paper that the number of moments of the optimal income process for which the LLN and the CLT apply depend on the number of finite moments possessed by the productivity shock. In the empirical literature this shock is often taken to be lognormal. In that case we have the remarkable conclusion that the LLN and the CLT hold for moments of all orders.

An extensive list of references for ergodicity of the Brock–Mirman model is given in Stachurski (2002). The majority of previous work has used restrictions on the support of the productivity shocks, which limits direct applicability to empirical macroeconomics. A notable exception is Mirman (1972).

LLN results for the stochastic Solow-Swan model were studied by Binder and Pesaran (1999) when the shock is bounded away from zero. LLN and CLT results for some stochastic growth models with unbounded shocks were given in Stachurski (2003), but the assumptions imposed on technology are too strict for the Brock–Mirman model. Evstigneev and Flåm (1997) and Amir and Evstigneev (2000) have studied CLT related properties of competitive equilibrium economies.

Geometric ergodicity has numerous theoretical and empirical applications in economics. As an example of the former, the *rate* at which stochastically growing economies tend to their steady state is a central component of the "convergence" debate; of the latter, geometric convergence is required by Duffie and Singleton (1993) for consistency of the Simulated Moments Estimator.

Geometric ergodicity also has applications to numerical procedures: When simulating time series drawn from a steady state distribution—as in the real business cycle literature, say—bounds on run-times depend on the rate of convergence for the distribution of the state variable to that steady state (Santos, 2003).

The primary mathematical reference for this paper is the monograph of Meyn and Tweedie (1993). We make use in particular of the powerful notion of V-uniform ergodicity. Much of that theory for aperiodic general-state Markov chains was developed only recently, by the same authors.

2. THE MODEL

All of the following assumptions are identical to Brock and Mirman (1972) apart from the distribution of the shock (see comments in the introduction). We can and do assume the existence of a single social planner, who implements a state-contingent savings policy to maximize the discounted sum of expected utilities. At the start of time t, the agent observes income y_t, which he or she divides between savings and consumption. Savings is added one-for-one to the existing capital stock. For simplicity we are going to assume that depreciation is total: current savings and capital stock are identified. Labor is supplied inelastically, and we normalize the total quantity to one.

After the time t investment decision is made a shock ε_t is drawn by nature and revealed to the agent. Production then takes place, yielding at the start of next period output

$$y_{t+1} = f(k_t)\,\varepsilon_t,$$

(1)

The sequence $(\varepsilon_t)_{t=0}^{\infty}$ is uncorrelated; f describes technology. We can think of the current shock ε_t as being realized *during* the production process. As a result, (y_t, k_t) and ε_t are independent.

Assumption 2.1. The function $f: [0, \infty) \to [0, \infty)$ is strictly increasing, strictly concave, continuously differentiable and satisfies

$$(2) \qquad f(0) = 0, \quad \lim_{k \to 0} f'(k) = \infty, \quad \lim_{k \to \infty} f'(k) = 0.$$

Assumption 2.2. The shock ε is distributed according to ψ, a density on $[0, \infty)$. The density ψ is continuous and strictly positive on the interior of its domain. In addition, the moments $\mathbb{E}(\varepsilon^p)$ and $\mathbb{E}(\varepsilon^{-1}) = \mathbb{E}(1/\varepsilon)$ are both finite for some $p \in \mathbb{N}$.

For example, the entire class of lognormal distributions satisfies Assumption 2.2 for every $p \in \mathbb{N}$.

In earlier studies it was commonly assumed that the shock ε only took values in a closed interval $[a, b] \subset (0, \infty)$. In this case $\mathbb{E}(\varepsilon^p)$ and $\mathbb{E}(1/\varepsilon)$ are automatically finite. For unbounded shocks the last two restrictions can be interpreted as bounds on the size of the right and left hand tails of ψ respectively. Without such bounds the stability of the economy is jeopardized.

The larger p can be taken in Assumption 2.2, the tighter the conclusions of the paper will be. For example, we prove that the Law of Large Numbers holds for all moments of the optimal process up to order p, and the Central Limit Theorem holds for all moments up to order q, where $q \le p/2$.

To formalize uncertainty, let each random variable ε_t be defined on a fixed probability space $(\Omega, \mathscr{F}, \mathbf{P})$, where Ω is the set of outcomes, $\mathscr{F}$ is the set of events $E \subset \Omega$, and $\mathbf{P}$ is a probability. By definition, $\mathbf{P}\{a \le \varepsilon_t \le b\} = \int_a^b \psi(z)dz$ for all a, b and t. The notation $\mathbb{E}_{\mathbf{P}}$ means integration with respect to $\mathbf{P}$.

A feasible savings policy is a (Borel) function π from $[0, \infty)$ to itself such that $0 \le \pi(y) \le y$ for all y. The set of all feasible policies will be denoted by Π. Each $\pi \in \Pi$ defines a Markov process on $(\Omega, \mathscr{F}, \mathbf{P})$ for

income via the recursion

$$(3) \qquad\qquad y_{t+1} = f(\pi(y_t))\,\varepsilon_t.$$

The problem for the agent is

$$(4) \qquad \max_{\pi \in \Pi} \mathbb{E}_{\mathbf{P}}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \quad c_t := y_t - \pi(y_t),$$

where, for given π, the sequence $(y_t)_{t=0}^{\infty}$ is determined by (3). The number $\beta \in (0,1)$ is the discount factor, and u is the period utility function.

Assumption 2.3. The function $u \colon [0,\infty) \to [0,\infty)$ is strictly increasing, bounded, strictly concave, continuously differentiable, and satisfies

$$(5) \qquad\qquad \lim_{c \to 0} u'(c) = \infty.$$

It is now very well known that there is a unique solution to (4) in Π, which for notational convenience we again refer to simply as π. This optimal policy is continuous and nondecreasing. In addition, consumption $y - \pi(y)$ is also increasing in income. The policy is interior, in the sense that $0 < \pi(y) < y$ for all $y > 0$. We take all these facts as given. For proofs see Mirman and Zilcha (1975).

Once an initial condition for income is specified, the optimal policy and the recursion (3) completely define the process $(y_t)_{t=0}^{\infty}$ for income. Suppose for now that the initial condition y_0 is a random variable, with distribution equal to some density φ_0 on $[0,\infty)$, independent of the productivity shocks. It can then be shown that the distribution of y_t is a density φ_t for all t, and

$$(6) \qquad \varphi_{t+1}(y') = \int p(y,y')\varphi_t(y)dy, \quad t \geq 0,$$

$$(7) \qquad \text{where} \quad p(y,y') := \psi\left(\frac{y'}{f(\pi(y))}\right)\frac{1}{f(\pi(y))}.$$

Heuristically, $p(y,y')dy'$ is the probability of moving from income y to income y' in one period. Equation (6) states that the probability of

being at y' next period is the probability of moving to y' via y, summed accross all y, weighted by the probability that current income is equal to y. All of the above is discussed at some length in Stachurski (2002).

It is perhaps more natural to regard y_0 as a single point, rather than a random variable with a density. In this case, provided $y_0 > 0$, one can take $\varphi_1(\cdot) = p(y_0, \cdot)$, a density, and the remaining sequence of densities is then defined inductively via (6). Let us agree to write $\varphi_t^{y_0}$ for the t-th element so defined.

A density φ^* is called stationary for the optimal process if it satisfies

$$(8) \qquad \varphi^*(y') = \int p(y, y')\varphi^*(y)dy, \quad \forall y.$$

It is clear from (6) and (8) that if y_t has distribution φ^*, then so does y_{t+n} for all $n \in \mathbb{N}$. A density satisfying (8) is also called a stochastic steady state. At such a long-run equilibrium the *probabilities* are stationary over time, even though the state variable is not.

3. Results

The fundamental result of Brock and Mirman (1972) is ergodicity. That is, for the optimal process there is a unique stationary distribution φ^*, which under the current assumptions will be a density, and $\|\varphi_t^y - \varphi^*\| \to 0$ as $t \to \infty$ for all $y > 0$. Here $\| \cdot \|$ is the L_1 distance.[1]

Our first result strengthens this to geometric ergodicity. In the statement of the theorem and much of what follows, we use the function

$$(9) \qquad V(y) := \frac{1}{y} + y^p, \quad p \text{ as defined in Assumption 2.2.}$$

The role of V is comparable to that of a Lyapunov function. With this definition we can state the first of our results.

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[1]That is, $\|\varphi_t - \varphi^*\| = \int |\varphi_t - \varphi^*|$. Brock and Mirman (1972) used a slightly weaker topology.

Theorem 3.1. *Let Assumptions 2.1–2.3 hold, and let $(y_t)_{t=0}^\infty$ be defined by (3), where π is the optimal policy. Then $(y_t)_{t=0}^\infty$ is geometrically ergodic. Precisely, there is a constant $\varrho \in (0,1)$ and an $R < \infty$ such that*

$$\|\varphi_t^y - \varphi^*\| \leq \varrho^t R V(y), \quad \forall t \in \mathbb{N},\ y > 0, \tag{10}$$

where φ^* is the unique stationary distribution for $(y_t)_{t=0}^\infty$.

For h a real function on the state space, define the random variable $S_n(h) := \sum_{t=1}^n h \circ y_t$. The LLN and CLT results are as follows.

Theorem 3.2. *Let the hypotheses of Theorem 3.1 hold, and let φ^* be the stationary distribution. If $h \colon (0, \infty) \to \mathbb{R}$ is any Borel function satisfying $|h| \leq V$, then the Law of Large Numbers holds for h. That is,*

$$\mathbb{E}_{\varphi^*}(h) := \int h \, d\varphi^* < \infty, \quad and \quad \lim_{n \to \infty} \frac{S_n(h)}{n} = \mathbb{E}_{\varphi^*}(h). \tag{11}$$

If in addition $h^2 \leq V$, then the Central Limit theorem also holds for h. Precisely, there is a constant $\sigma^2 \in [0, \infty)$ such that

$$\frac{S_n(h - \mathbb{E}_{\varphi^*}(h))}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2). \tag{12}$$

In the statement of the theorem the symbol $\xrightarrow{d}$ means convergence in distribution. If $\sigma^2 = 0$ then the right hand side of (12) is interpreted as the probability measure concentrated on zero. Also, $(h - \mathbb{E}_{\varphi^*}(h))(y_t) := h(y_t) - \mathbb{E}_{\varphi^*}(h)$.

These results are in a rather convenient form. In particular, since $x^p \leq V(x)$ we see that all moments of the income process up to order p satisfy the LLN, and all moments up to order q satisfy the CLT, where q is the largest integer such that $2q \leq p$.

The proof centers on establishing that the optimal process is V-uniformly ergodic for V specified by (9), where V-uniform ergodicity is defined in Meyn and Tweedie (1993, Chapter 16). Essentially this requires

geometric drift towards a subset of the state space which satisfies a certain minorization condition. All of these properties are shown to be satisfied from the model primitives and the restrictions implied by optimizing behaviour.

4. Proofs

It is simplest in what follows to take the state space for the Markov process $(y_t)_{t=0}^{\infty}$ to be $(0, \infty)$ rather than $[0, \infty)$. Since the optimal process is interior and the shock is distributed according to a density, $(y_t)_{t=0}^{\infty}$ remains in $(0, \infty)$ with probability one provided that $y_0 > 0$, which is always assumed to be true. The dynamics when $y_0 = 0$ are completely trivial so we can neglect to analyze them.

The following definitions are necessary. Let $\mathscr{B}$ be the Borel sets on $(0, \infty)$, let $\mathscr{M}$ be the (Borel) measures on $((0, \infty), \mathscr{B})$, and let $\mathcal{P}$ be all $\nu \in \mathscr{M}$ with $\nu(0, \infty) = 1$. For $B \in \mathscr{B}$ let $\mathbf{1}_B$ denote the indicator function of B. Proofs of lemmas are given in the appendix.

Definition 4.1. The optimal process $(y_t)_{t=0}^{\infty}$ defined by (3) is called μ-**irreducible** for $\mu \in \mathcal{P}$ if

$$\mathbf{P}\{y_t \in B \text{ for some } t \in \mathbb{N}\} > 0, \quad \forall y_0 > 0, \ B \in \mathscr{B} \text{ with } \mu(B) > 0.$$

In other words, $(y_t)_{t=0}^{\infty}$ visits every set of positive μ-measure from every starting point.

Lemma 4.1. *Under Assumptions 2.1–2.3, the optimal process is μ-irreducible for any $\mu \in \mathcal{P}$ absolutely continuous with respect to Lebesgue measure.*

Definition 4.2. Following Meyn and Tweedie (1993, Chapter 5), a set $C \in \mathscr{B}$ is called **small** with respect to the transition probability p defined in (7) if there is a nontrivial $\nu \in \mathscr{M}$ such that

$$(13) \qquad\qquad \int_B p(x, y) \geq \nu(B), \quad \forall B \in \mathscr{B}, \ x \in C.$$

In fact their definition is a little weaker than this—but all small sets in the sense of (13) are small in the sense of Meyn and Tweedie.

Definition 4.3. Let V be as in (9). The optimal process $(y_t)_{t=0}^{\infty}$ is called V**-uniformly ergodic** (Meyn and Tweedie, 1993, Chapter 16) if

$$(14) \qquad \sup_{y>0} \left\{ \frac{\|\varphi_t^y - \varphi^*\|}{V(y)} \right\} \to 0 \text{ as } t \to \infty.$$

Almost all the results derived in this paper follow from

Proposition 4.1. *Under Assumptions 2.1–2.3, the optimal process* $(y_t)_{t=0}^{\infty}$ *is* V*-uniformly ergodic.*

Proof. By Meyn and Tweedie (1993, Theorem 16.0.1), $(y_t)_{t=0}^{\infty}$ is V-uniformly ergodic whenever it is μ-irreducible for some $\mu \in \mathcal{P}$, aperiodic, and there is a "petite" set $C \in \mathscr{B}$, a $\varrho > 0$ and a $N < \infty$ such that

$$(15) \quad \int V(y)p(x,y)dy - V(x) \leq -\varrho V(x) + N\mathbf{1}_C(x), \quad \forall x \in (0,\infty).$$

We have not defined the notions of petite sets or aperiodicity. See Meyn and Tweedie (1993, Chapter 5) for both definitions. However, small sets in the sense of Definition 4.2 are a special case of petite sets, so in what follows establishing that a set is small will establish that it is petite. Discussion of aperiodicity is given below.

By Meyn and Tweedie (1993, Lemma 15.2.8), the drift condition (15) holds for a petite set if there is are positive constants $\lambda < 1$ and $b < \infty$ such that

$$(16) \qquad \int V(y)p(x,y)dy \leq \lambda V(x) + b, \quad \forall x \in (0,\infty).$$

and, in addition, V is "unbounded off petite sets," which in turn means that $\{x : V(x) \leq n\}$ is petite for every $n \in \mathbb{N}$ (Meyn and Tweedie, 1993, Chapter 8). The next two lemmas establish that (16) holds, and moreover, that that $\{x : V(x) \leq n\}$ is small and hence petite for each n.

Lemma 4.2. *There are positive constants $\lambda < 1$ and $b < \infty$ such that* (16) *holds.*

Lemma 4.3. *The set $\{x : V(x) \leq n\}$ is small for each $n \in \mathbb{N}$. Moreover, the optimal process $(y_t)_{t=0}^{\infty}$ is aperiodic.*

These two results complete the proof of Proposition 4.1.   □

Theorem 3.1 now follows immediately from Proposition 4.1 by Meyn and Tweedie (1993, Theorem 16.0.1, Part (ii)).

It remains to establish Theorem 3.2. By Meyn and Tweedie (1993, Theorem 17.0.1, Part (i)), the LLN holds for h provided that $(y_t)_{t=0}^{\infty}$ is positive Harris and $\int |h| d\varphi^* < \infty$. By positive Harris is meant that $(y_t)_{t=0}^{\infty}$ has an invariant distribution and is Harris recurrent. For a definition of Harris recurrent see Meyn and Tweedie (1993, Chapter 9). For our purposes we note that by the same reference, Proposition 9.1.8, Harris recurrence holds when $\{x : V(x) \leq n\}$ is small for each $n \in \mathbb{N}$, and that there is a small set $C \subset (0, \infty)$ such that

$$(17) \qquad \int V(y) p(x, y) dy \leq V(x), \quad \forall x \notin C.$$

We have already shown that $\{x : V(x) \leq n\}$ is small for each $n \in \mathbb{N}$ in Lemma 4.3. Regarding (17), let $C := \{x : V(x) \leq N\}$, where $N \in \mathbb{N}$ satisfies $N \geq b/(1 - \lambda)$, and λ and b are the constants defined in Lemma 4.2. If $x \notin C$, then $V(x) > b/(1 - \lambda)$, and hence

$$\int V(y) p(x, y) dy - V(x) \leq \lambda V(x) + b - V(x) \leq 0,$$

where the first inequality is from Lemma 4.2. Therefore $(y_t)_{t=0}^{\infty}$ is positive Harris, and it remains only to show that $\int |h| d\varphi^* < \infty$. Clearly it is sufficient to show that $\int V d\varphi^*$ is finite.

To see that this is the case, pick any intitial condition $y_0 = x$. By the recursion (6) and Lemma 4.2 we have

$$
\begin{aligned}
\int V(y)\varphi_t^x(y)dy &= \int V(y)\left[\int p(z,y)\varphi_{t-1}^x(z)dz\right]dy \\
&= \int \left[\int V(y)p(z,y)dy\right]\varphi_{t-1}^x(z)dz \\
&\leq \int \left[\lambda V(z) + b\right]\varphi_{t-1}^x(z)dz = \lambda\int V(z)\varphi_{t-1}^x(z)dz + b.
\end{aligned}
$$

Iterating backwards in the same way gives us the bound

$$
(18) \qquad \int V(y)\varphi_t^x(y)dy \leq \lambda^t V(x) + \frac{b}{1-\lambda} \leq M := V(x) + \frac{b}{1-\lambda}.
$$

Now set $K_n := \mathbf{1}_{[1/n,n]}$. By (18) we have

$$
\int K_n V(y)\varphi_t^x(y)dy \leq M, \quad \forall\, t, n.
$$

Note that the product $K_n V$ is bounded, so, as L_1 convergence implies weak convergence, taking the limit with respect to t gives

$$
\int K_n V(y)\varphi^*(y)dy \leq M, \quad \forall\, n.
$$

Now taking limits with respect to n and using the Monotone Convergence theorem gives $\int V d\varphi^* < \infty$.

We have now established the LLN part of Theorem 3.2. The CLT component is immediate from Meyn and Tweedie (1993, Theorem 17.0.1, Parts (ii)–(iv)), given that $(y_t)_{t=0}^\infty$ has already been shown to be V-uniformly ergodic. The constant σ^2 is given by

$$
\sigma^2 = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\varphi^*}[(S_n(h - \mathbb{E}_{\varphi^*}(h))^2].
$$

Appendix A

Proof of Lemma 4.1. Let μ be any probability on $(0,\infty)$ with a density. Now take any $B \subset (0,\infty)$ with positive μ-measure and any $y_0 \in (0,\infty)$.

It is easy to check that the set $[f(\pi(y_0))]^{-1} \cdot B$ has positive Lebesgue measure, so that from Assumption 2.2 we have

$$\mathbf{P}\{y_1 \in B\} = \int_{\{z: f(\pi(y_0))z \in B\}} \psi(z)dz = \int_{[f(\pi(y_0))]^{-1} \cdot B} \psi(z)dz > 0.$$

$\square$

Proof of Lemma 4.2. From Stachurski [2002, Eq. (22), p. 46] there are positive constants $\lambda_1 < 1$ and $b_1 < \infty$ such that

(19) $$\int \frac{1}{y} p(x,y) dy \le \lambda_1 \frac{1}{x} + b_1, \quad \forall\, x > 0.$$

(In that paper, Stachurski (2002, Assumption 3) requires that $\mathbb{E}(1/\varepsilon) < 1$. But if u, f and ε satisfy Assumutions 2.1–2.3 above then so do u, $(1/a)f$ and $\hat{\varepsilon} := a\varepsilon$ for any $a > 0$. Moreover, these two economies are identical, as is clear from (1). Since in this paper $\mathbb{E}(1/\varepsilon) < \infty$ holds, we are free to choose a such that $\mathbb{E}(1/\hat{\varepsilon}) = (1/a)\mathbb{E}(1/\varepsilon) < 1$.)

Next we prove that there are positive constants $\lambda_2 < 1$ and $b_2 < \infty$ such that

(20) $$\int y^p p(x,y) dy \le \lambda_2 x^p + b_2, \quad \forall\, x > 0.$$

First choose $\gamma \in (0,1)$ so that $\gamma^p \mathbb{E}(\varepsilon^p) < 1$. For such a γ we can find a $w < \infty$ such that for all $x > w$, $f(x) \le \gamma \cdot x$. (This follows from concavity of f and $f'(\infty) = 0$.) For all $x \in (0, w]$ we have $f(\pi(x)) \le f(x) \le f(w)$, and hence

$$\int y^p p(x,y) dy = \int [f(\pi(x))z]^p \psi(z) dz \le f(w)^p \mathbb{E}(\varepsilon^p), \quad \forall x \le w.$$

On the other hand, $x > w$ implies $f(\pi(x)) \le f(x) \le \gamma x$, so

$$\int y^p p(x,y) dy = \int [f(\pi(x))z]^p \psi(z) dz \le \gamma^p \mathbb{E}(\varepsilon^p) x^p, \quad \forall x > w.$$

Setting $\lambda_2 := \gamma^p \mathbb{E}(\varepsilon^p)$ and $b_2 := f(w)\mathbb{E}(\varepsilon^p)$ gives (20). Finally, combining (19) and (20) gives (16) when $\lambda := \min(\lambda_1, \lambda_2)$ and $b := b_1 + b_2$. $\square$

Proof of Lemma 4.3. For the first part of the lemma, evidently it is sufficient to prove that every closed interval $C := [a, b] \subset (0, \infty)$ is small. Since the optimal policy π is interior and continuous, and the density ψ is strictly positive and continuous, it follows from the representation (7) that p is continous and strictly positive on $(0, \infty) \times (0, \infty)$. Therefore we can find a $\delta > 0$ such that

$$p(x, y) \geq \delta, \quad \forall\, (x, y) \in C \times C.$$

$$\therefore \quad \int_B p(x, y) dy \geq \int_B \delta \mathbf{1}_C(y) dy, \quad \forall\, x \in C.$$

Since the measure $\nu(B) := \int_B \delta \mathbf{1}_C(y) dy$ is nontrivial it follows that C is ν-small.

Regarding aperiodicity, the existence of a ν-small set C in the sense of Definition 4.2 is equivalent to strong aperiodicity in the sense of Meyn and Tweedie (1993, Chapter 5), provided that $\nu(C) > 0$. Clearly $\nu(C) > 0$ holds for the previous construction. Strong aperiodicity implies aperiodicity. $\qquad \square$

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CENTER FOR OPERATIONS RESEARCH AND ECONOMETRICS, UNIVERSITÉ CATHOLIQUE DE LOUVAIN, 34 VOIE DU ROMAN PAYS, B-1348 LOUVAIN-LA-NEUVE, BELGIUM