THE UNIVERSITY OF MELBOURNE

Department of Economics

Working Paper Series

The Employed, the Unemployed, and the Unemployable: Directed Search with Worker Heterogeneity

Suren Basov, Ian King and
Lawrence Uren

March 2010

Research Paper Number 1094

ISSN: 0819-2642
ISBN: 978 0 7340 4447 1

Department of Economics
The University of Melbourne
Parkville VIC 3010
www.economics.unimelb.edu.au

The Employed, the Unemployed, and the Unemployable: Directed Search with Worker Heterogeneity*

Suren Basov Ian King Lawrence Uren†

March 15, 2010

Abstract

We examine the implications of worker heterogeneity on the equilibrium matching process, using a directed search model. Worker abilities are selected from a general distribution, subject to some weak regularity requirements, and the firms direct their job offers to workers. We identify conditions under which some fraction of the workforce will be "unemployable": no firm will approach them even though they offer positive surplus. For large markets we derive a simple closed form expression for the equilibrum matching function. This function has constant returns to scale and two new terms, which are functions of the underlying distribution of worker productivities: the percentage of unemployable workers, and a measure of heterogeneity (κ).The equilibrium unemployment rate is increasing in κ and, under certain circumstances, is increasing in the productivity of highly skilled workers, despite endogenous entry. A key empirical prediction of the theory is that $\kappa \geq 1$. We examine this prediction, using data from several countries.

JEL Codes: C78, J41, J64.

Key words: Directed search, worker heterogeneity, unemployment.

*We would like to thank Jim Albrecht, Chris Edmond, John Quiggin, and Susan Vroman for helpful discussions, and participants in seminars at the University of Melbourne, and Queensland University. The usual disclaimer applies.

†Suren Basov: La Trobe University, Ian King and Lawrence Uren: The University of Melbourne.

1 Introduction

The distribution of worker productivities is a key factor in determining the outcomes of the labor market, and macroeconomies more generally. In a general sense, it is well known that this distribution can affect the matching process, the unemployment rate, the types of unemployment observed, and the distribution of wages received by workers. To understand these relationships as clearly as possible, though, a theory of the labor market is required which places worker heterogeneity at the heart of the analysis, and which draws out the implications of this heterogeneity as equilibrium phenomena.

Directed search theory is a natural candidate for this type of analysis because rational selection across a variety of alternatives is its core feature. The location choice, in directed search settings, is made by buyers, who choose across sellers. The central insight of directed search was identified in Peters (1984): buyers, when considering which seller to approach, trade off two factors – the payoff from the good, and the probability of successful purchase. Thus, sellers that offer higher payoffs will typically attract more buyers and a lower probability, from the point of view of buyers, of a successful purchase. When applied to the labor market, the implications of this insight depend on which side of the market is interpreted as buyer and seller. For example, Montgomery (1991), and Burdett, Shi, and Wright (2001), modelled firms as sellers, and workers as buyers of jobs; whereas Julien, Kennes, and King (2000) modelled the market the other way, with workers interpreted as sellers, and firms as buyers, of labor. Thus, when focussing on the implications of worker (rather than firm) heterogeneity, it is most natural to use the latter approach, where firms choose across a distribution of worker types.[1]

One of the more attractive features of directed search theories of the labor market is that, in their simplest forms, they produce equilibrium matching processes that have many of the characteristics of the matching functions that are commonly supported by empirical work: matches are increasing in vacancies and unemployment, concave, and (with large markets) have constant returns to scale (Julien, Kennes, and King (2000)). In models with homogeneous workers, of this sort, however, this equilibrium matching process has no parameters in it – a fact

[1] In the simplest case with heterogeneous workers and homogeneous firms, if workers act as buyers and choose which firms to approach, then heterogeneity plays no role at all in the matching process. See Julien, Kennes, and King (2005) for further details.

that weakens its performance empirically, relative to, for example, Cobb-Douglas matching functions (Petrongolo and Pissarides (2001)). Directed search models do exist now that have parameters in the implied equilibrium matching processes (see, for example, Albrecht *et al* (2004), where the number of applications plays this role), and worker heterogeneity can play a role here, at least in principle [2]

In this paper, we examine the implications of worker heterogeneity using a directed search model based on Julien, Kennes, and King (2000), but with a non-degenerate distribution of worker productivities. We characterize the unique symmetric mixed strategy equilibrium of this model, under different of assumptions about the nature of the distribution of worker types. We start with the simplest case: binomial distributions. We then go on to consider arbirtrary distributions with continuous supports. In both cases, we characterize the nature of the equilibrium and draw out comparative static properties, as the underlying distribution of worker types changes. We also provide simple closed form solutions for the equilibrium matching process, in both cases, where the matching process is a function of statistics from the underlying distribution of worker productivities. The performance of this matching process is then assessed, using data from the United States and other countries.

We find that, for each different distribution of worker types, we can identify two different classes of workers: those who will be approached with positive probability, and those who will not. Thus, we identify the set of "unemployable" workers: those who will never be approached by firms, even though they have positive surplus. The intuition for their existence, in equilibrium, follows directly from the central insight of directed search, mentioned above. When firms choose which workers to approach, they are influenced not only by the potential productivity a worker offers, but also by the probability of being able to hire the worker – the expected payoff in equilibrium. With arbitrary distributions of worker types, in principle, one can identify cut-off values of worker types (at the lower end) where firms would prefer to take a chance on being able to hire more productive workers, even if they

[2] Petrongolo and Pissarides (2001) suggest a variety of ways in which parameters could be introduced into urn-ball matching functions, which are the mathematical form that many equilibrium matching processes in directed search models take. Although their analysis is not, strictly speaking, directed search, it is very suggestive about the effects of heterogeneity in directed search. See, also, Stevens (2007), for an alternative approach to providing microfoundations for the matching function.

could hire the lower productivity ones with probability one. Thus, the presence of high productivity workers introduces the possibility that some low productivity workers will simply never be offered jobs. Moreover, increasing the productivity of the best workers may, *ceteris paribus*, increase the fraction of workers at the lower end that are unemployable in equilibrium and increase the unemployment rate more generally.

We also identify three conceptually distinct effects that heterogeneity has on the equilibrium matching in the model. The first is the *decreased employability effect*, which arises due to the potential existence of unemployables introduced by heterogeneity. The second is the *increased miscoordination effect*. Minimization of the unemployment rate requires that each firm assign equal probability to approaching each worker. This occurs in the equilibrium with homogeneous workers, but not with heterogeneous ones, where more productive workers are assigned higher probabilities (However, the equilibrium we consider here is constrained-efficient in the sense that it maximizes expected total surplus.) The third effect is the *decreased effective labor force effect*: the probability of being employed is higher for those workers who fall into the category of being "employable", with heterogeneity. However, we show that, overall, heterogeneity increases the equilibrium unemployment rate.

The implied matching function has two terms in it that are functions of the underlying distribution of worker productivities. The first is the fraction of the workforce that is unemployable, q, and the second is a measure of heterogeneity, κ, that we construct. This second measure, according to the theory, must be weakly greater than one (the lower bound, with homogeneous workers). We examine whether or not this holds true in the data by considering two different empirical exercises. In the first exercise, we treat κ as a variable and use the matching function to generate a series $\{\kappa_t\}$. In the second exercise, we treat κ as a parameter and estimate its implied value by estimating the matching function in the way that is standard in the literature. The first method provides unambiguous support for the theory, while the second provides mixed support.

The remainder of the paper is organized as follows. Section 2 introduces the model with finite numbers of agents. In Section 3, we consider the case with arbitrarily large numbers of workers and firms, both with a fixed ratio of the two and with endogenous entry and derive most of the theoretical results. Section

4 identifies constrained efficient allocations in the model. Section 5 presents the empirical results. Concluding remarks are given in Section 6.

2 The Model with Finite Numbers of Agents

Consider a model where $M \in \mathbb{N}$ homogenous vacancies choose across $N \in \mathbb{N}$ heterogenous workers, and where each worker $i = 1, 2, ..., N$ has productivity y_i and $y_1 \geq y_2 \geq ... \geq y_N > 0$. Assume that firms choose whether to approach a particular worker after observing the vector of workers productivities.[3] Assume, further, that each firm has only one vacancy and can approach only one worker to fill it. If a firm is alone when it approaches worker i it makes a take it or leave it offer to the worker and captures all the surplus from the relationship. If several firms approach the same worker they Bertrand compete for her services. In this case, the worker's wage will be equal to her productivity and the firm will capture zero surplus.[4]

In a symmetric mixed strategy Nash equilibrium worker i is visited with the same probability π_i by all firms. For a firm, in this equilibrium, the expected payoff from visiting worker i is therefore the probability of being alone when approaching that worker, multiplied by the worker's productivity: $(1-\pi_i)^{M-1}y_i$. If both workers

[3] Thus, as in Julien, Kennes, and King (2000), this is a model with multiple job applications, where each worker applies to all firms, and firms can perfectly observe each worker's productivity level. See Albrecht, Gautier, and Vroman (2006), Galenianos and Kircher (2009), and Kircher (2009) for analyses of models where workers apply only to a fraction of firms.

[4] Julien, Kennes, and King (2006)) show that this wage determination mechanism is equivalent to the "Mortensen rule" (Mortensen (1982)), and can be thought of as the outcome of an auction through which workers sell their labor, and where they set their reserve wage equal to their outside option. In this static game, the outside option is normalized to zero. Julien, Kennes, and King (2000) show that the Nash equilibrium reserve wage converges to the outside option as the market becomes large.

We chose this mechanism because of its simplicity: the value of the wage rate is easy to find (either zero or y). However, due to the starkness of the induced wage distribution, this may lead one to question the realism of this distribution. As a check, we also considered an alternative mechanism: generalized Nash bargaining, where, after firms were assigned to workers, each worker (with multiple firms assigned) bargains sequentially with the assigned firms. This leads to a more realistic wage distribution (see Camera and Selcuk (2009) for a discussion of similar mechanisms) but is significantly more cumbersome to work with. In this paper, the emphasis is more on the properties of the equilibrium matching process than on the wage distribution *per se*. In large markets (the focus of the analysis in this paper) the matching processes for the two different wage determination mechanisms converge. Details of the comparison are available upon request.

i and j are visited with positive probabilities in the equilibrium then the firms should earn the same expected payoff from visiting the workers, i.e.

$$(1-\pi_i)^{M-1} y_i = (1-\pi_j)^{M-1} y_j. \tag{1}$$

Equation (1) implies that in equilibrium more productive workers are visited with higher probabilities. Therefore, in general, there exists a number $K \leq N$ such that $\pi_i > 0$ for $i = 1, ..., K$ and $\pi_i = 0$ for $i > K$. That is, all workers $i = K+1, ..., N$ are *unemployable* in the sense that all firms assign zero probability to visiting them, even though they have positive surplus associated with them. All other workers $i = 1, ..., K$ are employable but, at the end of the day, will be either employed or unemployed according to the visit probabilities π_i. In general, for worker i, the probability of being unemployed is $(1-\pi_i)^M$.

The values of the equilibrium probabilities π_i can be found through the following algorithm.

Step 1. Set $K = N$ and use (1) together with the normalization condition

$$\sum_{i=1}^{K} \pi_i = 1 \tag{2}$$

to find

$$\pi_i = \frac{\sum_{j=1}^{K} (y_i/y_j)^{\frac{1}{M-1}} - (K-1)}{\sum_{j=1}^{K} (y_i/y_j)^{\frac{1}{M-1}}}. \tag{3}$$

Step 2. If $\pi_K > 0$ then (3) gives the equilibrium probabilities. Otherwise set $\pi_K = 0$, and $K = N-1$ and go to step 1.

Note that this process will eventually halt. Indeed, it is easy to see that for $K = 2$

$$\pi_i = \frac{y_i^{\frac{1}{M-1}}}{y_i^{\frac{1}{M-1}} + y_j^{\frac{1}{M-1}}} > 0, \text{ for } i = \overline{1,2},\ j \neq i. \tag{4}$$

In particular, formula (4) implies that in any equilibrium at least two workers are visited. Indeed, if all firms visit only the most productive worker, they are guaranteed to earn zero payoffs, while the deviator will earn payoff y_2. Therefore, all the firms visiting the same worker cannot be the equilibrium. Therefore, we have arrived at the following proposition.

Proposition 1 *The matching game with finite numbers of players possesses a symmetric mixed strategy equilibrium in which the K most productive workers are visited with positive probabilities, while the $(N-K)$ least productive workers are visited with probability zero. The number K satisfies $2 \leq K \leq N$ and can be found together with matching probabilities if one follows steps 1 and 2 described above.*

The expected number of matches is given by:

$$x = \sum_{i=1}^{N} [1 - (1 - \pi_i)^M]. \tag{5}$$

where the probabilities π_i are determined by the algorithm given above. This formula for the matching function holds true for any distribution of worker productivies, and any finite numbers of workers and firms. While this formula is useful for some purposes, it is not particularly amenable either to comparative static analysis of the effects of heterogeneity, or to econometric estimation. To find formulas that are more useful for these purposes, we will examine equilibria in limit large economies.

Before doing so, though, it is useful to identify the equilibrium entry condition. Given a fixed cost of entry $k > 0$, entry will occur up to the point where

$$(1 - \pi_i)^{M-1} y_i = (1 - \pi_j)^{M-1} y_j = k \tag{6}$$

for all i in which $\pi_i > 0$, as determined by the above algorithm.

3 The Model with Large Numbers of Agents

In this section, we analyze equilibrium allocations in the economy in the limiting case when $M, N \to \infty$ in such a way that their ratio $\phi = M/N$ remains fixed, and characterize the equilibrium matching process as a function of the underlying distribution of worker productivities. To do so, it is instructive to consider different cases, with increasing degrees of generality. We start with the simplest possible case, where there are only two possible values of y.

3.1 Binomial Productivity Distributions

Suppose that productivities of workers come from a binomial distribution over $\{y_H, y_L\}$ with $y_H \geq y_L > 0$ and $\Pr(y = y_H) = p$. We first prove that, given any fixed number M of vacancies, if there are sufficiently many high productivity workers in a symmetric equilibrium, low productivity workers are visited with zero probability. Let q denote the fraction of the workforce that is employable. There are, therefore, two cases to consider: $q = p$ when only high productivity workers are employable, and $q = 1$ when all are.

Let N_L and N_H be the number of low productivity and high productivity workers, respectively, in the sample. Hence, from the algorithm given above, the probability that a low productivity worker is visited is

$$\pi_L = \max(0, \frac{N_H(y_L/y_H)^{1/(M-1)} - (N_H - 1)}{N_H(y_L/y_H)^{1/(M-1)} + N_L}). \tag{7}$$

Therefore if

$$N_H \geq \frac{1}{1 - (y_L/y_H)^{1/(M-1)}} \tag{8}$$

then $\pi_L = 0$.

Dividing both sides of (8) by N, taking into account that

$$(y_L/y_H)^{1/(M-1)} = 1 + \frac{1}{M} \ln \frac{y_L}{y_H} + o_p(\frac{1}{M}) \tag{9}$$

and, noting that $\phi = M/N$, one obtains, after some rearrangement, that, asymptotically, $\pi_L = 0$ if and only if

$$p \geq \frac{\phi}{\ln(y_H/y_L)}. \tag{10}$$

In particular, if competition for workers is strong, i.e.

$$\phi > p \ln(y_H/y_L) \tag{11}$$

then $q = 1$ and all workers are approached at the equilibrium.[5]

We consider each case in turn. Assume, first, that condition (10) holds, so that $q = p$ and all low productivity workers are unemployable. The expected number of matches is given by

$$x(N) = \sum_{i=1}^{N_H} [1 - (1 - \pi_H)^M]. \tag{12}$$

Define the matching rate

$$\xi(p, \phi) = p \lim_{N \to \infty} \frac{x(N)}{N}. \tag{13}$$

Then we obtain the equilibrium matching function:

$$\xi(p, \phi) = p(1 - \exp(-\frac{\phi}{p})). \tag{14}$$

This matching function is a straightforward generalization of the urn-ball matching function discussed in Petrongolo and Pissarides (2001), and derived in directed search models with homogeneous populations in Julien, Kennes, and King (2000) and Burdett, Shi, and Wright (2001). Indeed, with a homogenous population ($p = 1$), one obtains the familiar function: $\xi(1, \phi) = (1 - \exp(-\phi))$ Notice that the more general matching function ((14), preserves the key properties of the function in the homogeneous case. In particular, in levels, it has constant returns to scale in M and N. We can also see, very clearly, the effects of heterogeneity on the number of matches. First, notice that the matching rate is independent of the values of y_L and y_H, it depends only on the relative numbers of each type of agent, summarized in the value of p. With heterogeneity, $p < 1$, and has two effects on the probability of employment. Low productivity workers become unemployable and are not visited at all in equilibrium, however the probability of employment conditional on being a high type, $1 - \exp(-\frac{\phi}{p})$, increases relative to the case with a homogenous population. The total effect of heterogeneity is, however, to decrease

[5] The same result can be obtained by assuming that every firm approaches each high productivity worker with probability $\pi_H = 1/N_H$ and requiring that it is unprofitable to deviate and approach a low productivity worker, i.e. that $(1 - 1/N_H)^{M-1} y_H > y_L$.

employment:

$$\frac{\partial \xi(p,\phi)}{\partial p} = 1 - \frac{\frac{\phi}{p}+1}{\exp(\frac{\phi}{p})} > 0 \tag{15}$$

and, therefore, $\xi(p,\phi)$ is maximized at $p = 1$.

Now let us consider the other case, where condition (8) , for a fixed population or (10) for the infinite population does not hold. In this case, $q = 1$ and both types of workers are approached with positive probabilities in the equilibrium. The visitation probabilities are given by:

$$\pi_L = \frac{N_H(y_L/y_H)^{1/(M-1)} - (N_H - 1)}{N_H(y_L/y_H)^{1/(M-1)} + N_L} \tag{16}$$

$$\pi_H = \frac{N_L(y_H/y_L)^{1/(M-1)} - (N_L - 1)}{N_L(y_H/y_L)^{1/(M-1)} + N_H}. \tag{17}$$

In the limit, with large values of M and N, one obtains

$$\pi_L = \frac{1}{N}(1 - \frac{N_H}{M}\ln\frac{y_H}{y_L}) + o_p(\frac{1}{N}) \tag{18}$$

$$\pi_H = \frac{1}{N}(1 + \frac{N_L}{M}\ln\frac{y_H}{y_L}) + o_p(\frac{1}{N}). \tag{19}$$

Let

$$\kappa = (1-p)(\frac{y_H}{y_L})^p + p(\frac{y_L}{y_H})^{1-p}. \tag{20}$$

The matching rate, in this case, is:

$$\xi(\phi,\kappa) = 1 - \kappa\exp(-\phi). \tag{21}$$

Thus, as before, the matching function has constant returns to scale in M and N but, in this case, is not independent of the values of y_L and y_H. The value of κ depends on the relative value y_L/y_H, and the proportion of workers with high productivity: p. It is easy to show that, for any value of p, κ is minimized where $y_L/y_H = 1$, that is, in the homogeneous case – in which case, $\kappa = 1$, and we recover the homogeneous matching rate as the special case:

$$\xi(0,\phi;y_H,y_L)=\xi(1,\phi;y_H,y_L)=\xi(p,\phi;y,y)=1-\exp(-\phi). \tag{22}$$

The function κ is a measure of the heterogeneity of the underlying distribution of worker productivities, and is strictly increasing in the distance between y_L and y_H. Thus, it is clear from (21) that heterogeneity reduces the matching rate in equilibrium.

3.1.1 Vacancy Entry

We now consider the properties of the equilibrium when vacancies can enter by paying a fixed cost $k>0$. Once again, there are two cases to consider: when some workers are unemployable ($q=p$) and when not ($q=1$) We start with the first case.

Entry with Unemployable Workers .

Here, from (6), we have:

$$(1-\pi_H)^{M-1}y_H=k \tag{23}$$

where $\pi_H=1/N_H$. In the large market, this becomes:

$$y_H\exp(-\phi/p)=k \tag{24}$$

Solving this equation for the tightness ratio ϕ, we get:

$$\phi=p\,(\ln y_H-\ln k) \tag{25}$$

Now, recall condition (10), which must hold when low productivity workers are unemployable. Re-rewriting this condition, we have:

$$\phi\le p\,(\ln y_H-\ln y_L) \tag{26}$$

The above two conditions imply the following:

$$y_L\le k \tag{27}$$

This tells us that low productivity workers will be unemployable if and only their productivity level is below the cost of vacancy creation. We summarize this result in a proposition.

Proposition 2 *In the binomial model with entry, a worker is employable if and only if $y > k$.*

Let us now consider, then, the case where all workers are employable; that is, when $q = 1$.

Entry without Unemployable Workers .

In this case, the entry condition (6) becomes:

$$(1-\pi_L)^{M-1} y_L = (1-\pi_H)^{M-1} y_H = k \tag{28}$$

where π_L and π_H are given in (18) and (19) respectively. From here, using (19), we have

$$y_H \left[1 - \frac{1}{N}(1 + \frac{N_L}{M} \ln \frac{y_H}{y_L})\right]^{M-1} = k \tag{29}$$

In the limit, this becomes

$$y_H \exp\left[-\phi - (1-p)\ln(y_H/y_L)\right] = k \tag{30}$$

Solving for ϕ, we get:

$$\phi = p(\ln y_H - \ln y_L) + (\ln y_L - \ln k) \tag{31}$$

Lemma 1 *In the binomial model with entry, if all workers are employable then, in equilibrium, $\partial\phi/\partial p > 0$, $\partial\phi/\partial y_H > 0$, $\partial\phi/\partial y_L > 0$, $\partial\phi/\partial k < 0$.*

Proof. This follows directly from (31). ∎

3.1.2 The Equilibrium Unemployment Rate

In general, the equilibrium unemployment rate is given by $1 - \xi$. The value of ξ depends on whether or not low productivity workers are unemployable.

When Low Productivity Workers are Unemployable In this case, $q = p$ and the matching rate ξ is given in (14). In this case, the equilibrium unemployment rate is:

$$U = 1 - p + p \exp(-\phi) \tag{32}$$

where ϕ is given in (25). Making the substitution, we have:

$$U = 1 - p + p \left(\frac{k}{y_H} \right) \tag{33}$$

Clearly, from (33), the unemployment rate has two components in this case. The first component, $1-p$, captures the fraction of workers who are unemployable. The second term captures those who are employable (in the sense that each firm assigns positive probability to approaching them), but unemployed due to the fact that, ex post, they were not approached by any firms. Note that, in the homogeneous case, when $p = 1$, then $U = k/y_H$, as is standard.

Proposition 3 *In the binomial model, if low productivity workers are unemployable ($q = p$) then equilibrium unemployment is increasing in k and decreasing in y_H and p.*

Proof. Follows directly from (33). ■

The effects of increases in k and y_H on unemployment are entirely straightforward. Increases in the entry cost k reduces profitability, entry, and, thus, increases unemployment. Increases in productivity y_H have precisely the opposite effect. Increases in p have two effects: they reduce the fraction of the workforce that is unemployable (driving down unemployment), but increase the fraction that are subject to the possibility of coordination unemployment. Since unemployable workers are unemployed with probability one, and employable workers are unemployed with probability less than one, the overall effect is negative.

When Low Productivity Workers are Employable In this case, $q = 1$, and the matching rate is given in (21). In this case, the equilibrium unemployment rate is:

$$U = \kappa \exp(-\phi) \tag{34}$$

Where $\kappa = \kappa(y_H, y_L, p)$ is given in (20) and $\phi = \phi(y_H, y_L, p, k)$ is given in (31).

Proposition 4 *In the binomial model, if low productivity workers are employable ($q = 1$) then equilibrium unemployment is increasing in k, decreasing in y_L and p. The effect of y_H on unemployment depends on the value of y_H/y_L. When $y_H \approx y_L$, unemployment is decreasing in y_H. Beyond a critical point of y_H/y_L, unemployment is increasing in y_H.*

Proof. From (34), using (20) and (31) we obtain:

$$\frac{\partial \ln U}{\partial k} = \frac{1}{k} > 0 \tag{35}$$

$$\frac{\partial \ln U}{\partial y_L} = -\frac{1-p}{\kappa y_L}(\kappa + p(\frac{y_H}{y_L})^p(1 - \frac{y_L}{y_H})) < 0 \tag{36}$$

$$\frac{\partial \ln U}{\partial p} = -\frac{1}{\kappa}(\frac{y_H}{y_L})^{p-1}(\frac{y_H}{y_L} - 1) < 0 \tag{37}$$

$$\frac{\partial \ln U}{\partial y_H} = \frac{p}{\kappa y_H}((1-p)(\frac{y_H}{y_L})^{p-1}(\frac{y_H}{y_L} - 1) - \kappa) \tag{38}$$

It is straightforward to show that $\partial \ln U/\partial y_H < 0$ when $y_H \approx y_L$. Also, this derivative changes sign when y_H/y_L crosses the value ζ, where ζ is determined as the unique solution to:

$$(1-p)(\zeta^p - \zeta^{p-1}) = \kappa. \tag{39}$$

■

These results are quite intuitive. The effects of increasing vacancy costs, k, are entirely straightforward. They do not affect heterogeneity (κ) but simply reduce the profits of entry, thus reducing entry, and thereby increasing unemployment, as in the previous case. Increases in the output of low productivity workers y_L have two effects, both of which work in the same direction on unemployment. The first is to increase average productivity, thus encouraging entry and reducing unemployment. Secondly, increases in y_L compress the distribution of worker productivities, reducing the measure of heterogeneity, κ, and thereby increasing the efficiency of the matching process and reducing unemployment. Increases in the fraction of workers with high productivity, p, also has two effects. The first, again, increases average productivity (thereby inducing further entry and reducing unemployment). The second affects the degree of heterogeneity (κ). It is straightforward to show that the function $\kappa(p)$ is inverse U-shaped, with minimum values of 1 at $p = 0$ and $p = 1$. However, the overall effect of any increase in p on unemployment is always negative.

Increases in the output of high productivity workers y_H also have two effects on unemployment. As with increases in y, these increase average worker productivity, and induce entry. However, when y_H increases, it spreads the distribution of productivity, and increases κ, reducing the efficiency of matching. These two effects move in opposite directions. When heterogeneity is small, the first effect dominates, and unemployment is decreasing in y_H. However, as heterogeneity increases, the second effect dominates the first, and further increases in y_H actually increase unemployment.

We now turn to consider more general skill distributions, where productivities are sampled from an arbitrary distribution with c.d.f. $F(\cdot)$. Since the set of piecewise constant distribution functions, which correspond to random variables with finite support, is everywhere dense in the set of all distribution cases, we will start with considering the case when the productivity can take only a finite set of values $\{y_1, ..., y_I\}$ with $\Pr(y = y_i) = p_i$ and then generalize to the case of the arbitrary distribution.

3.2 Distributions with Finite Supports

Consider the case when the set of all possible productivities is finite. Without loss of generality assume $y_1 > y_2 > ... > y_I$. Let N_i be the number of workers with productivity y_i and

$$\sum_{j=1}^{I} N_j = N. \tag{40}$$

Formula (3) implies:

$$\pi_i = \frac{\sum_{j=1}^{K} N_j (y_i/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{K} N_j + 1}{\sum_{j=1}^{K} (y_i/y_j)^{\frac{1}{M-1}}}, \tag{41}$$

where K is the lowest productive worker visited at equilibrium with positive probability. If

$$\sum_{j=1}^{I} N_j (y_I/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{I} N_j + 1 > 0, \tag{42}$$

then all types are visited with positive probability. Otherwise, K is found from

$$\sum_{j=1}^{K} N_j (y_K/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{K} N_j + 1 > 0 \geq \sum_{j=1}^{K+1} N_j (y_{K+1}/y_j)^{\frac{1}{M-1}} - \sum_{j=1}^{K+1} N_j + 1. \tag{43}$$

Taking the limit as population increases, and proceeding in the same way as above, one obtains:

$$\pi_i = \frac{1}{K}(1 - \sum_{j=1}^{K} \frac{N_j}{M} \ln \frac{y_j}{y_i}) + o_p(\frac{1}{N}). \tag{44}$$

In general there exists a critical productivity y^* such that workers are approached with positive probability if $y_i \geq y^*$ and are not approached otherwise. We will determine this productivity shortly. Given y^* for the large population, the fraction of workers who are employable is $1 - F(y^*)$, i.e.

$$p \lim_{N\to\infty} \frac{K}{N} = 1 - F(y^*), \tag{45}$$

which implies

$$\pi_i = \frac{1}{N}(\frac{1}{1 - F(y^*)} - \frac{E(\ln y | y \geq y_K) - \ln y_i}{\phi}), \tag{46}$$

provided $y_i > y_K$ and $\pi_i = 0$ otherwise.

The critical productivity y^* is determined by:

$$\frac{\phi}{1 - F(y^*)} = E(\ln y | y \geq y^*) - \ln y^*. \tag{47}$$

Example Let us calculate y^* for the binary productivity model above. First, assume that $y^* \in (y_L, y_H)$, i.e. low productivity workers are not approached in equilibrium. Then $1 - F(y^*) = p$ and

$$\frac{\phi}{p} = \ln y_H - \ln y^* \Leftrightarrow y^* = y_H \exp(-\frac{\phi}{p}). \tag{48}$$

Note that $y^* \in (y_L, y_H)$ if and only if condition (10) holds. Otherwise, $y^* \leq y_L$, which implies $1 - F(y^*) = 1$. Adding $0 = \ln y_L - \ln y_L$ to the right hand side of equation (47) and rearranging one obtains

$$\ln y^* = \ln y_L + (p \ln \frac{y_H}{y_L} - \phi), \tag{49}$$

which is consistent with the assumption that $y^* \leq y_L$.

3.3 Distributions with Continuous Supports

It is easy to see that formulae (46) and (47) continue to hold if the productivities are selected from an arbitrary distribution $F(\cdot)$ with support $[\underline{y}, \overline{y}]$, where $\infty > \overline{y} \geq \underline{y} \geq 0$. Indeed, given such a distribution there exists a sequence of piecewise constant distributions $\{F_n(\cdot)\}_{n=0}^{\infty}$ such that

$$\lim_{n \to \infty} F_n(y) = F(y), \tag{50}$$

for all y in the support of $F(\cdot)$. For every distribution $F_n(\cdot)$ formulae (46) and (47) imply

$$\pi_{in} = \frac{1}{N}(\frac{1}{(1 - F_n(y_n^*))} - \frac{E_n(\ln y | y \geq y_{Kn}) - \ln y_i}{\phi}), \tag{51}$$

provided $y_i > y_{Kn}$ and $\pi_i = 0$ otherwise. In the case of an absolutely continuous distribution with compact support , the difference between y_{Kn} and y_n^* disappears in the limit of a large population and the critical productivity y_n^* is determined by:

$$\frac{\phi}{1-F(y_n^*)} = E_n(\ln y|y \geq y_n^*) - \ln y_n^*. \tag{52}$$

Here E_n denotes expectation calculated using $F_n(\cdot)$. Taking the limit as $n \to \infty$ one obtains (46) and (47), where expectations are calculated using $F(\cdot)$.

Let us now find the matching function. To start, note that

$$\frac{x}{N} = \sum_{i=1}^{K} \frac{N_i}{N}\{1 - [1 - \frac{1}{N}(\frac{1}{1-F(y^*)} - \frac{E(\ln y|y \geq y^*) - \ln y_i}{\phi})]\}^{N\phi}. \tag{53}$$

Equation (53) can be rewritten as:

$$\frac{x}{N} = \sum_{i=1}^{K} p_i(1 - \exp(-\frac{\phi}{1-F(y^*)} + E(\ln y|y \geq y^*) - \ln y_i)) + \frac{1}{N}o_p(\frac{1}{N}). \tag{54}$$

Passing to the probability limit one obtains

$$\xi = (1 - F(y^*))(1 - E(\frac{1}{y}|y \geq y^*)\exp(-\frac{\phi}{1-F(y^*)} + E(\ln y|y \geq y^*)), \tag{55}$$

where, as before, y^* is defined as solution to the equation

$$\frac{\phi}{1-F(y^*)} = E(\ln y|y \geq y^*) - \ln y^*, \tag{56}$$

provided it is greater than $\underline{y}$ and $y^* = \underline{y}$ otherwise.[6]

Using equation (56) one can write (55) in a form

$$\xi = (1 - F(y^*))(1 - y^* E(\frac{1}{y}|y \geq y^*)), \tag{57}$$

which implies

$$\xi \geq 0. \tag{58}$$

[6] It is straightforward to check that, for the case of binary distribution above, formulae (55)-(56) reduce to (14) if condition (10) is satisfied and (21) otherwise.

It is useful to introduce, again, the notation q for the fraction of workers who are employable. In this case:

$$q = 1 - F(y^*), \tag{59}$$

Also, define:

$$\kappa = E(\frac{1}{y}|y \geq y^*)\exp(E(\ln y|y \geq y^*)). \tag{60}$$

Then the matching rate can be expressed as:

$$\xi = q(1 - \kappa\exp(-\frac{\phi}{q})). \tag{61}$$

Notice that this matching function, once again, has constant returns to scale in M and N, and collapses down to the familiar function for homogeneous workers. In this case, $q = \kappa = 1$. Outside of this special case, to be able to assess the effects of heterogeneity, we must first assess the magnitude of κ in the general case.

Proposition 5 *For any distribution of skills,* $\kappa \geq 1$*.*

Proof. Let us consider two individuals with Bernoulli utility functions:

$$u_1 = -\frac{1}{y},\ u_2 = \ln y, \tag{62}$$

respectively. Both utility functions belong to CRRA class with coefficients of relative risk aversion:

$$r(u_1) = 2,\ r(u_2) = 1, \tag{63}$$

therefore the first individual is more risk averse than the second. This implies that for any lottery L

$$CE(u_1, L) \leq CE(u_2, L), \tag{64}$$

where $CE(u, L)$ denotes the certainty equivalent of lottery L for the individual with Bernoulli utility function $u(\cdot)$. Moreover equality is obtained if and only if lottery L is degenerate (see, Mas-Colell, Whinston, and Green, 1995, Ch.6). Consider the following lottery L^*: draw value y from distribution $F(\cdot)$, if $y \geq y^*$ pay the individual y, otherwise discard y and make a new draw. Then one can write κ as:

$$\kappa = -Eu_1(L^*) * \exp(Eu_2(L^*)). \tag{65}$$

Using the definition of certainty equivalent:

$$Eu(L) = u(CE(u, L)) \tag{66}$$

one can transform (65) into:

$$\kappa = \frac{CE(u_2, L)}{CE(u_1, L)} \geq 1, \tag{67}$$

where the last inequality follows from (64). Moreover, $\kappa = 1$ if and only if all the population of employable workers is homogeneous. ■

Using this Proposition and (58) one can obtain the following bounds for the matching function:

$$0 \leq \xi \leq q(1 - \exp(-\frac{\phi}{q})) \leq 1 - \exp(-\phi), \tag{68}$$

where the second inequality holds as equality if and only if all the population of employable workers is homogeneous.

Formula (61) allows us to identify three effects of heterogeneity on the matching function in general. The *decreased employability effect* signified by the term q in front of the brackets works to depress total employment, due to the fact that some workers are now not approached at all. The *increased miscoordination effect,* signified by parameter κ, tends to depress employment due to the fact that all firms are more likely to approach the most productive workers. This effect tends to decrease probability of employment conditional on the worker been employable. Finally, the *decreased effective labor force effect* signified by q in the denominator of the exponential factor works to increase the conditional probability of employment conditional on being employable. As we argued above, the net effect of heterogeneity on the total matching probability is negative. The effect of heterogeneity on the conditional probability of employment conditional on being employable:

$$1 - \kappa \exp(-\frac{\phi}{q}) \tag{69}$$

is, however, ambiguous. It becomes unambiguously positive if $\kappa = 1$, i.e. if all the population of employable workers is homogeneous, and unambiguously negative if

$q = 1$, i.e. the entire population is employable.

For the binary distribution of skills considered above, $\kappa = 1$, $q = p$ if (10) is satisfied and $\kappa = (1-p)(\frac{y_H}{y_L})^p + p(\frac{y_L}{y_H})^{1-p}$, $q = 1$ otherwise.

Finally, one can extend results of this Section to distributions with unbounded support, since any distribution with unbounded support can be represented as a limit of distributions with compact support.

3.3.1 Vacancy Entry

We now consider the properties of the equilibrium when vacancies can enter, with cost $k > 0$.

Proposition 6 *With entry, $y^* = k$.*

Proof. With finite numbers of workers,for any worker i, entry continues until profits are driven to zero:

$$(1 - \pi_i (y_i))^{M-1} y_i = k \tag{70}$$

where

$$\pi_i = \frac{1}{N}(\frac{1}{q} - \frac{E(\ln y | y \geq y^*) - \ln y_i}{\phi}) \tag{71}$$

In the limit, this entry condition becomes:

$$\exp\left(-\phi\left(\frac{1}{q} - \frac{E(\ln y|y \geq y^*) - \ln y_i}{\phi}\right)\right) y_i = k \tag{72}$$

This implies:

$$\exp(-\phi/q) \exp(E(\ln y|y \geq y^*)) = k \tag{73}$$

Taking logarithms of both sides:

$$-\phi/q + (E(\ln y|y \geq y^*)) = \ln k \tag{74}$$

Rearranging, and recalling that $q = 1 - F(y^*)$, we have:

$$\frac{\phi}{1 - F(y^*)} = (E(\ln y|y \geq y^*)) - \ln k \tag{75}$$

If $k \geq \underline{y}$ then k is a feasible value of y^*. Comparing (75) with the definition of y^* given in (56), we have: $y^* = k$.

If, alternatively, $k \leq \underline{y}$. then k is not a feasible value of y^*. In this case, y^* hits the lower bound $\underline{y}$. ■

3.3.2 The Equilibrium Unemployment Rate

As before, the equilibrium unemployment rate is given by $U = 1 - \xi$. Making the substitution from (61), and rearranging slightly, we have:

$$U = 1 - q + q\kappa \exp(-\phi/q) \tag{76}$$

where q, κ, and ϕ are given in (59), (60), and (56), respectively.

With entry, since $k \geq \underline{y}$, from Proposition 6, we know that $y^* = k$. These three equations then become:

$$q = 1 - F(k) \tag{77}$$

$$\kappa = E(\frac{1}{y}|y \geq k) \exp(E(\ln y|y \geq k)). \tag{78}$$

$$\phi = (1 - F(k)) \left[E\left(\ln y|y \geq k\right) - \ln k\right] \tag{79}$$

Thus, the unemployment rate is purely a function of the cost of entry, k, and the distribution of worker productivities $F(\cdot)$. Examining (76) we can see that, as before, there are two qualitatively different types of unemployment in this model. The first term $(1-q)$ represents the fraction of the workforce that is unemployable. (Notice that this is increasing in k.) The second term $(q\kappa \exp(-\phi/q))$ represents the fraction of the workforce that is employable (q) but not employed *ex post*, due to the stochastic nature of the matching process. The size of this term depends not only on the size of q, but also on market tightness (ϕ) and the measure of heterogeneity (κ), all of which are functions of k and $F(\cdot)$.

Substituting (77), (78), and (79) into (76), and collecting terms, gives us:

$$U(k) = F(k) + (1 - F(k))\, E(\frac{1}{y}|y \geq k)k \tag{80}$$

Proposition 7 *Equilibrium unemployment increases with vacancy costs k.*

Proof. Notice that

$$(1-F(k))E(\frac{1}{y}|y \geq k) = \int_k^{\overline{y}} \frac{f(y)}{y} dy. \tag{81}$$

Therefore, from(80), we have

$$\frac{\partial U}{\partial k} = f(k) + \int_k^{\overline{y}} \frac{f(y)}{y} dy - k\frac{f(k)}{k} = \int_k^{\overline{y}} \frac{f(y)}{y} dy > 0, \tag{82}$$

Which proves the proposition. ■

The total effect of increases in k on unemployment is unambiguous, although expression (82) contains terms with different signs. Intuitively, an increase in k reduces the entry of vacancies in the usual way, increasing unemployment. However, when $y^* = k$, an increase in k will increase y^*. This has two effects. First, it increases the fraction of workers that are unemployable (thus increasing unemployment); second, it reduces the heterogeneity among those who are employable – pushing down the unemployment rate for those workers. Overall, however, the first two effects dominate the third.

Our next objective is to investigate how the equilibrium unemployment reacts to increase in risk. First, we must provide an appropriate definition of increase in risk for this environment.[7]

Definition 1 *Let w and z be random variables with non-negative support. We say that z is obtained from w by* ***a radial increase in risk*** *if there exists a variable ε such that w and ε and independent and*

$$z = w\varepsilon \tag{83}$$

Note that definition of ε implies that it has non-negative support. To get some intuition concerning the above definition it is useful to re-write formula (83) in

[7] We are grateful to John Quiggin for suggesting the appropriate definition of the "more risky" for this environment.

logarithms, i.e.

$$\log z = \log w + \log \varepsilon. \tag{84}$$

Formula (84) shows that $\log z$ is obtained from $\log w$ by combination of a shift and a mean-preserving spread in risk.[8] Two cases are of a particular interest. If

$$E(\log \varepsilon) = 0 \tag{85}$$

then $\log z$ is obtained by a mean preserving spread in risk from $\log w$. Since by Jensen's inequality

$$E(\log \varepsilon) \leq \log(E\varepsilon), \tag{86}$$

where equality holds if and only if the distribution of ε is degenerate, in that case $Ez \leq Ew$. Another interesting special case is that of mean preserving radial increase in risk, i.e. the case when $E\varepsilon = 1$. In that case

$$E(\log z) \leq E(\log w). \tag{87}$$

The term "increase in risk" is justified by observation that

$$Var(\log z) = Var(\log w) + Var(\log \varepsilon) \geq Var(\log w). \tag{88}$$

Observe, also that

$$Var(z) = Var(w)(E\varepsilon^2) + (Ew^2)Var(\varepsilon), \tag{89}$$

which implies that for a mean preserving radial increase in risk

$$Var(z) \geq Var(w). \tag{90}$$

Below we will also assume $\underline{y} = 0$ and $\overline{y} = 0$. Now we a ready to formulate our next proposition.

Proposition 8 *Let the random variable y denote distribution of workers skills and the random variable z be obtained from y by a non-degenerate radial increase in risk, i.e. z satisfies (83) for some random non-degenerate variable ε. Furthermore,*

[8] Note that if y and ε and independent, so are $\log y$ and $\log \varepsilon$.

assume both random variables y and z have probability densities and the following expectations: $E(\frac{1}{y})$, $E(\ln y)$, $E(\frac{1}{\varepsilon})$, $E(\ln\varepsilon)$ are finite. Then there exists $k^* > 0$ such that for any $k < k^*$ radial increase in risk will lead to increase in miscoordination, i.e. increase in κ.

Proof. Let $F(\cdot)$ be the cdf of random variable y and $H(\cdot)$ be the of random variable z. We will prove that

$$\kappa(H) \geq \kappa(F), \tag{91}$$

where in agreement with equation (67), $\kappa(Q)$ is defined by[9]

$$\kappa(Q) = \frac{CE(u_1, Q)}{CE(u_2, Q)}, \tag{92}$$

for any arbitrary distribution Q and

$$u_1 = \ln y, u_2 = -\frac{1}{y}. \tag{93}$$

First, consider the case $k = 0$.[10] Let $f(\cdot)$ be the probability density of y and $g(\cdot)$ be probability density of ε. Then $h(\cdot)$, the probability density of z, is given by:

$$h(y) = \int_0^\infty \frac{g(t)}{t} f(\frac{y}{t})dt. \tag{94}$$

To compute $\kappa(H)$ note that

$$\ln CE(u_1, H) = \int_0^\infty \ln y(\int_0^\infty \frac{g(t)}{t} f(\frac{y}{t})dt)dy. \tag{95}$$

[9] We the dependence of κ on the distribution explicit, below.

[10] Though this case is not interesting by itself, since $k = 0$ implies $\phi = \infty$ and $\xi = 1$, i.e. full employment, κ is still formally well defined. By continuity argument, we will be able to extend our result for the economically interesting case of small but positive entry costs.

To evaluate integral (95) let us make a change of variables: $y = zt$, then

$$\ln CE(u_1, H) = \int_0^\infty \ln z f(z) dz + \int_0^\infty \ln t g(t) dt = \ln CE(u_1, F) + \ln CE(u_1, G), \quad (96)$$

which implies

$$CE(u_1, H) = CE(u_1, F) * CE(u_1, G). \quad (97)$$

Similar logic implies

$$CE(u_2, H) = CE(u_2, F) * CE(u_2, G) \quad (98)$$

and therefore

$$\kappa(H) = \kappa(F) * \kappa(G), \quad (99)$$

where

$$\kappa(G) = \frac{CE(u_1, G)}{CE(u_2, G)} > 1. \quad (100)$$

The above argument implies that

$$\kappa(H) > \kappa(F) \quad (101)$$

for $k = 0$. Since equation (64) implies that $\kappa(Q)$ is continuous in k for any random variable that has a density, the inequality will hold for sufficiently small k and the proposition follows. ■

So far we have studied the effect of a radial increase in risk on the measure of miscoordination, κ. Note that this effect is independent on expectation of ε. To study its effect on unemployment one must also study how it affects ϕ. This is summarized in the following Lemma.

Lemma 2 *Let the random variable y denote the distribution of worker productivities and random variable z be obtained from y by a non-degenerate radial increase in risk, i.e. z satisfies (83) for some random non-degenerate variable ε with*

$$E \ln \varepsilon < 0. \quad (102)$$

Furthermore, assume both random variables y and z have probability densities and *the expectations $E(\ln y)$, $E(\ln z)$.are finite. Then there exists $k^{**} > 0$ such that for all $k < k^{**}$ radial increase in risk will lead to decrease in the equilibrium firm to worker ratio ϕ.*

Note that condition (102) will be satisfied for a mean-preserving radial increase in risk.

Proof. Formula (84) applied to the pair of random variables z and y implies that $Ez < Ey$. Since

$$\omega(k) = E(\ln w | w > k) \tag{103}$$

is continuous in k for any random variable w with density and

$$\omega(0) = E(\ln w) \tag{104}$$

is finite for $w \in \{y, z\}$ there exists $k^{**} > 0$ such that for all $k < k^{**}$

$$E(\ln z | z > k) > E(\ln y | y > k) \tag{105}$$

and the Proposition follows from (79). ■

Notice that, when all workers are employable: $k \leq \underline{y}$, $F(k) = 0$, $q = 1$, and

$$\kappa = E_F(\frac{1}{y}) \exp(E_F(\ln y)). \tag{106}$$

$$\phi = E_F (\ln y) - \ln k \tag{107}$$

$$U = E_F(\frac{1}{y})k, \tag{108}$$

where we made explicit the dependence of expectation on distribution F. Let G be obtained from F by a mean preserving spread of risk. Then

$$E_G(\frac{1}{y}) > E_F(\frac{1}{y}) \text{ and } E_G(\ln y) > E_F(\ln y) \tag{109}$$

Thus, such a transformation increases our measures of heterogeneity, entry, and unemployment. The general effect of an increase in heterogeneity on unemployment is summarized in the next Proposition.

Proposition 9 *Let random variable y denote distribution of workers skills and random variable z be obtained from y by a non-degenerate radial increase in risk, i.e. z satisfies (83) for some random non-degenerate variable ε with*

$$E \ln \varepsilon < 0. \tag{110}$$

Furthermore, assume both random variables y and z have probability densities and the expectations $E(\frac{1}{y})$, $E(\ln y)$, $E(\frac{1}{\varepsilon})$, $E(\ln \varepsilon)$ are finite Then there exists $\overline{k} > 0$ such that for all $k < \overline{k}$ radial increase in risk will lead to increase in unemployment

Proof. Let us choose k^*, k^{**} from the proofs of Propositions 8 and 9 respectively, set $\overline{k} = \min(k^*, k^{**})$. Then the Proposition follows from (79). ■

In particular, the Proposition implies that an average productivity preserving radial increase in heterogeneity of workers' skills will lead to an increase in unemployment for sufficiently low entry costs.

Corollary 1 *Under the assumptions of Proposition 9 there exists ε such that $E\varepsilon > 1$ but unemployment increases for sufficiently small k. That is, it is possible for increases in average productivity to increase equilibrium unemployment.*

Proof. It is sufficient to provide an example. Consider $\ln \varepsilon \sim N(-1, 2)$. Then $E \ln \varepsilon = -1$ and $E\varepsilon = e > 1$.Then the result follows from Proposition 9. ■

Thus, as in the binomial case considered above, circumstances exist when increases in the average productivity of workers may increase the equilibrium unemployment rate, despite the implied increase in vacancy entry, due to the increase in the visit probabilities placed on highly productive workers and the consequent reduction in the visit probabilities of less productive workers. However, at this point, it is worthwhile to stress the efficiency of these equilibria.

4 Constrained Efficiency

In this Section we identify the allocations that are constrained efficient, for given M and N and compare them to the equilibrium matching found in Section 2.. First, notice that unconstrained efficiency requires that M most productive workers should be matched to the firms. In particular, if $M > N$ ($\phi > 1$) unconstrained efficiency requires zero unemployment rate. Such matching can be easily achieved if the firms are allowed to move sequentially. If firms move simultaneously, however, such an allocation is not feasible. We will employ the following notion of constrained efficiency. Assume each firm send a request for a worker to a clearing house. The clearing house send request to worker i with probability π_i. A worker who receives just one request accepts the job at zero wage, a worker who receives more than one request accepts a job and bids her wage up to her productivity, while the worker who gets no requests remain unemployed.

We assume that the clearing house cannot keep track of the workers who already received requests. Therefore, the only control variables the clearing house has are probabilities π_i, which are selected to maximize the social surplus of the matches, i.e. the clearing house solves:

$$\begin{array}{c} \max\limits_{\pi} \sum\limits_{i=1}^{N} (1-(1-\pi_i)^M) y_i \\ s.t. \sum\limits_{i=1}^{N} \pi_i = 1, \pi_i \geq 0 \end{array} , \tag{111}$$

where $\pi = (\pi_1, ..., \pi_N)$ is the vector of probabilities.

Definition 2 *Matching probabilities are constrained efficient if they solve problem (111).*

The Lagrangian for this problem is:

$$L = \sum_{i=1}^{N} (1-(1-\pi_i)^M) y_i - \lambda (\sum_{i=1}^{N} \pi_i - 1) + \sum_{i=1}^{N} \mu_i \pi_i. \tag{112}$$

The Kunh-Tucker conditions are

$$\left\{ \begin{array}{c} M(1-\pi_i)^{M-1} y_i - \lambda + \mu_i = 0 \\ \sum\limits_{i=1}^{N} \pi_i = 1, \ \pi_i \geq 0, \ \mu_i \geq 0 \\ \mu_i \pi_i = 0 \end{array} \right. . \tag{113}$$

To analyze system (113) notice that

$$\frac{\partial^2 L}{\partial y_i \partial \pi_i} = M(1-\pi_i)^{M-1} > 0, \tag{114}$$

therefore efficient probabilities are increasing in productivities. Therefore, there exists $K \leq N$ such that $\pi_i > 0$ for $i = 1, ..., K$ and $\pi_i = 0$ for $i > K$.

Lagrange multiplies are given by:

$$\begin{cases} \lambda = M(1-\pi_K)^{M-1} y_K \\ \mu_i = 0 \text{ for } i = \overline{1, K} \\ \mu_i = M[(1-\pi_K)^{M-1} y_K - y_i] > 0 \text{ for } i = \overline{K+1, N} \end{cases}. \tag{115}$$

Notice that the constraint efficient probabilities coincide with the equilibrium matching probabilities. That is, equilibrium matching is constrained efficient. This constrained efficiency result holds true even with finite numbers of agents, due to the particular wage determination mechanism we use in this paper. It remains true in the limiting large economy, and with entry.[11]

5 Empirical Evidence

In this section we take some of the implications of our theory directly to the data. The theory provides us with the following specification for a matching function, equation (61):

$$\xi = q\left(1 - \kappa \exp\left(-\frac{\phi}{q}\right)\right)$$

where q represents the per cent of employable workers, κ is the measure of heterogeneity, and ϕ is the vacancy-unemployment rate.

A simple test of the theory is to solve (61) for κ to obtain

$$\kappa_t = \exp(\frac{\phi_t}{q_t})\frac{q_t - \xi_t}{q_t} \tag{116}$$

and to compare each κ_t with one.

[11] These last two points are very standard in directed search models, and the proof is omitted here. Details are available from the authors upon request.

Alternatively, one can also estimate the model, treating κ as a parameter. Assuming that κ is constant requires making some ad hoc assumptions on the distribution; however, the exercise is still useful, since it allows us to construct a confidence interval for a weighted average value of κ and test formally that it is greater than one. The exercise also proves useful for countries where the quality of individual data is poor and individual observation of κ_t are to noisy for any useful comparisons with theory to be made.

5.1 Treating κ as a Variable

The theory suggests that κ is a function of the distribution of worker productivity and costs of vacancy creation and potentially time varying. If we take the long-term unemployment to unemployment ratio as the ratio of the unemployable workers in the economy it is possible to use equation (116) to generate the implied value of κ given the observed data. Doing so gives us an idea of how κ_t varies over time. We conduct this exercise for the US, using the available data from 2001Q1-2007Q1. Our data comes from a variety of sources: Shimer (2007) uses the Current Population Survey to extract an estimate for the rate at which workers transition from unemployment to employment adjusted for time aggregation.[12] The number of vacancies is taken from the JOLTS data set and unemployment data is available from the Bureau of Labor Statistics. Figure 1 illlustrates the series $\{\kappa_t\}$ generated from this exercise. Clearly, although there is significant variation in κ over time we find that it exceeds a value of one in all periods, in accordance with the theory.

In principle, the standard errors of κ can be estimated from

$$d\kappa_t = h_\phi d\phi_t + h_\xi d\xi_t + h_q dq_t, \tag{117}$$

where

$$h(\phi_t, \xi_t, q_t) = \exp(\frac{\phi_t}{q_t})\frac{q_t - \xi_t}{q_t} \tag{118}$$

and $d\phi_t$, $d\xi_t$, dq_t are the measurement errors.

[12] This data was constructed by Robert Shimer. For additional details, please see Shimer (2007) and his webpage http://home.uchicago.edu/ shimer/data/flows/.

5.2 Treating κ as Parameter

We begin with an examination of the matching function in the United States. The model in this paper is static, but when estimating a directed search matching function it is necessary to take a stand on the length of a time period during which firms can attract workers. For the purposes of our estimation, we set a period equal to one month. That is, firms are allowed to approach one worker on a monthly basis.[13] Given the instantaneous rate of matching derived by Shimer (2007), we are able to extract the probability that a worker forms a job match within a particular month.

With our available data, we are able to estimate the parameters of our matching function using non-linear least squares. There are two possible methods of proceeding. First, we can take the ratio of long term unemployed as a proxy for size of the unemployable population $1 - q$, and allow this to vary over time, according to the data. [14] Alternatively, we could treat q as another parameter. We consider each approach in turn.

5.2.1 Treating q as a Variable

There are potentially two sources of error in this procedure. Some of the long-term unemployed workers may actually be employable and some of the short-term unemployed may be unemployable.[15] The calibration of q seems a reasonable activity to undertake for a couple of reasons. First, it may reduce the standard errors associated with our variable of prime interest, κ. Second, our model does not restrict q to be constant over time.

Thus, initially, we calibrate q_t using unemployment by duration data. Then κ

[13] Selecting a shorter time period will lower the probability of finding a job within a period. In the data, this lower probability of job finding is equivalent to an increase in frictions and will tend to increase our estimates of κ providing stronger support for a model with heterogeneity relative to a model with worker homogeneity. We do not see selecting a longer time period than a month as reasonable. In particular, we expect firms to be able to send out applications at a rate of at least one per month.

[14] For long-term unemployed we take workers who have been unemployed for six months or longer. For European countries we take workers who have been unemployed for a year or longer as a measure of unemployable. These differences reflect data constraints.

[15] It is difficult to know how important these biases are. We do know that given the job finding rates observed in the data, the probability that a worker with the average rate of job finding remains unemployed after six months is about 10 per cent.

is estimated as the solution to the following problem:

$$\min_{\kappa} \sum_{t=0}^{T} \left(\xi_t - q_t \left(1 - \kappa \exp \left(-\frac{\phi_t}{q_t} \right) \right) \right)^2 .$$

It is straightforward to show that the estimate, $\widehat{\kappa}$ will satisfy:

$$\widehat{\kappa} = \frac{\sum_{t=0}^{T} q_t^2 \kappa_t}{\sum_{t=0}^{T} q_t^2}, \tag{119}$$

where κ_t is the true value of time series κ at period t, given by (116). Our theory predicts that $\widehat{\kappa} > 1$ The results are provided in Table 1 and confirm that prediction. This result is strongly statistically significant and supportive of the presence of worker heterogeneity:

Table 1: Estimation of κ

Parameter	κ
Estimate	1.26410
Std Error	0.02843

5.2.2 Treating q as a Parameter

Here, we attempt to estimate both κ and q as parameters. That is, they are selected to solve the following problem:

$$\min_{\{\kappa,q\}} \sum_{t=0}^{T} \left(\xi_t - q \left(1 - \kappa \exp \left(-\frac{\phi_t}{q} \right) \right) \right)^2 . \tag{120}$$

However, if true value of q differs from period to period this procedure will produce is downward biased estimate for κ under reasonable assumptions. To see it, note that solving problem (120) leads to

$$\widehat{\kappa} = \frac{1}{T}\sum_{t=0}^{T}\widetilde{\kappa}_t \tag{121}$$

$$\widehat{q} = \frac{\sum_{t=0}^{T} w_t(\widehat{q})q_t}{\sum_{t=0}^{T} w_t(\widehat{q})} \tag{122}$$

$$w_t(\widehat{q}) = (1-\widehat{\kappa}\exp(-\frac{\phi_t}{\widehat{q}}))(1-\widehat{\kappa}\exp(-\frac{\phi_t}{\widehat{q}}))(1+\frac{\phi_t}{\widehat{q}}) \tag{123}$$

$$\widetilde{\kappa}_t = \exp(\frac{\phi_t}{\widehat{q}})\frac{\widehat{q}-\xi_t}{\widehat{q}} = h(\phi_t,\xi_t,\widehat{q}). \tag{124}$$

Note that if value of $\widehat{\kappa}$ is not too big then all weights $w_t(\widehat{q})$ are positive. Moreover, for reasonable realizations of q_t and ϕ_t[16] function $h(\phi_t,\xi_t,\cdot)$ is convex. Therefore

$$\widetilde{\kappa}_t = h(\phi_t,\xi_t,\widehat{q}) \leq \frac{\sum_{t=0}^{T} w_t(\widehat{q})\kappa_t}{\sum_{t=0}^{T} w_t(\widehat{q})}, \tag{125}$$

where true value κ_t is defined by equation (116). Therefore, the estimate for the average value of κ is downward biased, which is likely to cause a downward bias for $\widehat{q}$. The estimation below, which produces value of $\widehat{\kappa}$ below one and significantly underestimates $\widehat{q}$ confirms these observations.

Table 2: Estimation of κ and q

Parameter	κ	q
Estimate	0.84106	0.32144
Std Error	0.06438	0.00794

[16] One needs $2q_t > \phi_t$

5.3 Cross-country Evidence

The above results outlines the evidence from the United States. It is possible to examine other countries. Shimer (2007) and Elsby, Hobijn and Sahin (2008) illustrate how to use data on unemployment duration to infer job finding probabilities. We collect data on unemployment by duration and job vacancies for a set of European countries from the Eurostat Database. This allows us to extract estimates of the matching rate for the following European countries: Finland, Greece, the Netherlands, Portugal, Slovenia and the United Kingdom.[17]

Table 3: Estimation of κ

Parameter	κ
Finland	1.28507 (0.03614)
Greece	1.18829 (0.03998)
Netherlands	2.2661 (0.2255)
Portugal	1.006612 (0.008548)
Slovenia	1.20873 (0.02292)
United Kingdom	2.59312 (0.09025)

We examine the case where the ratio of short-term to total unemployment is used as a measure of employability. The results are provided in Table 3 with standard errors provided in brackets. Under this specification the cross-country data is relatively in suggesting that $\kappa > 1$ with the exception of Portugal where the implied value of κ is consistent with a directed search model of the labour market, but without heterogeneity in worker productivity.

We repeat our previous analysis and estimate using the implied value of κ and q for a broad range of countries. These results are displayed in Table 4. Here the results are more mixed. For a set of countries the estimated value of κ remains greater than one but for a Portugal and Greece, κ is estimated to be less than one. A feature of this specification is that the percentage of employable workers in most specifications remains surprisingly small.

[17] Our choice of countries is generally restricted by the necessity of having vacancy data and unemployment by duration data for a number of years. Spain was removed from the sample due to the fact that the definition of a vacancy changes in mid-sample.

Table 4: Cross-country evidence on κ and the Fraction of Employable Workers

Country	κ	q
Finland	1.107046 (0.120810)	0.138962 (0.004186)
Greece	0.290263 (0.310822)	0.060405 (0.003556)
Netherlands	50.921362 (15.943657)	0.081429 (0.004255)
Portugal	0.71082 (0.10850)	0.08995 (0.01250)
Slovenia	1.218025 (0.444972)	0.073154 (0.009908)
United Kingdom	9.494712 (3.451678)	0.129374 (0.003291)

5.4 Directed Search versus Cobb-Douglas Matching Function

In much of the applied labour search literature, a reduced form Cobb-Douglas matching function is often used. The advantages of this specification is that it fits the data well but this comes at the expense of not having explicit microfoundations. As noted by Petrongolo and Pissarides (2001), the standard directed search matching function tends to perform poorly relative to the Cobb-Douglas matching function. These matching functions are non-nested so it is difficult to compare to these models directly. Returning to the US data, Figure 2 presents how the predicted values of the different models compare to the actual data.

There are a couple of points of interest. First, the directed search model with worker production heterogeneity performs significantly better in matching the data than a model without heterogeneity. However, relative to a Cobb-Douglas matching function, the elasticity of the job finding rate with respect to the vacancy unemployment rate is too large. To show this, Figure 3 illustrates the observed response of the job finding rate to changes in the vacancy-unemployment rate and the fitted values implied by our different models for the United States. The elasticity of job finding in response to changes in the vacancy-unemployment rate is too high in directed search models.

It is straightforward to take our implied matching functions and derive the elasticity of matching with respect to the ϕ. Doing so we find,

$$\epsilon_{DS,1} = \frac{\phi}{\exp(\phi) - 1} \tag{126}$$

$$\epsilon_{DS,2} = \frac{\phi\kappa}{q(\exp(\frac{\phi}{q}) - \kappa)} \tag{127}$$

where ϵ is the elasticity of the job finding rate with respect to ϕ with subscript $DS, 1$ corresponding to the case of directed search with homogenous workers and $DS, 2$ with heterogeneous workers. Note that when $\kappa \to 1$ and $q \to 1$ that $\epsilon_{DS,2} \to \epsilon_{DS,1}$.

Over our short sample the average vacancy-unemployment rate is just over 0.5 in the United States. From the data, the estimated elasticity of job finding with respect to ϕ is 0.37. For the directed search model with homogenous workers, the elasticity is 0.77when $\phi = 0.5$. We can also think about how worker heterogeneity will affect this elasticity. There are two effects. First,introducing worker heterogeneity may increase κ above one which tends to increase elasticity. This in itself tends to increase the elasticity. Second, it generates unemployable workers that may reduce q below one, which may reduce the elasticity. In our unrestricted regression of κ and q we find $\kappa = 0.82$ and $q = 0.4$. This combination of (κ, q) implies from (6) an elasticity of 0.38 which matches the data well - but only by having a value of $\kappa < 1$.

6 Conclusions and Extensions

In this study we have explored the implications of worker heterogeneity on matching and unemployment when firms direct their search. We have used the simplest model possible in the sense that the environment is static, heterogeneity is one-dimensional, and firms are homogeneous. However, we have allowed for considerable generality with regard to the distribution of worker productivities. By doing so, we have identified conditions under which some proportion of the workforce will not be approached in equilibrium. Plausibly, these workers, which we call "unemployables" correspond, in the real world to long-term unemployed workers. They are qualitatively different from workers who will be approached with positive probability, but are unlucky *ex post*, and, thus, simply unemployed. In equilibrium, unemployment is a prospect facing workers of all levels of skill –

but only those workers at the lower end of the distribution are unemployable, and have no realistic hope of being employed.

We also found that a simple equilibrium matching function emerges in this environment, which is a generalization of the matching function that comes out of directed search models with homogeneity, but has two extra terms: the fraction of unemployable workers and a statistic, κ, that measures the degree of heterogeneity in the distribution. This allowed us to consider the effects of this heterogeneity on the equilibrium matching rate and unemployment, in a relatively straightforward way. In particular, we found that, under certain conditions, an increasing the productivity of high-skilled workers will increase the equilibrium unemployment rate – despite the fact that it encourages entry by firms. Finally, we tested one of the key predictions of the theory ($\kappa \geq 1$) and found support for this in the data.

Several immediate extensions come to mind, which we find interesting. First, in a dynamic version of this model, with aggregate shocks, one might expect that the fraction of workers who are unemployable would vary over the cycle – and, potentially, with more volatility than the fraction of workers who are simply not lucky enough to find work. Secondly, introducing firm heterogeneity alongside worker heterogeneity seems worthy of study. Finally, allowing for investments on either side of the market, and thus, potentially endogenous heterogeneity clearly would be of interest.

References

[1] Albrecht, James, Pieter Gautier, Serene Tan, and Susan Vroman (2004), "Matching with Multiple Applications Revisited", *Economics Letters*, 84, 311-314.

[2] Albrecht, James, Pieter Gautier, and Susan Vroman, (2006) "Equilibrium Directed Search with Multiple Applications", *Review of Economic Studies*, 73, 869-891.

[3] Burdett, Ken, Shouyong Shi and Randall Wright, (2001) "Pricing and Matching with Frictions", *Journal of Political Economy*, 107, 1060-1085.

[4] Camera, Gabriele, and Cemil Selcuk, (2009), "Price Dispersion with Directed Search", *Journal of the European Economic Association*, 7, 1193-1224.

[5] Elsby, M., B. Hobijn, and A. Sahin. (2008), "Unemployment Dynamics in the OECD", *NBER* Working Paper 14617, 2008.

[6] Galenianos, Manolis, and Philipp Kircher, (2009), "Directed Search with Multiple Job Applications", *Journal of Economic Theory*, 144, 445-471.

[7] Julien, Benoit, John Kennes and Ian King (2000) "Bidding for Labor", *Review of Economic Dynamics*, 3, 619-649.

[8] Julien, Benoit, John Kennes and Ian King (2005), "Ex Post Bidding and Efficient Coordination Unemployment", *Canadian Journal of Economics*, 38, 174-193.

[9] Julien, Benoit, John Kennes and Ian King (2006) "The Mortensen Rule and Efficient Coordination Unemployment", *Economics Letters*, 90, 149-155

[10] Kircher, Philipp, (2009), "Efficiency of Simultaneous Search", *Journal of Political Economy*, 117, 861-913.

[11] Mas-Colell, Andreu, Michael Whinston, and Jerry Green (1995), *Microeconomic Theory*, Oxford University Press: New York.

[12] Montgomery, James, (1991) "Equilibrium Wage Dispersion and Interindustry Wage Differentials", *Quarterly Journal of Economics*, 106, 163-179.

[13] Peters, Michael, (1984), "Equilibrium with Capacity Constraints and Restricted Mobility", *Econometrica*, 52, 1117-1129.

[14] Petrongolo, Barbara, and Christopher. Pissarides, (2001) "Looking into the Black Box: A Survey of the Matching Function", *Journal of Economic Literature,* 39, 390-431

[15] Shimer, Robert, (2007), "Reassessing the Ins and Outs of Unemployment. *NBER* Working Paper 13421.

[16] Stevens, Margaret (2007), "New Microfoundations for the Aggregate Matching Function", *International Economic Review*, 48, 847-868.

Figure 1

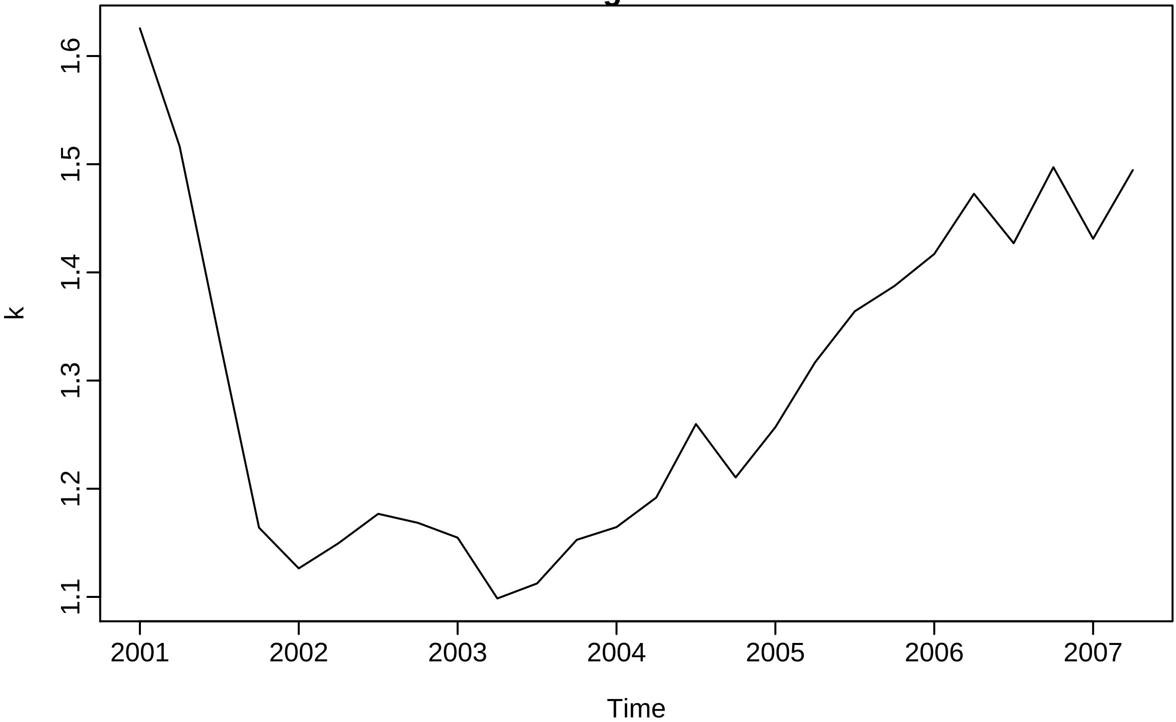


Figure 2

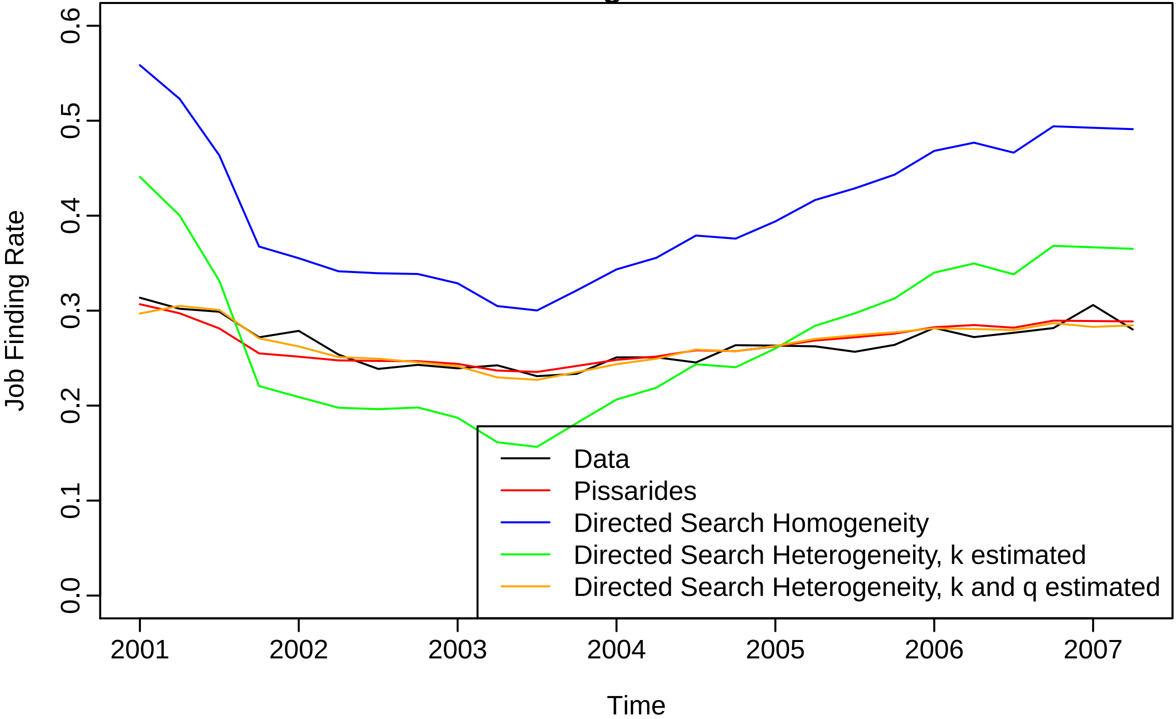
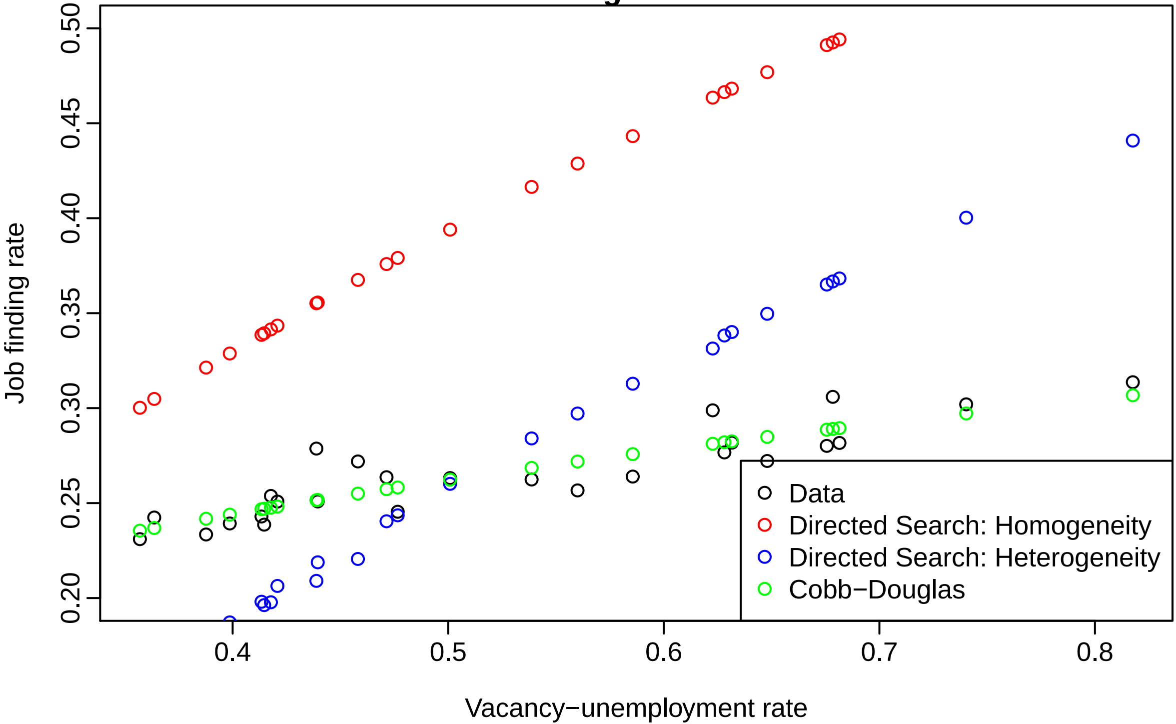

Job Finding Rate
0.0
0.1
0.2
0.3
0.4
0.5
0.6
2001
2002
2003
2004
2005
2006
2007
Time
Data
Pissarides
Directed Search Homogeneity
Directed Search Heterogeneity, k estimated
Directed Search Heterogeneity, k and q estimated

Figure 3



Job finding rate
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.4
0.5
0.6
0.7
0.8
Vacancy−unemployment rate
Data
Directed Search: Homogeneity
Directed Search: Heterogeneity
Cobb−Douglas