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Optimal Lending Contracts with Long Run Borrowing Constraints

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Abstract

This paper discusses two ways to amend the optimal lending contract under asymmetric information studied in Clementi and Hopenhayn (2006) to change its long-run implications so that firm growth and exit driven by borrowing constraints exist in the long run. One way assumes that the entrepreneur has a lower discount factor than the bank, and the other assumes the bank has limited commitment. The optimal lending contracts under each variation closely resemble each other.

Keywords: Optimal lending contract, Borrowing constraints, Asymmetric information, Limited commitment, Impatient entrepreneur

JEL code: G3, L2, D21

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1 Introduction

A considerable body of empirical evidence suggests that many firms are constrained in their borrowing and borrowing constraints can be important determinants of firm dynamics.[1] Recent attempts to analyse, theoretically, the impact of borrowing constraints on firm dynamics (Quadrini (2004), Albuquerque and Hopenhayn (2004), and Clementi and Hopenhayn (2006)) have used the optimal contract design framework – where borrowing constraints exist as a feature of an optimal long-term lending contract subject to financial frictions such as asymmetric information or limited commitment issues. This approach has been very successful in generating short-to-medium-run implications of borrowing constraints. For example, the optimal lending contract under asymmetric information studied in Clementi and Hopenhayn (2006) (hereafter, CH) implies that smaller firms are more financially constrained, grow faster, but has higher probability of being liquidated. These features are consistent with the qualitative properties of firm growth and survival documented in the empirical literature.[2]

However, all of these models imply that, in the long run, borrowing constraints cease to bind for firms, and hence there is no long run firm growth or exit driven by borrowing constraints. For instance, in CH, the evolution of equity values of the firm (the state variable of the recursive contract) has two absorbing states: either the firm is liquidated, or it grows in finite periods to the point where it reaches its unconstrained efficient size and will never be liquidated. This seems counterfactual: many firms face borrowing constraints even though they are large and have been in existence for long periods of time. Sensitivity of investment to cash flow is widely documented in empirical studies using data for publicly traded firms which are relatively large and mature.[3] In particular, the seminal study of Fazzari, Hubbard and Peterson (1988) finds that this sensitivity is statistically significant even for the group of firms with the largest capital stock, suggesting that large firms could also be financially constrained. Hu and Schiantarelli (1998) even find evidence that larger firms are more likely to be financially constrained.

This paper demonstrates that it is relatively straightforward to amend the CH contract to change its long-run implications so that a non-degenerate stationary distribution of firm sizes (equity values) exists, with borrowing constraints binding in the long run. In particular, two different variations to the CH framework are considered – both of which yield similar results. The first is inspired by Aiyagari and Williamson (1999), in the context of risk sharing with private information, who show that different discount rates for the principal and the agent with the agent being more impatient can generate non-degenerate stationary distributions of expected utilities for the agent. Here, we assume a lower discount factor for the entrepreneur than for the bank and derive an analogous result. The assumption of impatient entrepreneur is also consistent with a higher exit rate generally observed for non-financial firms than for banks. If non-financial firms have higher exit rates than financial firms, they are expected to discount future cash flows more heavily. The second variation relaxes the assumption that

[1]For a survey of this literature, see Hubbard (1998) and Stein (2003).

[2]For empirical studies on firm dynamics, see Evans (1987), Hall (1987), Dunne, Roberts and Samuelson (1989), and Davis, Haltiwanger and Schuh (1996).

[3]See, for example, Fazzari, Hubbard and Peterson (1988), Gilchrist and Himmelberg (1995), Hu and Schiantarelli (1998).

the bank can fully commit. In CH, the entrepreneur has limited commitment in the sense that it has limited liability for repayments, while the bank is able to keep promises under all circumstances. In particular, once the firm grows to the unconstrained stage (this will be achieved in finite periods if the firm is not liquidated at an early stage) the bank will advance the unconstrained efficient amount of working capital to the firm while receiving zero repayment in every period onwards. Although full commitment on the part of the bank (the principal) is standard in the literature, it implies *unlimited* liability of the bank in the context. So in the second variation we allow the possibility that banks can renege on the contract if the expected discounted flow of payments to the bank implied by the contract fall below a critical level. The introduction of this limited commitment, once again, implies a non-degerate stationary distribution of firm sizes.

Intuitively, in CH, it is optimal for the firm to "front load": to repay all revenues to the bank until the equity of the firm grows to the unconstrained stage and, thereafter, to make zero repayments while the bank advances the unconstrained efficient amount of working capital to the firm in every period. (The term "equity" represents the firm's claim to future cash flows that the investment project will deliver, or the firm's stake in the joint venture. It grows as the firm makes repayments to the bank, or in other words, as the firm puts deposits in the bank.) If, however, the entrepreneur is impatient relative to the bank then the interest rate the firm earns on its deposits at the bank is lower than its time preference rate, so that the firm would not find it optimal to accumulate deposits to the level required by the CH contract for reaching the unconstrained stage. Similarly, the limited commitment assumption for the bank is equivalent to putting an upper bound on the firm's deposits at the bank, suggesting that the firm, fearing that the bank may renege, does not want to deposit too much at the bank. Both assumptions prevent the firm from accumulating deposits at the bank to the unconstrained level implied by the CH contract. As a consequence, the optimal lending contract under each variation implies long-run borrowing constraints, driving firm growth and exit in the long run.

Considering that the value functions and the policy functions associated with the two contracts exhibit the same patterns, we conduct a numerical comparison to see whether the two variations are structurally identical. The numerical results show that there are some structural differences in the computed value functions and policy functions, but the differences are modest compared with the similarities. In particular, the two contracts imply roughly identical stationary distributions of equity values for the firm.

This paper contributes to the literature on dynamic contracting in two different ways. First, it formally establishes the result that different discount rates can generate non-degenerate distributions of firm sizes in an otherwise standard CH model. The intuition for this result, as mentioned above, reflects results found in the risk-sharing models of Williamson (1998) and Aiyagari and Williamson (1999, 2000), and in work by Monge-Naranjo (2009), using a continuous time version of the contracting problem with limited commitment of the firm studied in Albequerque and Hopenhayn (2004). This paper is also the first to study the implications of limited commitment on the part of the principal for optimal contract design. Most studies in the literature assume that the agent may or may not fully commit, but the principal has full commitment.[4] This study shows that the introduction of limited commitment on the principal's side can have a significant impact on the long-run properties of optimal contracts.

The remainder of the paper is organized as follows. Sections 2 and 3 characterise the optimal lending contracts under each variation. Section 4 discusses the connections between the two assumptions and presents a numerical comparison between the two contracts. Section 5 concludes. All proofs for the lemmas and propositions are provided in the Technical Appendix.

2 Limited Commitment of the Bank

In this section we characterise the optimal contract with limited commitment of the bank. we start with this modification since the setup of the contracting problem is quite similar to the CH contract such that certain properties of the optimal contract are more straightforward to establish. We first briefly review the contracting environment, then give the recursive formulation of the contracting problem, and finally characterise the properties of the optimal contract.

2.1 The Contracting Environment

Here is a brief review of the contracting environment, as described in CH. Time is discrete, and the time horizon is infinite. At time 0 the entrepreneur (firm) has a project which requires a fixed initial investment $I_0 > 0$ and a per-period investment of working capital. Once in operation, the project is subject to revenue shocks θ_t in each period, where $\theta_t \in \{H, L\}$ with prob$\{\theta = H\} = p$. If $\theta_t = H$, the project produces revenues $R(k_t)$, where k_t is the amount of working capital invested in period t, and revenues are zero if $\theta_t = L$. The function R is assumed to be continuous, uniformly bounded from above, strictly increasing, strictly concave and continously differentiable. Revenue shocks are independent over time, and their realizations in each period are assumed to be private information for the entrepreneur. At the beginning of every period the project can be liquidated to generate a scrap value $S \geq 0$. The entrepreneur's initial wealth is given by $M < I_0$, implying that financial services from a lender (bank) are needed to finance the project. At time 0, the bank offers a take-it-or-leave-it contract to the entrepreneur, under which the bank helps finance the initial investment of the project and per-period working capital in exchange for repayment from the entrepreneur. In every period the entrepreneur is assumed to be liable for repayment only to the extent of current revenues, i.e., the entrepreneur has limited liability for repayments. The entrepreneur and the bank are both risk neutral.

CH discusses properties of the optimal contract under two crucial assumptions: the

[4] For example, Green (1987), Atkeson and Lucas (1992), Spear and Srivastava (1987) and many others assume full commitment for both parties, while Phelan (1995), Krueger and Uhlig (2006), and Albuquerque and Hopenhayn (2004) assume lack of commitment on the agent's side. In fact Phelan (1995) also allows the principal to walk away from the contract at some cost. This can be interpreted as a form of limited commitment for the principal.

entrepreneur and the bank discount future cash flows at the same discount factor $\delta \in (0,1)$, and both parties are able to commit to the long-term contract. This section characterises the optimal contract under the assumption that the bank has limited commitment to the contract. That is, the bank would renege on the contract without punishment if the value of the contract to itself from any period onward falls below some critical level. This critical value, denoted as $\underline{B}$, represents the value of some outside option for the bank. $\underline{B}$ is assumed to be greater than the continuation value of the CH contract to the bank when the firm grows to the unconstrained stage, which is given by $\tilde{B} \equiv -\frac{k^*}{1-\delta}$, where k^* denotes the unconstrained efficient amount of working capital that would be advanced by the bank if the informational friction is absent, i.e., $k^* = \text{argmax}_k \; pR(k) - k$.[5] It is also natural to assume that $\underline{B}$ is less than the scrap value of the project S such that the limited commitment constraint for the bank is satisfied when the bank liquidates the firm to claim the scrap value. These assumptions are summarised in Assumption 1.

Assumption 1 $\tilde{B} < \underline{B} < S$.

2.2 A recursive formulation of the contract

As discussed in CH, the presence of private information gives rise to a long-term contract offered by the bank, which, conditional on the history of reports of the entrepreneur $h^t = (\hat{\theta}_0, \hat{\theta}_1, ..., \hat{\theta}_t)$, specifies a contingent policy of liquidation probabilities $\alpha_t(h^{t-1})$, transfers from the bank to the entrepreneur in case of liquidation $X_t(h^{t-1})$, input of working capital $k_t(h^{t-1})$, and repayment from the entrepreneur to the bank in case of no liquidation $\tau_t(h^t)$ to maximize the value of the contract to the bank (expected discounted net cash flow accrued to the bank).

Without the limited commitment assumption for the bank, the contracting problem possesses a recursive formulation, taking the entrepreneur's value entitlement (the value of the contract to the entrepreneur) at the beginning of a period, V, as the state variable. Following CH, V is also called the equity value of the firm throughout the discussion in the sense that it is the firm's claim to future cash flows or the firm's stake in the investment project. With limited commitment of the bank, the optimal contract must satisfy that the value of the contract to the bank at the beginning of any period is bounded below by $\underline{B}$ conditional on any history of reports of the entrepreneur. Note that the CH contract does not satisfy this constraint: when V reaches $\tilde{V} \equiv \frac{pR(k^*)}{1-\delta}$, the value of the contract to the bank is $\tilde{B}$, strictly less than $\underline{B}$ by Assumption 1. Intuitively, the limited commitment constraint is equivalent to putting an upper bound on V, since the total value of the contract is bounded above by $\tilde{W} \equiv \frac{pR(k^*)-k^*}{1-\delta}$, the maximum expected discounted profit the project can yield. Any value entitlement to the entrepreneur greater than this upper bound cannot be faithfully promised by the bank, since a violation of the limited commitment constraint would lead the bank to renege on the contract. Denote this upper bound on feasible value entitlements to the entrepreneur as $\bar{V}^{\text{LC}}$, where 'LC' stands for 'limited commitment', and take it as given for the

[5] In CH, once the firm reaches the unconstrained stage, the bank advances the unconstrained efficient amount of working capital k^* to the firm while receives zero repayment in every period such that the continuation value of the contract to the bank is given by $-\frac{k^*}{1-\delta}$.

moment. Then the optimal contract has a recursive formulation with $V \leq \bar{V}^{\mathrm{LC}}$ as the state variable. This argument is consistent with Phelan (1995), where he considers an insurance contract between a firm and an agent with privately observed endowments assuming that both parties can walk away from the contract at the beginning of a period (with or without cost), and shows that the two limited commitment constraints can boil down to a restriction on the set of feasible continuation utilities for the agent such that the efficient contract is recursive.

Note that V is bounded below by zero because the limited liability constraint ensures the entrepreneur a non-negative net cash flow in every period. For a given $V \in [0, \bar{V}^{\mathrm{LC}}]$, the bank's problem is to choose the choice variables to maximize $B(V)$, the value of the contract to itself, or equivalently, to maximize the total value of the contract, $W(V) \equiv V + B(V)$. Since both parties have the same discount factor, using $W(V)$ rather than $B(V)$ as the objective function can simplify the formulation. This will not be the case in Section 3, where the two parties are assumed to have different discount factors.

A first decision to be made in a period is whether to liquidate the project, obtaining the scrap value S, or keep it in operation. If the project is not scrapped, the decision at the continuation stage is to choose the amount of working capital, repayment to the bank, and etc. For $V \in [0, \bar{V}^{\mathrm{LC}}]$, a recursive formulation for the liquidation problem is given by

$$
\begin{aligned}
(P_1^{\mathrm{LC}}) \quad W(V) \quad &= \quad \max_{\alpha \in [0,1], X, V_c} \left\{ \alpha S + (1-\alpha)\hat{W}(V_c) \right\} \\
s.t. \qquad & \quad V = \alpha X + (1-\alpha) V_c \qquad\qquad (1) \\
& \quad X \geq 0, V_c \geq 0.
\end{aligned}
$$

Here, α is the liquidation probability. As discussed in CH, a stochastic liquidation is optimal due to the non-convexity introduced by a constant scrap value. X is the transfer from the bank to the entrepreneur in case of liquidation. V_c is the value entitlement to the entrepreneur contingent upon continuation, and $\hat{W}(V_c)$ is the value of the contract at the continuation stage. Eq. (1) is a promise-keeping constraint, stating that the contract delivers an expected value equal to V to the entrepreneur such that the bank's promise to the entrepreneur is fulfilled.

A recursive formulation for the continuation problem is given by

$$
\begin{aligned}
(P_2^{\mathrm{LC}}) \quad \hat{W}(V) \quad &= \quad \max_{k,\tau,V^H,V^L} \left\{ pR(k) - k + \delta\{pW(V^H) + (1-p)W(V^L)\} \right\} \\
s.t. \qquad & \quad V = p(R(k) - \tau) + \delta\{pV^H + (1-p)V^L\} \qquad (2) \\
& \quad \tau \leq \delta(V^H - V^L), \qquad (3) \\
& \quad \tau \leq R(k), \qquad (4) \\
& \quad V^H \leq \bar{V}^{\mathrm{LC}}, \qquad (5) \\
& \quad V^L \leq \bar{V}^{\mathrm{LC}}, \qquad (6) \\
& \quad k, \tau, V^H, V^L \geq 0.
\end{aligned}
$$

Here the state variable V is the value entitlement to the entrepreneur contingent on continuation, k is the amount of working capital advanced by the bank, τ is the repayment to the

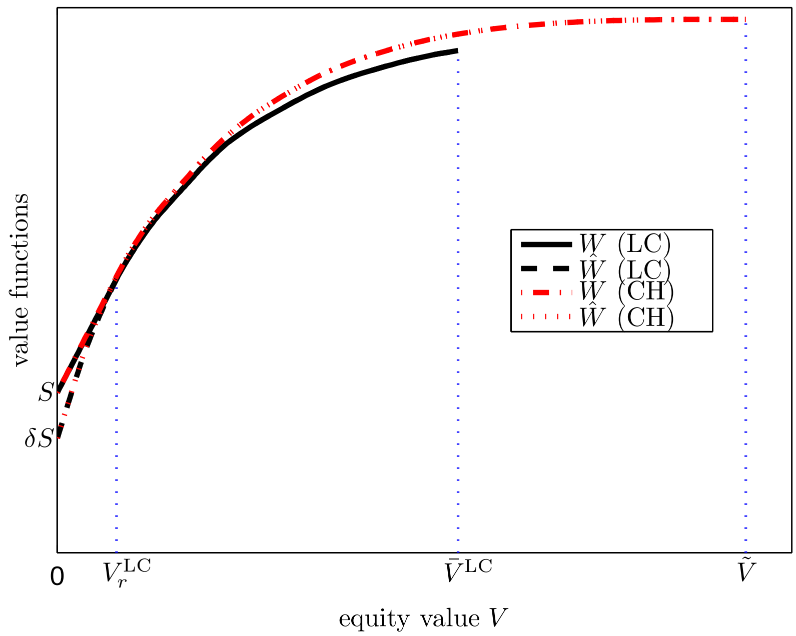

Figure 1: Value functions for the contract with limited commitment

bank if a high revenue shock is reported (repayment is zero if a low shock is reported), and V^H and V^L are the continuation value entitlements to the entrepreneur at the beginning of next period if a high or a low shock is reported respectively. Eq. (2) is the promising-keeping constraint. Eq. (3) is the short version of the incentive compatibility constraint

$$R(k) - \tau + \delta V^H \geq R(k) + \delta V^L.$$

This constraint ensures that the entrepreneur truthfully reports when a high shock is realized. Note that the entrepreneur cannot misreport when a low shock is realized. Eq. (4) is the limited liability constraint for the entrepreneur. Eq. (5) and (6) are imposed to ensure that the limited commitment constraint for the bank is satisfied.

2.3 Properties of the optimal contract

Notice that the contracting problem shares the same recursive formulation as the CH contract except that the limited commitment constraints (5) and (6) are imposed. As shown below, the contract shares a lot of the properties of the CH contact, but due to the limited commitment constraint imposed, it also exhibits some crucial differences.

First, given $\bar{V}^{\mathrm{LC}}$, the value functions $W(V)$ and $\hat{W}(V)$ exhibit identical properties as in CH. That is, there exist unique $W(V)$ and $\hat{W}(V)$ that satisfy (P_1^{LC}) and (P_2^{LC}), and $W(V)$ and $\hat{W}(V)$ are continuous, (weakly) concave and strictly increasing on $[0, \bar{V}^{\mathrm{LC}}]$. However, due to the extra constraints imposed by limited commitment of the bank, $W(V)$ and $\hat{W}(V)$ are lower than the corresponding value functions of the CH contract for all $0 < V \leq \bar{V}^{\mathrm{LC}}$, as illustrated in Fig. 1.[6] Using the properties of the value functions, we can establish the existence and uniqueness of $\bar{V}^{\mathrm{LC}}$, which has been taken as given so far. From the discussions earlier, $\bar{V}^{\mathrm{LC}}$ must satisfy

$$B_{[0,\bar{V}^{\mathrm{LC}}]}(\bar{V}^{\mathrm{LC}}) \equiv W_{[0,\bar{V}^{\mathrm{LC}}]}(\bar{V}^{\mathrm{LC}}) - \bar{V}^{\mathrm{LC}} = \underline{B}, \tag{7}$$

where the subscript $[0, \bar{V}^{\mathrm{LC}}]$ is imposed to highlight the state space associated with the value functions. If $B_{[0,\bar{V}^{\mathrm{LC}}]}(\bar{V}^{\mathrm{LC}}) < \underline{B}$, the limited commitment constraint for the bank is violated. If $B_{[0,\bar{V}^{\mathrm{LC}}]}(\bar{V}^{\mathrm{LC}}) > \underline{B}$, a higher value than $\bar{V}^{\mathrm{LC}}$ can be promised to the entrepreneur, in which case $\bar{V}^{\mathrm{LC}}$ is not an upper bound on feasible value entitlements to the entrepreneur. Assumption 1 implies that if such a $\bar{V}^{\mathrm{LC}}$ exists, it must satisfy $0 < \bar{V}^{\mathrm{LC}} < \tilde{V}$. Exploiting (7), Lemma 1 establishes the existence and uniqueness of $\bar{V}^{\mathrm{LC}} \in (0, \tilde{V})$.

Lemma 1 *There exists a unique $0 < \bar{V}^{LC} < \tilde{V}$ satisfying (7).*

The liquidation problem (P_1^{LC}) shares identical properties as the CH contract. In short, there exists a stochastic liquidation region, $[0, V_r^{\mathrm{LC}}]$, where $0 < V_r^{\mathrm{LC}} \leq \bar{V}^{\mathrm{LC}}$ (Lemma 3 below shows that $V_r^{\mathrm{LC}} < \bar{V}^{\mathrm{LC}}$). For $V \in [0, V_r^{\mathrm{LC}}]$, it is optimal to give the entrepreneur a lottery with values of $X = 0$ in case of liquidation and $V_c = V_r^{\mathrm{LC}}$ in case of continuation. The probability of liquidation, $\alpha(V)$, is linearly decreasing in the entrepreneur's value entitlement on the liquidation region. The total value of the contract on this region is given by a linear combination of S and $\hat{W}(V_r^{\mathrm{LC}})$, with weights $\alpha(V)$ and $1 - \alpha(V)$ respectively. These properties are summarized in Proposition 1.

Proposition 1 *There exists $0 < V_r^{LC} \leq \bar{V}^{LC}$, such that*

(i) $\alpha(V) = 1 - \frac{V}{V_r^{LC}}$ *for* $V \in [0, V_r^{LC}]$, *and* $\alpha(V) = 0$ *for* $V \in [V_r^{LC}, \bar{V}^{LC}]$;

(ii) $X(V) = 0$ *for* $V \in [0, \bar{V}^{LC}]$;

(iii) $V_c(V) = V_r^{LC}$ *for* $V \in [0, V_r^{LC}]$, *and* $V_c(V) = V$ *for* $V \in [V_r^{LC}, \bar{V}^{LC}]$;

(iv) $W(V) = \alpha(V)S + (1 - \alpha(V))\hat{W}(V_r^{LC})$, *for* $V \in [0, V_r^{LC}]$, *and* $W(V) = \hat{W}(V)$ *for* $V \in [V_r^{LC}, \bar{V}^{LC}]$.

It's not made clear in CH, but it's worth noting that V_r^{LC} is the maximum V that satisfies $\hat{W}'(V) = \frac{\hat{W}(V)-S}{V}$. By (i) and (iv) of Proposition 1, $W(V) = S + \frac{\hat{W}(V_r^{\mathrm{LC}})-S}{V_r^{\mathrm{LC}}} V$, or equivalently, $W(V) = S + \hat{W}'(V_r^{\mathrm{LC}})\, V$. So W is linear on $[0, V_r^{\mathrm{LC}}]$ with $W'(V)$ given by $\hat{W}'(V_r^{\mathrm{LC}})$ for $V \in [0, V_r^{\mathrm{LC}}]$. This result will be useful to establish Lemma 3 below.

Now consider problem (P_2^{LC}). By Proposition 1(iii), the state variable V lies in $[V_r^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$ in equilibrium. But we consider a larger state space $[0, \bar{V}^{\mathrm{LC}}]$ to characterise the solution of

[6] (P_1^{LC}) and (P_2^{LC}) are solved numerically and Fig. 1-4 plot the computed value functions and policy functions. The corresponding value functions and policy functions for the CH contract with the same parameterization are also plotted for comparison. The parameter values used to produce Fig. 1-4 is as follows: $\delta = \frac{1}{1.4}$, $\pi = 0.5$, $S = 2.3625 \cdot 10^3$, $\underline{B} = -222.8$, $R(k) = Ak^{\alpha}$, $\alpha = 0.4$, and A is chosen such that $k^* = 1500$. The qualitative features of the contract do not hinge on this specific parameterization.

(P_2^{LC}). First, since W is continuous and concave, and R is continuous and strictly concave, the policy function for capital advancement $k(V)$ is single-valued and continuous by standard arguments of dynamic programming. For a given $V \in [0, \bar{V}^{\mathrm{LC}}]$, if the limited commitment constraints (5) and (6) are not binding, (P_2^{LC}) would have the same formulation as the continuation problem in CH and as a result would share identical properties. The following lemma defines such a region.

Lemma 2 *There exists $0 < V_1^{LC} \leq \bar{V}^{LC}$, such that $V^H(V_1^{LC}) = \bar{V}^{LC}$. For $0 \leq V < V_1^{LC}$, (5) and (6) are not binding.*

It can be shown that $V_r^{\mathrm{LC}} < V_1^{\mathrm{LC}} < \bar{V}^{\mathrm{LC}}$. This relationship will be established after we summarize the properties of the optimal contract on $[0, V_1^{\mathrm{LC}}]$ in Proposition 2.

Proposition 2 *For $V \in [0, V_1^{LC}]$,*

(i) $\tau(V) = R(k(V))$, i.e., the limited liability constraint (4) is binding;

(ii) $k(V) < k^*$, and the incentive compatibility constraint (3) is binding;

(iii) $V^L(V) < V < V^H(V)$ for $V > 0$; $V^H(V)$ is strictly increasing and $V^L(V)$ is non-decreasing.

The repayment policy stated in Part (i) implies zero consumption for the entrepreneur, i.e., a zero dividend policy for the firm as long as V^H is lower than its upper bound $\bar{V}^{\mathrm{LC}}$ (see Fig. 4). This allows the equity value of the firm to reach its upper bound in the shortest possible time. Part (ii) says that the firm is borrowing constrained in the sense that the amount of working capital advanced by the bank is strictly less than its unconstrained efficient level k^*, which would be achieved if the bank can also observe the revenue shocks of the firm (see Fig. 2). Part (iii) implies that the bank promises the entrepreneur a higher beginning-of-next-period value entitlement if a high shock is reported today, and a lower value entitlement if a low shock is reported today (see Fig. 3). Such report-dependent future value entitlements are crucial for inducing a truthful report from the entrepreneur when a high shock is realized. Since V^H is less than $\bar{V}^{\mathrm{LC}}$ on this region, it is strictly increasing. V^L is nondecreasing on this region. Since V^L is bounded below by zero, there might exist a region of V where V^L is zero. All these properties resemble those of the CH contract on the region $[0, \tilde{V})$.[7] In fact, Fig. 1-4 show that for $V \in [0, V_1^{\mathrm{LC}}]$ the value functions and policy functions of the limited commitment contract are almost identical to those of the CH contract.

The following lemma establishes the relationship among V_r^{LC}, V_1^{LC} and $\bar{V}^{\mathrm{LC}}$.

Lemma 3 *$V_r^{LC} < V_1^{LC} < \bar{V}^{LC}$.*

For $V > V_1^{\mathrm{LC}}$, the limited commitment constraint (5) becomes binding. So the optimal contract exhibits different features from CH, which are summarized in Proposition 3.

[7] More precisely, the properties described in Proposition 2 are identical to those of the CH contract on a region defined as $[0, V_1]$, where V_1 is the minimum V that satisfies $V^H(V) = \tilde{V}$. For $V \in (V_1, \tilde{V}]$, $\tau(V) = R(k(V))$ is optimal but not necessary.

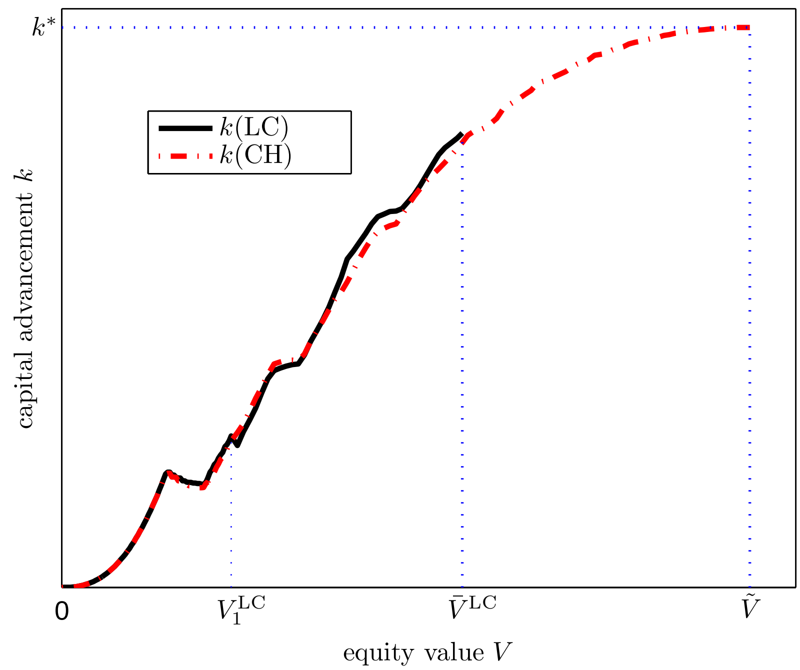

Figure 2: Capital advancement for the contract with limited commitment

Proposition 3 *For* $V \in [V_1^{LC}, \bar{V}^{LC}]$,

(i) $V^H(V) = \bar{V}^{LC}$, *i.e., the limited commit constraint (5) is binding;*

(ii) $k(V) < k^*$, *and the incentive compatibility constraint (3) is binding;*

(iii) $V^L(V) < V$, *and* $V^L(V)$ *is strictly increasing in* V*;*

(iv) $\tau(V) > 0$ *and strictly decreasing in* V*;*

(v) there exists $V_2 \in [V_1^{LC}, \delta\bar{V}^{LC})$ *such that for* $V > V_2$, *the limited liability constraint (4) is not binding, and* $k(V)$ *is non-decreasing.*

Part (i) actually establishes that the limited commitment constraint (5) is binding for $V > V_1^{\mathrm{LC}}$. Part (ii) imply that the borrowing constraints hold throughout the state space $[0, \bar{V}^{\mathrm{LC}}]$, while in CH, borrowing constraints cease to hold when V reaches $\tilde{V}$. This difference is clearly demonstrated in Fig. 2. Part (v) establishes the monotonicity of capital input $k(V)$ on a subset of the state space. Although monotonicity cannot be established throughout the whole state space, it's clear from Fig. 2 that $k(V)$ tends to increase with V, suggesting that borrowing constraints tend to relax as the firm grows bigger.

Part (iii) is the most crucial result that distinguishes the current contract from CH and leads to firm dynamics in the long run. Note that (iii), together with Proposition 2(iii), states that $V^L(V) < V$ for all feasible values of V, including the highest value $\bar{V}^{\mathrm{LC}}$ (see

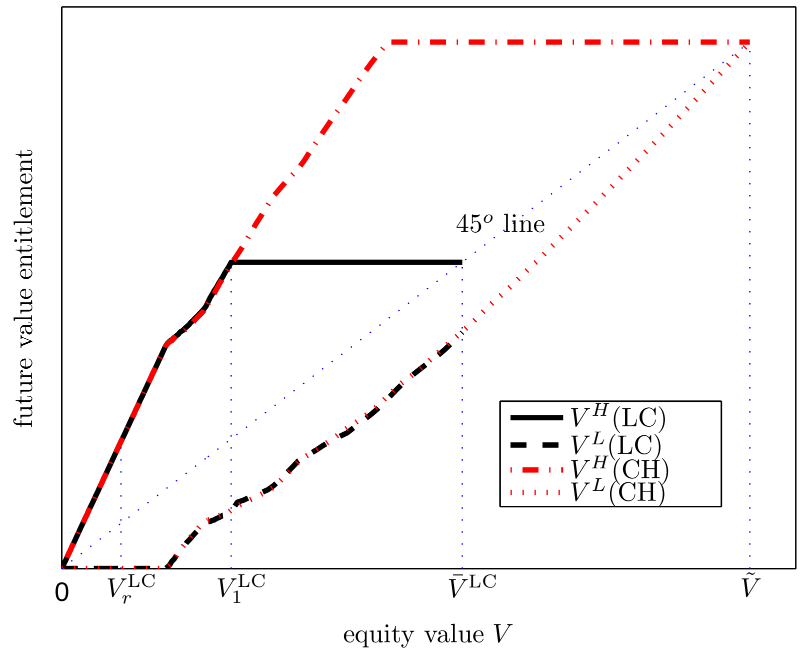

Figure 3: Equity dynamics for the contract with limited commitment

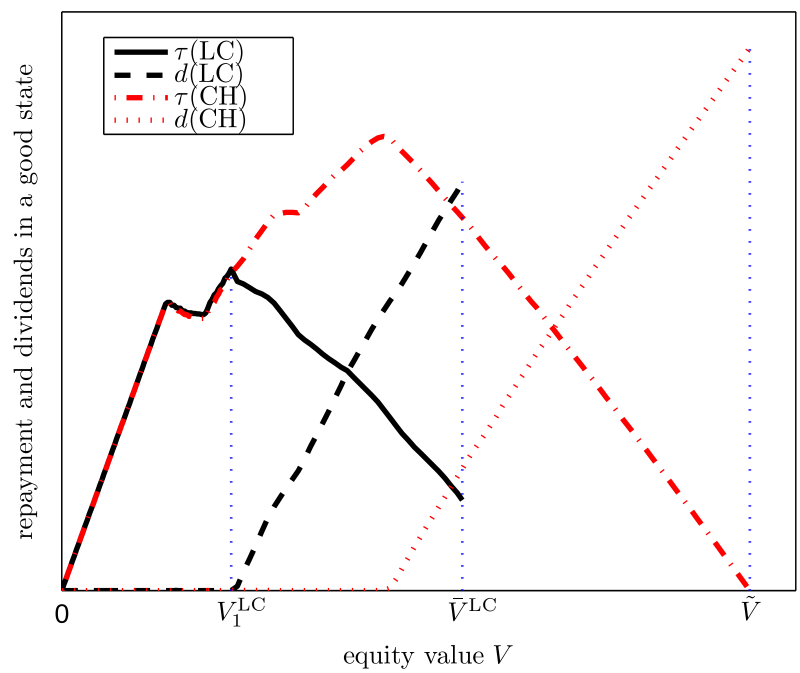

Figure 4: Repayment and dividends for the contract with limited commitment

Fig. 3). This implies that starting from any non-zero level, the equity value of the firm can reach any level in its state space $[0, \bar{V}^{\mathrm{LC}}]$, in particular, it can fall down to the liquidation region $[0, V_r^{\mathrm{LC}}]$ following a sequence of low revenue shocks. As a result, the firm is borrowing constrained throughout its life-cycle and faces a positive probability of being eventually liquidated. Since the policy functions V^H and V^L are single-valued, the equity dynamics dictated by the contract, as shown in Fig. 3, implies a unique stationary distribution of equity values for the firm that exhibit mobility within the ergodic set $[0, \bar{V}^{\mathrm{LC}}]$. Therefore there are endogenous firm exit and firm growth in the long run. This constitutes a crucial difference from CH, where $V^H(\tilde{V}) \geq \tilde{V}$ and $V^L(\tilde{V}) = \tilde{V}$, implying that once a firm's equity value reaches $\tilde{V}$ it will never fall down such that the firm ceases to be borrowing constrained and will never be liquidated.[8] As stated there, the evolution process of equity values has two absorbing states, $V = 0$ and $V \geq \tilde{V}$: "Eventually, either the first one is reached and the firm is liquidated, or the second one is reached and borrowing constraints cease forever". As a consequence, there are no borrowing constraints or firm dynamics in the long run.

The optimal repayment and dividends policies also exhibit important differences from the CH contract. Part (iv), together with Proposition 2(i), implies that the repayment of the firm to the bank when a high shock is reported is always greater than zero as long as $V > 0$. But $\tau(V)$ is strictly decreasing on $[V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$, suggesting that the bank charges a lower repayment from the firm once V^H cannot be further raised to reward the firm for truthful report of a high shock. Part (v) implies that the dividends of the firm $d(V) \equiv R(k(V)) - \tau(V) > 0$ for $V > V_2$, and $d(V)$ is strictly increasing on $[V_2, \bar{V}^{\mathrm{LC}}]$ since $k(V)$ is strictly increasing and $\tau(V)$ is strictly decreasing. So as its equity value reaches some level on $[V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$, the firm starts to pay dividends if a high shock is realized and the amount of dividends is strictly increasing in the firm's equity value. These features are illustrated in Fig. 4. Recall that an optimal repayment policy described in CH is for the firm to transfer all the revenues to the bank and pay zero dividends for $V < \tilde{V}$ and at $\tilde{V}$ the firm pays zero repayment to the bank and all revenues are paid as dividends. This obviously contrasts with the repayment and dividends policies here implied by (iv) and (v). However, this is not the unique optimal repayment policy for the CH contract. Another optimal repayment policy and the corresponding dividends policy are shown in Fig. 4.[9] Note that they share similar patterns as the optimal repayment and dividends policies for the limited commitment contract. That is, the firm transfers all revenues to the bank and pays zero dividends as long as V^H is less than its upper bound, and once V^H reaches its upper bound, the repayment

[8] In Fig. 3, for the CH contract, V^H higher than $\tilde{V}$ is set to $\tilde{V}$ for illustration purpose.

[9] The repayment and dividends policies for the CH contract shown in Fig. 4 are obtained by imposing an upper bound $\tilde{V}$ for the state variable, i.e., imposing a state space $[0, \tilde{V}]$ for the contract. In CH, no upper bound for V is imposed in deriving the properties of the optimal contract despite the fact that the expected discounted cash flow to the firm is naturally bounded by $\tilde{V}$, the maximum expected discounted revenue the project can yield. As a result, the policy functions for V^H and τ are not uniquely determined once V^H reaches $\tilde{V}$. Imposing the state space $[0, \tilde{V}]$ would not change the optimal policy functions for k and V^L for any $V \in [0, \tilde{V}]$, and also would not change V^H and τ as long as $V^H \leq \tilde{V}$ (these functions are uniquely determined in the CH contract), but has the advantage to yield unique V^H and τ once V^H reaches $\tilde{V}$. Since the limited commitment contract has a state space $[0, \bar{V}^{\mathrm{LC}}]$, it seems more reasonable to compare the repayment and dividends policies with the corresponding policies of the CH contract with the state space $[0, \tilde{V}]$.

declines while the dividends increases. However, the important difference remains: for the CH contract, repayment becomes zero once the firm's equity value reaches certain level, while repayment is strictly positive throughout the state space for the limited commitment contract. This feature of the repayment policy ensures that the limited commitment contract delivers a value to the bank not lower than $\underline{B}$.

3 Impatient Entrepreneur

This section characterises the optimal contract assuming that the entrepreneur has a lower discount factor $\delta^e < \delta$, that is, the entrepreneur discounts future cash flows more heavily than the bank or in other words the entrepreneur is more impatient than the bank.

3.1 A recursive formulation

Again, the contracting problem has a recursive formulation, taking the value of the contract to the entrepreneur V as the state variable. An upper bound for V is the maximum expected discounted cash flow the project can deliver for the entrepreneur, $\tilde{V}^{\text{IE}} \equiv \frac{pR(k^*)}{1-\delta^e} < \tilde{V}$, where 'IE' stands for 'impatient entrepreneur'. For a given $V \in [0, \tilde{V}^{\text{IE}}]$, the contracting problem is to specify contract terms to maximize $B(V)$, the value of the contract to the bank.

The recursive formulation for the liquidation problem is then given by

$$
\begin{aligned}
(P_1^{\text{IE}}) \quad B(V) \quad &= \max_{\alpha, X, V_c} \; \alpha(S - X) + (1-\alpha)\hat{B}(V_c) \\
s.t. \quad &V = \alpha X + (1-\alpha)V_c \qquad (8) \\
&\alpha \in [0,1], X \geq 0, V_c \in [0, \tilde{V}^{\text{IE}}].
\end{aligned}
$$

And the recursive formulation for the continuation problem is given by

$$
\begin{aligned}
(P_2^{\text{IE}}) \quad \hat{B}(V) \quad &= \max_{k,\tau,V^H,V^L} \; p\tau - k + \delta[pB(V^H) + (1-p)B(V^L)] \\
s.t. \quad &V = p(R(k) - \tau) + \delta^e[pV^H + (1-p)V^L] \qquad (9) \\
&\tau \leq \delta^e(V^H - V^L) \qquad (10) \\
&\tau \leq R(k) \qquad (11) \\
&k \geq 0, \tau \geq 0, 0 \leq V^H, V^L \leq \tilde{V}^{\text{IE}}.
\end{aligned}
$$

3.2 Properties of the Optimal Contract

First, by standard arguments of dynamic programming, there exist unique $B(V)$ and $\hat{B}(V)$ that satisfy (P_1^{IE}) and (P_2^{IE}). Further, since R is continuous and strictly concave, (P_1^{IE}) and (P_2^{IE}) define a concave dynamic programming such that $B(V)$ and $\hat{B}(V)$ are (weakly) concave, and the policy function $k(V)$ is single-valued and continuous. It's also easy to see that at $V = 0$, $B = S$, $\hat{B} = \delta S$ and $\alpha = 1$, and at $V = \tilde{V}^{\text{IE}}$, $k = k^*$, $\tau = 0$, $V^H = V^L = \tilde{V}^{\text{IE}}$ such that $B \geq \hat{B} \geq \tilde{B}$.

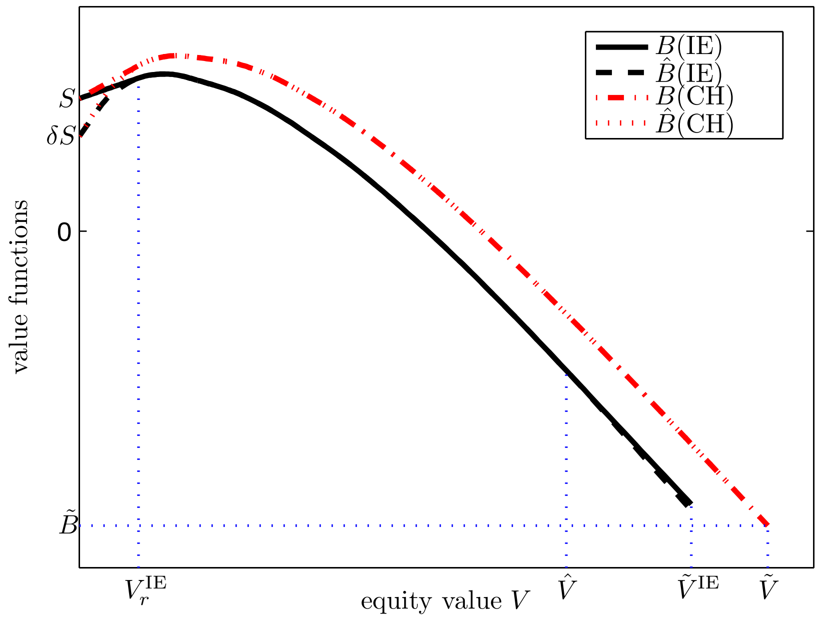

Figure 5: Value functions for the contract with impatient entrepreneur

Fig. 5 plots the value functions. Note that $B(V)$ and $\hat{B}(V)$ are lower than the corresponding functions implied by the CH contract for all $V \in (0, \tilde{V}^{\mathrm{IE}}]$, implying that a lower discount factor for the entrepreneur reduces the value of the contract to the bank and consequently reduces the total value of the contract at all levels of $V > 0$.[10] This property is similar as the limited commitment contract discussed earlier. Also notice from Fig. 5 that there exists a region of V such that the value of the contract to the bank increases with the value of the contract to the firm, so both parties would find it beneficial to renegotiate the contract once the firm's equity value evolves to this region. Therefore like the original CH contract, the contract with impatient entrepreneur is not renegotiation-proof. The limited commitment contract discussed in Section 2 is also not renegotiation-proof, though it's not clearly shown in Fig. 1.[11]

To characterise other properties of the contract, we impose the following assumption.

Assumption 2 $S < \tilde{V}^{IE} + \tilde{B}$.

Assumption 2 implies that $S < \tilde{V}^{\mathrm{IE}} + B(\tilde{V}^{\mathrm{IE}})$, since $B(\tilde{V}^{\mathrm{IE}}) \geq \tilde{B}$. So this assumption is to ensure the scrap value S strictly less than the total value of the contract at $\tilde{V}^{\mathrm{IE}}$ so

[10] Fig. 5-8 plot the computed value functions and policy functions for the contract with the same set of parameters as those producing Fig. 1-4, plus $\delta^e = 0.95 \cdot \delta$. The corresponding functions for the CH contract are also plotted for comparison.

[11] If we plot the value of the contract to the bank in Fig. 1, a similar pattern as in Fig. 5 would be obtained. Renegotiation-proof contracts with repeated moral hazard are studied in Wang (2000) and Quadrini (2004), where renegotiation-proofness is obtained by imposing some lower bound on attainable expected utilities of the agent.

that liquidation in the current period is not optimal when the firm's equity value reaches its upper bound. It is comparable to the assumption in CH that $S < \tilde{W}$, with $\tilde{W}$ being the total value of the contract at $\tilde{V}$.

3.2.1 Characterising the solution to $(P_1^{\mathbf{IE}})$

Similar to the discussions for the limited commitment contract, we will divide the state space into several regions and characterise the solutions to (P_1^{IE}) and (P_2^{IE}) on different regions respectively. To discuss the solution to (P_1^{IE}), we partition the domain of V into three regions (see Fig. 5): (i) a region where liquidation in current period is possible, $0 \leq V \leq V_r^{\mathrm{IE}}$; (ii) a region where there is no liquidation in current period and the total value of the contract contingent upon no liquidation is increasing, $V_r^{\mathrm{IE}} \leq V \leq \hat{V}$, where $\hat{V}$ satisfies $\hat{B}'(\hat{V}) = -1$; and (iii) $\hat{V} < V \leq \tilde{V}^{\mathrm{IE}}$. We first characterise the crucial point $\hat{V}$. Using the solution to (P_2^{IE}) at $\bar{V}^{\mathrm{IE}}$ which is known to us, Lemma 4 shows that $\hat{B}'(\tilde{V}^{\mathrm{IE}}) \leq -1$ such that a $0 < \hat{V} \leq \tilde{V}^{\mathrm{IE}}$ exists. Then Proposition 4 summarizes the properties of the optimal liquidation problem on regions (i) and (ii). Using these properties, Lemma 5 shows that $\hat{V} < \tilde{V}^{\mathrm{IE}}$. Finally, the solution to (P_1^{IE}) on region (iii) is briefly discussed.

Lemma 4 *$\hat{B}'(\tilde{V}^{IE}) \leq -1$.*

Since $\hat{B}'(0) > 0$ and $\hat{B}'(\tilde{V}^{\mathrm{IE}}) \leq -1$, by the concavity of $\hat{B}$ there exists a unique $0 < \hat{V} \leq \tilde{V}^{\mathrm{IE}}$ such that $\hat{B}'(\hat{V}) = -1$ and $\hat{B}'(V) > -1$ for $V < \hat{V}$. Note that the total value of the contract contingent upon continuation, $V + \hat{B}(V)$, is strictly increasing on $[0, \hat{V}]$. As a result, the solution to the liquidation problem (P_1^{IE}) on $[0, \hat{V}]$ shares similar properties as those of the CH contract and the limited commitment contract. Proposition 4 summarizes these properties in the current context.

Proposition 4 *There exists $V_r^{IE} \in (0, \hat{V})$ such that*

(i) $\alpha(V) = 1 - \frac{V}{V_r}$ for $V \in [0, V_r^{IE}]$ and $\alpha(V) = 0$ for $V \in [V_r^{IE}, \hat{V}]$.

(ii) $X(V) = 0$ for $V \in [0, \hat{V}]$.

(iii) $V_c(V) = V_r^{IE}$ for $V \in [0, V_r^{IE}]$ and $V_c(V) = V$ for $V \in [V_r^{IE}, \hat{V}]$.

(iv) $B(V) = S + \frac{\hat{B}(V_r^{IE}) - S}{V_r^{IE}} V$ (or equivalently, $B(V) = S + \hat{B}'(V_r^{IE}) V$) on $[0, V_r^{IE}]$, and $B(V) = \hat{B}(V)$ for $V \in [V_r^{IE}, \hat{V}]$.

(v) $V + B(V)$ and $V + \hat{B}(V)$ are both strictly increasing on $[0, \hat{V}]$.

A proof for Proposition 4 is straightforward and is thus skipped in the Technical Appendix. But it's worth characterising V_r^{IE} in the current context. V_r^{IE} is defined as the maximum V that satisfies $\hat{B}'(V) = \frac{\hat{B}(V) - S}{V}$. Since $\hat{B}(0) < S < \tilde{V}^{\mathrm{IE}} + \hat{B}(\tilde{V}^{\mathrm{IE}})$, where the second '$<$' follows Assumption 2, the concavity of $\hat{B}(V)$ implies that a unique $V_r^{\mathrm{IE}} \in (0, \tilde{V}^{\mathrm{IE}}]$ exists. Note that $\hat{B}'(V_r^{\mathrm{IE}}) \geq \frac{\hat{B}(\tilde{V}^{\mathrm{IE}}) - S}{\tilde{V}^{\mathrm{IE}}} > \frac{\hat{B}(\tilde{V}^{\mathrm{IE}}) - (\tilde{V}^{\mathrm{IE}} + \hat{B}(\tilde{V}^{\mathrm{IE}}))}{\tilde{V}^{\mathrm{IE}}} = -1$, so $V_r^{\mathrm{IE}} < \hat{V}$. Using these properties, it can be shown that $\hat{V}$ is strictly less than $\tilde{V}^{IE}$. This is established in Lemma 5.

Lemma 5 $\hat{V} < \tilde{V}^{IE}$.

For $V \in (\hat{V}, \tilde{V}^{\mathrm{IE}}]$, $\alpha(V)$ is not necessarily zero so that $B(V)$ and $\hat{B}(V)$ are not necessarily equal (see Fig. 5). To see this, consider (P_1^{IE}) for two cases: $\alpha = 0$ and $\alpha > 0$. If $\alpha = 0$, the value of the contract to the bank is $\hat{B}(V)$. If $\alpha > 0$, since $V + \hat{B}(V)$ attains its maximum at $\hat{V}$, setting $V_c = \hat{V}$ is optimal. Then the value to the bank is given by $\alpha S + (1-\alpha)[\hat{V} + \hat{B}(\hat{V})] - V$, which is obtained by substituting out αX using (8). It's clear that if $\hat{V} + \hat{B}(\hat{V}) > V + \hat{B}(V)$ (this only holds with weak inequality by the weak concavity of $\hat{B}$), a small positive α exists such that $\alpha S + (1-\alpha)[\hat{V} + \hat{B}(\hat{V})] - V > \hat{B}(V)$, i.e., zero liquidation probability is not optimal. A sufficient condition for the strict inequality to hold is the strict concavity of $\hat{B}$ on $[\hat{V}, \tilde{V}^{\mathrm{IE}}]$, which is not easy to establish in the current context. However, we will not spend further time on this because the properties of the optimal contract on this region are irrelevant to the long run dynamics of the firm, as will become clear in the subsection below.

3.2.2 Characterising the solution to $(P_2^{\mathbf{IE}})$

In this subsection, we discuss the properties of the optimal capital advancement, repayment policy and equity dynamics by examining (P_2^{IE}). Again, we divide the state space into several regions, and on each region we show that certain constraints of (P_2^{IE}) are binding such that the problem can be reduced and certain properties can be identified.

First, as in Section 2, we want to define a region of V on which the optimal contract shares similar properties as in CH. In particular, the optimal repayment to the bank equals the firm's total revenue in a period, i.e., the limited liability constraint (11) is binding. An investigation of (P_2^{IE}) reveals that as long as V^H is less than some critical value, raising both V^H and the repayment τ can strictly increase the value of the contract to the bank such that the limited liability constraint can be binding. This critical value for V^H, denoted as $\bar{V}^{\mathrm{IE}}$, is defined as the minimum V that satisfies $B'(V) = -\frac{\delta^e}{\delta}$. That is, $B'(\bar{V}^{\mathrm{IE}}) = -\frac{\delta^e}{\delta}$, $B'(V) > -\frac{\delta^e}{\delta}$ for $V < \bar{V}^{\mathrm{IE}}$, and $B'(V) \leq -\frac{\delta^e}{\delta}$ for $V > \bar{V}^{\mathrm{IE}}$. By the definition of V_r^{IE}, $B'(V_r^{\mathrm{IE}}) > 0 > B'(\bar{V}^{\mathrm{IE}})$. It follows from the concavity of B that $\bar{V}^{\mathrm{IE}} > V_r^{\mathrm{IE}}$. On the other hand, by Proposition 4, $B(V) = \hat{B}(V)$ for $V \in [V_r^{\mathrm{IE}}, \hat{V}]$ such that $B'(\hat{V}) = \hat{B}'(\hat{V}) = -1 < -\frac{\delta^e}{\delta}$. So $\bar{V}^{\mathrm{IE}} < \hat{V}$.

Then the threshold of V, denoted as V_1^{IE}, is defined as the minimum V that satisfies $V^H(V) = \bar{V}^{\mathrm{IE}}$. Since $V^H(0) = 0$, $V_1^{\mathrm{IE}} > 0$. For $V \in [0, V_1^{\mathrm{IE}}]$, since both the incentive compatibility constraint (10) and the limited liability constraint (11) are binding, the contract shares similar properties as the limited commitment contract on $[0, V_1^{\mathrm{LC}}]$. Before we formally establish these properties in Proposition 5, we first prove the following result, which will be useful in the proof of Proposition 5.

Lemma 6 $(1-\delta^e)V + (1-\delta)B(V) \leq pR(k^*) - k^*$, *for all* $V \in [0, \bar{V}^{IE}]$.

Lemma 6 is comparable to the result in CH that the total value of the contract under all contingencies is no more than its unconstrained efficient level that would be achieved if the informational friction is absent.[12] This property holds for all V in CH by the monotonicity of

[12] In CH, the total value of the contract with symmetric information is given by $\tilde{W} \equiv \frac{pR(k^*)-k^*}{1-\delta}$, which provides an upper bound for the value of the contract with asymmetric information. That is, $W(V) = V + B(V) \leq \frac{pR(k^*)-k^*}{1-\delta}$ or equivalently $(1-\delta)V + (1-\delta)B(V) \leq pR(k^*) - k^*$ for all V.

W. In the current context, an analogue to this property can be shown to hold for $V \in [0, \bar{V}^{\mathrm{IE}}]$.

The properties of the solution to (P_2^{IE}) on $[0, V_1^{\mathrm{IE}}]$ are summarized in Proposition 5 and demonstrated in the policy functions plotted in Fig. 6-8.

Proposition 5 *For any $V \in [0, V_1^{IE}]$,*

(i) $\tau(V) = R(k(V))$, i.e., the limited liability constraint (11) is binding;

(ii) $k(V) < k^*$, and the incentive compatibility constraint (10) is binding;

(iii) $V^H(V) > V$ for $V > 0$ and $V^H(V)$ is strictly increasing; $V^L(V) < V^H(V)$ for $V > 0$, and $V^L(V)$ is non-decreasing.

A difference from the properties of the limited commitment contract as described in Proposition 2 is that when a low shock is reported today, the entrepreneur's value entitlement for next period is not necessarily lower than today's value entitlement. That is, $V^L(V) < V$ does not hold throughout the region $[0, V_1^{\mathrm{IE}}]$. Due to more heavy discounting, the entrepreneur has an incentive to demand a higher future value entitlement even when a low shock is reported today. This would be also desirable to the bank if an increase in the firm's value entitlement delivers a higher value to the bank, which can happen at lower values of V. The smaller δ^e is relative to δ, the more likely this can be the case.[13]

The following lemma establishes a similar relationship among V_r^{IE}, V_1^{IE}, and $\bar{V}^{\mathrm{IE}}$ as described in Lemma 3.

Lemma 7 *$V_r^{IE} < V_1^{IE} < \bar{V}^{IE}$.*

Since $\bar{V}^{\mathrm{IE}} < \hat{V}$, we have $V_r^{\mathrm{IE}} < V_1^{\mathrm{IE}} < \bar{V}^{\mathrm{IE}} < \hat{V}$. This relationship among the threshold values is clearly demonstrated in Fig. 7.

We have shown in Proposition 5 that the firm is borrowing constrained if its equity value $V \in [0, V_1^{\mathrm{IE}}]$. Then it's natural to ask whether there exists a level of V such that the firm is no longer borrowing constrained when its equity value reaches this level. In CH, this threshold is given by $\tilde{V}$. For the current contract, this value turns out to be the $\hat{V}$ defined earlier, as will become clear below. For $V \in [V_1^{\mathrm{IE}}, \hat{V})$, it can be shown that the incentive compatibility constraint continues to be binding, but the limited liability constraint is not necessarily binding. The properties of the contract on this region are characterised in Proposition 6 and illustrated in Fig. 6-8.

Proposition 6 *For any $V \in [V_1^{IE}, \hat{V})$,*

(i) $k(V) < k^*$, and the incentive compatibility constraint (10) is binding.

(ii) $B'(V^H(V)) = -\frac{\delta^e}{\delta}$; it is optimal to set $V^H(V) = \bar{V}^{IE}$;

(iii) $V^L(V) < V$, and $V^L(V)$ is strictly increasing in V;

(iv) $\tau(V) > 0$ and $\tau(V)$ is strictly decreasing in V.

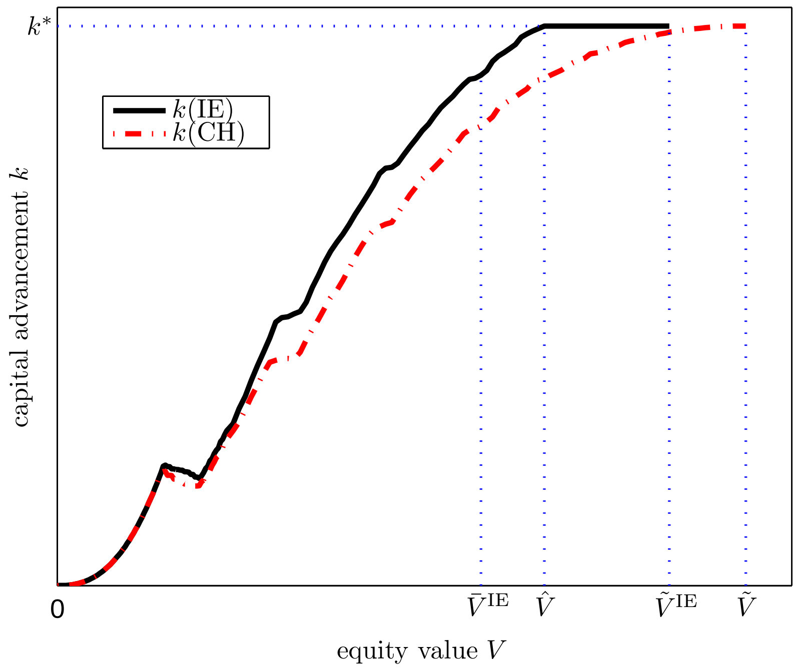

Figure 6: Capital advancement for the contract with impatient entrepreneur

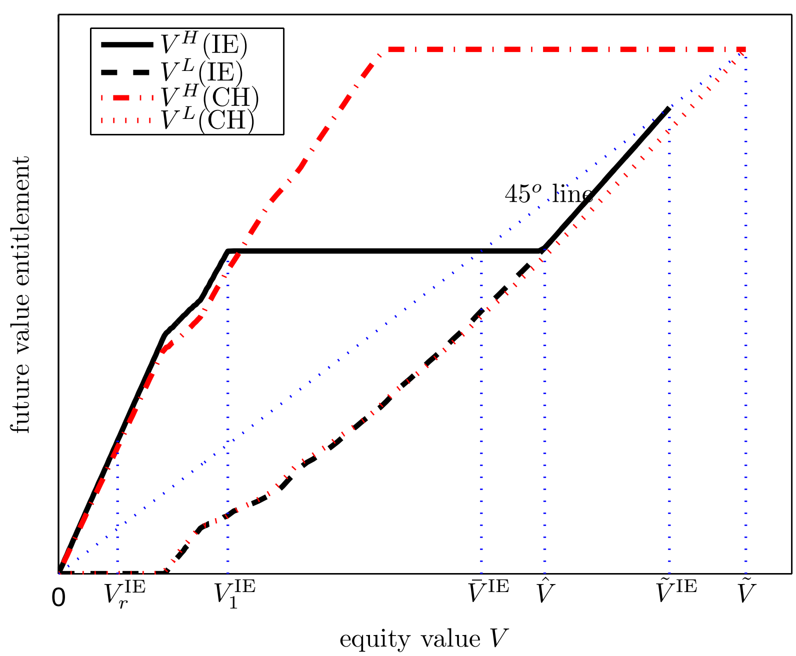

Figure 7: Equity dynamics for the contract with impatient entrepreneur

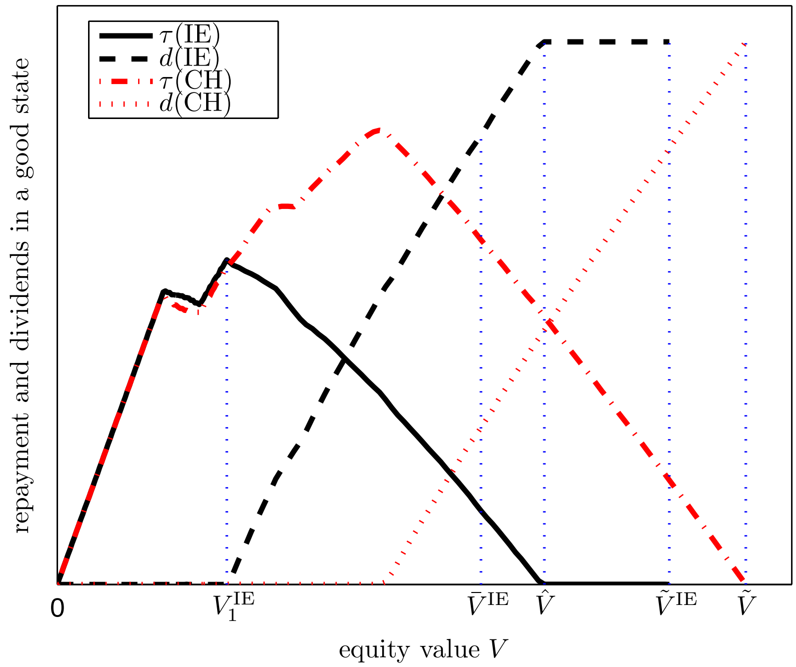

Figure 8: Repayment and dividends for the contract with impatient entrepreneur

It's clear that the properties described in Proposition 6 are very similar to those described in Proposition 3 for the limited commitment contract on $[V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$. In particular, the optimal policy functions for V^H and V^L, as shown in Fig. 7, imply a non-degenerate stationary distribution of equity values. To see this, note that $\bar{V}^{\mathrm{IE}} \leq V^H < \hat{V}$ on $[V_1^{\mathrm{IE}}, \hat{V})$ by (ii), despite the fact that V^H may not be unique on this region.[14] So the V^H curve must intersect the 45 degree line at some $V \in [\bar{V}^{\mathrm{IE}}, \hat{V})$. While the V^H curve need not intersect the 45 degree line uniquely on $[V_1^{\mathrm{IE}}, \hat{V})$, there will be a smallest value of V such that $V^H(V) = V$. For instance, the smallest intersection is $\bar{V}^{\mathrm{IE}}$ if the optimal policy is $V^H = \bar{V}^{\mathrm{IE}}$ on $[V_1^{\mathrm{IE}}, \hat{V})$. In general it may not equal to $\bar{V}^{\mathrm{IE}}$ since the optimal V^H is not necessarily unique. However, to avoid too many notations, we still denote this smallest intersection as $\bar{V}^{\mathrm{IE}}$. For all $V \in [0, \bar{V}^{\mathrm{IE}}]$, $V^H(V) \leq \bar{V}^{\mathrm{IE}}$ by construction, and $V^L(V) < \bar{V}^{\mathrm{IE}}$ by (iii). Therefore $[0, \bar{V}^{\mathrm{IE}}]$ constitutes an ergodic set, and there exists a unique stationary distribution of V on $[0, \bar{V}^{\mathrm{IE}}]$ that exhibits mobility. This argument is consistent with Aiyagari and Williamson (1999), where they study optimal risk sharing under private information and show that, with impatient agent, the optimal contract implies a unique non-degenerate stationary distribution for expected utilities of the agent.

For $V \in [\hat{V}, \tilde{V}^{\mathrm{IE}}]$, $k(V) = k^*$, i.e., the firm ceases to be borrowing constrained. To see this, suppose that $k(V) < k^*$ for some $V \in [\hat{V}, \tilde{V}^{\mathrm{IE}}]$. Then the first-order condition

[13] This result can be seen from equation (21) in the Technical Appendix. Note that with $\delta^e < \delta$, if $\hat{B}'(V) > 0$, it's possible that $B'(V^L(V)) \leq B'(V)$ such that $V^L(V) \geq V$.

[14] Since B is weakly concave, there might exist $\bar{V}^{\mathrm{IE}} < V < \hat{V}$ also satisfying $B'(V) = -\frac{\delta^e}{\delta}$.

for $k(V)$ is given by $pR'(k(V)) = \frac{1}{-\lambda+\gamma/p}$ (see Eq. (22) in the Technical Appendix), where λ and γ are Lagrangian multipliers on (9) and (11) respectively. By the concavity of $\hat{B}$, $\hat{B}'(V) \leq \hat{B}'(\hat{V}) = -1$, so $\lambda = \hat{B}'(V) \leq -1$. Since $\gamma \geq 0$, $pR'(k(V)) \leq 1$. However $k(V) < k^*$ implies that $pR'(k(V)) > 1$. So $k(V) = k^*$ for all $V \in [\hat{V}, \tilde{V}^{\mathrm{IE}}]$. Since the firm is no longer borrowing constrained, by the concavity of B, $V^H(V) = V^L(V)$ is optimal for $V \in [\hat{V}, \tilde{V}^{\mathrm{IE}}]$, implying that $\tau(V) = 0$ on this region (see Fig. 8).

Notice that $\{\tilde{V}^{\mathrm{IE}}\}$ is also an ergodic set, but as long as the initial equity value of the firm is less than $\tilde{V}^{\mathrm{IE}}$, the stationary distribution of equity values implied by the contract will put zero probability on $\tilde{V}^{\mathrm{IE}}$. Note that the initial equity value of the firm V_0 has to satisfy $\hat{B}(V_0) \geq I_0 - M$ for the bank to be willing to enter the contractual relationship with the firm. This participation constraint ensures that $V_0 < \tilde{V}^{\mathrm{IE}}$.[15] Hence the contract with impatient entrepreneur implies a unique stationary distribution of equity values with ergodic set $[0, \bar{V}^{\mathrm{IE}}]$. On $[0, \bar{V}^{\mathrm{IE}}]$, it's clear that the contract with impatient entrepreneur exhibits very similar properties as the limited commitment contract with state space $[0, \bar{V}^{\mathrm{LC}}]$, as established in the propositions and demonstrated in the figures. In particular, endogenous borrowing constraints as well as firm growth and exit exist in the long run.

4 A Comparison between Limited Commitment and Impatient Entrepreneur

From the discussions in Section 2 and 3, we see that the contract with limited commitment of the bank and the contract with impatient entrepreneur share very similar properties. In comparison to the original CH contract, both modifications lead to a downward shift in the value functions; the evolution process of the firm's equity values under both modifications has a unique stationary distribution exhibiting mobility; the firm is financially constrained throughout its life-cycle and has a positive probability of being liquidated eventually; and the firm makes a positive repayment to the bank if a high revenue shock is realized throughout its life-cycle.

In the optimal contracting literature, it is typically assumed that the principal has full commitment in the sense that it can fully honor their contracts under all circumstances. This study is the first to assume limited commitment on the principal's side and explore its implications for optimal contract design. The results discussed earlier show that limited commitment for the bank yields properties that closely resemble those from impatient entrepreneur, suggesting that the two assumptions are closely related. On the one hand, both assumptions prevent the firm from accumulating too much deposits at the bank such that the firm's equity value is bounded above by $\bar{V}^{\mathrm{LC}}$ and $\bar{V}^{\mathrm{IE}}$ respectively in equilibrium. On the other hand, both assumptions put a lower bound on the value of the contract to the bank: limited commitment for the bank explicitly puts a lower bound while impatient entrepreneur implicitly puts a lower bound on the value of the contract to the bank. To see the latter, note from Fig. 5 that the value of the contract to the bank is bounded below by $B(\bar{V}^{\mathrm{IE}})$. For

[15] Note that $\hat{B}(\tilde{V}^{\mathrm{IE}}) = -k^* + \delta B(\tilde{V}^{\mathrm{IE}}) \leq -k^* + \delta(\tilde{W} - \tilde{V}^{\mathrm{IE}}) = -\frac{k^*}{1-\delta} + \delta \left[\frac{1}{1-\delta} - \frac{1}{1-\delta^e}\right] pR(k^*) < 0 < I_0 - M$ if δ^e is not too small relative to δ.

a given lower bound $\underline{B}$, a δ^e can be found such that the contract with impatient entrepreneur implies the same lower bound on value to the bank. Recall that the impatient entrepreneur assumption stems from a lower interest rate on bank deposits than the time preference rate of the firm, where the latter reflects the average return on investment opportunities the firm has access to. So the assumption of impatient entrepreneur is consistent with the empirical observation that bank deposits typically pay a lower return than other investment assets. The close resembleness between these two assumptions imply that the assumption of limited commitment for the bank may be well rooted in the practice of banking.

Considering the similarities and connections discussed above, it's interesting to ask whether these two variations of the original CH contract are structurally identical. That is, for a given $\underline{B}$, whether there exists a δ^e such that the contract with impatient entrepreneur yields identical value functions, policy functions, and stationary distribution of equity values to the contract with limited commitment of the bank, or equivalently, for a given δ^e, whether there exists a $\underline{B}$ such that the limited commitment contract yields identical value functions, policy functions and stationary distribution to the contract with impatient entrepreneur. If not, how do they differ under the two modifications? Since both contracting problems cannot be solved analytically, there are no analytical results to these questions. So a numerical comparison between the two contracts is conducted to give some insights.

To compare the two contracts, we take the contract with impatient entrepreneur that is solved earlier and illustrated in Fig. 5-8 as given. Denote the computed $\bar{V}^{\mathrm{IE}}$ as $\bar{V}$. Then using the same parameterization for the limited commitment contract, we try to find $\underline{B}$ such that the limited commitment contract yields a $\bar{V}^{\mathrm{LC}}$ equal to $\bar{V}$. That is, for a given δ^e, we iterate on $\underline{B}$ such that the limited commitment contract yields the same ergodic set of equity values as the contract with impatient entrepreneur. The computed value functions and policy functions for the two contracts are plotted in Fig. 9-12.

First, Fig. 11 shows that the policy functions for V^H and V^L implied by the two contracts are very close to each other, implying that the two contracts would generate a similar stationary distribution of equity values. However, there does exist minor differences in the equity dynamics. Notice that V^H is a bit higher under the contract with impatient entrepreneur, suggesting that a higher future value entitlement is required to induce truthful report from the entrepreneur due to more heavy discounting by the entrepreneur.

The value functions and other policy functions exhibit more significant differences. Fig. 9 shows that the total value of the contract with impatient entrepreneur is consistently lower than its corresponding value with limited commitment, implying that the value of the contract to the bank is also consistently lower with impatient entrepreneur. The reason underlying this result is clearly shown in Fig 10, where the amount of working capital advanced from the bank is found to be higher under the contract with impatient entrepreneur and the difference is more significant for higher equity values, while the repayment to the bank, as shown in Fig. 12, is not so much different from the limited commitment contract. Fig. 12 also shows that once the firm starts to pay dividends, the dividends under the contract with impatient entrepreneur are consistently higher than dictated by the limited commitment contract. This explains why the capital advancement is higher with impatient entrepreneur. Since the entrepreneur discounts future cash flows more heavily, a higher current consumption (dividends of the firm) is preferred, which requires a larger amount of working capital

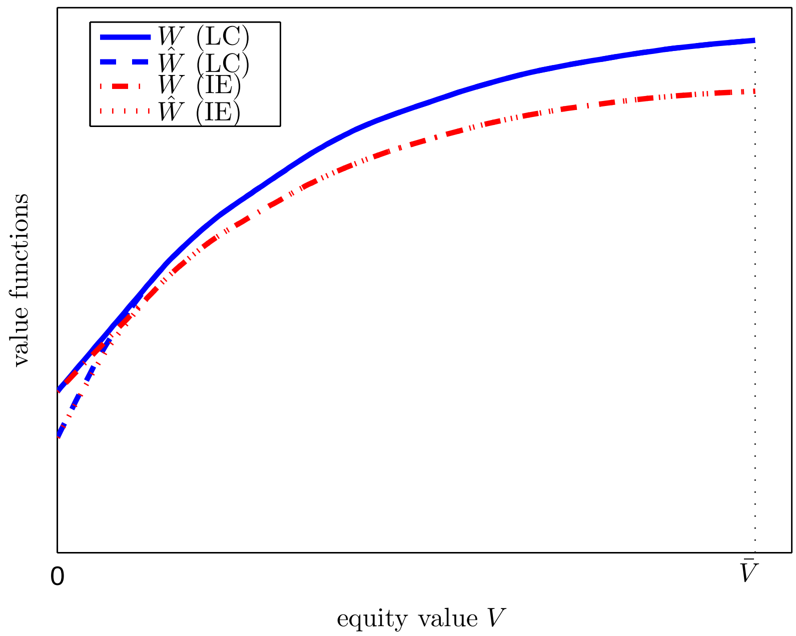

Figure 9: Value functions for the contracts with limited commitment and impatient entrepreneur

to yield higher current revenues.

The numerical results clearly show that the two contracts are not equivalent to each other. There are structural differences in the computed value functions and policy functions. However, as demonstrated in Fig. 9-12, the differences are modest compared with the similarities between the two contracts. A natural subsequence research is to calibrate the two versions of the model and quantitatively evaluate their differences and similarities and the resultant quantitative implications for firm dynamics.[16]

5 Conclusions

The CH model provides a very useful framework for analysing the impact of endogenous borrowing constraints on firm dynamics. While the original version of the model implies the non-existence of borrowing constraints in the long-run, it can be adapted in relatively straightforward ways to remove this implication. This paper illustrates two different ways of doing this, and draws out their implications for the optimal lending contract and the implied

[16] In Quadrini (2004) and Clementi and Hopenhayn (2006), the optimal contract under asymmetric information is parameterized to examine its implications for firm growth and failure in relation to firm size and age. They find that on average the sensitivity of investment to cash flows decreases with firm size and age, firm survival rate increases while exit rate decreases with firm size and age, and the mean and variance of the growth rate decreases with firm size and age. These qualitative properties still hold under both variations studied here.

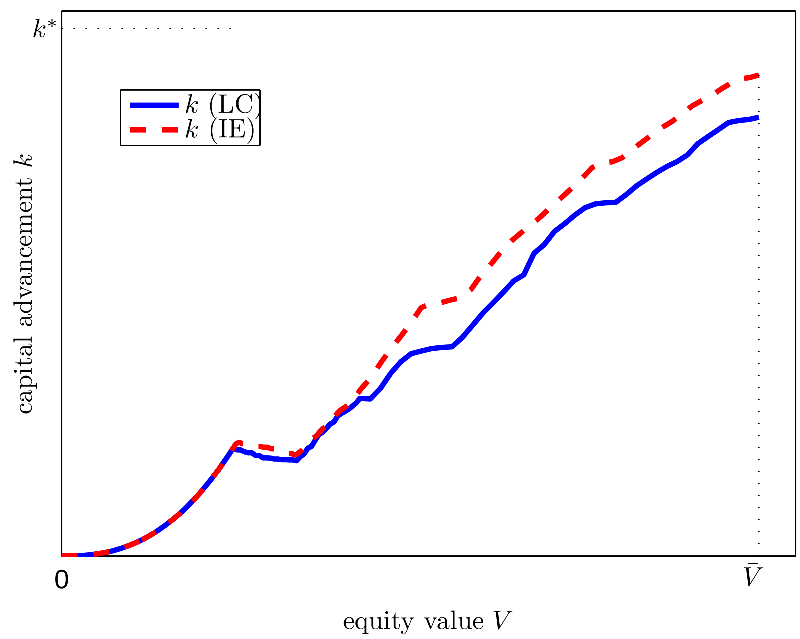

Figure 10: Capital advancement for the contracts with limited commitment and impatient entrepreneur

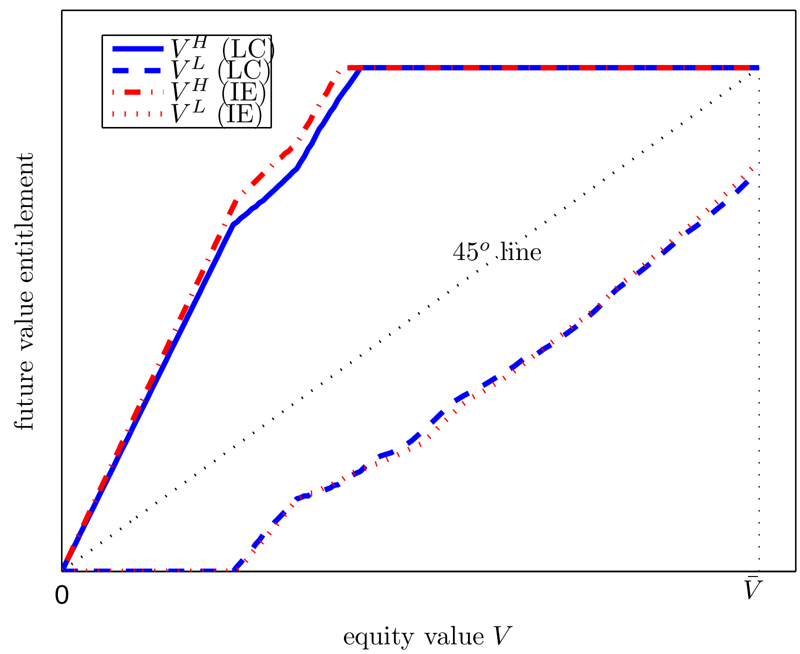

Figure 11: Equity dynamics for the contracts with limited commitment and impatient entrepreneur

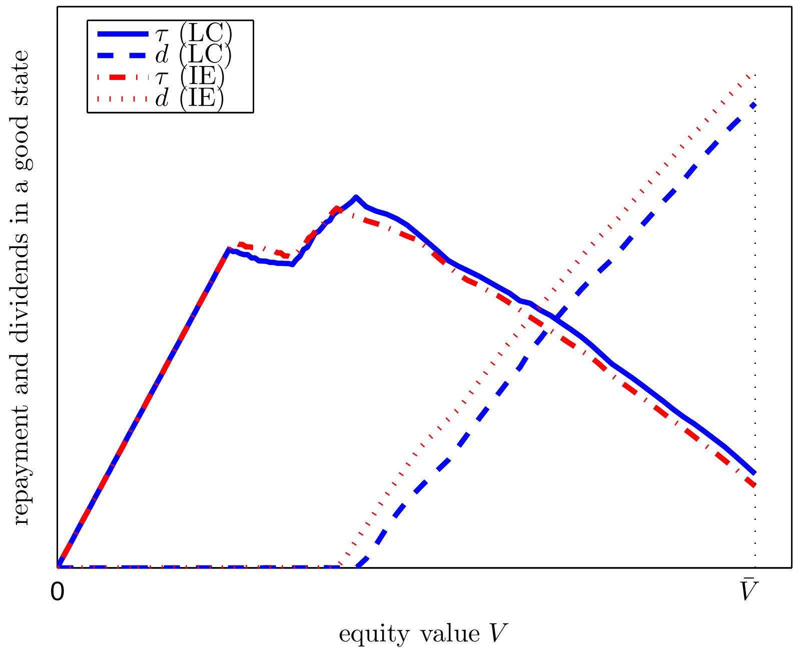

Figure 12: Repayment and dividends for the contracts with limited commitment and impatient entrepreneur

firm dynamics. The implications are quite similar theoretically and numerical comparisons also show them to be close quantitatively. This suggests that a certain amount of robustness exists in the framework.

Since the two modified contracts imply long run borrowing constraints and firm dynamics, they can easily be incorporated into industry or general equilibrium settings to study the quantitative implications of endogenous financing constraints for industry dynamics or the aggregate economy. Existing quantitative studies on the aggregate consequences of endogenous financing constraints include Cooley, Marimon and Quadrini (2004) and Smith and Wang (2006), where the former incorporates an optimal lending contract with limited commitment of firms while the latter incorporates a CH type optimal lending contract into a general equilibrium model. In both studies, exogenous exit of firms is introduced to avoid degeneracy in the long run. This paper provides two alternative methods of avoiding this degeneracy, and it would be interesting to compare their empirical implications.

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Technical Appendix

Proof of Lemma 1 : Consider a sequence of candidates for $\bar{V}^{\text{LC}}$, $\{\bar{V}_0, \bar{V}_1, \bar{V}_2, \ldots\}$. Let $\bar{V}_0 = \tilde{V}$. Then from CH, $W_{[0,\bar{V}_0]}(\bar{V}_0) = \tilde{W}$, so $B_{[0,\bar{V}_0]}(\bar{V}_0) = \tilde{W} - \tilde{V} = \tilde{B} < \underline{B}$ (Assumption 1). Since $B_{[0,\bar{V}_0]}(0) = W_{[0,\bar{V}_0]}(0) - 0 = S > \underline{B}$ (Assumption 1). By the continuity of $B_{[0,\bar{V}_0]}$, there exists the highest point $\bar{V}_1 \in (0, \bar{V}_0)$ such that $B_{[0,\bar{V}_0]}(\bar{V}_1) = \underline{B}$. Since $\bar{V}_1 < \bar{V}_0$, $W_{[0,\bar{V}_1]}(\bar{V}_1) \leq W_{[0,\bar{V}_0]}(\bar{V}_1)$. So $B_{[0,\bar{V}_1]}(\bar{V}_1) \leq B_{[0,\bar{V}_0]}(\bar{V}_1) = \underline{B}$. Again, there exists the highest point $\bar{V}_2 \in (0, \bar{V}_1]$ such that $B_{[0,\bar{V}_1]}(\bar{V}_2) = \underline{B}$. Continuing this process defines a non-increasing sequence $\{\bar{V}_0, \bar{V}_1, \bar{V}_2, \ldots\}$, with $B_{[0,\bar{V}_i]}(\bar{V}_{i+1}) = \underline{B}$. This sequence converges to a unique limit $\bar{V}^{\text{LC}} < \tilde{V}$, which satisfies $B_{[0,\bar{V}^{\text{LC}}]}(\bar{V}^{\text{LC}}) = \underline{B}$. By construction, $B_{[0,V]}(V) < \underline{B}$ for $V > \bar{V}^{\text{LC}}$, i.e, any $V > \bar{V}^{\text{LC}}$ cannot be feasibly promised to the entrepreneur. On the other hand, $B_{[0,\bar{V}^{\text{LC}}]}(V) > \underline{B}$ for $V < \bar{V}^{\text{LC}}$, since $B_{[0,\bar{V}^{\text{LC}}]}(V)$ is concave and $B_{[0,\bar{V}^{\text{LC}}]}(0) = S > \underline{B}$. In other words, any $V < \bar{V}^{\text{LC}}$ can be feasibly promised. So $\bar{V}^{\text{LC}}$ provides a unique upper bound on feasible value entitlements to the entrepreneur. □

Proof of Lemma 2 : First, $V^H(0) = 0 < \bar{V}^{\text{LC}}$. Now we show that $V^H(\bar{V}^{\text{LC}}) = \bar{V}^{\text{LC}}$ by contradiction. Consider (P_2^{LC}) with $V = \bar{V}^{\text{LC}}$. Suppose $V^H(\bar{V}^{\text{LC}}) < \bar{V}^{\text{LC}}$. Then (4) is not binding, otherwise, (2) would imply that $\bar{V}^{\text{LC}} = \delta(pV^H(\bar{V}^{\text{LC}}) + (1-p)V^L(\bar{V}^{\text{LC}})) < \delta\bar{V}^{\text{LC}}$. Now that (4) is not binding, increasing $\tau(V)$ and $V^H(V)$ in a way that keeps all constraints hold can strictly increase the total value of the contract, since W is strictly increasing. This is a contradiction to optimality. So $V^H(\bar{V}^{\text{LC}}) = \bar{V}^{\text{LC}}$. By the continuity of $V^H(V)$, there exists $0 < V_1^{\text{LC}} \leq \bar{V}^{\text{LC}}$ such that $V^H(V_1^{\text{LC}}) = \bar{V}^{\text{LC}}$, and $V^H(V) < \bar{V}^{\text{LC}}$ for $V < V_1^{\text{LC}}$. By (3), $V^L(V) \leq V^H(V)$, so that $V^L(V) < \bar{V}^{\text{LC}}$ for $V < V_1^{\text{LC}}$. □

Proof of Proposition 2 : (i) For $V = 0$, this is obvious. Consider (P_2^{LC}) for an arbitrary $0 < V < V_1^{\text{LC}}$. From Lemma 2, $V^H(V) < \bar{V}^{\text{LC}}$. If $\tau(V) < R(k(V))$, since W is strictly increasing, total value of the contract can be strictly increased by increasing $\tau(V)$ and $V^H(V)$ in a way that keeps all constraints hold. So $\tau(V) = R(k(V))$ for any $V < V_1^{\text{LC}}$. By the continuity of τ and k, the equality also holds for $V = V_1^{\text{LC}}$.

(ii) With (4) binding, $k(V) < k^*$ directly follows the proof of Proposition 2 in CH, and a binding (3) follows the proof of Lemma 2 in CH.

(iii) With (4) binding, $V^L(V) < V < V^H(V)$ follows Proposition 4 in CH, and Proposition 5 there establishes that both $V^H(V)$ and $V^L(V)$ are non-decreasing in V. Actually, we can show that $V^H(V)$ is strictly increasing in the current context. Since both (3) and (4) are binding, (P_2^{LC}) can be rewritten as

$$\begin{aligned} \hat{W}(V) \quad &= \max_{V^H, V^L \geq 0} \left\{ p\delta(V^H - V^L) - R^{-1}\left(\delta(V^H - V^L)\right) + \delta\{pW(V^H) + (1-p)W(V^L)\} \right\} \\ s.t. \quad & V = \delta\{pV^H + (1-p)V^L\}. \end{aligned} \tag{12}$$

The first-order condition for $V^H(V)$ is given by

$$W'(V^H(V)) \leq \hat{W}'(V) - \left[1 - \frac{R^{-1'}[\delta(V^H(V) - V^L(V))]}{p}\right], \text{ with equality if } V^H(V) > 0. \tag{13}$$

Consider $V, V' \in [0, V_1^{\mathrm{LC}}]$, $V < V'$. If $V = 0$, $V^H(V) = 0 < V^H(V')$. Consider $V > 0$, suppose that $V^H(V) \geq V^H(V')$. Then the concavity of W implies that $W^{'}(V^H(V)) \leq W^{'}(V^H(V'))$. Also, (12) implies that $V^L(V) < V^L(V')$, and hence $\delta(V^H(V) - V^L(V)) > \delta(V^H(V') - V^L(V'))$. By the strict convexity of R^{-1},

$$R^{-1^{'}}\left[\delta(V^H(V) - V^L(V))\right] > R^{-1^{'}}\left[\delta(V^H(V') - V^L(V'))\right].$$

It follows from (13) that $\hat{W}'(V) < \hat{W}'(V')$, a contradiction to $\hat{W}$ being concave and $V < V'$. So $V^H(V) < V^H(V')$ for $V < V'$, and thus $V^H(V)$ is strictly increasing on $[0, V_1^{\mathrm{LC}}]$. $\square$

Proof of Lemma 3 : First, by (iii) of Proposition 2, $V^L(V_1^{\mathrm{LC}}) < V^H(V_1^{\mathrm{LC}}) = \bar{V}^{\mathrm{LC}}$. So

$$V_1^{\mathrm{LC}} = \delta\left[p\bar{V}^{\mathrm{LC}} + (1-p)V^L(V_1^{\mathrm{LC}})\right] < \delta\bar{V}^{\mathrm{LC}} < \bar{V}^{\mathrm{LC}}.$$

Now suppose $V_1^{\mathrm{LC}} \leq V_r^{\mathrm{LC}}$. Consider (P_2^{LC}) for $V = V_1^{\mathrm{LC}}$. Since both (3) and (4) are binding, the problem can be reduced to

$$\begin{aligned}
\hat{W}(V) \quad &= \max_{k\geq 0}\{pR(k) - k + \delta\{pW(V^H) + (1-p)W(V^L)\} \\
s.t. \quad & V^H = \frac{V + (1-p)R(k)}{\delta},\ V^L = \frac{V - pR(k)}{\delta} \geq 0.
\end{aligned}$$

Since $0 < k(V_1^{\mathrm{LC}}) < k^*$, it satisfies the first-order condition

$$pR'(k) - 1 + (1-p)pR'(k)[W'(\bar{V}^{\mathrm{LC}}) - W'(V^L)] = 0.$$

Since $pR'[k(V_1^{\mathrm{LC}})] > 1$, $W'(\bar{V}^{\mathrm{LC}}) < W'(V^L(V_1^{\mathrm{LC}}))$. By the Envelope condition,

$$\hat{W}^{'}(V_1^{\mathrm{LC}}) = pW^{'}(\bar{V}^{\mathrm{LC}}) + (1-p)W^{'}(V^L(V_1^{\mathrm{LC}})) < W^{'}(V^L(V_1^{\mathrm{LC}})).$$

Since $V^L(V_1^{\mathrm{LC}}) < V_1^{\mathrm{LC}} \leq V_r^{\mathrm{LC}}$ and W is linear on $[0, V_r^{\mathrm{LC}}]$, $W^{'}(V^L(V_1^{\mathrm{LC}})) = \hat{W}^{'}(V_r^{\mathrm{LC}})$. So we have $\hat{W}^{'}(V_1^{\mathrm{LC}}) < \hat{W}^{'}(V_r^{\mathrm{LC}})$, which contradicts $\hat{W}$ being concave and the assumption that $V_1^{\mathrm{LC}} \leq V_r^{\mathrm{LC}}$. So $V_1^{\mathrm{LC}} > V_r^{\mathrm{LC}}$. $\square$

Proof of Proposition 3 :

(i) By construction, $V^H(V_1^{\mathrm{LC}}) = \bar{V}^{\mathrm{LC}}$, and it has been proved that $V^H(\bar{V}^{\mathrm{LC}}) = \bar{V}^{\mathrm{LC}}$ in the proof of Lemma 2. Now suppose there exists $V_0 \in (V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}})$, such that $V^H(V_0) < \bar{V}^{\mathrm{LC}}$. Then following Proposition 2, (4) and (3) are both binding at V_0. By the arguments in the proof of Proposition 2(iii), $V^H(V_0) > V^H(V_1^{\mathrm{LC}}) = \bar{V}^{\mathrm{LC}}$. So $V^H(V) = \bar{V}^{\mathrm{LC}}$ for all $V \in [V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$.

(ii) This holds at V_1^{LC}. Consider any $V \in (V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$. If $\tau(V) = R(k(V))$, then it follows from the proof of Proposition 2(ii) that $k(V) < k^*$. If $\tau(V) < R(k(V))$, consider two cases. First, if $\tau(V) = 0$, then $V^L(V) = \bar{V}^{\mathrm{LC}}$, and the promising keeping constraint (2) becomes $V = pR(k(V)) + \delta\bar{V}^{\mathrm{LC}}$. It follows that $pR(k(V)) \leq (1-\delta)\bar{V}^{\mathrm{LC}} < (1-\delta)\tilde{V} = pR(k^*)$, and hence $k(V) < k^*$. Second, if $\tau(V) > 0$, then $V^L(V) < \bar{V}^{\mathrm{LC}}$. It follows that if $k(V) = k^*$, a slight decrease in $k(V)$ and a slight increase in $V^L(V)$ can achieve a higher surplus while keeping all constraints in (P_2^{LC}) still hold. Thus, $k(V) < k^*$ for all $V \in [V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$.

With $k < k^*$, (3) is binding on $[V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$, otherwise, a slight increase in k and τ in a way that keeps $R(k) - \tau$ unchanged can strictly increase the total surplus.

(iii) For $V \in [V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$, since (3) is binding, it follows from (2) that $k = R^{-1}\left(\frac{V-\delta V^L}{p}\right)$, and (4) can be rewritten as $\delta\bar{V}^{\mathrm{LC}} - V \le (1-p)R(k)$. So (P_1^{LC}) and (P_2^{LC}) can be reduced to

$$
\begin{aligned}
(P^{LC}) \quad W(V) \quad &= \max_{V^L \in [0, \bar{V}^{\mathrm{LC}}]} \left\{ V - \delta V^L - R^{-1}\left(\frac{V-\delta V^L}{p}\right) + \delta\{pW(\bar{V}^{\mathrm{LC}}) + (1-p)W(V^L)\} \right\} \\
s.t. \quad & \delta\bar{V}^{\mathrm{LC}} - V \le (1-p)\frac{V-\delta V^L}{p}. \qquad (14)
\end{aligned}
$$

The first order condition for V^L is given by

$$
W'(V^L) \ge \frac{1 - R^{-1'}\left(\frac{V-\delta V^L}{p}\right)\frac{1}{p}}{1-p} + \frac{\gamma}{p}, \quad \text{with equality if} \quad V^L < \bar{V}^{\mathrm{LC}}, \qquad (15)
$$

where $\gamma \ge 0$ is the Lagrangian multiplier on (14). By the Envelope condition,

$$
W'(V) = 1 - R^{-1'}\left(\frac{V-\delta V^L}{p}\right)\frac{1}{p} + \frac{\gamma}{p}. \qquad (16)
$$

Note that $V_1^{\mathrm{LC}} = \delta[p\bar{V}^{\mathrm{LC}} + (1-p)V^L(V_1^{\mathrm{LC}})] < \delta\bar{V}^{\mathrm{LC}}$. So V can fall in $[V_1^{\mathrm{LC}}, \delta\bar{V}^{\mathrm{LC}})$ or $[\delta\bar{V}^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$. If $V \in [\delta\bar{V}^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$, (14) is not binding since $k(V) > 0$. Then $\gamma = 0$ and it follows from (15) and (16) that $W'(V^L(V)) \ge \frac{W'(V)}{1-p} > W'(V)$. The concavity of W implies that $V^L(V) < V$, so (15) holds with equality. To show that $V^L(V)$ is strictly increasing on $[\delta\bar{V}^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$, suppose there exist $V < V' \in [\delta\bar{V}^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$ such that $V^L(V) \ge V^L(V')$. Then $V - \delta V^L(V) < V' - \delta V^L(V')$ such that $R^{-1'}\left(\frac{V-\delta V^L(V)}{p}\right) < R^{-1'}\left(\frac{V'-\delta V^L(V')}{p}\right)$. Then it follows from (15) that $W'(V^L(V)) > W'(V^L(V'))$, a contradiction to W being concave and $V^L(V) \ge V^L(V')$.

Now consider $V \in [V_1^{\mathrm{LC}}, \delta\bar{V}^{\mathrm{LC}})$. First it follows from (14) that $V^L(V) \le \frac{V - p\delta\bar{V}^{\mathrm{LC}}}{\delta(1-p)} < \frac{\delta\bar{V}^{\mathrm{LC}} - p\delta\bar{V}^{\mathrm{LC}}}{\delta(1-p)} = \bar{V}^{\mathrm{LC}}$. So (15) holds with equality. It follows from (15) and (16) that

$$
W'(V^L(V)) = \frac{p}{1-p}\left\{1 - R^{-1'}\left(\frac{V-\delta V^L(V)}{p}\right)\frac{1}{p}\right\} + W'(V). \qquad (17)
$$

By (ii), $k(V) < k^*$, so $R^{-1'}\left(\frac{V-\delta V^L(V)}{p}\right) < \frac{1}{R'(k^*)} = p$. Then it follows from (17) that $W'(V^L(V)) > W'(V)$, such that $V^L(V) < V$. Similarly as above, we can show that $V^L(V)$ is strictly increasing on $[V_1^{\mathrm{LC}}, \delta\bar{V}^{\mathrm{LC}})$. In summary, $V^L(V) < V$ and strictly increasing on $[V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$.

(iv) Since the incentive constraint (3) is binding, $\tau(V) = \delta\bar{V}^{\mathrm{LC}} - V^L(V)$. By (iii), $V^L(V) < V \le \bar{V}^{\mathrm{LC}}$, so $\tau(V) > 0$. And since $V^L(V)$ is strictly increasing in V, $\tau(V)$ is strictly decreasing in V.

(v) Consider (P^{LC}) formulated in part (iii) for $V \in [V_1^{\mathrm{LC}}, \bar{V}^{\mathrm{LC}}]$. Note that the limited liability constraint (14) is binding for $V = V_1^{\mathrm{LC}}$, but is not binding for $V \geq \delta\bar{V}^{\mathrm{LC}}$. By continuity, there exists $V_2 \in [V_1^{\mathrm{LC}}, \delta\bar{V}^{\mathrm{LC}})$, such that (14) is binding for $V \in [V_1^{\mathrm{LC}}, V_2]$, and not binding for $V \in (V_2, \bar{V}^{\mathrm{LC}}]$. For $V \in (V_2, \bar{V}^{\mathrm{LC}}]$, since $V^L(V)$ is strictly increasing and W is concave, $W'(V^L(V))$ is non-increasing. It follows from (15) that $R^{-1'}\left(\frac{V-\delta V^L}{p}\right)$ is non-decreasing in V, and hence $k(V) = R^{-1}\left(\frac{V-\delta V^L}{p}\right)$ is non-decreasing in V. $\square$

Proof of Lemma 4 : Consider (P_2^{IE}) for $V = \tilde{V}^{\mathrm{IE}}$, it's clear that $k(\tilde{V}^{\mathrm{IE}}) = k^*$, $\tau(\tilde{V}^{\mathrm{IE}}) = 0$, and $V^H(\tilde{V}^{\mathrm{IE}}) = V^L(\tilde{V}^{\mathrm{IE}}) = \tilde{V}^{\mathrm{IE}}$. Then it follows from the first-order condition with respect to k that $-1 + pR'(k^*)\left[-\lambda + \frac{\gamma}{p}\right] \geq 0$, where λ and $\gamma \geq 0$ are the Lagrangian multipliers on (9) and (11) respectively. Since $\tau(\tilde{V}^{\mathrm{IE}}) < R(k^*)$, $\gamma = 0$. Then $-\lambda \geq \frac{1}{pR'(k^*)} = 1$. By the Envelope condition, $\hat{B}'(\tilde{V}^{\mathrm{IE}}) = \lambda \leq -1$. $\square$

Proof of Lemma 5 : Consider (P_2^{IE}) for $V = \tilde{V}^{\mathrm{IE}}$. Since $V^L = \tilde{V}^{\mathrm{IE}}$, the first-order condition with respect to V^L implies that

$$B'(\tilde{V}^{\mathrm{IE}}) \geq \lambda\frac{\delta^e}{\delta} + \frac{\mu\delta^e}{\delta(1-p)} \geq \lambda\frac{\delta^e}{\delta} = \hat{B}'(\tilde{V}^{\mathrm{IE}})\frac{\delta^e}{\delta}, \tag{18}$$

where λ, $\mu \geq 0$ and $\gamma \geq 0$ are the Lagrangian multipliers on (9), (10), and (11) respectively. If $\hat{V} = \tilde{V}^{\mathrm{IE}}$, by Proposition 4, $B(V) = \hat{B}(V)$ for $V \in [V_r^{\mathrm{IE}}, \tilde{V}^{\mathrm{IE}}]$, and hence $B'(\tilde{V}^{\mathrm{IE}}) = \hat{B}'(\tilde{V}^{\mathrm{IE}}) = -1$. Since $\delta^e < \delta$, it follows from (18) that $-1 \geq -\frac{\delta^e}{\delta} > -1$. $\square$

Proof of Lemma 6 : First, we show that $V^H(\bar{V}^{\mathrm{IE}}) = \bar{V}^{\mathrm{IE}}$ is optimal. If $V^H(\bar{V}^{\mathrm{IE}}) < \bar{V}^{\mathrm{IE}}$, the limited liability constraint (11) must be binding. Otherwise, since $B'(V^H(\bar{V}^{\mathrm{IE}})) > -\frac{\delta^e}{\delta}$, increasing $\tau(\bar{V}^{\mathrm{IE}})$ by ϵ and $V^H(\bar{V}^{\mathrm{IE}})$ by $\frac{\epsilon}{\delta^e}$ would make all constraints still hold, while the change in the objective function is given by $p\epsilon(1 + B'[V^H(\bar{V}^{\mathrm{IE}})]\frac{\delta}{\delta^e})$, which is strictly positive. Then by (9), $\bar{V}^{\mathrm{IE}} = \delta^e\left[pV^H(\bar{V}^{\mathrm{IE}}) + (1-p)V^L(\bar{V}^{\mathrm{IE}})\right] < \bar{V}^{\mathrm{IE}}$. So we must have $V^H(\bar{V}^{\mathrm{IE}}) \geq \bar{V}^{\mathrm{IE}}$ so that $B'(V^H(\bar{V}^{\mathrm{IE}})) \leq -\frac{\delta^e}{\delta}$. If $B'(V^H(\bar{V}^{\mathrm{IE}})) < -\frac{\delta^e}{\delta}$, then slightly lowering $\tau(\bar{V}^{\mathrm{IE}})$ and $V^H(\bar{V}^{\mathrm{IE}})$ can keep all constraints hold while achieve a higher surplus for the bank. So $B'(V^H(\bar{V}^{\mathrm{IE}})) = -\frac{\delta^e}{\delta}$ and it is optimal to set $V^H(\bar{V}^{\mathrm{IE}}) = \bar{V}^{\mathrm{IE}}$. Therefore,

$$\begin{aligned}
\hat{B}(\bar{V}^{\mathrm{IE}}) &= p\tau(\bar{V}^{\mathrm{IE}}) - k(\bar{V}^{\mathrm{IE}}) + \delta\left\{pB(\bar{V}^{\mathrm{IE}}) + (1-p)B(V^L(\bar{V}^{\mathrm{IE}}))\right\} \\
&\leq p\tau(\bar{V}^{\mathrm{IE}}) - k(\bar{V}^{\mathrm{IE}}) + \delta\left\{pB(\bar{V}^{\mathrm{IE}}) + (1-p)\left[B(\bar{V}^{\mathrm{IE}}) + B'(\bar{V}^{\mathrm{IE}})(V^L(\bar{V}^{\mathrm{IE}}) - \bar{V}^{\mathrm{IE}})\right]\right\} \\
&= p\tau(\bar{V}^{\mathrm{IE}}) - k(\bar{V}^{\mathrm{IE}}) + \delta B(\bar{V}^{\mathrm{IE}}) + (1-p)\delta^e[\bar{V}^{\mathrm{IE}} - V^L(\bar{V}^{\mathrm{IE}})] \\
&= \left[pR(k(\bar{V}^{\mathrm{IE}})) - \bar{V}^{\mathrm{IE}} + \delta^e(p\bar{V}^{\mathrm{IE}} + (1-p)V^L(\bar{V}^{\mathrm{IE}}))\right] \\
&\qquad -k(\bar{V}^{\mathrm{IE}}) + \delta B(\bar{V}^{\mathrm{IE}}) + (1-p)\delta^e[\bar{V}^{\mathrm{IE}} - V^L(\bar{V}^{\mathrm{IE}})] \\
&= pR(k(\bar{V}^{\mathrm{IE}})) - k(\bar{V}^{\mathrm{IE}}) - (1-\delta^e)\bar{V}^{\mathrm{IE}} + \delta B(\bar{V}^{\mathrm{IE}}),
\end{aligned}$$

where the "$\leq$" follows from the concavity of B, and the second last "$=$" follows from the promise-keeping constraint (9). Since $V_r^{\mathrm{IE}} < \bar{V}^{\mathrm{IE}} < \hat{V}$, we have

$$B(\bar{V}^{\mathrm{IE}}) = \hat{B}(\bar{V}^{\mathrm{IE}}) \leq pR(k^*) - k^* - (1-\delta^e)\bar{V}^{\mathrm{IE}} + \delta B(\bar{V}^{\mathrm{IE}}),$$

i.e. $(1-\delta^e)\bar{V}^{\mathrm{IE}} + (1-\delta)B(\bar{V}^{\mathrm{IE}}) \leq pR(k^*) - k^*$. $\square$

Proof of Proposition 5 : (i) For $V = 0$, this is obvious. Consider (P_2^{IE}) with an arbitrary $0 < V < V_1^{\mathrm{IE}}$, then $V^H(V) < \bar{V}^{\mathrm{IE}}$ and $B'(V^H(V)) > -\frac{\delta^e}{\delta}$. If $\tau(V) < R(k(V))$, then by similar arguments in the proof of Lemma 6, increasing $\tau(V)$ and $V^H(V)$ can make all constraints still hold while strictly increase the surplus to the bank. So $\tau(V) = R(k(V))$ for $V < V_1^{\mathrm{IE}}$. By continuity, this also holds at V_1^{IE}.

(ii) Suppose there exists $V \in [0, V_1^{\mathrm{IE}}]$ such that $k(V) = k^*$. Then $\tau(V) = R(k^*) > 0$ and $V^H(V) > V^L(V) \geq 0$. If B is strictly concave on $[V^L(V), V^H(V)]$, slightly lowering $k(V)$ (and hence $\tau(V)$) and $V^H(V)$ while raising $V^L(V)$ in a way that keeps all constraints hold can strictly increase the value to the bank. If B is linear on $[V^L(V), V^H(V)]$, it follows from (i) that $pB(V^H) + (1-p)B(V^L) = B(pV^H + (1-p)V^L) = B(\frac{V}{\delta^e})$. It's easy to show that both V, $\frac{V}{\delta^e} \in [V^L(V), V^H(V))$, then the linearity of B implies that

$$B\left(\frac{V}{\delta^e}\right) = B(V) + B'(V)\left(\frac{V}{\delta^e} - V\right) > B(V) - \frac{\delta^e}{\delta}\left(\frac{V}{\delta^e} - V\right).$$

So we have

$$B(V) \geq \hat{B}(V) = pR(k^*) - k^* + \delta B\left(\frac{V}{\delta}\right) > pR(k^*) - k^* + \delta B(V) - V + \delta^e V,$$

i.e. $(1-\delta)B(V) + (1-\delta^e)V > pR(k^*) - k^*$. This violates Lemma 6.

With $k(V) < k^*$ and a binding (11), (10) must be binding, otherwise the surplus to the bank can be strictly increased by slightly increasing τ and k in a way that keeps $R(k) - \tau$ unchanged.

(iii) Binding (10) and (11) imply that $V^H(V) = \frac{V+(1-p)R(k(V))}{\delta^e}$ and $V^L(V) = \frac{V-pR(k(V))}{\delta^e}$. Since $k(V) > 0$ for $V > 0$, it's clear that $V^H(V) > V$ and $V^H(V) > V^L(V)$ for $V > 0$. Now for $V \in (0, V_1^{\mathrm{IE}}]$, rewrite (P_2^{IE}) as

$$\begin{aligned} \hat{B}(V) = & \max_{V^H, V^L \geq 0} \left\{ p\delta^e(V^H - V^L) - R^{-1}\left(\delta^e(V^H - V^L)\right) + \delta\{pB(V^H) + (1-p)B(V^L)\} \right\} \\ & s.t. \quad V = \delta^e\{pV^H + (1-p)V^L\}. \end{aligned} \tag{19}$$

Since $V^H(V) > 0$, $V^L(V) \geq 0$, $V^H(V)$ and $V^L(V)$ satisfies the first order conditions

$$B'(V^H) = \frac{\delta^e}{\delta}\left\{\hat{B}'(V) - \left[1 - \frac{R^{-1'}(\delta^e(V^H - V^L))}{p}\right]\right\}, \tag{20}$$

$$B'(V^L) \leq \frac{\delta^e}{\delta}\left\{\hat{B}'(V) + \frac{p}{1-p}\left[1 - \frac{R^{-1'}(\delta^e(V^H - V^L))}{p}\right]\right\}, \text{with equality if } V^L > 0, \tag{21}$$

To show that $V^H(V)$ is strictly increasing on $[0, V_1^{\mathrm{IE}}]$, suppose there exist $V, V' \in [0, V_1^{\mathrm{IE}}]$, $V < V'$, such that $V^H(V) \geq V^H(V')$. Then $B'(V^H(V)) \leq B'(V^H(V'))$. It follows from (19) that $V^L(V) < V^L(V')$, and hence $\delta^e(V^H(V) - V^L(V)) > \delta^e(V^H(V') - V^L(V'))$. By the strict convexity of R^{-1}, $R^{-1'}\left[\delta^e(V^H(V)-V^L(V))\right] > R^{-1'}\left[\delta^e(V^H(V')-V^L(V'))\right]$. Then (20) implies that $\hat{B}'(V) < \hat{B}'(V')$, a contradiction to $\hat{B}$ being concave and $V < V'$. So $V^H(V) < V^H(V')$ for $V < V'$, i.e., $V^H(V)$ is strictly increasing.

To show that $V^L(V)$ is non-decreasing on $[0, V_1^{\text{IE}}]$, consider $V, V' \in [0, V_1^{\text{IE}}]$, $V < V'$. If $V^L(V) = 0$, then $V^L(V) \le V^L(V')$. If $V^L(V) > 0$, then (21) holds with equality at V. Following the same arguments as above, we can show that a contradiction to the concavity of $\hat{B}$ would obtain if $V^L(V) > V^L(V')$. So $V^L(V) \le V^L(V')$ for $V < V'$, i.e., $V^L(V)$ is nondecreasing. □

Proof of Lemma 7 : First, by (iii) of Proposition 5, $V_1^{\text{IE}} < V^H(V_1^{\text{IE}}) = \bar{V}^{\text{IE}}$. Now suppose $V_1^{\text{IE}} \le V_r^{\text{IE}}$. Consider (P_2^{IE}) for $V = V_1^{\text{IE}}$. Since both (10) and (11) are binding, the problem is reduced to

$$\max_{k\in[0,k^*]} pR(k) - k + \delta\left[pB(V^H) + (1-p)B(V^L)\right],$$

where $V^H = \frac{V+(1-p)R(k)}{\delta^e}, V^L = \frac{V-pR(k)}{\delta^e}$. Since $0 < k(V_1^{\text{IE}}) < k^*$, it satisfies the first order condition

$$pR'(k(V_1^{\text{IE}})) - 1 + \frac{\delta p(1-p)R'(k(V_1^{\text{IE}}))}{\delta^e}[B'(\bar{V}^{\text{IE}}) - B'(V^L(V_1^{\text{IE}}))] = 0.$$

It follows that $B'(\bar{V}^{\text{IE}}) < B'(V^L(V_1^{\text{IE}}))$. By the Envelope condition,

$$\hat{B}'(V_1^{\text{IE}}) = pB'(\bar{V}^{\text{IE}}) + (1-p)B'(V^L(V_1^{\text{IE}})) < B'(V^L(V_1^{\text{IE}})).$$

It can be shown that $V^L(V_1^{\text{IE}}) < V_1^{\text{IE}} \le V_r^{\text{IE}}$, then Proposition 4(iv) implies that $B'(V^L(V_1^{\text{IE}})) = \hat{B}'(V_r^{\text{IE}})$. So we have $\hat{B}'(V_1^{\text{IE}}) < \hat{B}'(V_r^{\text{IE}})$, a contradiction to $\hat{B}$ being concave and the assumption that $V_1^{\text{IE}} \le V_r^{\text{IE}}$. So $V_1^{\text{IE}} > V_r^{\text{IE}}$.

But it's left out to show $V^L(V_1^{\text{IE}}) < V_1^{\text{IE}}$. If $V^L(V_1^{\text{IE}}) = 0$, this is obvious. Consider $V^L(V_1^{\text{IE}}) > 0$. Since $B'[V^H(V_1^{\text{IE}})] = -\frac{\delta^e}{\delta}$, it follows from (20) that $\hat{B}'(V_1^{\text{IE}}) = -\frac{R^{-1'}(\delta^e(V^H-V^L))}{p} < 0$. Then by (21), $B'(V^L(V_1^{\text{IE}})) > \hat{B}'(V_1^{\text{IE}}) \ge B'(V_1^{\text{IE}})$. By the concavity of B, $V^L(V_1^{\text{IE}}) < V_1^{\text{IE}}$. □

Proof of Proposition 6 :

We first write down the first-order conditions for (P_2^{IE}), which will be used intensively in the proof. For any $V > 0$, form the Lagrangian:

$$\begin{aligned} L &= p\tau - k + \delta[pB(V^H) + (1-p)B(V^L)] \\ &\quad +\lambda[V - p(R(k)-\tau) - \delta^e(pV^H + (1-p)V^L)] + \mu[\delta^e(V^H - V^L) - \tau] + \gamma(R(k) - \tau), \end{aligned}$$

where λ, $\mu \ge 0$ and $\gamma \ge 0$ are the Lagrangian multipliers on (9), (10), and (11) respectively.

Then the first-order conditions are given by

$$\frac{\partial L}{\partial k} = -1 - \lambda p R'(k) + \gamma R'(k) = -1 + pR'(k)\left[-\lambda + \frac{\gamma}{p}\right] \geq 0, = 0 \text{ if } k < k^*, \quad (22)$$

$$\frac{\partial L}{\partial \tau} = p + \lambda p - \mu - \gamma \leq 0, = 0 \text{ if } \tau > 0, \quad (23)$$

$$\frac{\partial L}{\partial V^H} = \delta p B'(V^H) - \lambda \delta^e p + \mu \delta^e \geq 0, = 0 \text{ if } V^H < \tilde{V}, \quad (24)$$

$$\frac{\partial L}{\partial V^L} = \delta(1-p)B'(V^L) - \lambda\delta^e(1-p) - \mu\delta^e \leq 0 \text{ if } V^L = 0, = 0 \text{ if } V^L \in (0, \tilde{V}), \geq 0 \text{ if } V^L = \tilde{V}, \quad (25)$$

$$\mu[\delta^e(V^H - V^L) - \tau] = 0, \mu \geq 0,$$
$$\gamma[R(k) - \tau] = 0, \gamma \geq 0.$$

(i) Consider two cases: the limited liability constraint (11) is binding or not. If (11) is binding, then following the proof of Proposition 5(ii), $k(V) < k^*$. If (11) is not binding, then $\gamma = 0$ in (22), i.e., $k(V)$ satisfies $-1 + pR'(k(V))(-\lambda) \geq 0$. By the Envelope condition, $\lambda = \hat{B}'(V) > -1$, since $V < \hat{V}$. So $pR'(k(V)) > 1$ such that $k(V) < k^*$.

With $k(V) < k^*$, it can be shown that (10) is binding. Suppose not. Then if (11) is binding, the surplus to the bank can be strictly increased by slightly increasing τ and k in a way that keeps $R(k) - \tau$ unchanged. If (11) is not binding, since $V^H > V^L$ and B is concave, increasing τ and V^L while decreasing V^H can strictly increase the bank's surplus with all constraints still hold.

(ii) First, by the definition of V_1^{IE}, $V^H(V) \geq \bar{V}^{\text{IE}}$ for $V \geq V_1^{\text{IE}}$. It then follows that $B'(V^H(V)) \leq -\frac{\delta^e}{\delta}$. If $B'(V^H(V)) < -\frac{\delta^e}{\delta}$, then slightly lowering $\tau(V)$ and $V^H(V)$ can keep all constraints hold while achieve a higher surplus for the bank. So $B'(V^H(V)) = -\frac{\delta^e}{\delta}$ for all $V \in [V_1^{\text{IE}}, \hat{V})$. Since $B'(\bar{V}^{\text{IE}}) = -\frac{\delta^e}{\delta}$, it is optimal to set $V^H(V) = \bar{V}^{\text{IE}}$ for all $V \in [V_1^{\text{IE}}, \hat{V})$.

(iii) We first show that $V^L(V) < V$ for $V \in (V_1^{\text{IE}}, \hat{V})$ (this holds at V_1^{IE}). By (24) and (25),

$$B'(V^H(V)) = \lambda\frac{\delta^e}{\delta} - \frac{\delta^e}{\delta}\frac{\mu}{p}, \quad B'(V^L(V)) = \lambda\frac{\delta^e}{\delta} + \frac{\delta^e}{\delta}\frac{\mu}{1-p}.$$

If $\mu = 0$, $B'(V^L(V)) = B'(V^H(V)) = -\frac{\delta^e}{\delta}$ such that $\lambda = -1$. Since $k(V) < k^*$, by (22), $pR'(k(V)) = \frac{1}{-\lambda + \frac{\gamma}{p}} = \frac{1}{1+\frac{\gamma}{p}} \leq 1$. However, $k(V) < k^*$ implies that $pR'(k(V)) > 1$. So $\mu > 0$, then $B'(V^L(V)) > \lambda\frac{\delta^e}{\delta}$. By the Envelope condition, $\lambda = \hat{B}'(V) = B'(V)$, where the latter '=' holds since $B(V) = \hat{B}(V)$ on $[V_1^{\text{IE}}, \hat{V})$. From the proof of Lemma 6, $\hat{B}'(V_1^{\text{IE}}) < 0$, so $B'(V) < 0$ for $V \in (V_1^{\text{IE}}, \hat{V})$. Thus $B'(V^L(V)) > B'(V)\frac{\delta^e}{\delta} > B'(V)$ and hence $V^L(V) < V$.

Next, we show that $V^L(V)$ is strictly increasing with V. For any $V \in [V_1^{\text{IE}}, \hat{V}]$, since $\tau(V) = \delta^e(V^H(V) - V^L(V)) > 0$, (23) implies that $\mu = p\left[1 + \lambda - \frac{\gamma}{p}\right]$. By (22), $\lambda - \frac{\gamma}{p} = -\frac{1}{pR'(k(V))}$. So it follows from (25) that

$$B'(V^L(V)) = \lambda\frac{\delta^e}{\delta} + \frac{\delta^e}{\delta}\frac{\mu}{1-p} = \frac{\delta^e}{\delta}\left\{B'(V) + \frac{p}{1-p}\left[1 - \frac{1}{pR'(k(V))}\right]\right\}, \quad (26)$$

where $k(V) = R^{-1}\left(\frac{V-\delta^e V^L(V)}{p}\right)$ by (9) and (10). Suppose there exist $V < V'$ such that $V^L(V) \geq V^L(V')$. It follows that $B'(V) \geq B'(V')$ and $k(V) < k(V')$. Since R is strictly concave, $R'(k(V)) > R'(k(V'))$. Then (26) implies that $B'(V^L(V)) > B'(V^L(V'))$, a contradiction to B being concave and $V^L(V) \geq V^L(V')$.

(iv) Since (10) is binding, $\tau(V) = \delta(V^H(V) - V^L(V)) > 0$. Since $V^H(V)$ is independent of V and $V^L(V)$ is strictly increasing in V, $\tau(V)$ is strictly decreasing on $[V_1^{\mathrm{IE}}, \hat{V})$. $\square$